

Graphs with maximum degree 6 are acyclically 11-colorable

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Abstract

An acyclic k -coloring of a graph G is a proper vertex coloring of G , which uses at most k colors, such that the graph induced by the union of every two color classes is a forest. In this note, we prove that every graph with maximum degree six is acyclically 11-colorable, thus improving the main result of [12].

1 Introduction

A *proper vertex coloring* of a graph $G = (V, E)$ is an assignment of colors to the vertices of G such that two adjacent vertices do not use the same color. A proper vertex coloring of a graph G is *acyclic* if G contains no bicolored cycles; in other words, the graph induced by every two color classes is a forest. The *acyclic chromatic number* of G , denoted by $\chi_a(G)$, is the smallest integer k such that G is acyclically k -colorable. Acyclic colorings were introduced by Grünbaum [6], who proved that every planar graph is acyclically 9-colorable and conjectured that 5 colors are enough. Mitchem [9] reduced the number of colors to 8, Albertson and Berman [2] to 7 colors and Kostochka [8] to 6 colors. Finally, in 1979, Borodin [3] proved that 5 colors are enough. This bound is tight since there exist 4-regular planar graphs [6] which are not acyclically 4-colorable. Concerning graphs with bounded maximum degree, Alon *et al.* [1] proved that asymptotically every graph with maximum degree Δ is acyclically colorable with $O(\Delta^{4/3})$ colors; moreover they exhibited graphs with maximum degree Δ with acyclic chromatic number at least $\Omega(\Delta^{4/3}/(\log \Delta)^{1/3})$. For small maximum degrees, it was proved by Skulrattanakulchai [10] that 4 colors are enough to acyclically color graphs with maximum degree 3 (this bound is tight because of K_4 , where K_n denotes the complete graph with n vertices). In 1979, Burstein [4] proved that every graph with maximum degree 4 is acyclically 5-colorable (this bound is tight because of K_5). It was proved by Fertin and Raspaud [5] that every graph of maximum degree 5 can be acyclically colored with 9 colors. Recently, this result was improved by Yadav *et al.* [11] who reduced the number of colors to 8, then by Kostochka (personal communication) who proved that 7 colors are enough. Yadav *et al.* [11] also proved that every graph of maximum degree 6 can be acyclically colored with 12 colors [12]. Here we improve this last result by proving that:

Theorem 1 *Every graph with maximum degree six is acyclically 11-colorable.*

The proof of this result is based on Lemma 1 (see below) and the existence of a certain kind of spanning trees in regular graphs that we call “good spanning trees”.

Lemma 1 *Every graph G with maximum degree at most 6 and minimum degree strictly less than 6 is acyclically 11-colorable.*

We now introduce some notations. The following terminology was introduced in [5]. A partial acyclic coloring of G is a coloring φ of a subset S of V such that φ is an acyclic coloring of

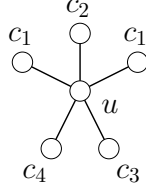


Figure 1: The list $L_u = (2, 1, 1, 1)$

$G[S]$ (the subgraph induced by S). A partial acyclic coloring using at most k colors is said to be a partial acyclic k -coloring of G . Let φ be a partial acyclic k -coloring of G and let v be an uncolored vertex of G . We say that a color c for v allows us to extend φ if the partial coloring φ' defined by $\varphi'(u) = \varphi(u)$ for any $u \in S$ and by $\varphi'(v) = c$ is a partial acyclic k -coloring of G . For a vertex $u \in V \setminus S$, we denote the set of colored neighbors of u by $N_c(u) = N(u) \cap S$ (where $N(u)$ is the set of the neighbors of u , i.e. $N(u) = \{v \in V(G) : uv \in E(G)\}$) and $\#cn(u) = |N_c(u)|$. We denote by $SC(N_c(u))$ the set of colors used by vertices in $N_c(u)$ and $\#dcn(u) = |SC(N_c(u))|$. Given a vertex u and a color c , let $n_c(u)$ be the number of vertices in $N_c(u)$ colored with the color c . For each vertex u , we set $L_u = (n_1, n_2, \dots, n_{\#dcn(u)})$ where each n_i denotes the number of times that a color appears in the neighborhood of u and $n_1 \geq n_2 \geq \dots \geq n_{\#dcn(u)}$. For example in Figure 1, we have: all the neighbors of u are colored (thus $\#cn(u) = 5$ and $N_c(u) = N(u)$), $SC(N_c(u)) = \{c_1, c_2, c_3, c_4\}$, $\#dcn(u) = 4$ and $L_u = (2, 1, 1, 1)$ (two neighbors colored c_1 , one colored with resp. c_2, c_3, c_4). Finally, we denote by $\Delta(G)$ and $\delta(G)$, the maximum and the minimum degree of the graph G , respectively. We use $\llbracket 1; n \rrbracket$ to denote the set of integers $\{1, 2, \dots, n\}$.

