Label-based Tree Representation

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LOCALITY '05
September 26, 2005

Brief view of the problem

Many (distributed) algorithms on graphs use spanning trees.

- Broadcast
- Convergecast
- Graph exploration
- *etc.*

Need for locality

Local knowledge of the tree.

Our goal

Minimize the total memory used to store the tree.

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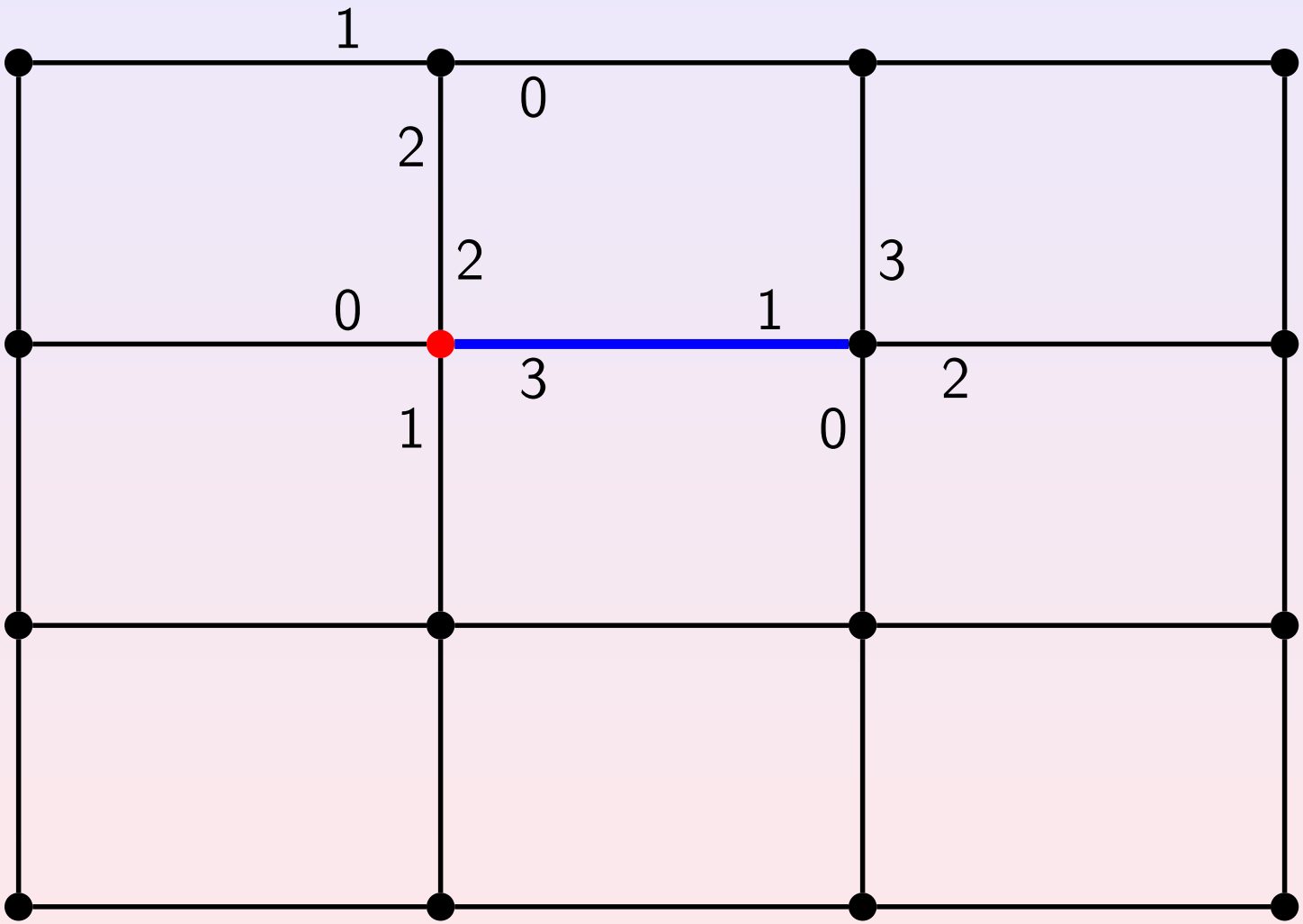
Local labeling of the edges



Tree representations (1)

All-port tree representation

- Label: all port numbers of the incident tree edges
- Two measures for a spanning tree T
  - Maximum used space: $M_{all}(T)$
  - Total used space: $S_{all}(T)$

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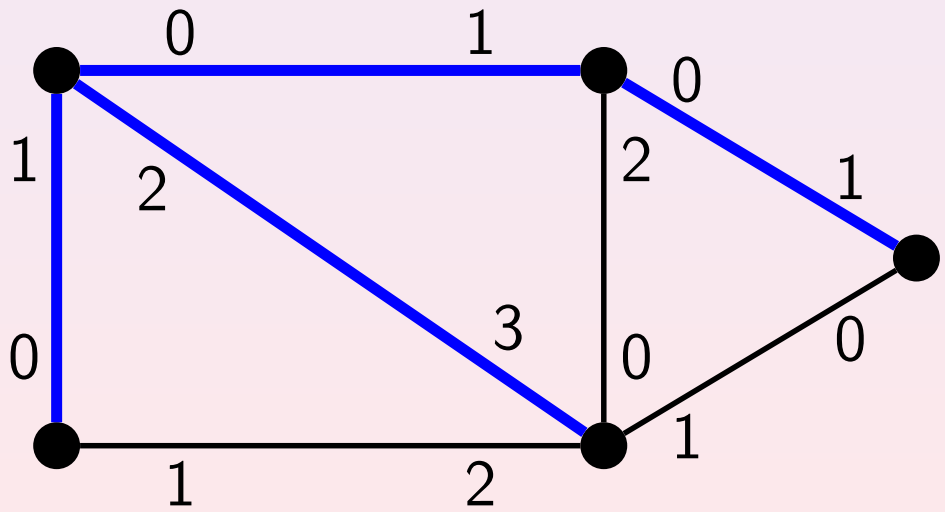

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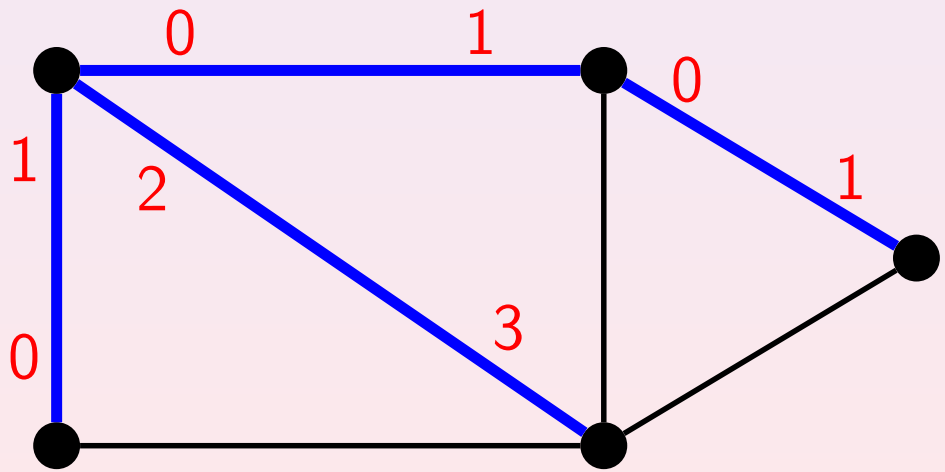

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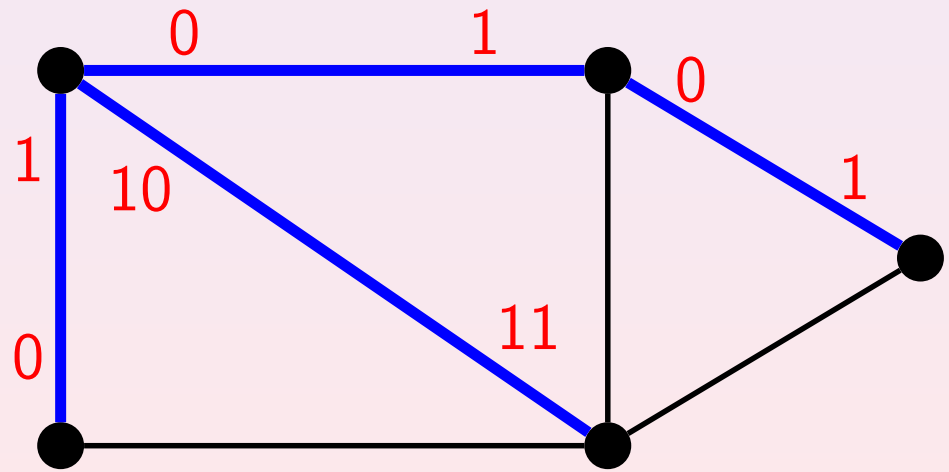

$M_{all}(T) = 4$ bits

$S_{all}(T) = 10$ bits

Tree representations (2)

Upward tree representation

- Label: port number leading to the root
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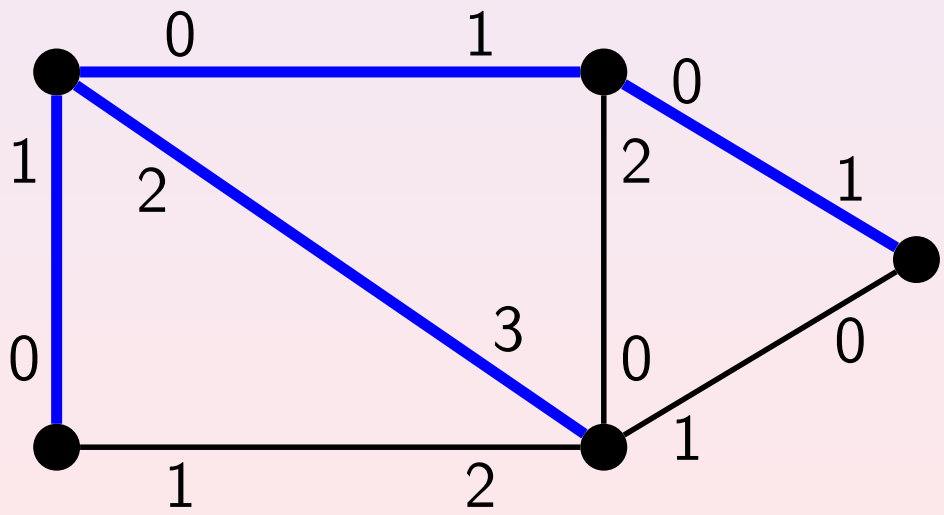

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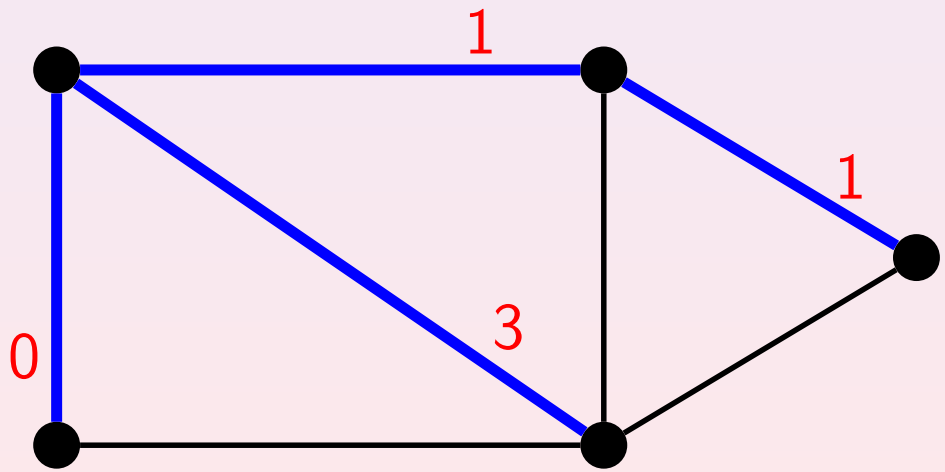

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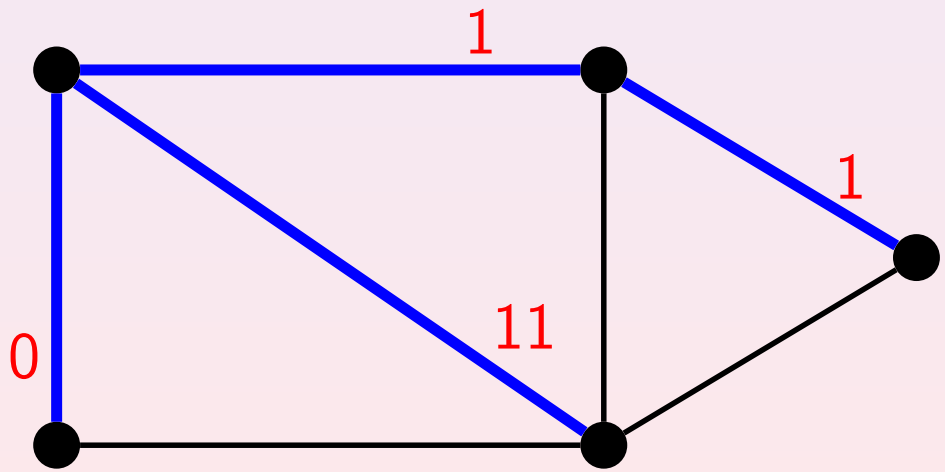

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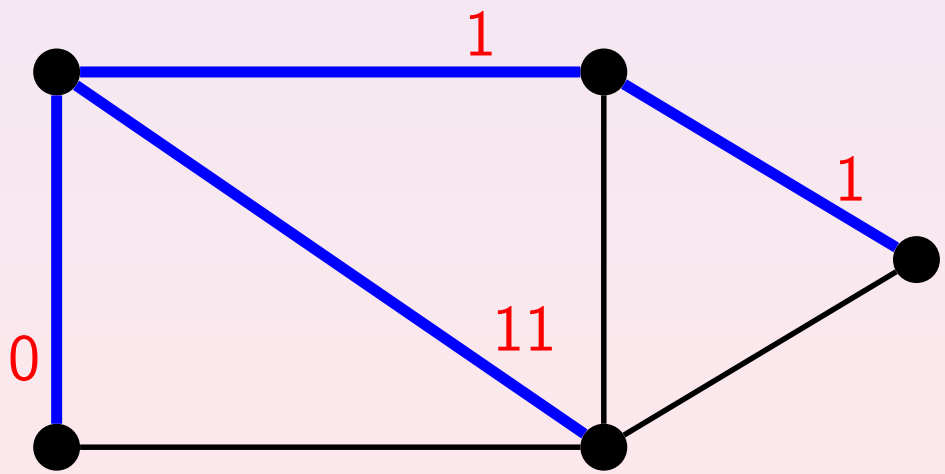

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Send a message to the root.

