

Euler Tour Lock-in Problem in the Rotor-Router Model

I choose pointers and you choose port numbers

Evangelos Bampas^{1,3} Leszek Gasieniec²
Nicolas Hanusse³ David Ilcinkas³
Ralf Klasing³ Adrian Kosowski^{3,4}

¹National Technical University of Athens, Greece

²University of Liverpool, UK

³CNRS / INRIA / Univ. of Bordeaux, France

⁴Gdańsk University of Technology, Poland

Journées Graphes et Algorithmes

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Definitions

Anonymous graphs / networks

- No (used) node labeling
- **Local port numbering** at node v from 1 to $\deg(v)$

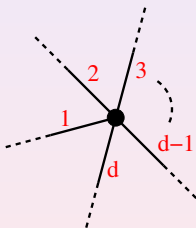
Mobile agent / robot / message / anything

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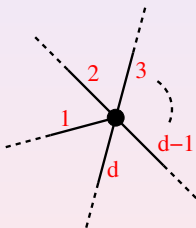
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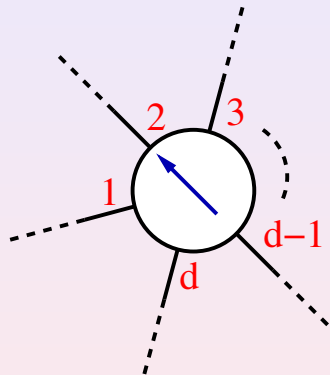
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Rotor-router mechanism

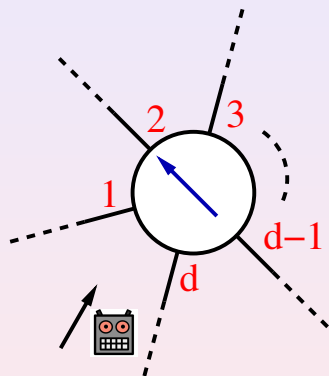


Very simple mechanism:

- Each node has a **pointer** ↗
- The agent follows the pointer and “increments” it with respect to the cyclic ordering \odot (induced by the port numbering)

Other names: Propp machine, Next-Port, Edge Ant Walk

Rotor-router mechanism

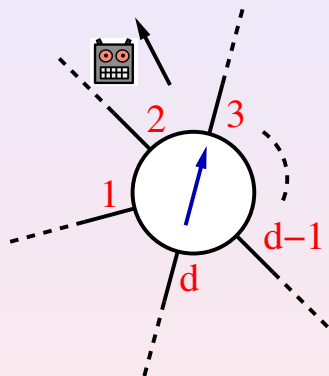


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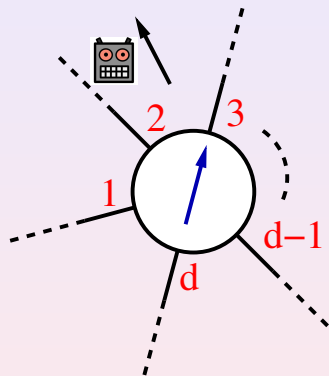


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Context

Known results

- The agent eventually **traverses each edge**
- The traversal stabilizes into an Euler tour: each edge is traversed once in each direction within a period
- Lock-in time: $\Theta(m \cdot D)$ (m : # edges, D : diameter)

Motivations / Applications

- Graph exploration (by a software agent / robot)
- Mutual exclusion
- Stabilisation of distributed processes
- Work and load balancing problems
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Related work

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- S. Bhatt, S. Even, D. Greenberg, and R. Tayar, *Journal of Graph Algorithms and Applications*, 2002.
- J.N. Cooper and J. Spencer, *Combinatorics, Probability and Computing*, 2006.
- B. Doerr and T. Friedrich, *Combinatorics, Probability and Computing*, 2009.
- A.S. Fraenkel, *Mathematics Magazine*, 1970.
- L. Gasieniec and T. Radzik, *Proc. WG*, 2008.
- V.B. Priezzhev, D. Dhar, A. Dhar, and S. Krishnamurthy, *Physics Review Letters*, 1996.
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Our results

Problem

How does the **lock-in time** depend on the **initial configuration** of the ports \circlearrowleft and pointers \nearrow ?