Section 2 is dedicated to the proofs of our results. In Section 3, we conclude with some questions related to the acyclic coloring of graphs with maximum degree six.

2 Proofs

The proof of Theorem 1 is divided in two cases depending on the minimum degree of the graph G :

- (1) Either $\delta(G) < 6$ and thus Lemma 1 allows us to find an acyclic coloring of G (Section 2.1),
- (2) or $\delta(G) = 6$ and thus the existence of a “good spanning tree” (Section 2.2) permits us to find an ordering of the vertices that allows to color the whole graph (Section 2.3).

Note that most of the proof techniques appears in [5, 10, 11, 12].

2.1 Proof of Lemma 1

In this section, we prove that:

If G is connected with $\Delta(G) \leq 6$ and $\delta(G) < 6$, then G is acyclically 11-colorable.

Let v be a vertex of degree $d(v) < 6$. Let T be a spanning tree of G rooted at v . We first define a post order walk on T , denoted by \prec , and then we color the vertices of G according to this order \prec . Let x_1, \dots, x_n be the vertices of G such that for every i, j , $1 \leq i < j \leq n$, $x_i \prec x_j$ and $x_n = v$. Observe that for all $1 \leq i \leq n$, x_i has at most five neighbors x_j with $j < i$. We will color the x_i 's successively using Lemmas 2, 3 and 4. The obtained coloring will be an acyclic 11-coloring of G .

We begin by an observation which is a follow-up of Observation 1 in [5]:

Observation 1 Let G be a graph of maximum degree 6 and let φ be a partial acyclic 11-coloring of G . Suppose v is an uncolored vertex of G . If all the neighbors of v use distinct colors, then by coloring v properly we extend φ . If a color c in $SC(N_c(v))$ appears $n_c(v) > 1$ times among the neighbors of v , then, in order to color v , we need to avoid at most $\lfloor 5n_c(v)/2 \rfloor$ distinct colors to prevent the creation of bicolored cycles going through v and the vertices colored with c .

Lemma 2 Let G be a graph of maximum degree 6 and let φ be a partial acyclic 11-coloring of G . Then, for any uncolored vertex u such that $\#cn(u) \leq 3$, there exists a color for u that allows us to extend φ .

Proof

First, suppose that no color is repeated among the neighbors of u , thus $L_u = (1)$, $L_u = (1, 1)$, or $L_u = (1, 1, 1)$. By Observation 1, u only needs to avoid the colors used by its neighbors, then there remains at least eight colors to color u .

Now, suppose that a color appears (at least) twice among the neighbors of u . Then, since $\#cn(u) \leq 3$, we have exactly three cases: $L_u = (2)$, $L_u = (2, 1)$, or $L_u = (3)$. When $L_u = (2)$, $L_u = (2, 1)$ (resp. $L_u = (3)$), then, by Observation 1, u needs to forbid five colors (resp. seven colors) in order to avoid the possible creation of bicolored cycles and at most two more colors (resp. one more color) in order to maintain the proper coloring. In each case, at least three choices remain to color u . □

Lemma 3 Let G be a graph of maximum degree 6 and let φ be a partial acyclic 11-coloring of G . Then, for any uncolored vertex u such that $\#cn(u) = 4$, there exists a color for u that allows us to extend φ .

Proof

Let G be a graph with $\Delta(G) \leq 6$, φ be a partial acyclic 11-coloring of G , and u be an uncolored vertex u such that $\#cn(u) = 4$. Let v_1, v_2, v_3, v_4 be its four colored neighbors.

