Distributedly Testing Cycle-Freeness

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GT Algorithmique Distribuée
January 12, 2015

Framework

Objective

Monitoring properties in large-scale distributed networks:

- Some nodes (possibly all) are queried
- Queried nodes execute a $O(1)$-round local distributed algorithm, producing a (small) local output
- A central authority gathers these local outputs and takes a global decision

Contexts of use

- Sensor networks with a base station
- Complexity theory in distributed computing

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Property / Distributed language

We consider:

Properties

- Graph properties: large expansion, cycle-freeness
- Properties on labels: existence of a unique leader
- Mixed properties: $(\Delta + 1)$-coloring, existence of a spanning tree

More formally, distributed languages

A distributed language L is a set of labeled graphs (G, ℓ)

- G is a connected undirected graph
- $\ell : V(G) \to \{0,1\}^*$ is a function that labels each node v with the label $\ell(v)$.

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Precisions on the model

Local algorithms

Use of the $\mathcal{LOCAL}$ model:

- Distinct IDs
- Synchronous rounds, simultaneous start
- No failures, no dynamicity
- No congestion: unbounded message to each neighbor

Central algorithm

The central authority

- receives as input the multiset of the local outputs computed by the nodes
- runs a deterministic algorithm.

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Types of decision

	#queried nodes	type of output	decision mechanism	gap	error	certificates
property testing	$o(n)$	$O(\log n)$ bits	algorithm	ϵ-far	yes	no
dist. decision	n	yes/no	$\forall$ yes / $\exists$ no	none	no	no
dist. testing	n	$O(\log n)$ bits	algorithm	none	no	no
dist. verification	n	yes/no	$\forall$ yes / $\exists$ no	none	no	yes
dist. certification	n	$O(\log n)$ bits	algorithm	none	no	yes

Distributed decision (cycle-freeness)

	#queried nodes	type of output	decision mechanism	success probability	certificates
dist. decision	n	yes/no	$\forall$ yes / $\exists$ no	impossible	no

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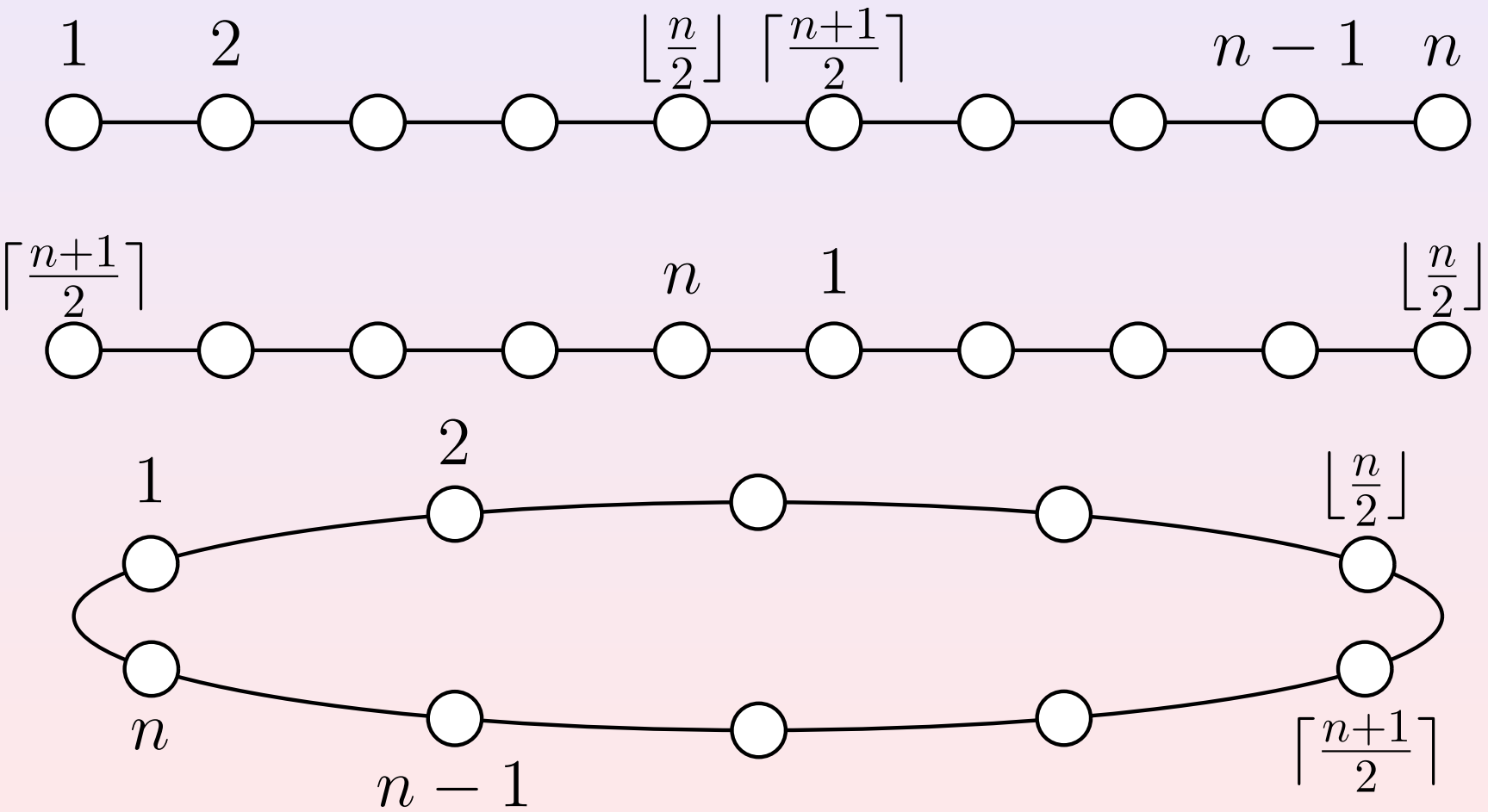