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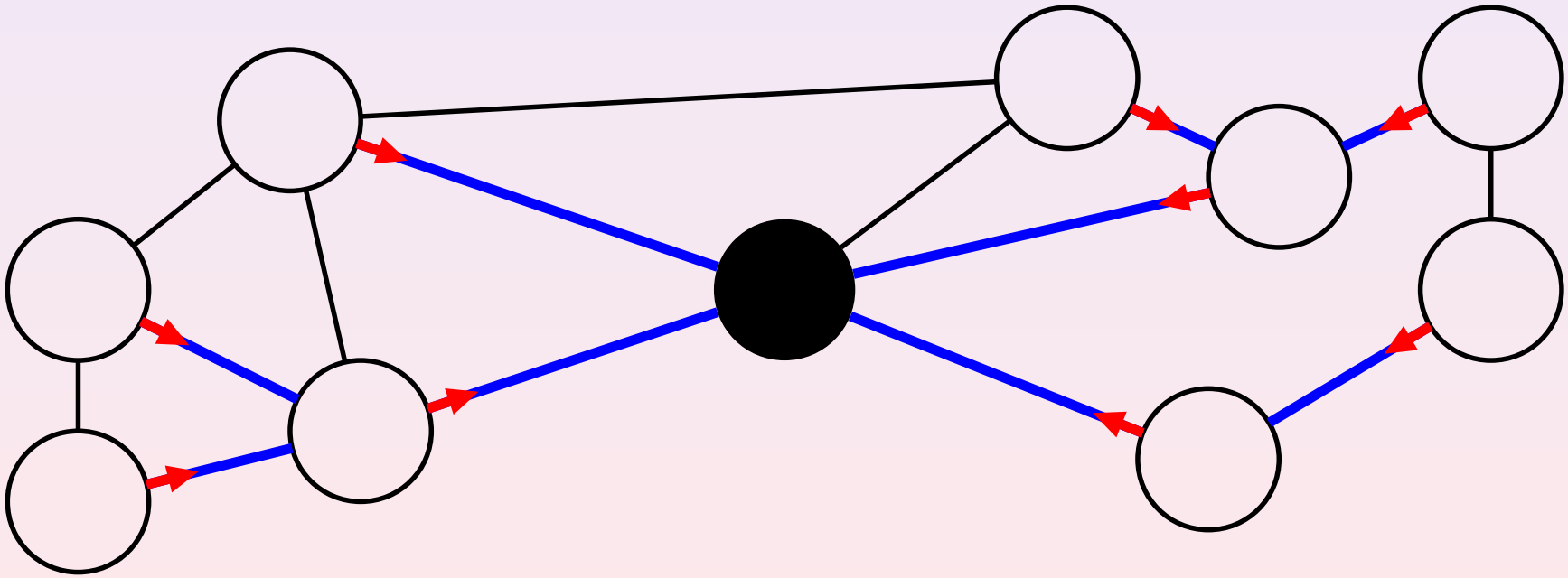

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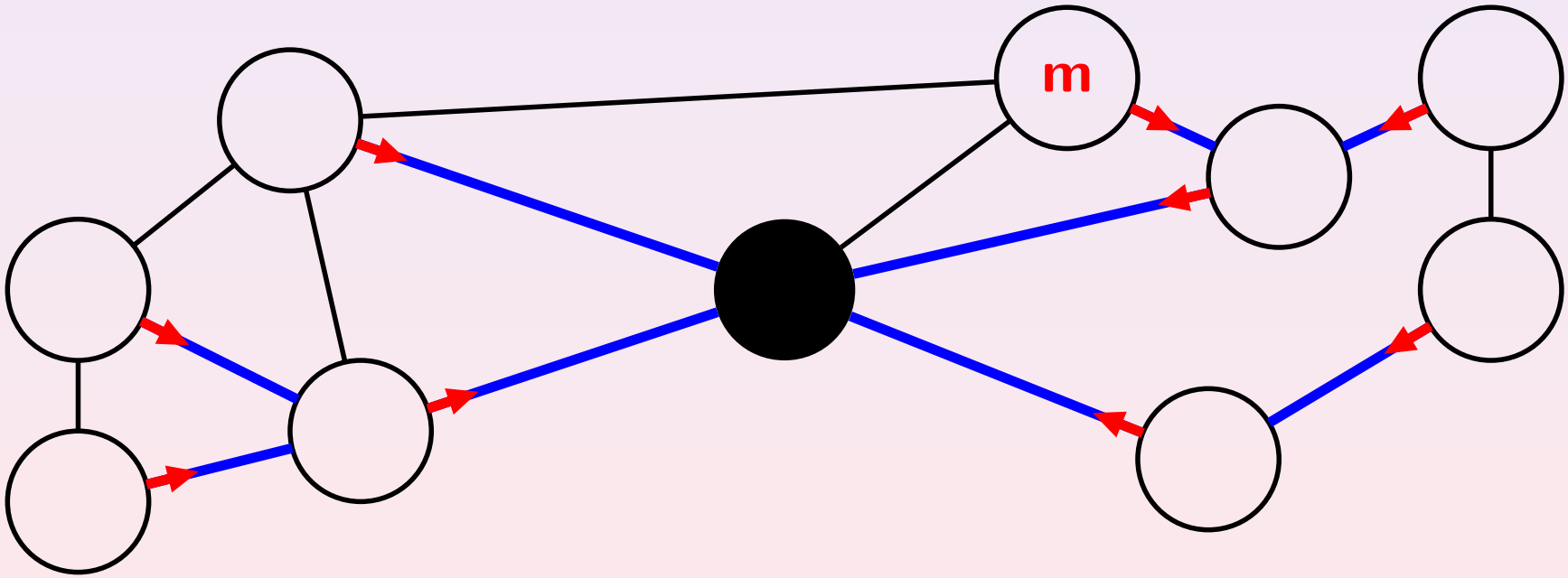

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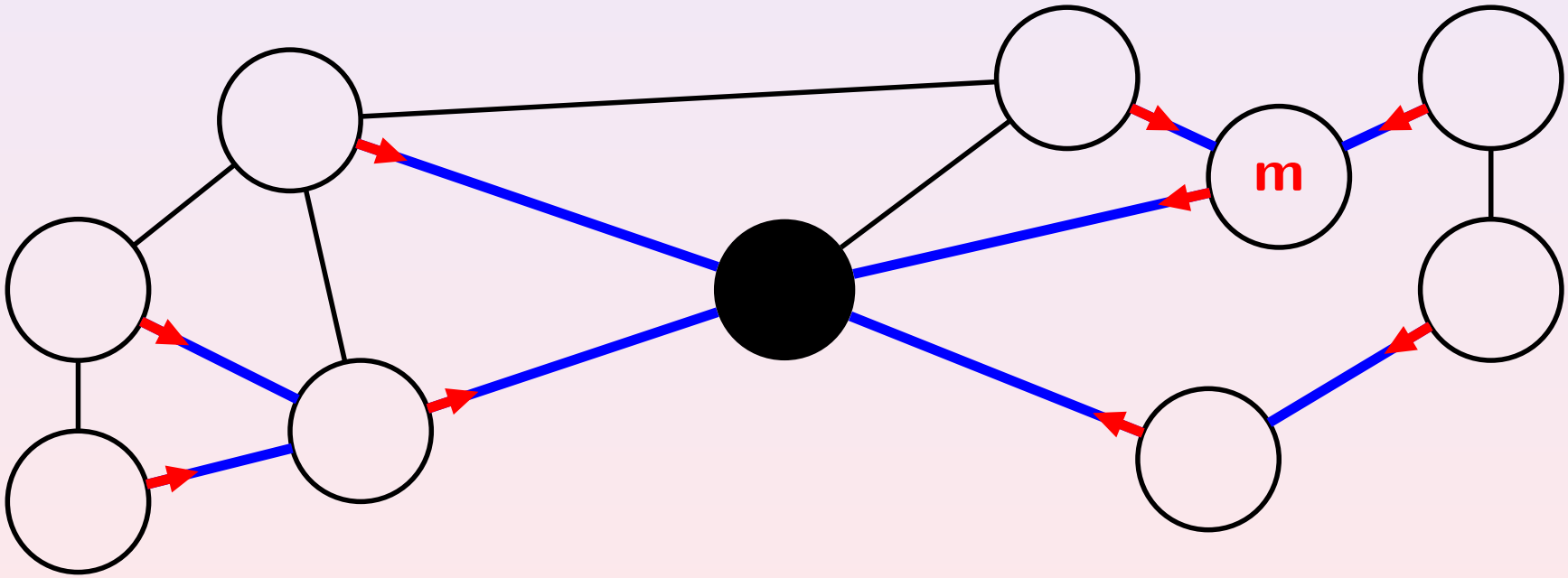

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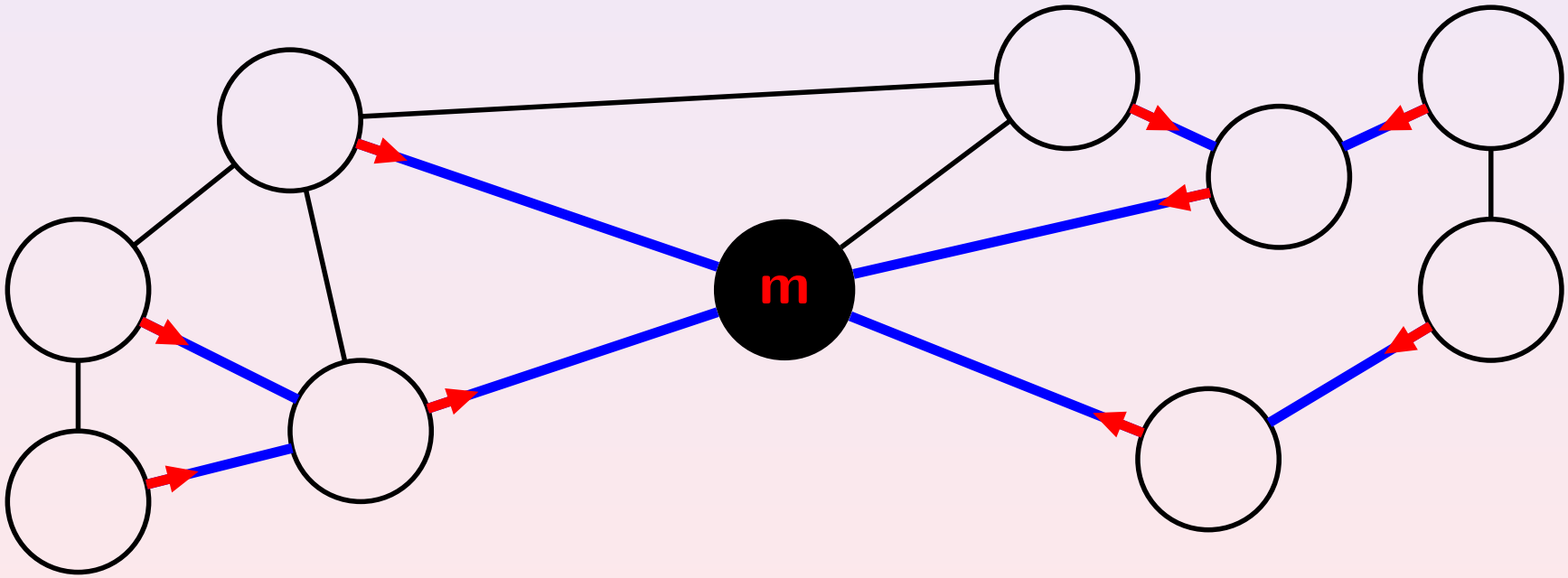

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Collecting information to the root

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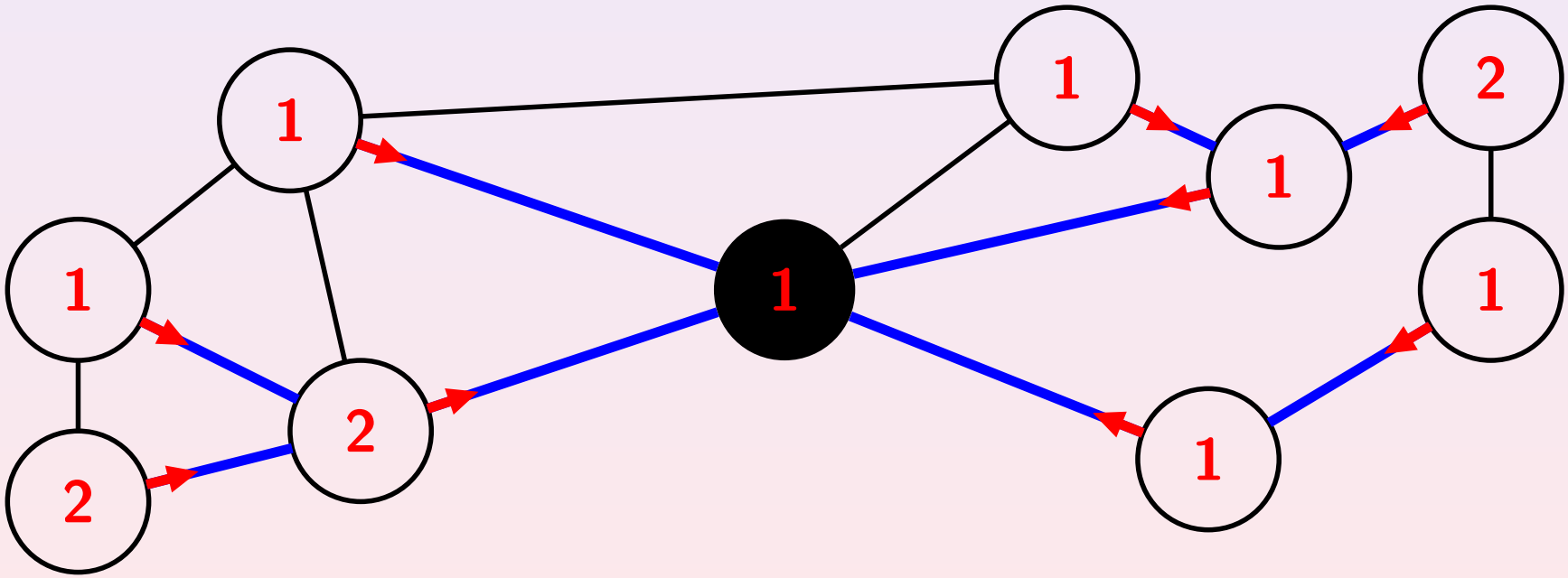