Scenario	Worst case	Best case
Case \mathcal{P} -all	$\Theta(m)$	$\Theta(m)$
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\mathcal{P} =Player, \mathcal{A} =Adversary

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- Consider any choice of ports \odot and pointers \nearrow
- Run virtually the agent for $\Theta(m \cdot D)$ steps
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Useful properties (known results)

Definitions

- **Phase:** Interval between two traversals of the first edge
- G_i : Graph induced by the edges traversed in Phase i
- **Saturated node:** Node whose incident edges are all traversed in both directions during the current phase

(Known) Properties

- Each arc is traversed at most once during a phase
- $G_i \subseteq G_{i+1}$
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- $\Rightarrow G_{i+1} \supseteq \text{neighborhood}(G_i)$



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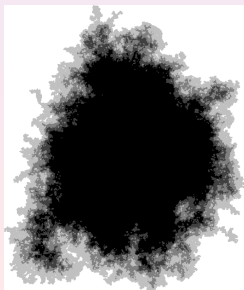
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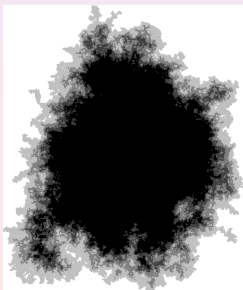
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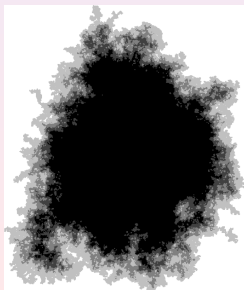
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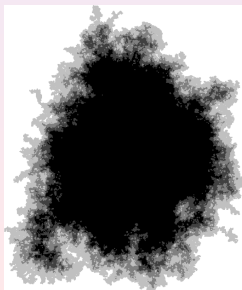
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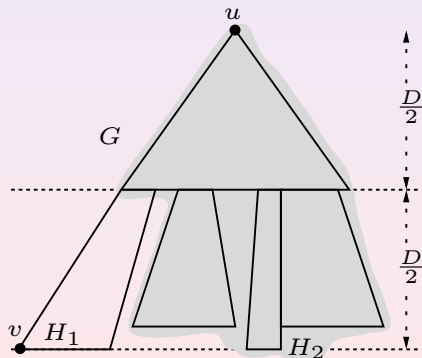
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Case \mathcal{A} -all	$\Theta(m \cdot D)$	$\Theta(m \cdot D)$

(Sketch of the) Proof

- Let $[u, v]$ be a diameter
- Make G_1 be the larger of H_1 and H_2
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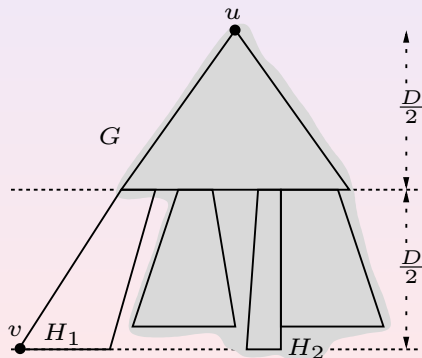


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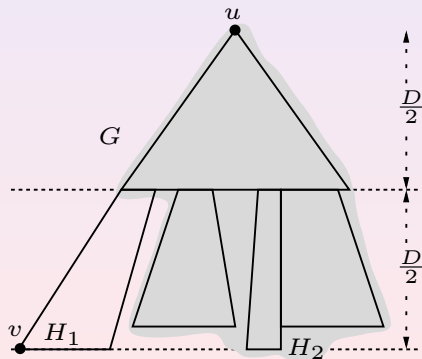
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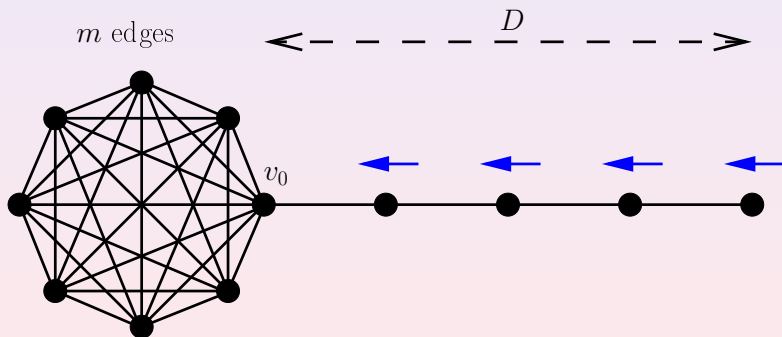