Distributed testing (cycle-freeness)

	#queried nodes	type of output	decision mechanism	success probability	certificates
dist. testing	n	$\leq \lceil \log \Delta \rceil$ bits	algorithm	deterministic	no

Local algorithm:

- outputs the degree of the node

Decision mechanism:

- YES $\Longleftrightarrow \sum$ outputs $= 2n - 2$

Main question answered by this work

Are $\log \Delta$ bits really necessary?

(The answer is essentially "yes".)

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Distributed verification (cycle-freeness)

	#queried nodes	type of output	decision mechanism	success probability	certificates
dist. verification	n	yes/no	$\forall$ yes / $\exists$ no	deterministic	$\Theta(\log n)$ bits

Certificates:

- choose an arbitrary node r
- node v's certificate is its distance to r

Local algorithm:

- YES $\Longleftrightarrow$ both are true
  - if cert. $= 0$, then all neighbors have cert. 1
  - if cert. is $x \neq 0$, then exactly one neighbor has cert. $x-1$ and all others have cert. $x+1$

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Distributed certification (cycle-freeness)

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dist. certification	n	2 bits	algorithm	deterministic	2 bits

Certificates:

- choose an arbitrary node r; node v's certificate is
  - if $v = r$, then 3
  - if $v \neq r$, then its distance to r modulo 3

Local algorithm:

- outputs the pair (b, b') where
  - if cert. $= 3$, then $b = 1$ and, $b' = 1$ iff all neighbors have cert. 1
  - if cert. is $x \neq 3$, then $b = 0$ and, $b' = 1$ iff exactly one neighbor has cert. $x - 1$ and all others have cert. $x + 1$

Decision mechanism:

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Our main result

For the cycle-freeness decision problem:

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dist. certification	n	2 bits	algorithm	deterministic	2 bits
dist. testing [this paper]	n	$\geq \lceil \log \Delta \rceil - 1$ bits	algorithm	deterministic	no

Model

$\mathcal{LOCAL}$ model

- Pairwise distincts IDs
- Synchronous fault-free rounds (& simultaneous wake-up)
- Messages of unlimited size

Distributed testing

- All nodes execute a t-round local distributed algorithm, producing a local output (t is a constant)
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Theorem to be proved

Every distributed tester for cycle-freeness in connected max.-degree-Δ graphs has output size at least $\lceil \log \Delta \rceil - 1$ bits.

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Usefulness of the identifiers

ID-oblivious

The algorithm's output does not depend on the IDs.

Order-invariant

The algorithm's output only depends on the relative order of the IDs.