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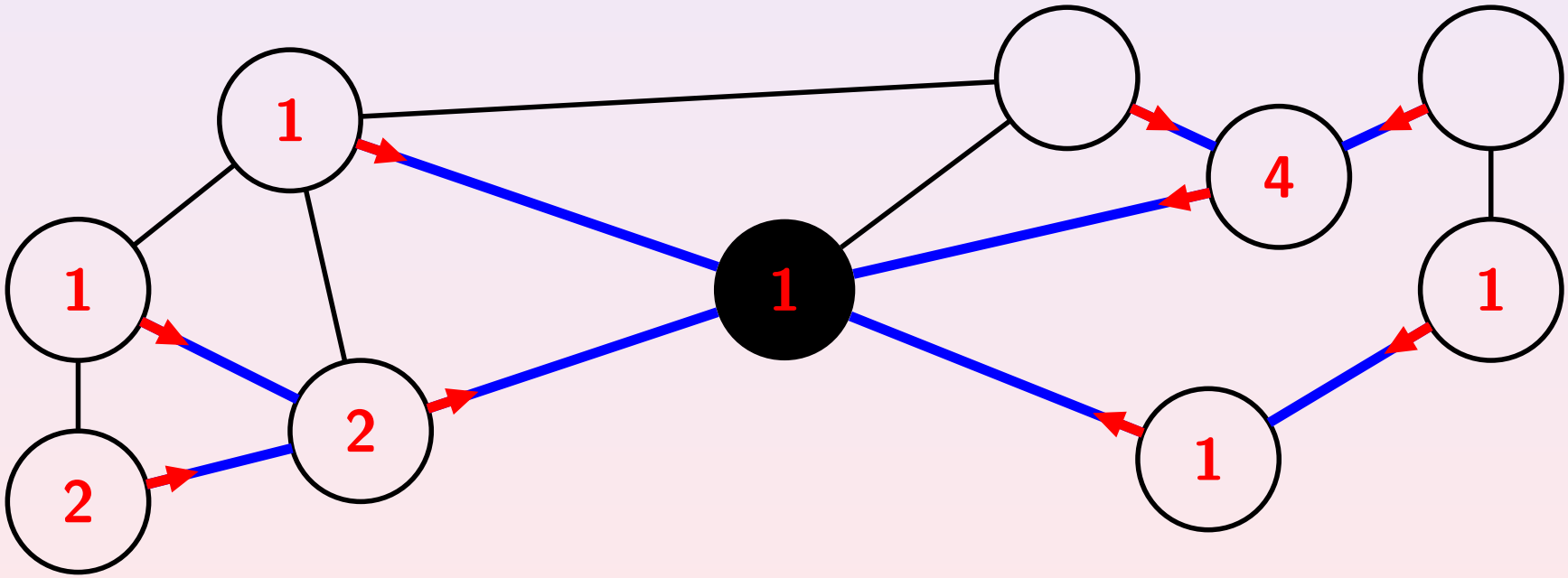

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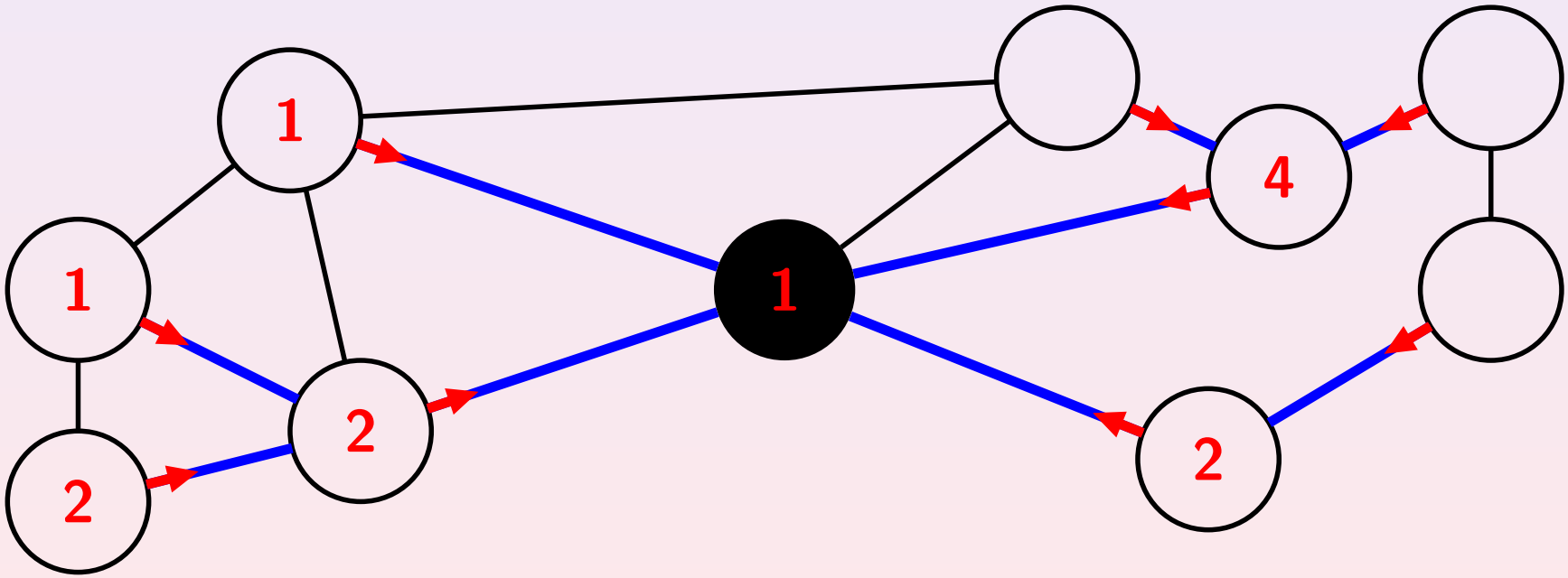

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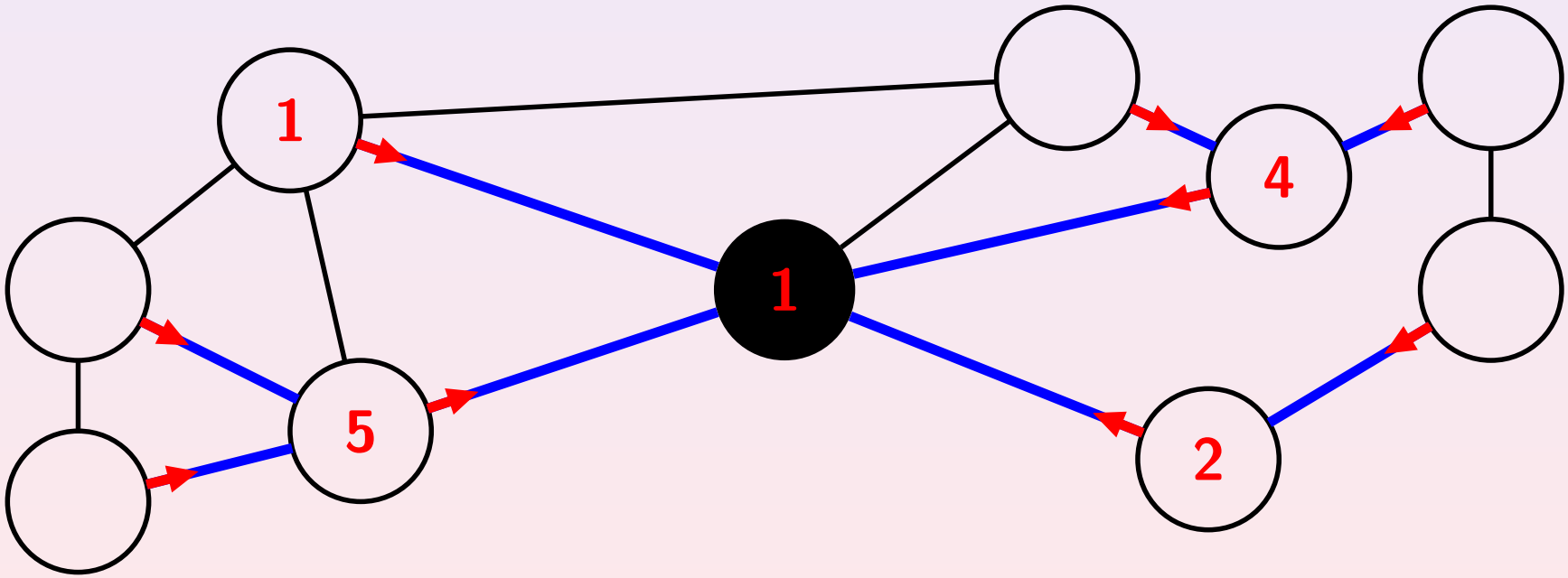

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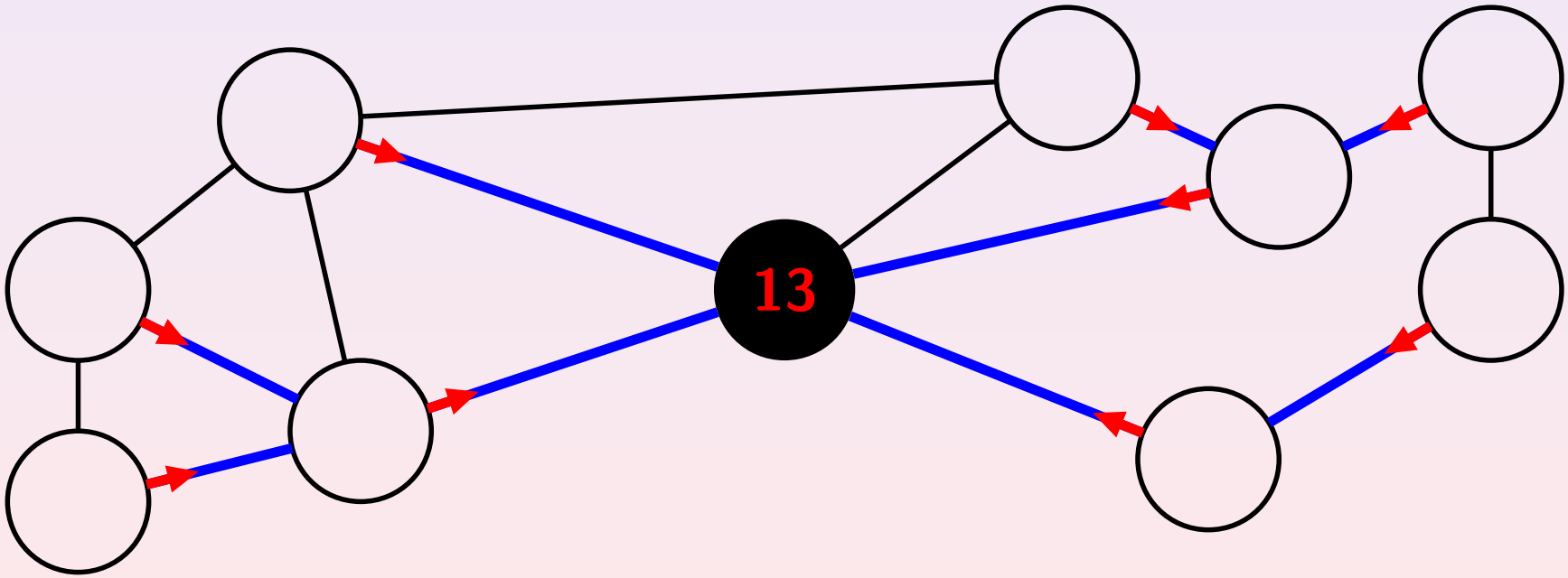
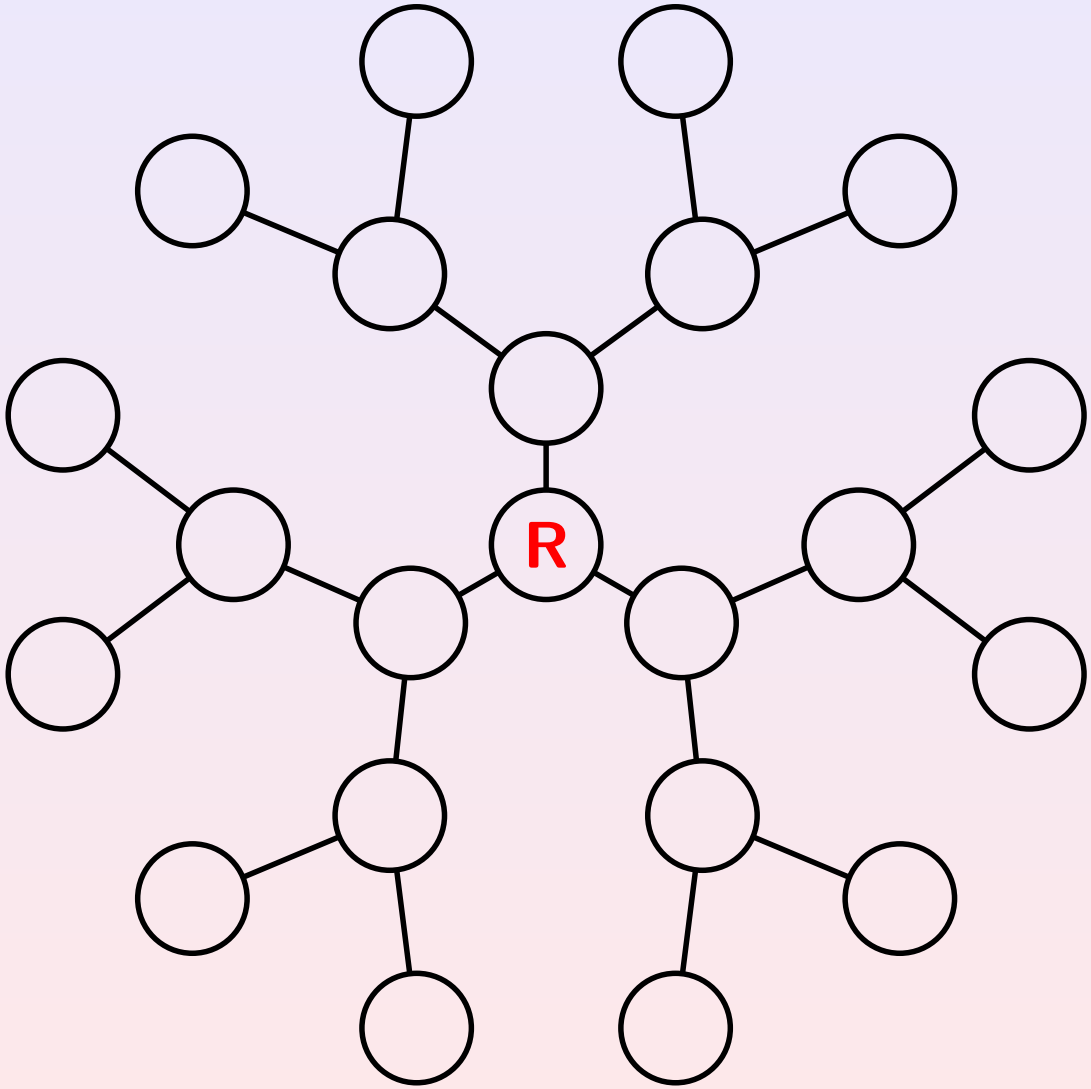