1. If no color is repeated among the neighbors of u i.e. $L_u = (1, 1, 1, 1)$, then there remains seven colors to color u , since u needs to avoid only the colors used by its four neighbors.
2. If exactly one color is repeated among the neighbors of u , then we have three cases $L_u = (2, 1, 1)$, $L_u = (3, 1)$ and $L_u = (4)$.
 - 2.1 Case $L_u = (2, 1, 1)$.
By Observation 1, u needs to forbid five colors to avoid the possible creation of bicolored cycles and three more colors to maintain the proper coloring. Then, three choices remain to color u .
 - 2.2 Case $L_u = (3, 1)$.
By Observation 1, u needs to forbid seven colors to avoid the possible creation of bicolored cycles and two more colors to maintain the proper coloring. Then, two choices remain to color u .
 - 2.3 Case $L_u = (4)$.
W.l.o.g. suppose that $\varphi(v_1) = \varphi(v_2) = \varphi(v_3) = \varphi(v_4) = 1$. Observe that if one of the v_i 's has $\#dcn(v_i) = 5$, say v_1 , then we can recolor v_1 with a color different from 1 and those in $SC(N_c(v_1))$. We obtain $L_u = (3, 1)$, a case that is solved in 2.2. So suppose that for $1 \leq i \leq 4$, $\#dcn(v_i) \leq 4$. Then, u needs to avoid at most eight ($\lfloor (4 \times 4)/2 \rfloor$) colors to maintain the acyclic coloring and one more color for the proper coloring. Then, two choices remain to color u .
3. If two colors are repeated among the neighbors of u , then $L_u = (2, 2)$. W.l.o.g. suppose that $\varphi(v_1) = \varphi(v_2) = 1$ and $\varphi(v_3) = \varphi(v_4) = 2$. Observe that if one of the v_i 's has $\#dcn(v_i) = 5$, say v_1 , then we can recolor v_1 with a color different from 1 and 2 and those

in $SC(N_c(v_1))$. We obtain $L_u = (2, 1, 1)$, a case that is solved in 2.1. So suppose that for $1 \leq i \leq 4$, $\#dcn(v_i) \leq 4$. Then, u needs to avoid at most eight ($\lfloor (2 \times 4)/2 \rfloor + \lfloor (2 \times 4)/2 \rfloor$) colors to maintain the acyclic coloring and two colors for the proper coloring. Then, one choice remains to color u .

□

Lemma 4 *Let G be a graph of maximum degree 6 and let φ be a partial acyclic 11-coloring of G . Then, for any uncolored vertex u such that $\#cn(u) = 5$, there exists a color for u that allows us to extend φ .*

Proof

Let G be a graph with $\Delta(G) \leq 6$, φ be a partial acyclic 11-coloring of G , and u be an uncolored vertex u such that $\#cn(u) = 5$. Let v_1, v_2, v_3, v_4, v_5 be its five colored neighbors, and for $1 \leq i \leq 5$ and $1 \leq j \leq 5$, let v_i^j be the five neighbors of v_i distinct from u .

1. If no color is repeated among the neighbors of u (thus $L_u = (1, 1, 1, 1, 1)$), then there remains six colors to color u .
2. If exactly one color is repeated among the neighbors of u , then we have four cases $L_u = (2, 1, 1, 1, 1)$, $L_u = (3, 1, 1, 1, 1)$, $L_u = (4, 1, 1, 1, 1)$, and $L_u = (5, 1, 1, 1, 1)$.

2.1 Case $L_u = (2, 1, 1, 1, 1)$.

By Observation 1, u needs to forbid five colors to avoid the possible creation of bicolored cycles and four more colors to maintain the proper coloring. Then, two choices remain to color u .

2.2 Case $L_u = (3, 1, 1, 1, 1)$.

By Observation 1, u needs to forbid seven colors to avoid the possible creation of bi-colored cycles and three more colors to maintain the proper coloring. Then, one choice remains to color u .

2.3 Case $L_u = (4, 1, 1, 1, 1)$.

W.l.o.g. assume that $\varphi(v_1) = \varphi(v_2) = \varphi(v_3) = \varphi(v_4) = 1$ and $\varphi(v_5) = 2$. Observe that if one of the v_i 's ($1 \leq i \leq 4$) has $\#dcn(v_i) = 5$, say v_1 , then we can recolor v_1 with a color different from 1 and 2 and those in $SC(N_c(v_1))$. We obtain $L_u = (3, 1, 1, 1, 1)$, a case that is solved in 2.2. So suppose that for $1 \leq i \leq 4$, $\#dcn(v_i) \leq 4$. Then, u needs to avoid at most eight ($\lfloor (4 \times 4)/2 \rfloor$) colors to maintain the acyclic coloring and two colors for the proper coloring. Then, one choice remains to color u .