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Sketch of the proof

- $< \lceil \log \Delta \rceil - 1$ bits $\Rightarrow$ same output for degrees i and $j > i$
- construction of two almost identical graphs
  - a tree with $x + z$, resp. y, nodes of degree i, resp. j
  - a non-tree with x, resp. $y + z$, nodes of degree i, resp. j

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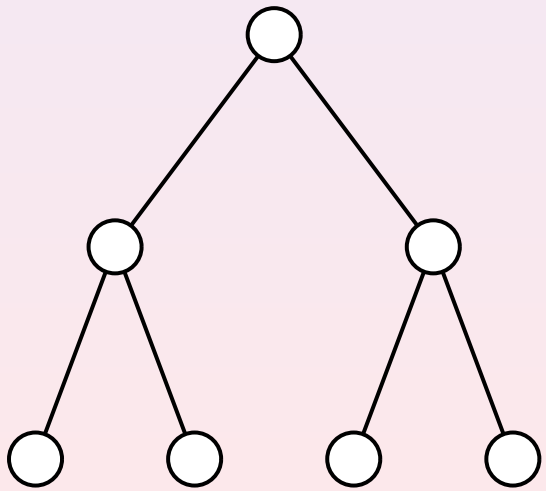

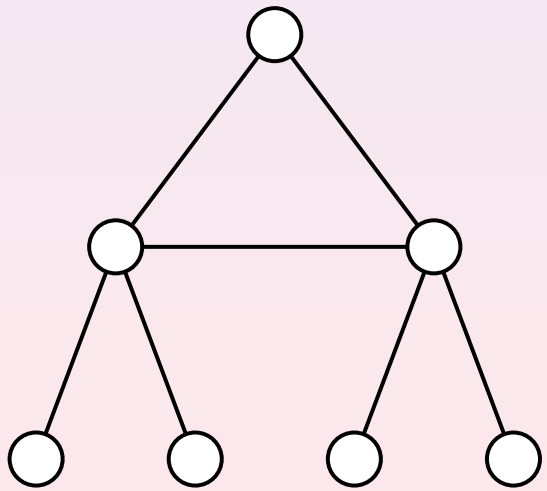

Generalizing to arbitrary (constant) t

Use of subdivided trees:

- replace each edge by a path of length $2t + 1$
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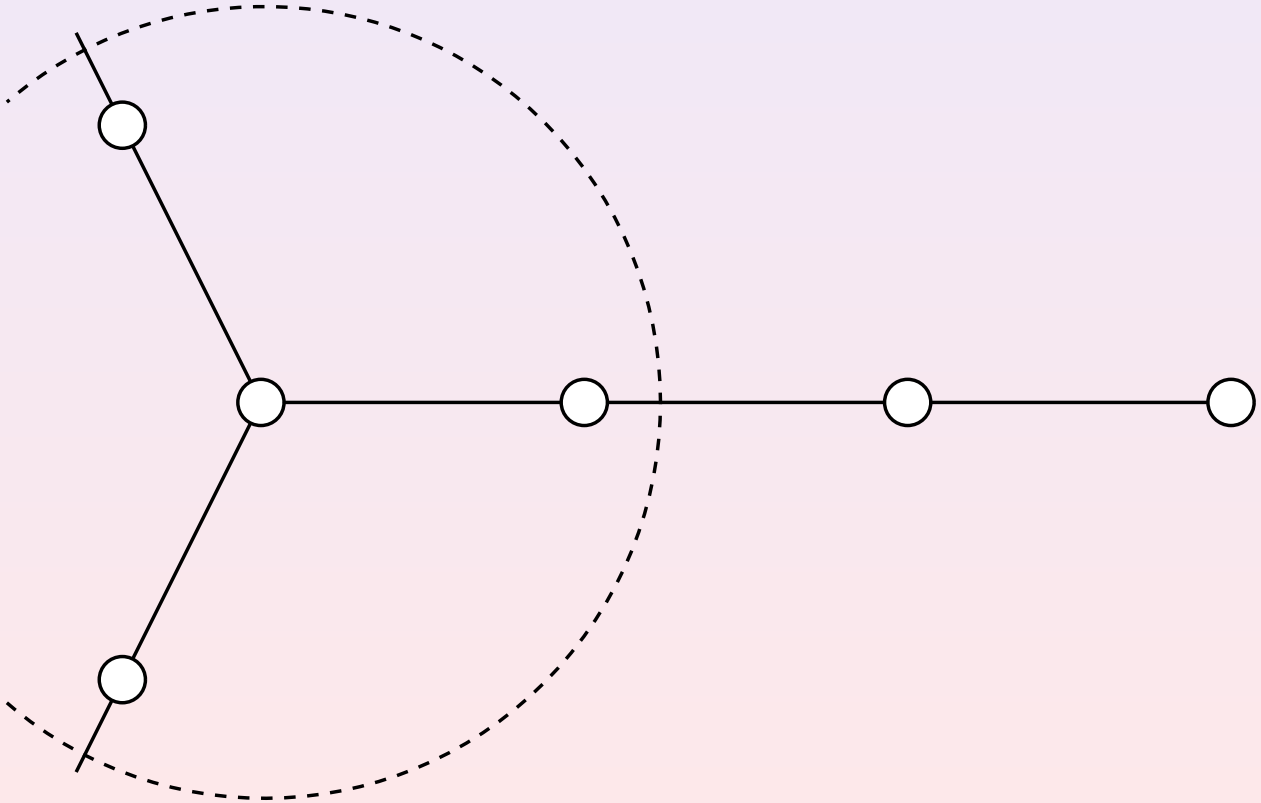