Applications: graph exploration (1)



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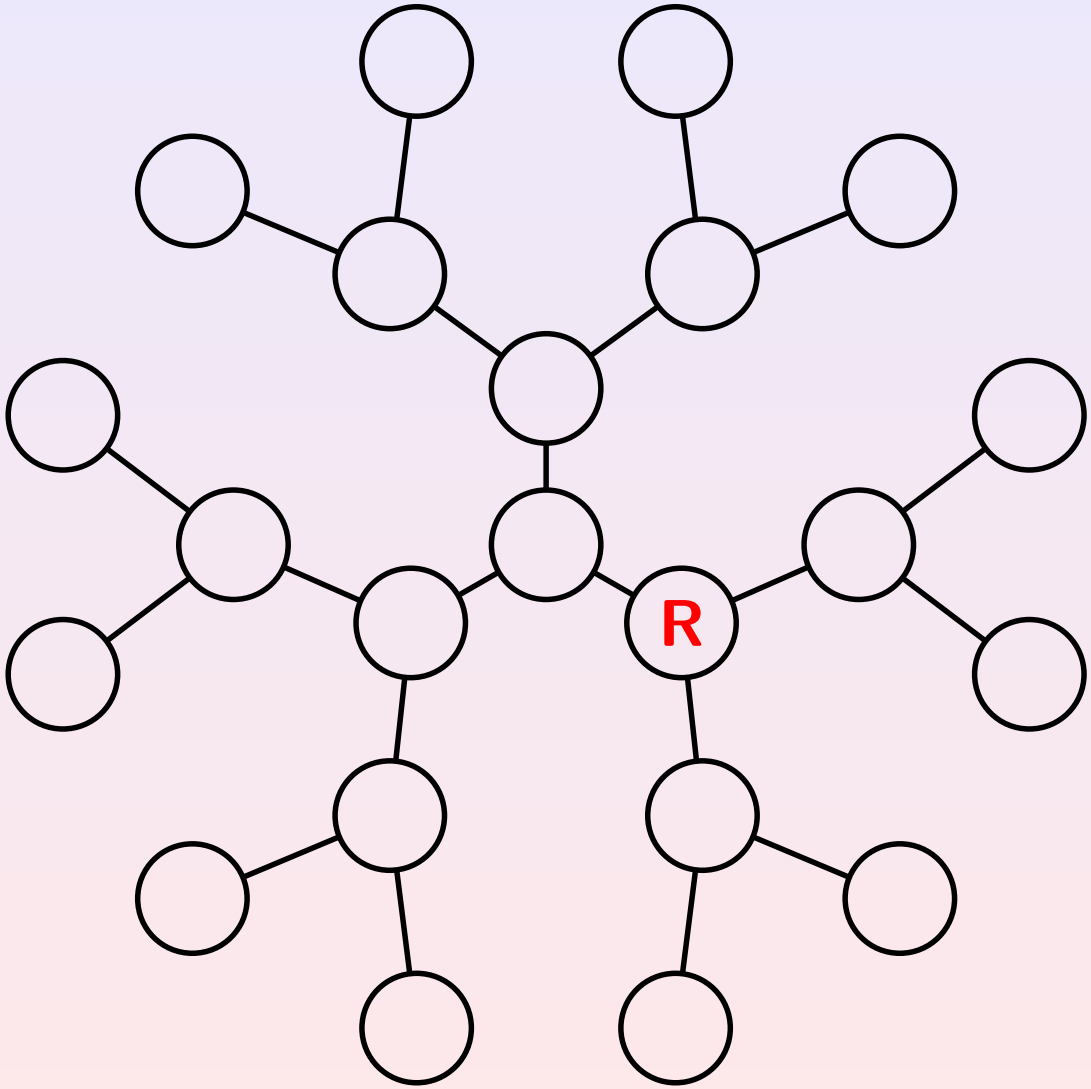

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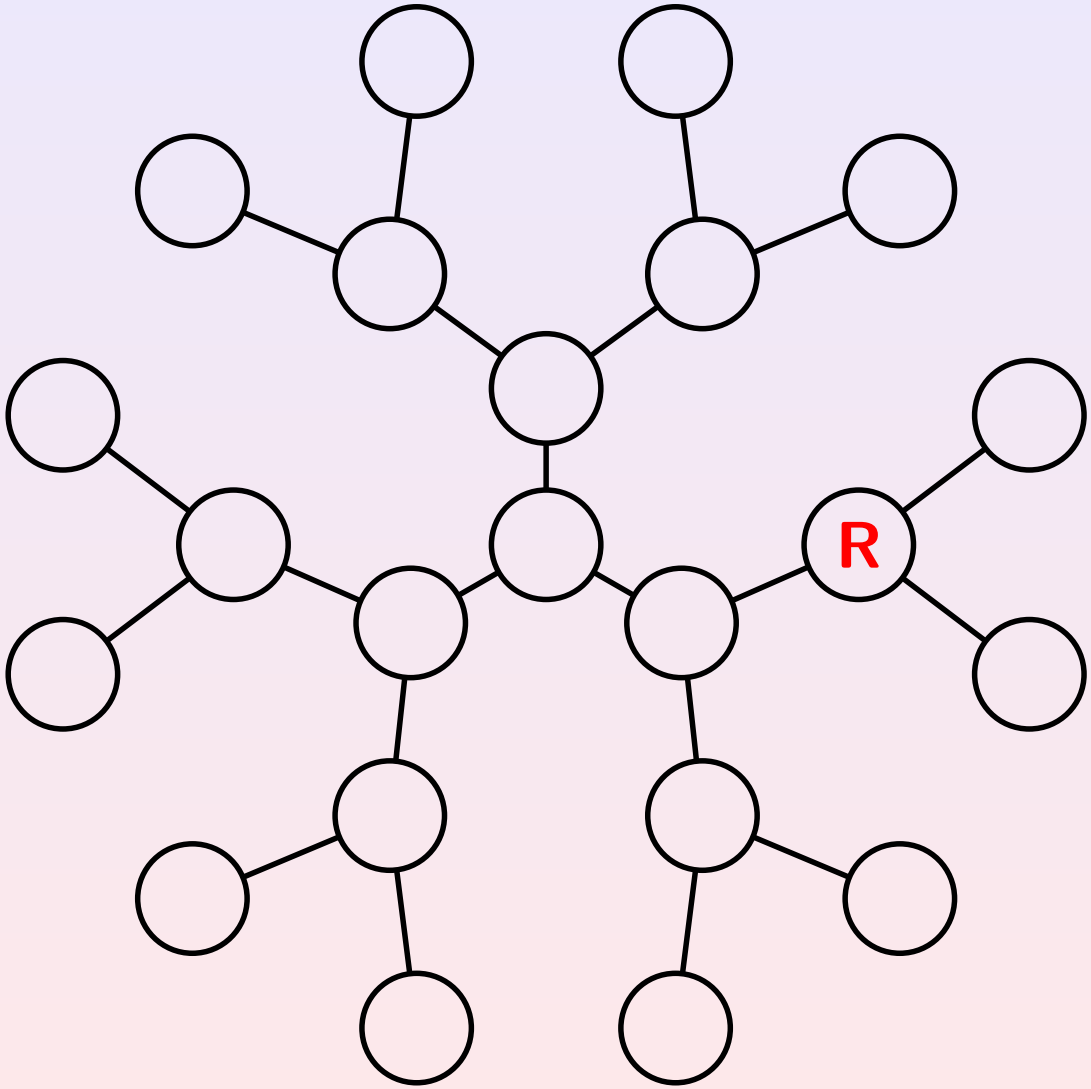

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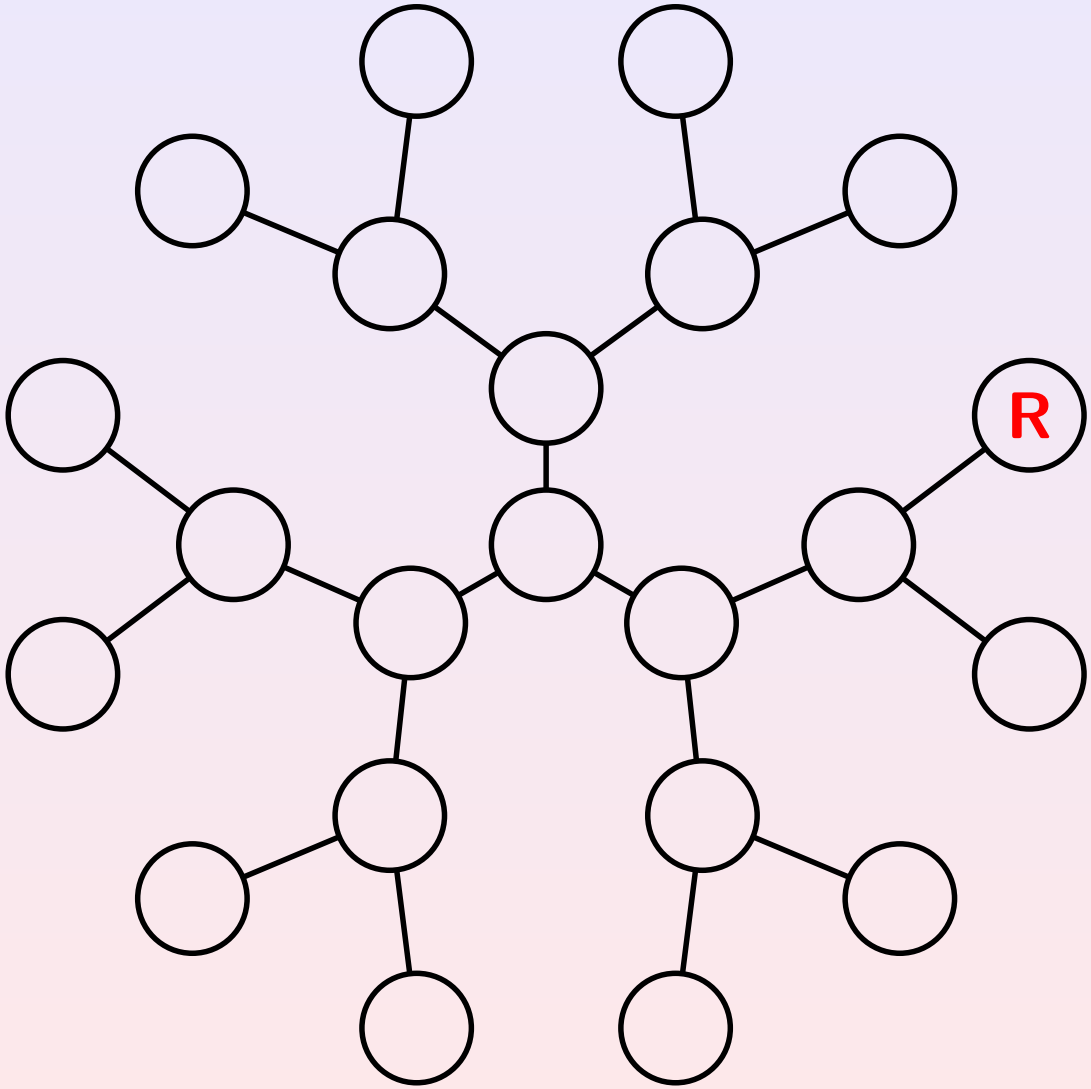