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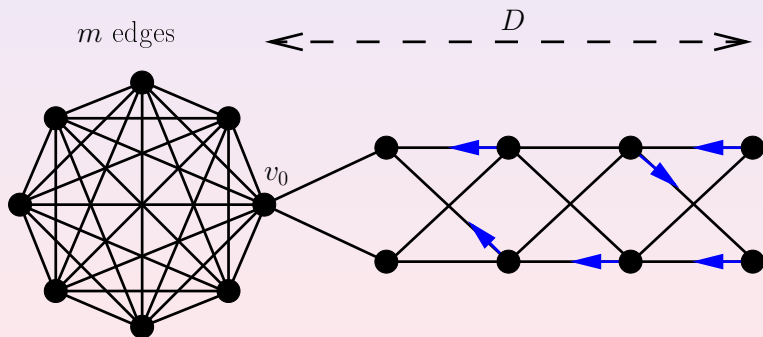
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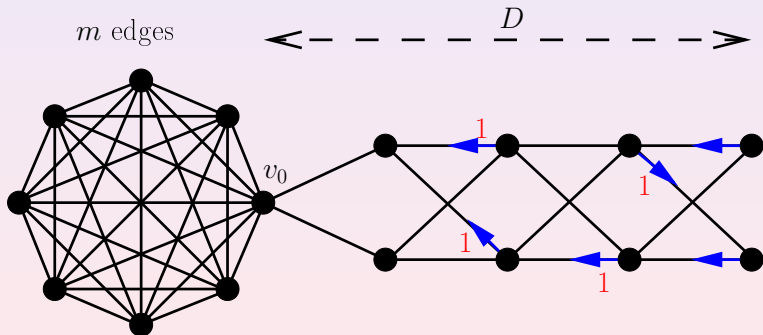
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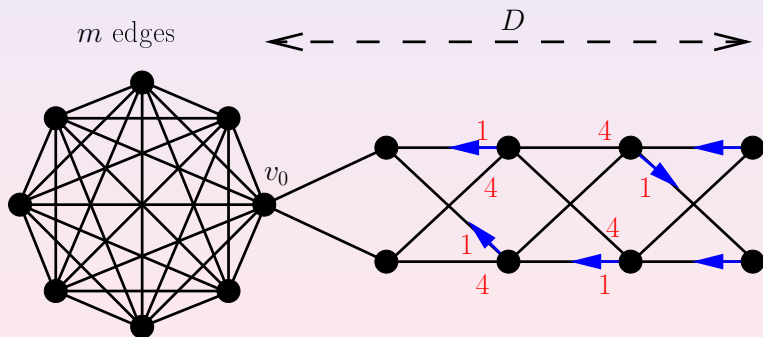
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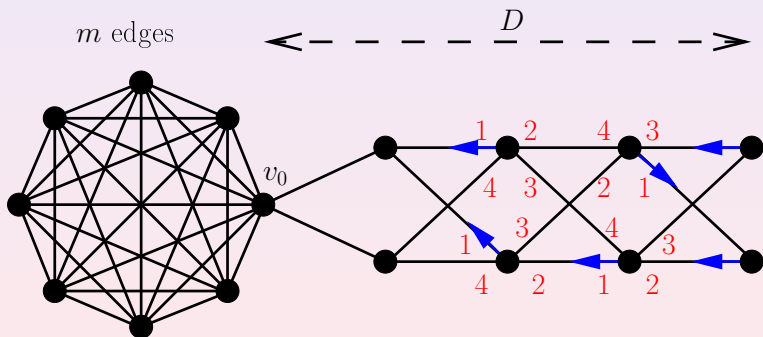
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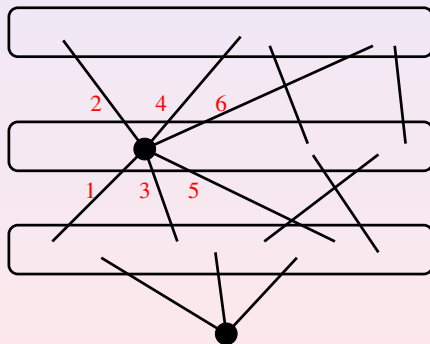
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Conclusion and perspectives

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Open problem

- Does there exist any graph of **large diameter** with lock-in time $O(m)$ in Case $\mathcal{P}(\circ)\mathcal{A}(\nearrow)$?

What if ports \circ and/or pointers \nearrow are set uniformly at random?

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