The case of the subdivided graphs

Lemma

Only four outputs are sufficient in subdivided graphs!

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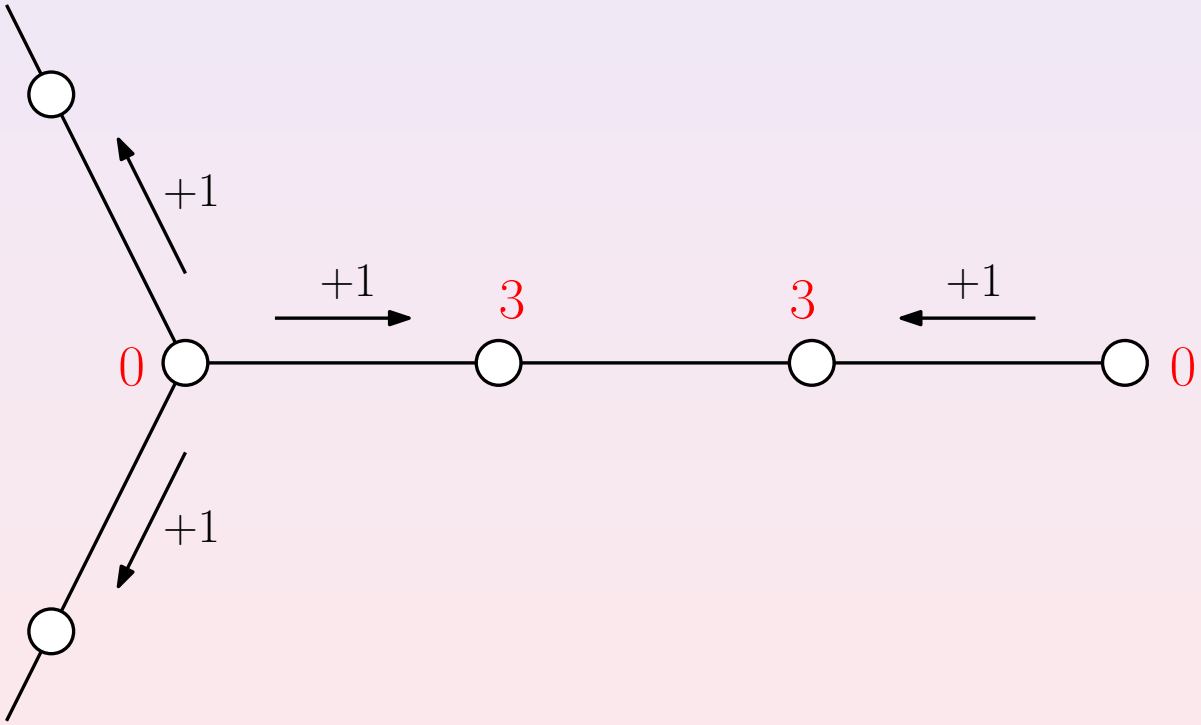

Nodes of degree different from 2 distribute their degree.

A solution

Hiding the trees into the forest!