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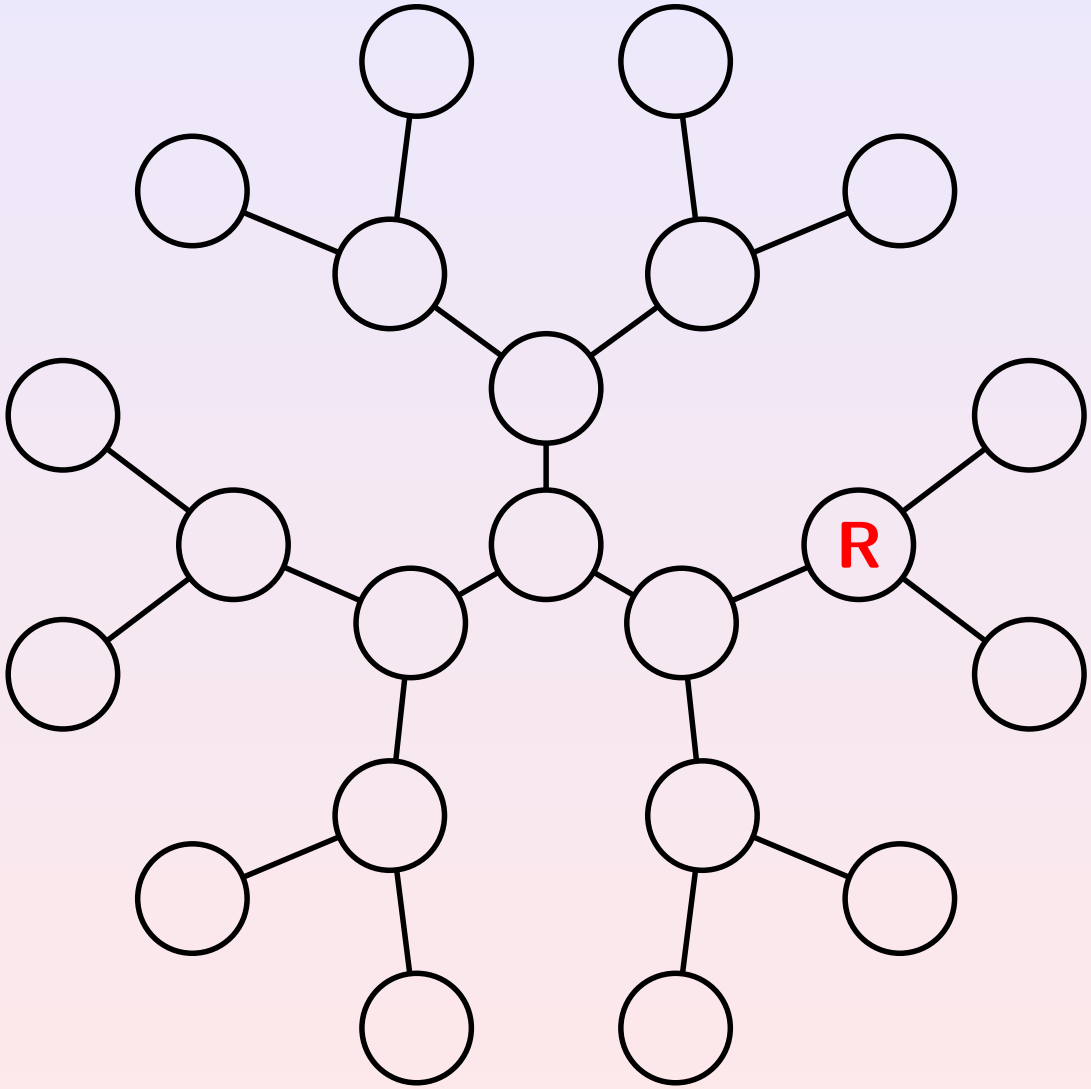

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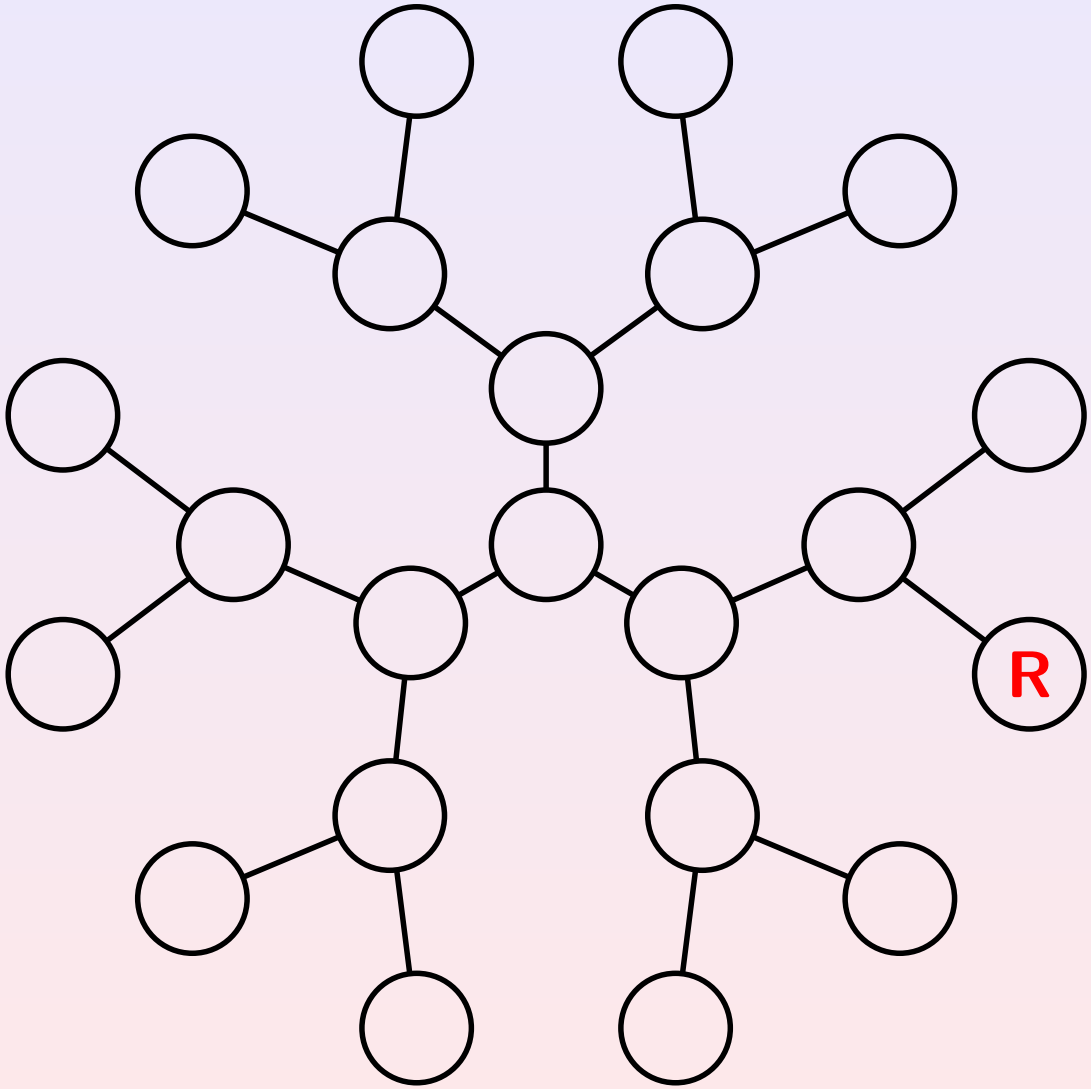

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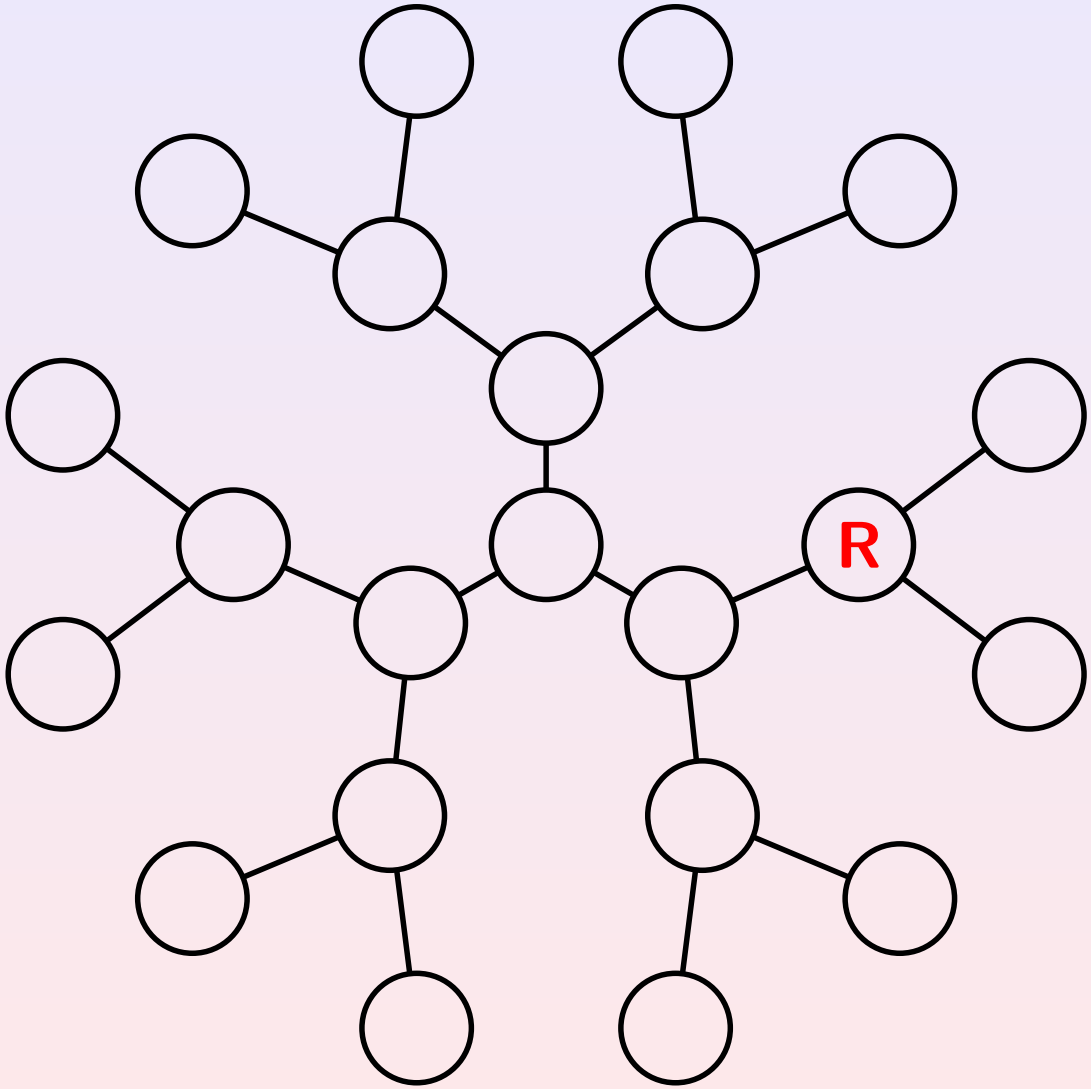

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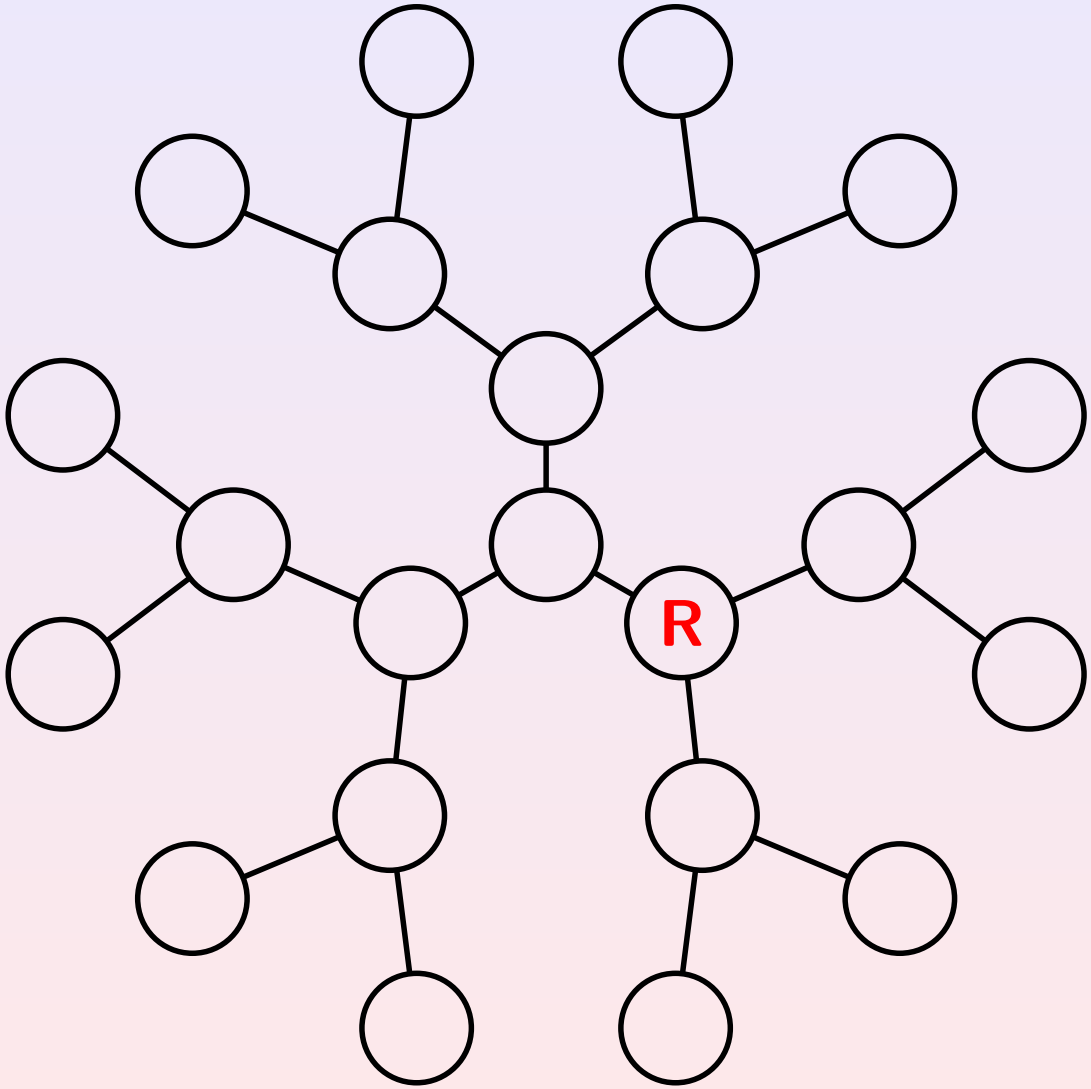