2.4 Case $L_u = (5, 1, 1, 1, 1)$.

W.l.o.g. suppose that $\varphi(v_1) = \varphi(v_2) = \varphi(v_3) = \varphi(v_4) = \varphi(v_5) = 1$. Observe that if one of the v_i 's has $\#dcn(v_i) = 5$, say v_1 , then we can recolor v_1 with a color different from 1 and those in $SC(N_c(v_1))$. We obtain $L_u = (4, 1, 1, 1, 1)$, a case that is solved in 2.3. So suppose that for $1 \leq i \leq 5$, $\#dcn(v_i) \leq 4$. Suppose there exists a colored neighbor v_i ($1 \leq i \leq 5$) of u , say v_1 , such that the colored neighbors of v_1 use at most three distinct colors. Thus, u needs to forbid at most nine ($\lfloor (3 + 4 \times 4)/2 \rfloor$) colors to avoid the possible creation of bicolored cycles and one more color to maintain the proper coloring. Then, one choice remains to color u . Suppose now, that for all $1 \leq i \leq 5$, $\#dcn(v_i) = 4$. We focus on v_1 . If $\#cn(v_1) = 4$ (i.e. $L_{v_1} = (1, 1, 1, 1)$), then we can recolor v_1 with a color c in $\llbracket 1; 11 \rrbracket \setminus \{1\} \setminus SC(N_c(v_1))$ (there remains at least six colors). Hence, L_u is modified and becomes $L_u = (4, 1, 1, 1, 1)$, a case that is solved in 2.3. So assume that $\#cn(v_1) = 5$ (i.e. $L_{v_1} = (2, 1, 1, 1, 1)$), and w.l.o.g. $\varphi(v_1^1) = \varphi(v_1^2) = 2$, $\varphi(v_1^3) = 3$, $\varphi(v_1^4) = 4$, and $\varphi(v_1^5) = 5$. We recolor v_1 with a color different from 1, 2, 3, 4, 5 and those of $SC(N_c(v_1^1))$. Hence, L_u is modified and becomes $L_u = (4, 1, 1, 1, 1)$, a case that is solved in 2.3.

3. If two colors are repeated among the neighbors of u , we have two cases $L_u = (2, 2, 1, 1, 1)$ or $L_u = (3, 2, 1, 1, 1)$.

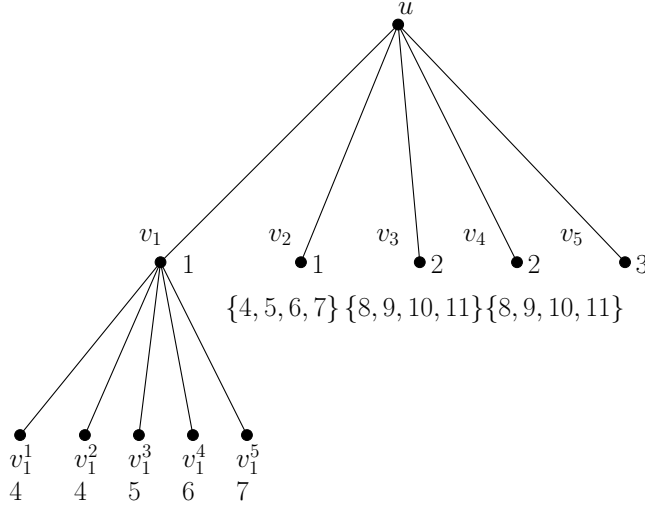


Figure 2: The list $L_u = (2, 2, 1)$

3.1 Case $L_u = (2, 2, 1)$.

W.l.o.g. suppose that $\varphi(v_1) = \varphi(v_2) = 1$; $\varphi(v_3) = \varphi(v_4) = 2$ and $\varphi(v_5) = 3$.