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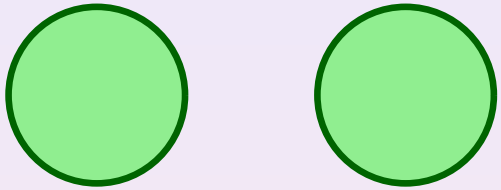
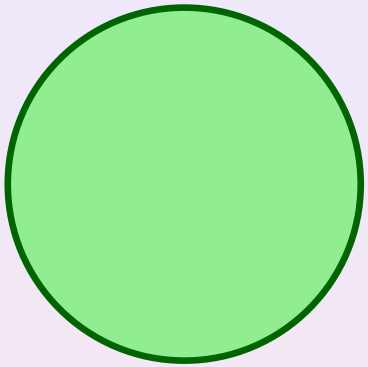
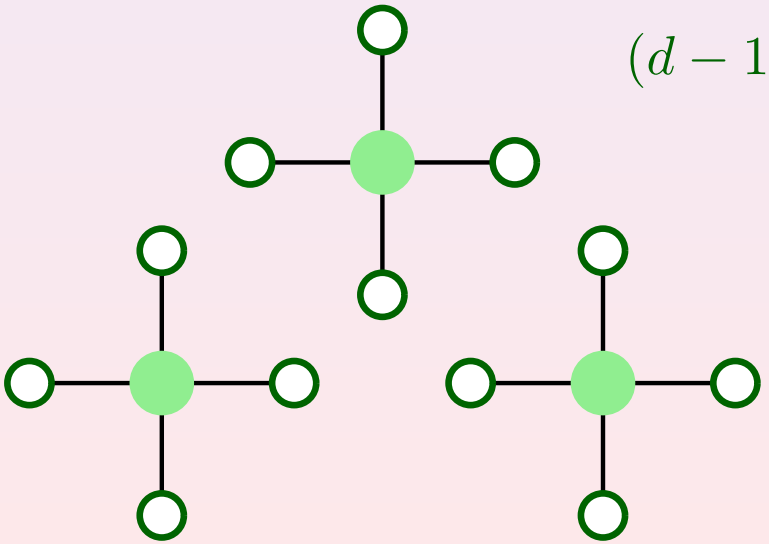
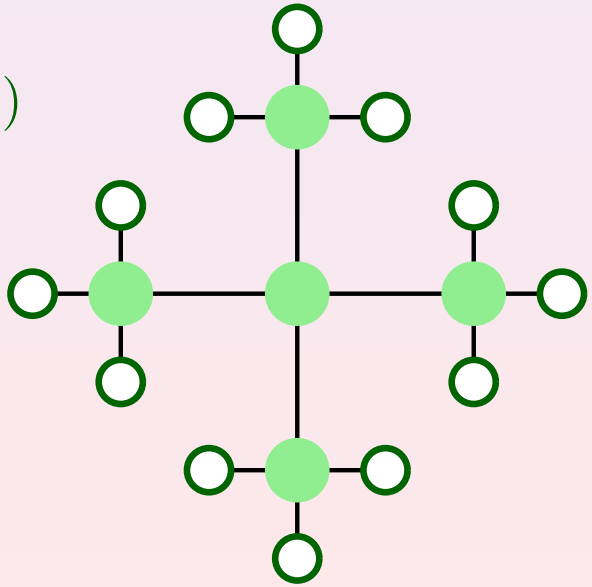