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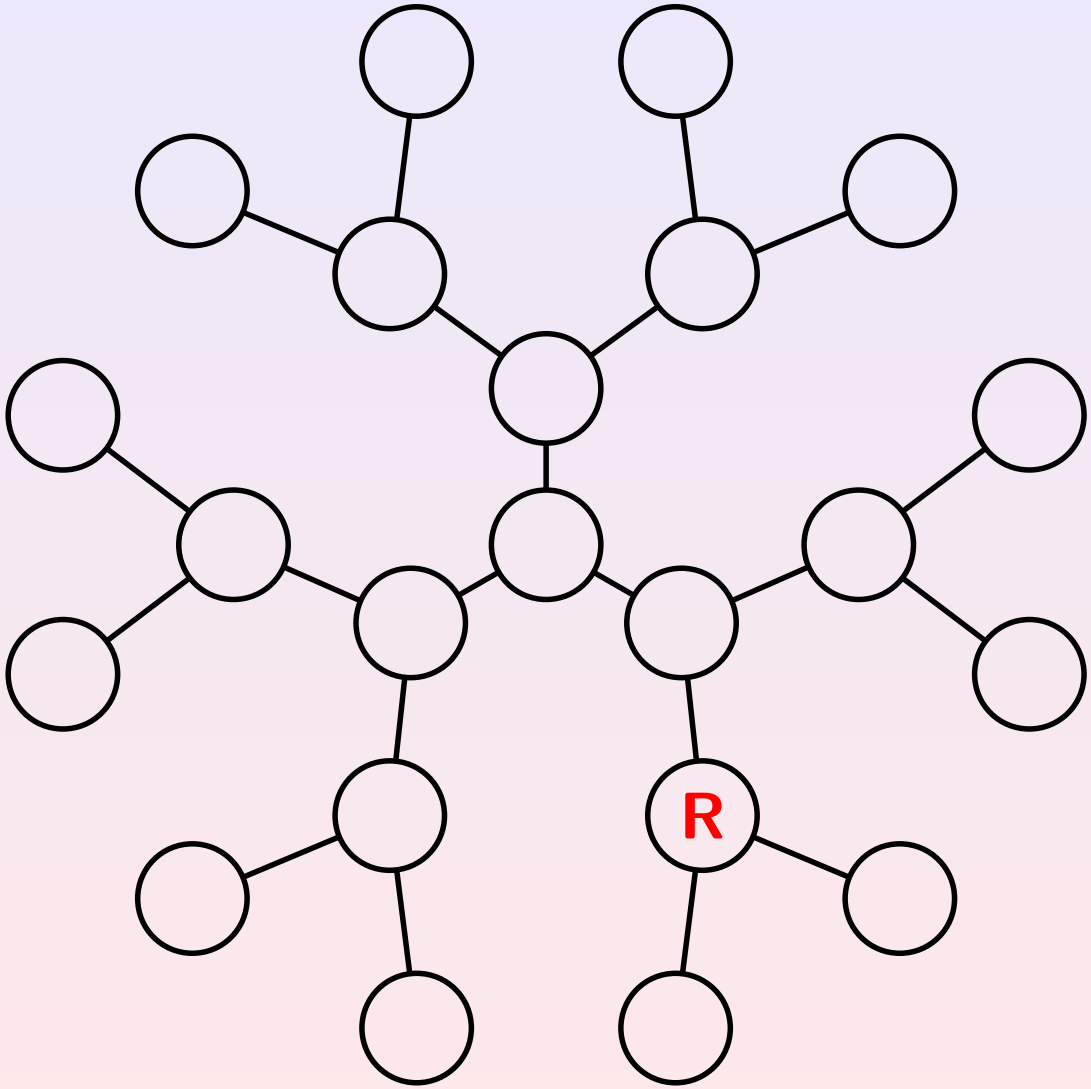

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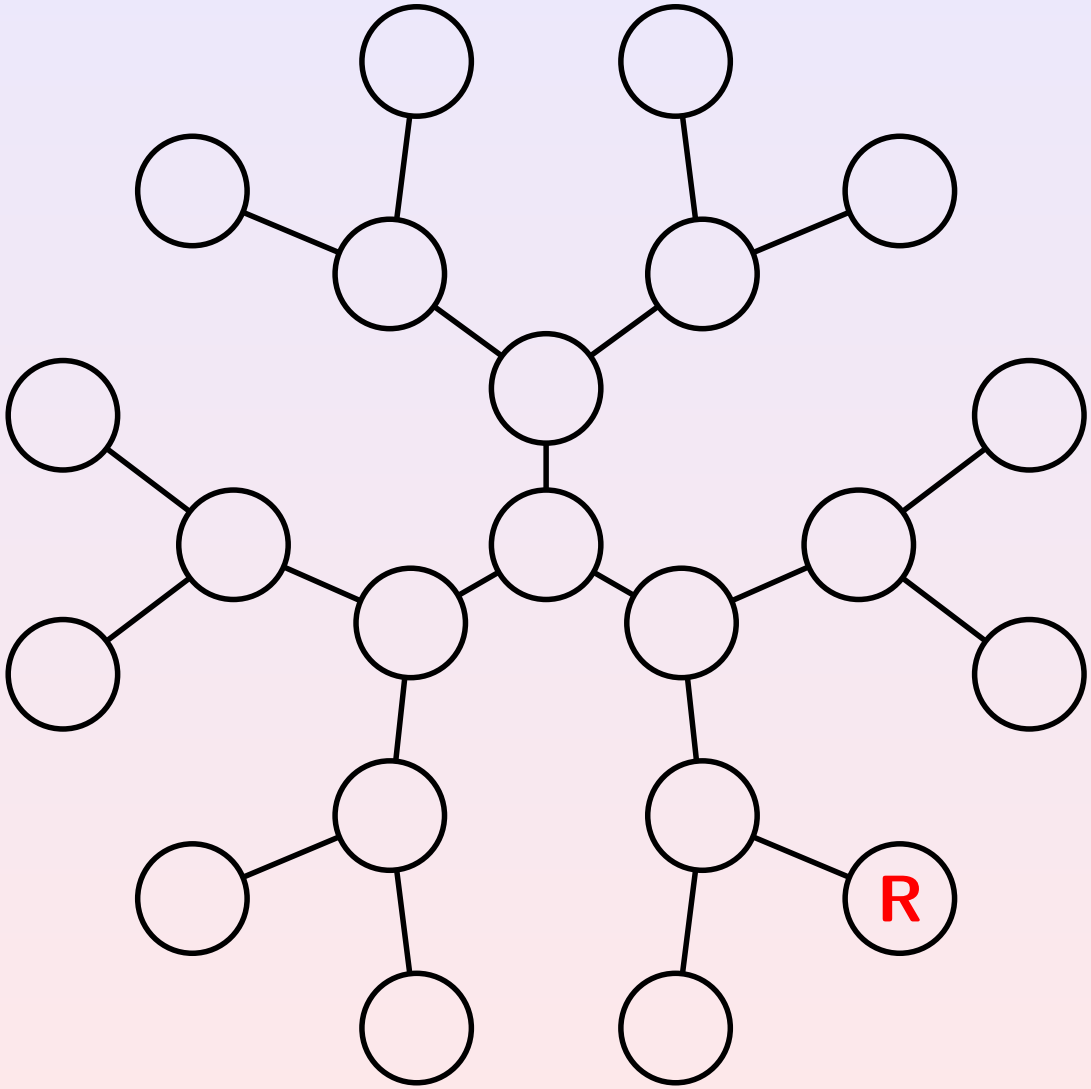

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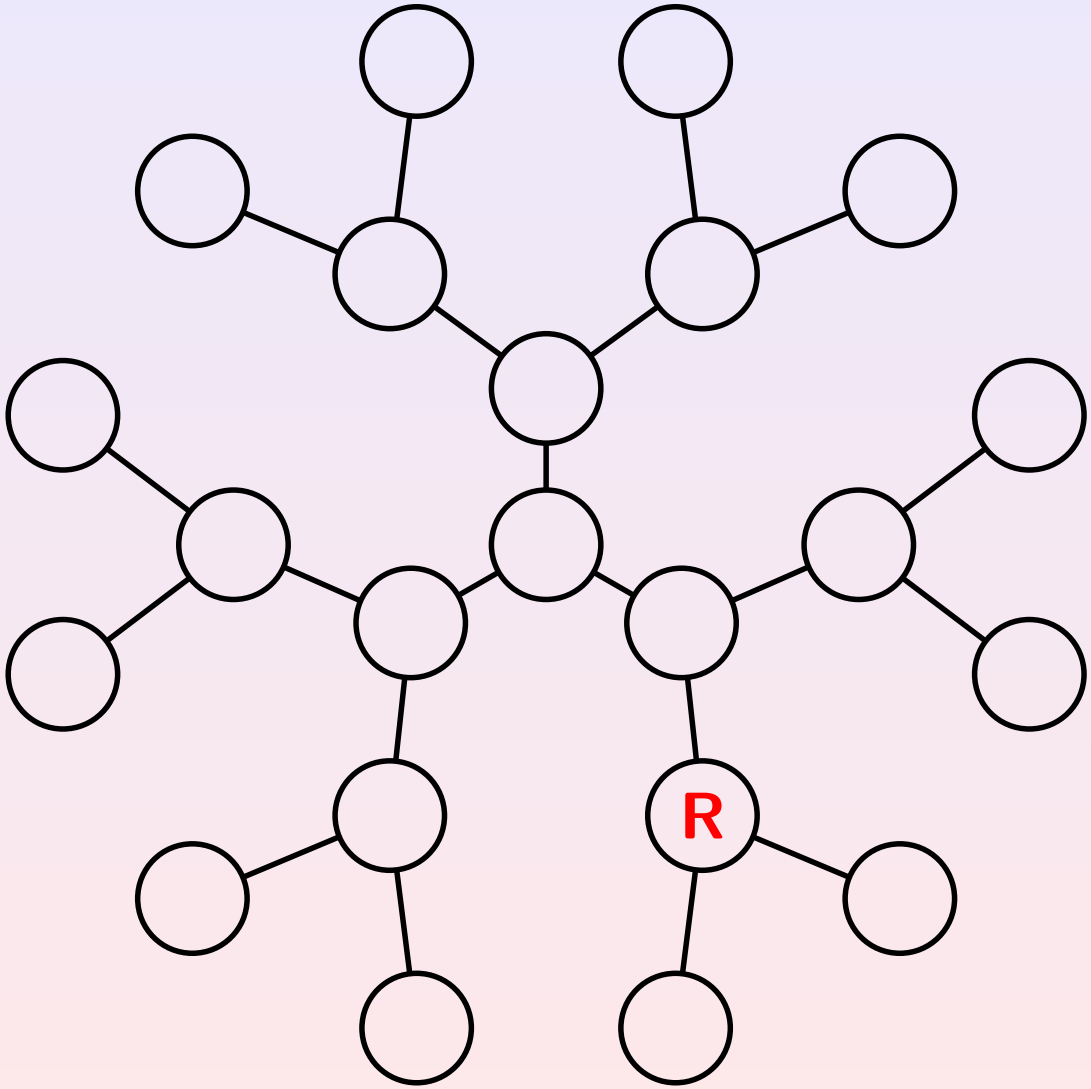

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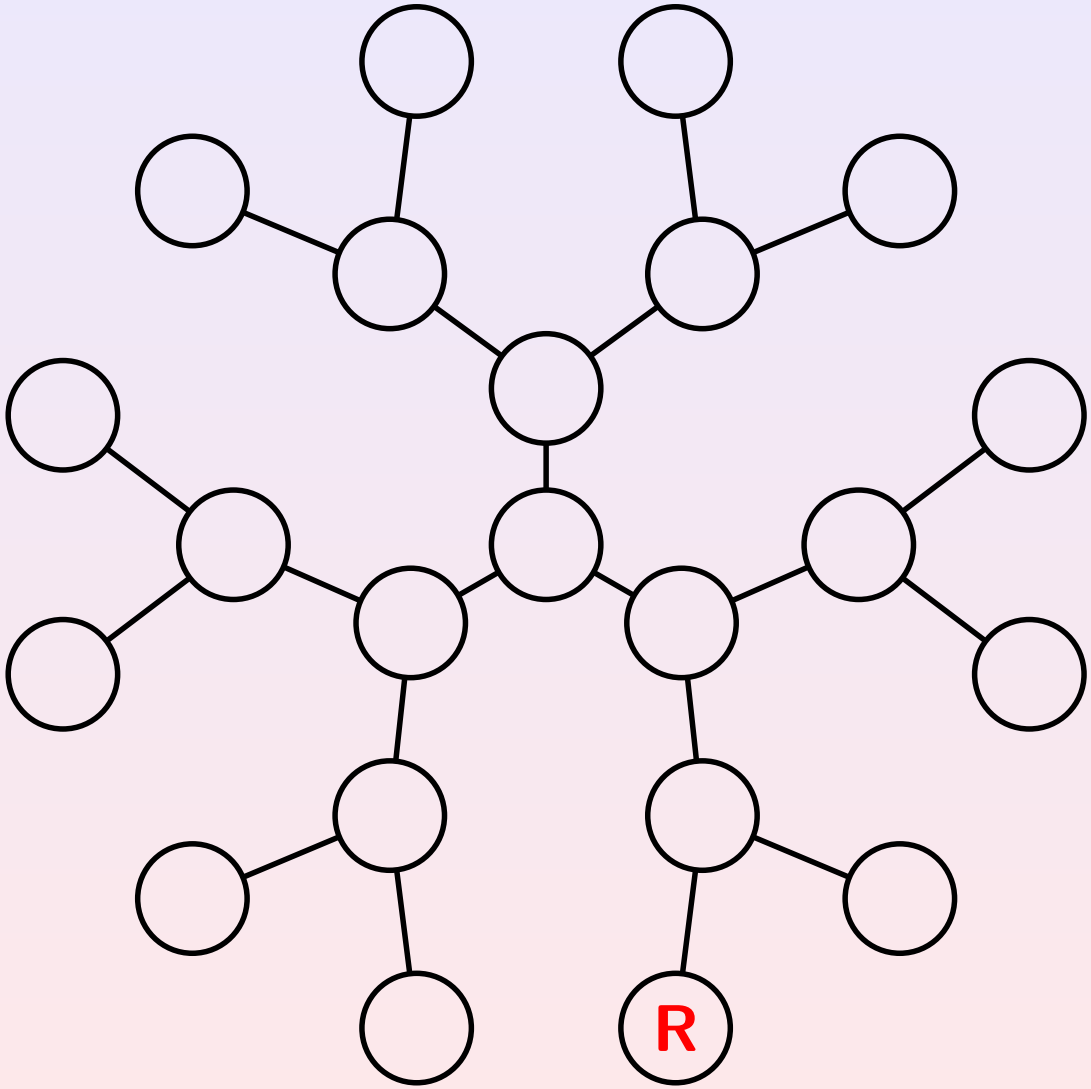