Observe that if one of the v_i 's ($1 \leq i \leq 4$) has $\#dcn(v_i) = 5$, say v_1 , then we can recolor v_1 with a color different from 1, 2, 3 and those in $SC(N_c(v_1))$. We obtain $L_u = (2, 1, 1, 1)$, a case that is solved in 2.1. So suppose that for $1 \leq i \leq 4$, $\#dcn(v_i) \leq 4$. Suppose there exists a colored neighbor v_i ($1 \leq i \leq 4$) of u , say v_1 , such that the colored neighbors of v_1 use at most three distinct colors. Thus, u needs to forbid at most seven ($\lfloor (3+4)/2 \rfloor + \lfloor (2 \times 4)/2 \rfloor$) colors to avoid the possible creation of bicolored cycles and three more colors to maintain the proper coloring. Then, one choice remains to color u . Suppose now, that for all $1 \leq i \leq 4$, $\#dcn(v_i) = 4$, then for all $1 \leq i \leq 4$, we have $L_{v_i} = (2, 1, 1, 1)$ or $L_{v_i} = (1, 1, 1, 1)$. W.l.o.g. $SC(N_c(v_1)) = SC(N_c(v_2)) = \{4, 5, 6, 7\}$ and $SC(N_c(v_3)) = SC(N_c(v_4)) = \{8, 9, 10, 11\}$ ($SC(N_c(v_1))$ and $SC(N_c(v_3))$ are distinct, otherwise there exists a color that extends φ to u). See Figure 2. We focus on v_1 and its neighborhood. We will try to recolor v_1 with a color different from 1 and 3: if we succeed, then we will obtain a new L_u solved previously (in 2.1 or 2.2); if not, then we will show that there exists a color for u that extends φ . If $\#cn(v_1) = 4$ (i.e. $L_{v_1} = (1, 1, 1, 1)$), then we recolor v_1 with a color different from $\llbracket 1; 7 \rrbracket$ and we are done. So assume that $\#cn(v_1) = 5$ (i.e. $L_{v_1} = (2, 1, 1, 1)$), and w.l.o.g. $\varphi(v_1^1) = \varphi(v_1^2) = 4$, $\varphi(v_1^3) = 5$, $\varphi(v_1^4) = 6$, and $\varphi(v_1^5) = 7$. We try to recolor v_1 with a color different from 1, 4, 5, 6, 7 and those of $SC(N_c(v_1^1))$. If there is a choice different from 3, then we are done. Otherwise this means that $SC(N_c(v_1^1) \setminus \{v_1\}) = \{2, 8, 9, 10, 11\}$. By applying the same reasoning on v_1^2 , either we can recolor v_1 , or $SC(N_c(v_1^2) \setminus v_1) = \{2, 8, 9, 10, 11\}$. In that case, we can color u with 4.

3.2 Case $L_u = (3, 2)$.

W.l.o.g. suppose that $\varphi(v_1) = \varphi(v_2) = \varphi(v_3) = 1$ and $\varphi(v_4) = \varphi(v_5) = 2$. Observe that if one of the v_i 's has $\#dcn(v_i) = 5$, say v_1 , then we can recolor v_1 with a color different from 1, 2 and those in $SC(N_c(v_1))$. We obtain $L_u = (2, 2, 1)$, a case that is solved in 3.1. Hence for $1 \leq i \leq 5$, we have $\#dcn(v_i) \leq 4$.

3.2.1 Suppose v_4 has $\#dcn(v_4) = 4$.

3.2.1.1 If $L_{v_4} = (1, 1, 1, 1)$, then we can recolor v_4 with a color different from 1, 2 and those in $SC(N_c(v_4))$. We obtain a new partial acyclic coloring of G where $L_u = (3, 1, 1)$, a case solved previously.

3.2.1.2 If $L_{v_4} = (2, 1, 1, 1)$, then we can recolor v_4 with a color different from 2, and those in $SC(N_c(v_4))$ and $SC(N_c(v_4^1))$ (where we suppose that the color on v_4^1 appears twice in $SC(N_c(v_4))$). We obtain a new partial acyclic coloring of G where $L_u = (3, 1, 1)$ or $(4, 1)$, cases solved previously.

The same argument holds for v_5 . Hence, we assume that $\#dcn(v_4) \leq 3$ and $\#dcn(v_5) \leq 3$.