$$2 \cdot (2\pi R) = 2\pi(2 \cdot R)$$

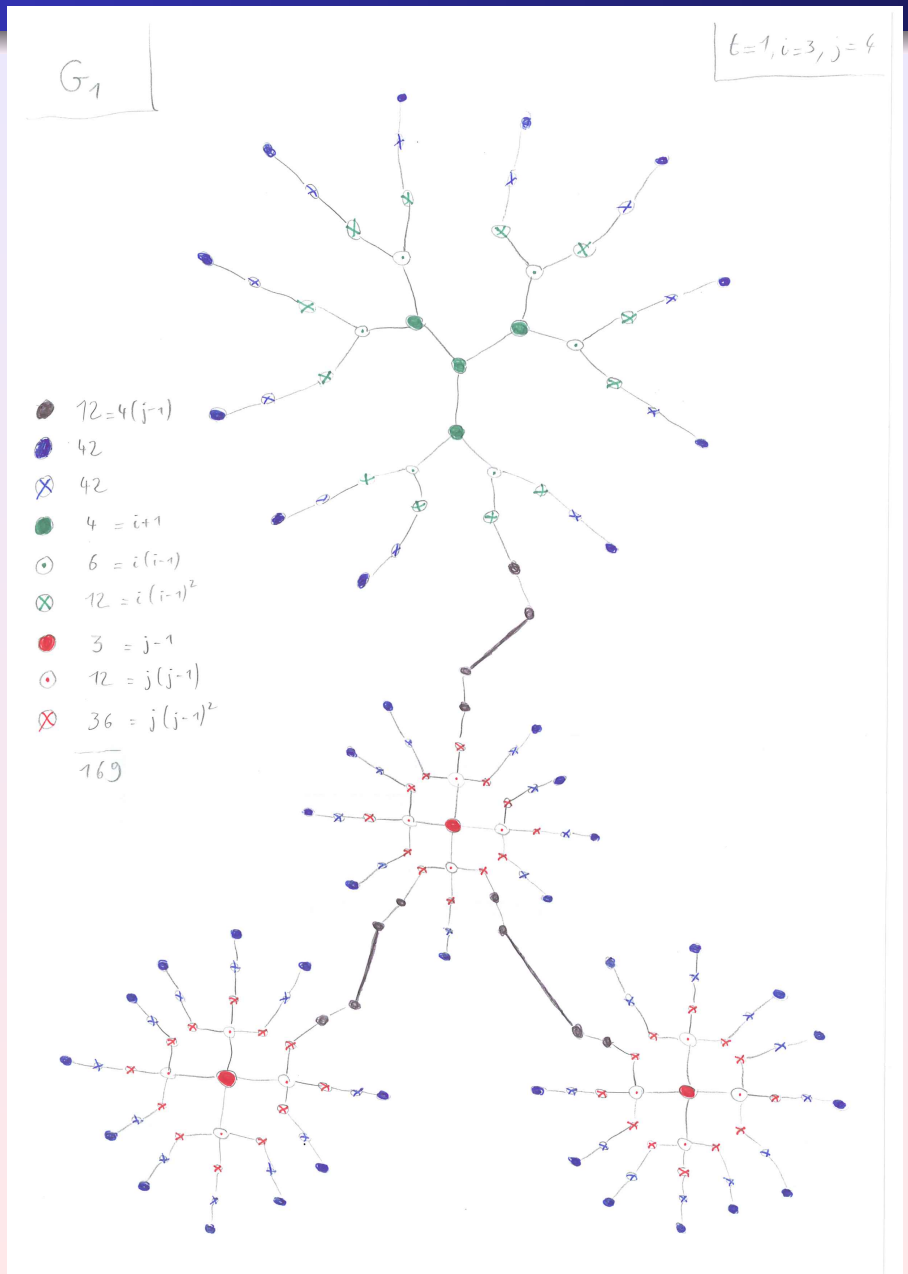


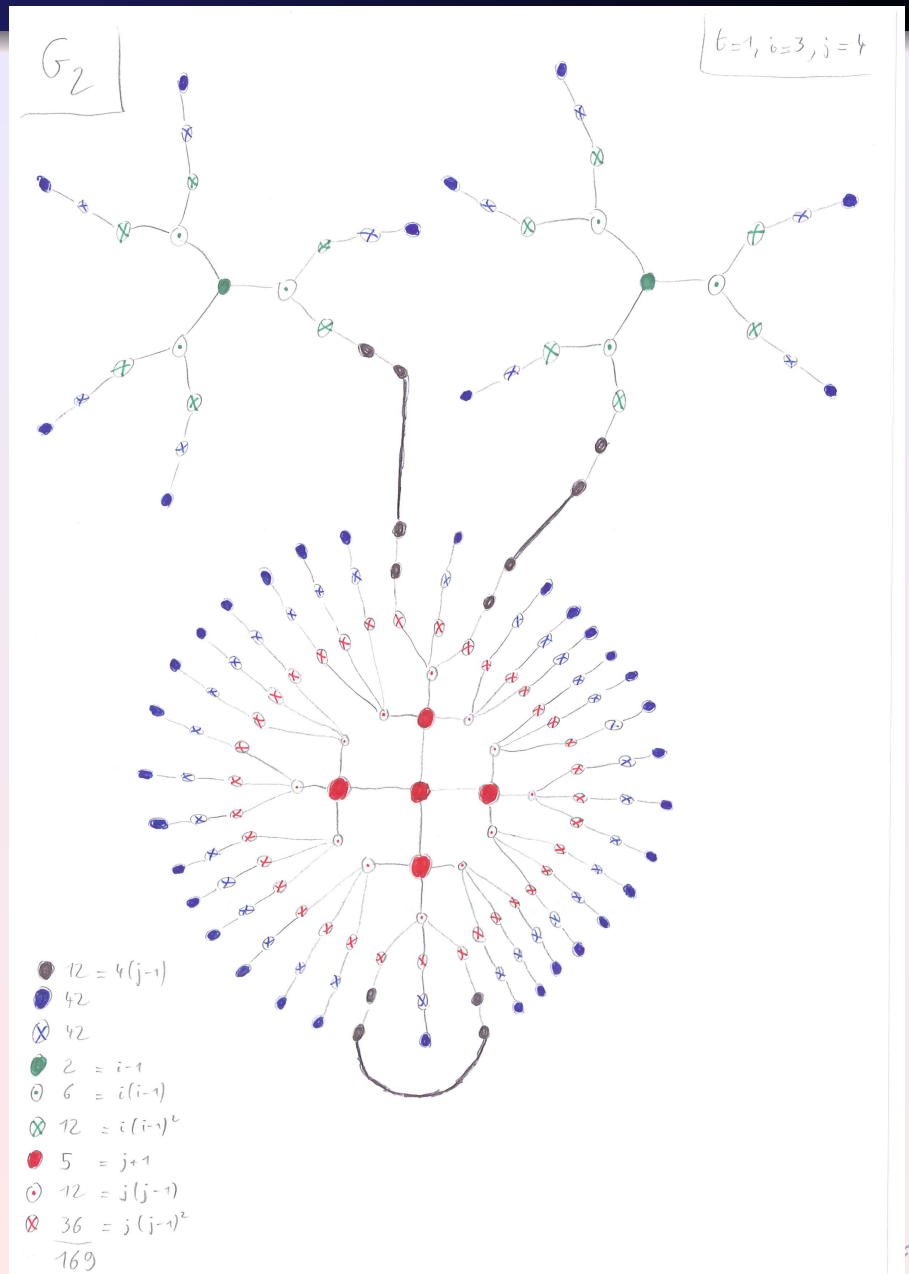


$$(d-1) \cdot d = d \cdot (d-1)$$



ID-oblivious solution

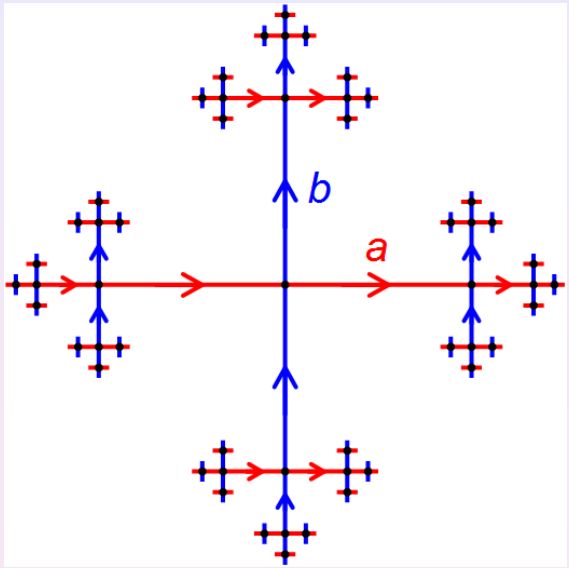




Towards an order-invariant solution

Definition: free group (wikipedia)

The free group F_S over a given set S consists of all expressions (a.k.a. words, or terms) that can be built from members of S, considering two expressions different unless their equality follows from the group axioms (e.g. $st = suu^{-1}t$, but $s \neq t$ for $s, t, u \in S$). The members of S are called generators of F_S.



Definition: linearly ordered group (wikipedia)

A linearly ordered group is a group G equipped with a total order "$\leq$", that is translation-invariant: Let $x, y, z \in G$, we say that $(G, \leq)$ is a left-ordered group if $x \leq y$ implies $z\,x \leq z\,y$.

An order-invariant solution

Theorem (at least from the 1940's)

Every free group is left-orderable.