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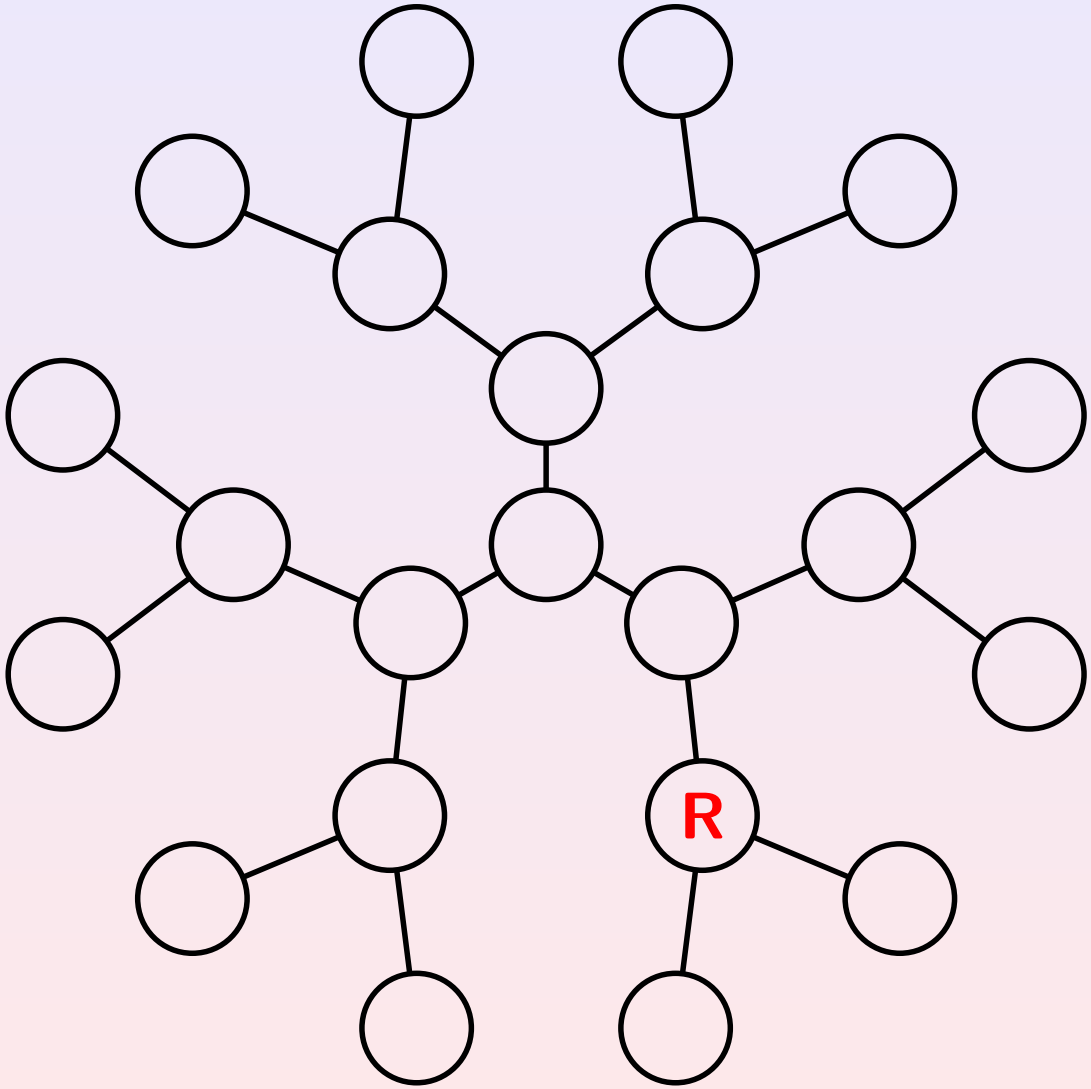

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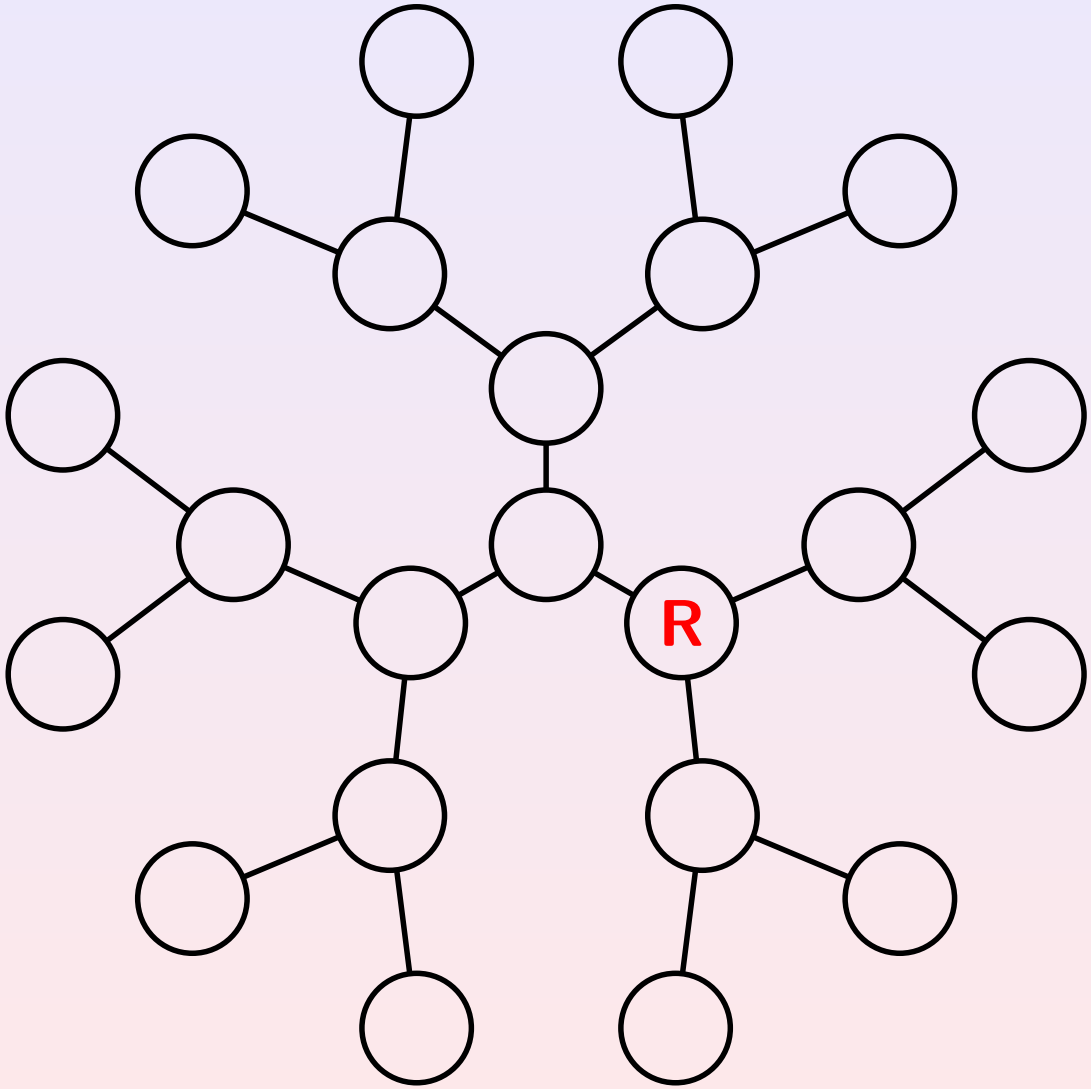

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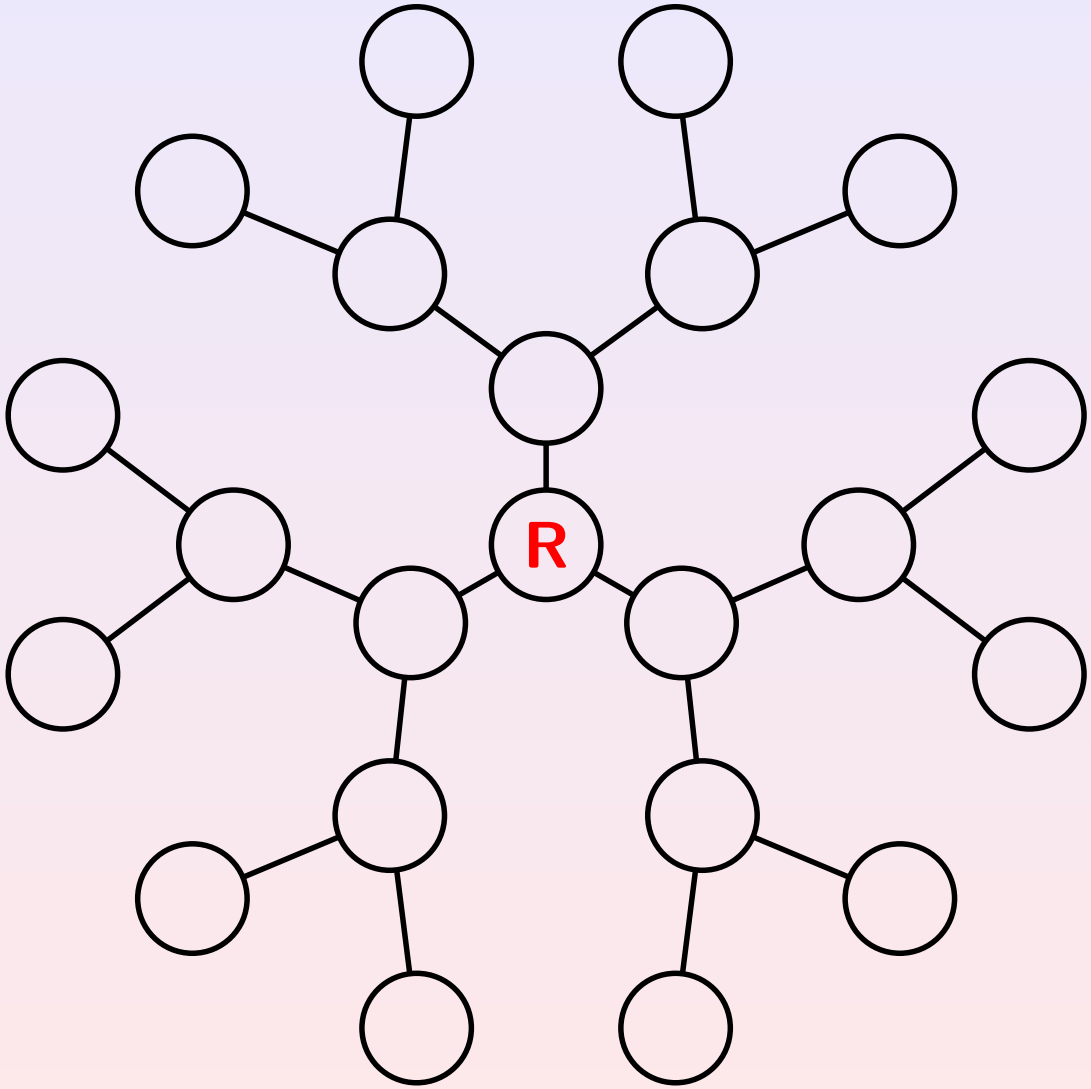

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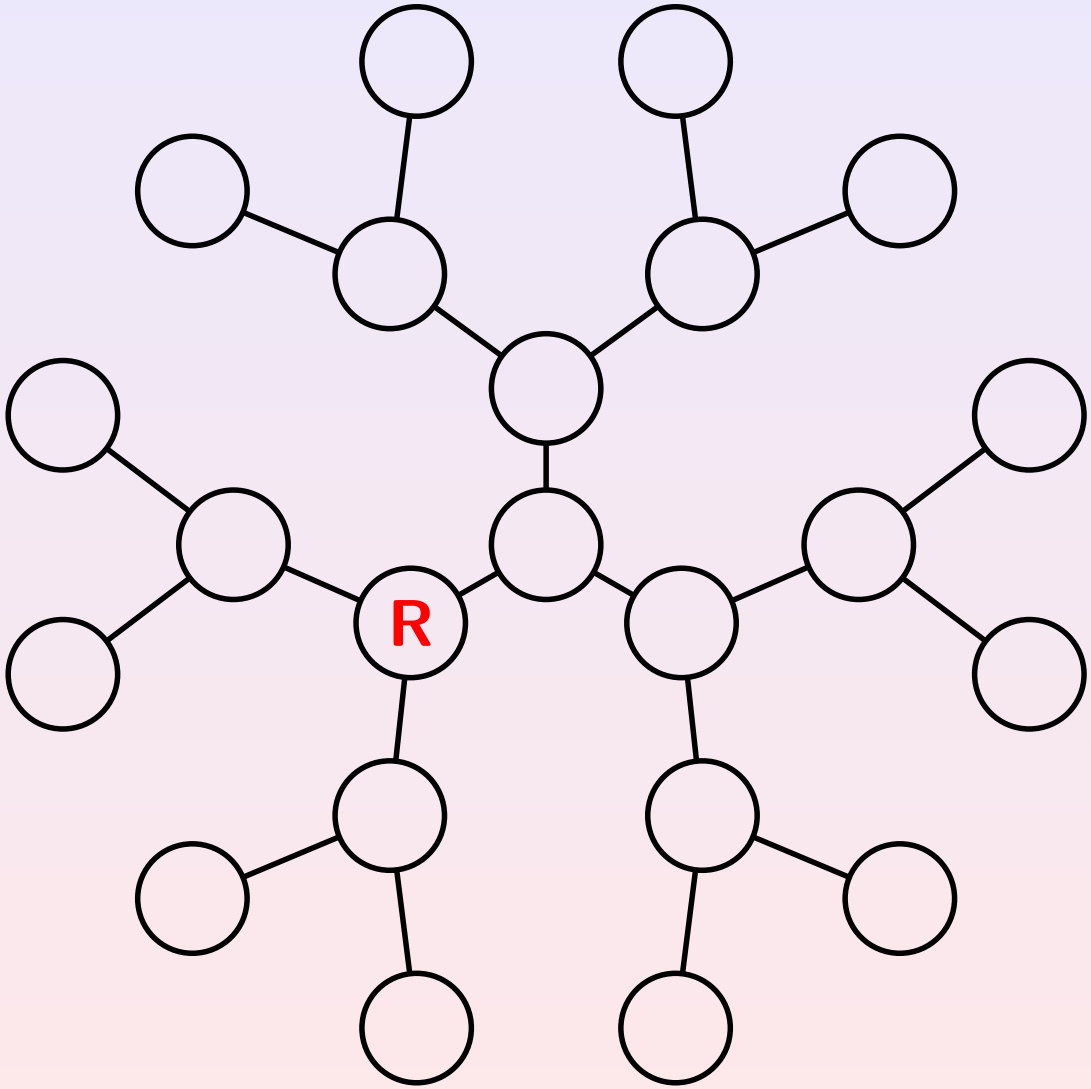
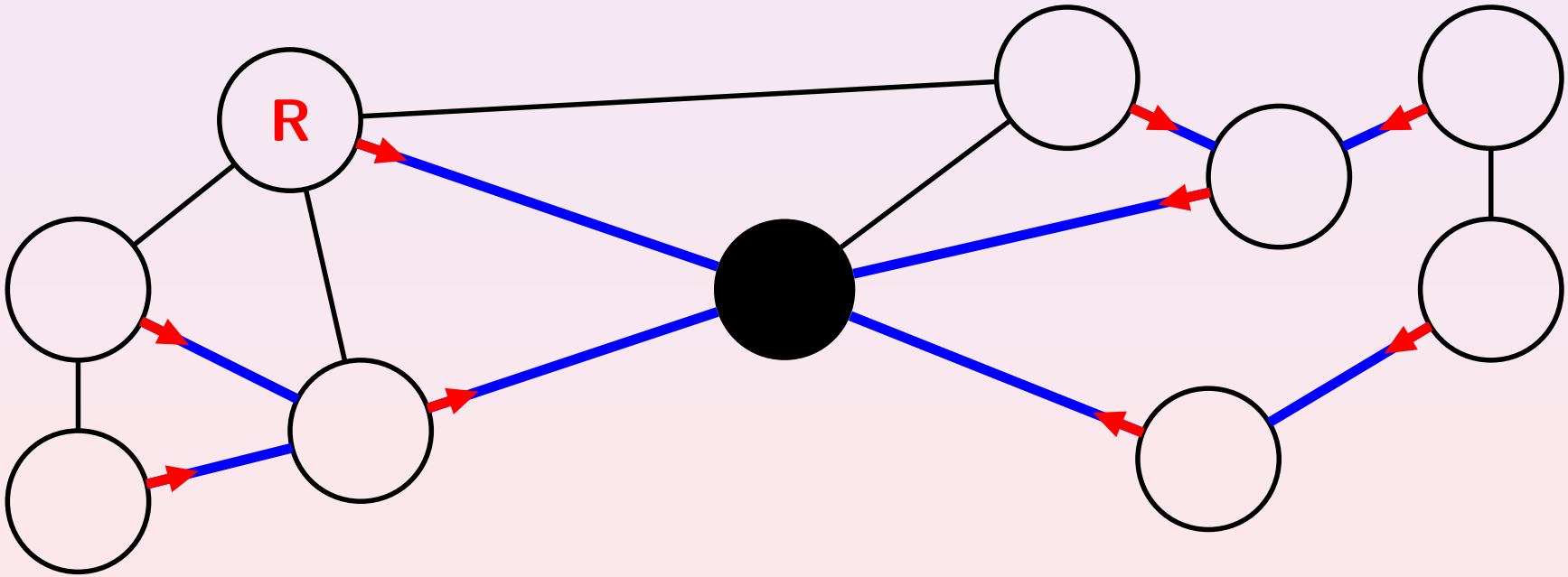