3.2.2 We claim that $\#dcn(v_i) = 4$ for $1 \leq i \leq 3$. To see this, suppose by contradiction, that $\#dcn(v_i) = 4$ for at most two v_i ($1 \leq i \leq 3$) (recall that $\#dcn(v_i) \leq 3$ for the others). Then, u needs to forbid at most eight ($\lfloor (2 \times 4 + 3)/2 \rfloor + \lfloor (2 \times 3)/2 \rfloor$) colors to avoid the possible creation of bicolored cycles and two more colors to maintain the proper coloring. Then, one choice remains to color u . Hence, assume that $\#dcn(v_1) = \#dcn(v_2) = \#dcn(v_3) = 4$. If one cannot find a choice of color for u , then there must be six bicolored cycles going through u , and v_1, v_2, v_3 . Also three bicolored cycles going through u and v_4, v_5 . Thus, w.l.o.g. we may assume that $SC(N_c(v_1)) \cup SC(N_c(v_2)) \cup SC(N_c(v_3)) = \{3, 4, 5, 6, 7, 8\}$, $SC(N_c(v_1)) = \{3, 4, 5, 6\}$, $SC(N_c(v_4)) = SC(N_c(v_5)) = \{9, 10, 11\}$ and each color i with $3 \leq i \leq 8$ (resp. $9 \leq i \leq 11$) appears at least once in exactly two different colored neighborhood of v_i with $1 \leq i \leq 3$ (resp. $4 \leq i \leq 5$). See Figure 3. We focus on v_1 and its neighborhood. We will try to recolor v_1 with a color different from 1 and 2: if we succeed, then we will obtain a new L_u solved previously; if not, then we will show that there exists a color for u that extends φ . If $\#cn(v_1) = 4$ (i.e. $L_{v_1} = (1, 1, 1, 1)$), then we recolor v_1 with a color different from $\llbracket 1; 6 \rrbracket$ and we are done. So assume that $\#cn(v_1) = 5$ (i.e. $L_{v_1} = (2, 1, 1, 1)$), and w.l.o.g. $\varphi(v_1^1) = \varphi(v_2^2) = 3$, $\varphi(v_3^3) = 4$, $\varphi(v_4^4) = 5$, and $\varphi(v_5^5) = 6$. We try to recolor v_1 with a color different from 1, 3, 4, 5, 6 and those of $SC(N_c(v_1^1))$. If there is a choice different from 2, then we are done. Otherwise we have $SC(N_c(v_1^1) \setminus \{v_1\}) = \{7, 8, 9, 10, 11\}$ and we try to recolor v_1 with a color different from 1, 3, 4, 5, 6 and those of $SC(N_c(v_1^2))$. If there is a choice different from 2, then we are done. If not, then we are sure that $SC(N_c(v_1^1)) = SC(N_c(v_1^2)) = \{7, 8, 9, 10, 11\}$, then we color u with 3, since 3 cannot appear in $SC(N_c(v_4)) \cup SC(N_c(v_5))$, and 3 cannot appear in both $SC(N_c(v_2))$ and $SC(N_c(v_3))$ (recall that each color i with $3 \leq i \leq 8$ appears at least once in exactly two distinct $N_c(v_j)$ with $1 \leq j \leq 3$).

This completes the proof of Lemma 1. \square

2.2 Good spanning trees

Let G be a Δ -regular connected graph. A *good spanning tree* of G is a spanning tree T such that T contains a vertex adjacent to $\Delta - 1$ leaves. We prove now the following:

Theorem 2 *Every regular connected graph admits a good spanning tree.*

Proof

Let G be a Δ -regular connected graph. Let v be a vertex of G and let u be a vertex at maximum distance from v , say distance k . Assume u is chosen such that it has the least number of neighbors which are at distance $k - 1$ from v . Let u_1 be a neighbor of u at distance $k - 1$ from v and let u_2, \dots, u_Δ be the other neighbors of u . Then we claim that $G_1 = G \setminus (N(u) \setminus \{u_1\})$ is connected. By contradiction, suppose G' is a connected component of G_1 that contains neither v nor u (noting that u and v are connected in G_1). Then every vertex of G' is at distance, in G , exactly k from v . Let x be any vertex of G' . Every neighbor of x at distance (in G) $k - 1$ from v is also a neighbor of u . But x is not adjacent to u_1 because otherwise it would be in the same connected component as u . Then x has less neighbors at distance (in G) $k - 1$ from v than u . This contradicts the choice of u .