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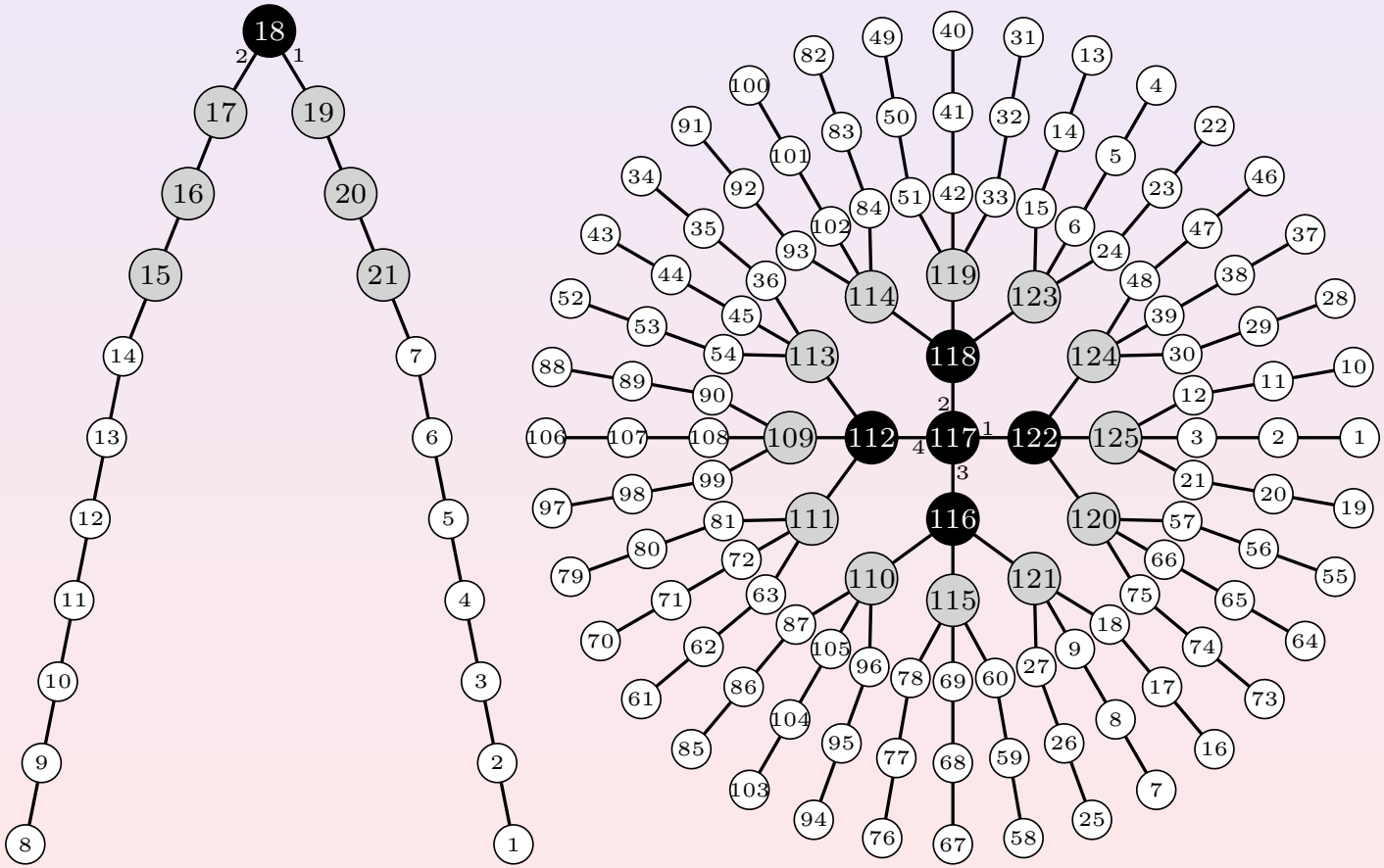

The final argument

Theorem (of independent interest)

$\exists$ a solution $\Longrightarrow$ $\exists$ an order-invariant solution

For every non-negative integers k, t, Δ, and every language $\mathcal{L}$ defined on connected graphs with maximum degree Δ, and k-valued domain, if there exists a t-round construction algorithm $\mathcal{A}$ for $\mathcal{L}$, then there is a t-round order-invariant construction algorithm $\mathcal{A}'$ for $\mathcal{L}$.

Thank you

for your attention