Now we construct a good spanning tree T of G as follows: first take any spanning tree T_1 of G_1 , and add to T_1 the edges uu_2, \dots, uu_Δ that cover the vertices u_2, \dots, u_Δ . \square

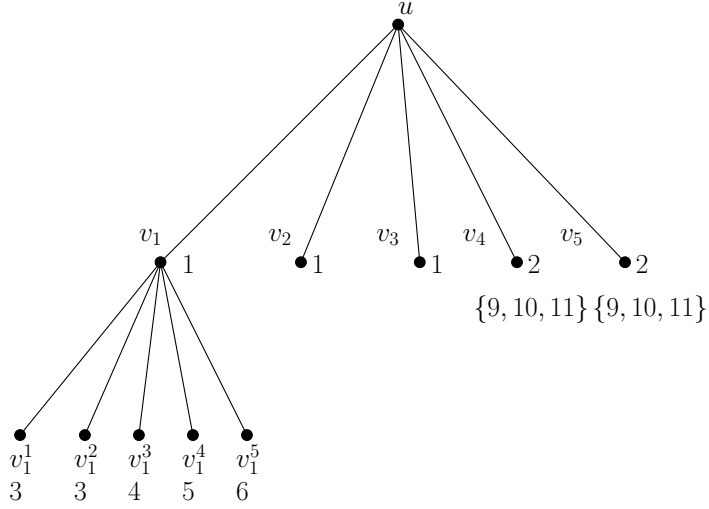


Figure 3: The list $L_u = (3, 2)$

2.3 Proof of Theorem 1

In this section we prove that:

Every graph with maximum degree six is acyclically 11-colorable.

Let G be a graph with maximum degree six. If $\delta(G) < 6$, then Lemma 1 completes the proof. Consider in the following that G is a 6-regular connected graph.

Let T be a good spanning tree of G (such a tree exists due to Theorem 2). Let x_n be a vertex adjacent to five leaves in T . We order the vertices of G from x_1 to x_n according to a post order walk of T rooted at x_n where x_1, x_2, x_3, x_4, x_5 are five leaves. First, we color x_1, x_2, x_3, x_4, x_5 with distinct colors and then we will successively color x_6, x_7, \dots, x_n . In order to color x_i with $6 \leq i \leq n-1$, we use Lemmas 2, 3 and 4, but we never recolor the vertices x_1, x_2, x_3, x_4, x_5 .

In Lemma 2, no recoloring is used. In the case $L_u = (4)$ of Lemma 3, v_1 cannot be x_1, x_2, x_3, x_4 or x_5 since v_1 has five colored neighbors and u is not x_n . Similarly in the case $L_u = (2, 2)$ of Lemma 3 and in the case $L_u = (4, 1)$ of Lemma 4, v_1 cannot be x_1, x_2, x_3, x_4 or x_5 . In the case $L_u = (5)$ (resp. $L_u = (2, 2, 1)$) of Lemma 4, observe that if one of the v_i 's has $\#dcn(v_i) = 5$ for $1 \leq i \leq 5$ (resp. $1 \leq i \leq 4$), then we can recolor this v_i because v_i cannot be x_1, x_2, x_3, x_4 or x_5 since v_i has five colored neighbors and u is not x_n . Hence for $1 \leq i \leq 5$ (resp. $1 \leq i \leq 4$), we have $\#dcn(v_i) \leq 4$. We focus first on v_1 . If v_1 is, say x_1 , then we just focus on v_2 instead of v_1 (since v_1 and v_2 have the same color, we are sure that v_2 is not x_2, x_3, x_4, x_5).

Consider now the case $L_u = (3, 2)$ of Lemma 4. Observe that if one of the v_i 's has $\#dcn(v_i) = 5$ for $1 \leq i \leq 5$, then we can recolor this v_i because v_i cannot be x_1, x_2, x_3, x_4 or x_5 since v_i has five colored neighbors and u is not x_n . Hence for $1 \leq i \leq 5$, we have $\#dcn(v_i) \leq 4$. We focus first on v_4 and v_5 . If one of them, say v_4 , is such that $\#dcn(v_4) = 4$ and v_4 is not an x_i (with $1 \leq i \leq 5$), then we apply 3.2.1. Otherwise, either (a) one of them is an x_i , say $v_4 = x_2$, and $\#dcn(v_5) \leq 3$, or (b) $\#dcn(v_4) \leq 3$ and $\#dcn(v_5) \leq 3$.

(a) Assume that $v_4 = x_2$ and $\#dcn(v_5) \leq 3$. We claim that $\#dcn(v_i) = 4$ for $1 \leq i \leq 3$. To see this, suppose by contradiction, that $\#dcn(v_i) = 4$ for at most two v_i ($1 \leq i \leq 3$). Then, u needs to forbid at most eight $(\lfloor (2 \times 4 + 3)/2 \rfloor + \lfloor (4 + 3)/2 \rfloor)$ colors to avoid the possible creation of bicolored cycles and two more colors to maintain the proper coloring. Then, one choice remains to color u . Thus, w.l.o.g. we may assume that $\varphi(v_1) = \varphi(v_2) = \varphi(v_3) = 1$, $\varphi(v_4) = \varphi(v_5) = 2$, $SC(N_c(v_1)) \cup SC(N_c(v_2)) \cup SC(N_c(v_3)) = \{3, 4, 5, 6, 7, 8\}$, $SC(N_c(v_1)) = \{3, 4, 5, 6\}$, $SC(N_c(v_4)) \supseteq$

$\{9, 10, 11\}$, $SC(N_c(v_5)) = \{9, 10, 11\}$ and each color i with $3 \leq i \leq 8$ (resp. $9 \leq i \leq 11$) appears at least once in exactly two different colored neighborhood of v_i with $1 \leq i \leq 3$ (resp. $4 \leq i \leq 5$). We focus on v_1 and its neighborhood. Note that if v_1 is, say x_1 , then we just focus on v_2 instead of v_1 (since v_1, v_2 and v_3 have the same color, we are sure that v_2 is not x_3, x_4, x_5). We will try to recolor v_1 with a color different from 1 and 2: if we succeed, then we will obtain a new L_u solved previously; otherwise we will show that there exists a color for u that extends φ . If $\#cn(v_1) = 4$ (i.e. $L_{v_1} = (1, 1, 1, 1)$), then we recolor v_1 with a color different from $\llbracket 1; 6 \rrbracket$ and we are done. So assume that $\#cn(v_1) = 5$ (i.e. $L_{v_1} = (2, 1, 1, 1)$), and w.l.o.g. $\varphi(v_1^1) = \varphi(v_1^2) = 3$, $\varphi(v_1^3) = 4$, $\varphi(v_1^4) = 5$, and $\varphi(v_1^5) = 6$. We try to recolor v_1 with a color different from 1, 3, 4, 5, 6 and those of $SC(N_c(v_1^1))$. If there is a choice different from 2, then we are done. Otherwise we try to recolor v_1 with a color different from 1, 3, 4, 5, 6 and those of $SC(N_c(v_1^2))$. If there is a choice different from 2, then we are done. Otherwise we are sure that $SC(N_c(v_1^1)) = SC(N_c(v_1^2)) = \{7, 8, 9, 10, 11\}$, then we color u with 3, since 3 can appear at most once in $SC(N_c(v_4)) \cup SC(N_c(v_5))$, and 3 cannot appear in both $SC(N_c(v_2))$ and $SC(N_c(v_3))$.

(b) Suppose now, $\#dcn(v_4) \leq 3$ and $\#dcn(v_5) \leq 3$. We apply 3.2.2. We focus on v_1 . If v_1 is, say x_1 , then we just focus on v_2 instead of v_1 (since v_1, v_2 and v_3 have the same color, we are sure that v_2 is not x_2, x_3, x_4, x_5).

At this point, we have an acyclic coloring of $G \setminus \{x_n\}$ such that x_1, x_2, x_3, x_4, x_5 use five distinct colors. Finally, to color x_n we have two cases:

Case $L_{x_n} = (1, 1, 1, 1, 1)$. By Observation 1, it suffices to color x_n properly (we have at least five remaining colors).

Case $L_{x_n} = (2, 1, 1, 1, 1)$. By Observation 1, x_n needs to forbid five colors to avoid the possible creation of bicolored cycles and five more colors to maintain the proper coloring. Then, one choice remains to color x_n .

That completes the proof of Theorem 1. □

3 Conclusion

In this paper, we improve the upper bound of $\chi_a(G)$ for graphs with maximum degree six. Until now, the best known lower bound is given by K_7 . It seems to be difficult to provide graphs that need more colors. We conclude with the two following questions:

Question 1 *Exhibit a graph G with $\Delta(G) = 6$ and $\chi_a(G) > 7$ (if such a graph exists).*

Question 2 *Prove that every graph with $\Delta(G) = 6$ has $\chi_a(G) < 11$.*

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