Applications: graph exploration (2)

Goal

A mobile entity has to traverse every edge of an unknown anonymous graph.



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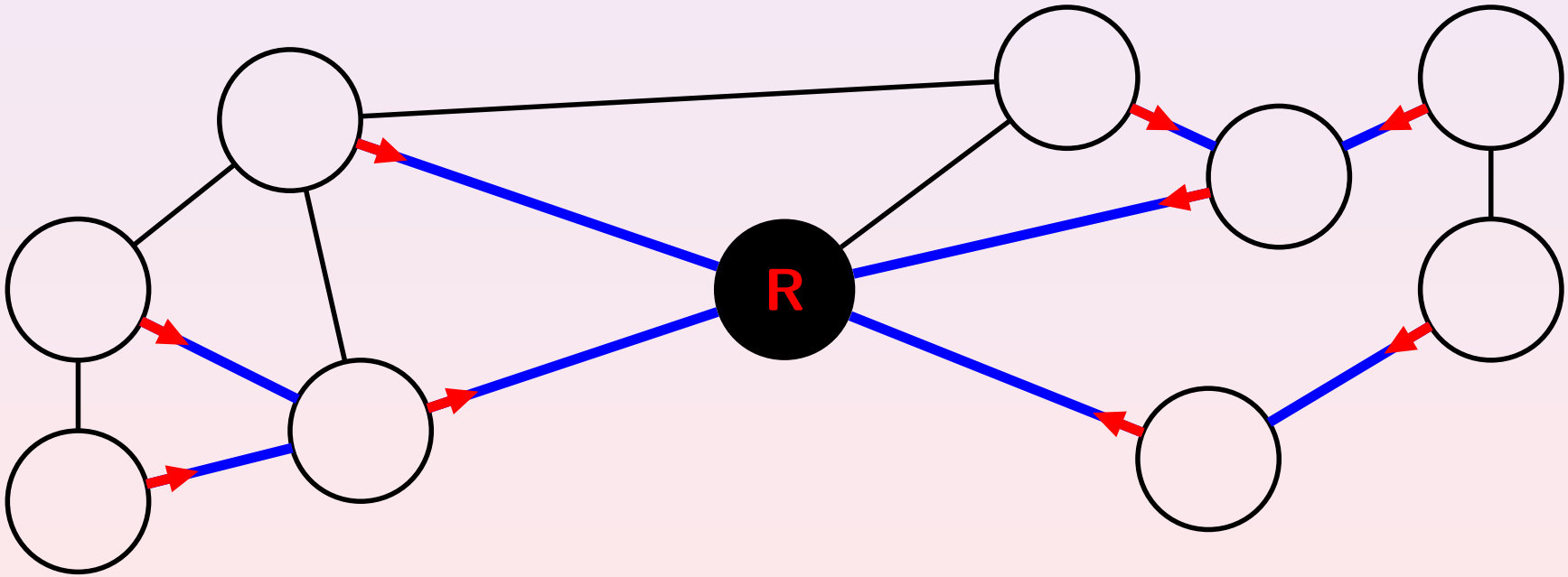

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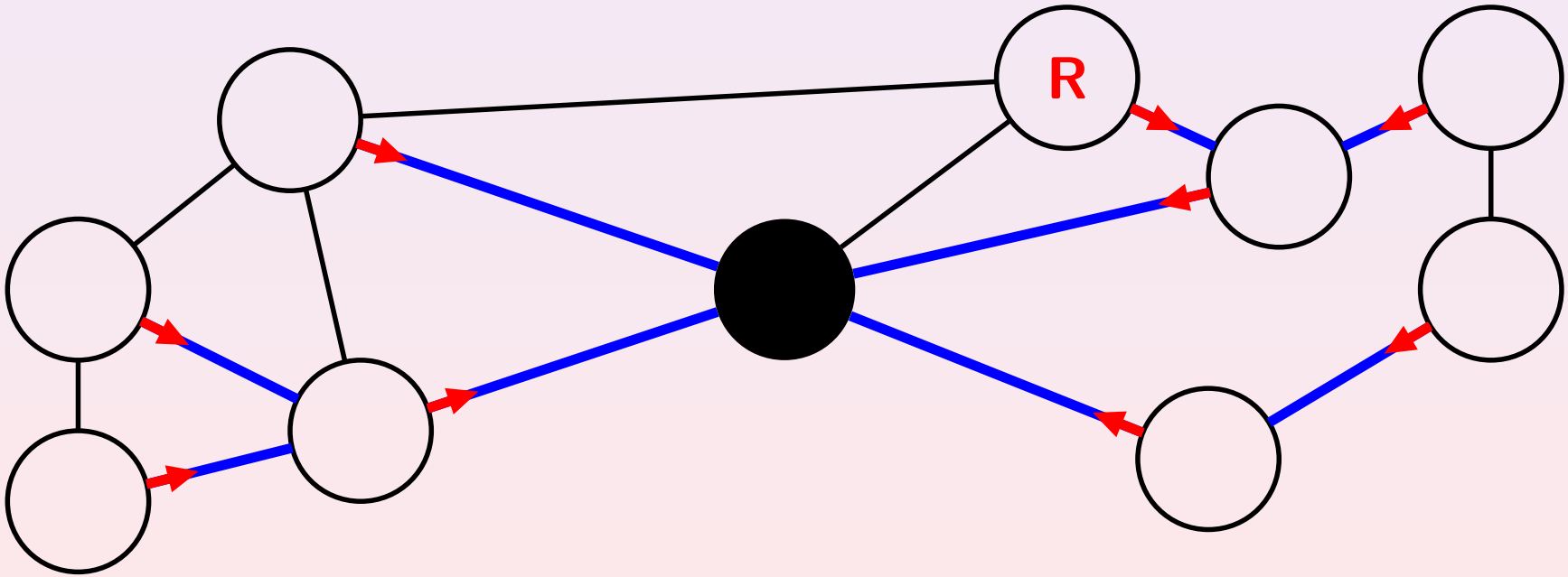

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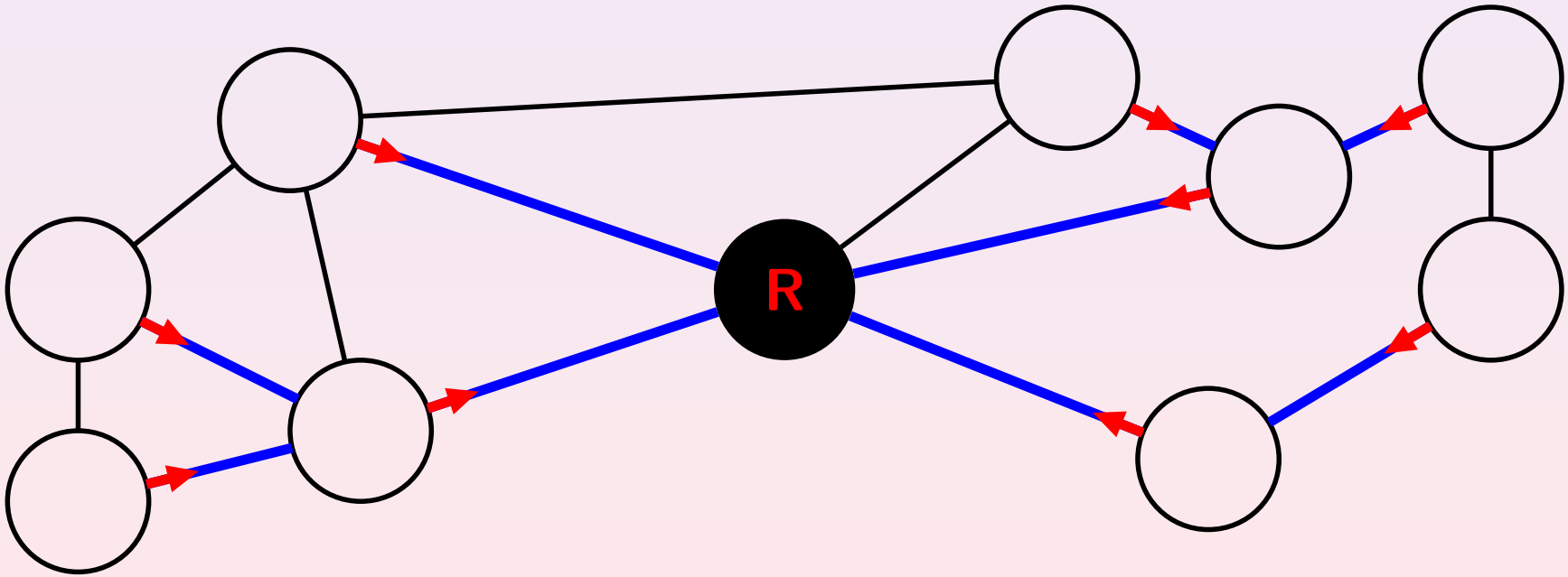

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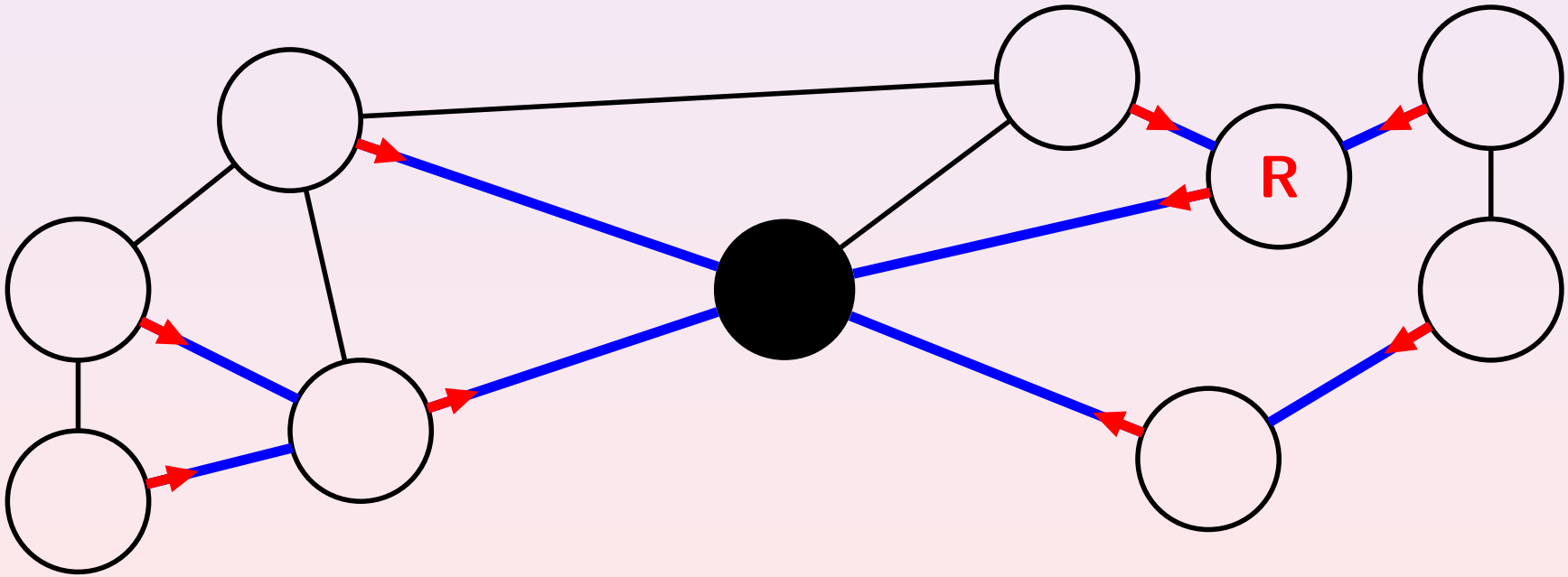

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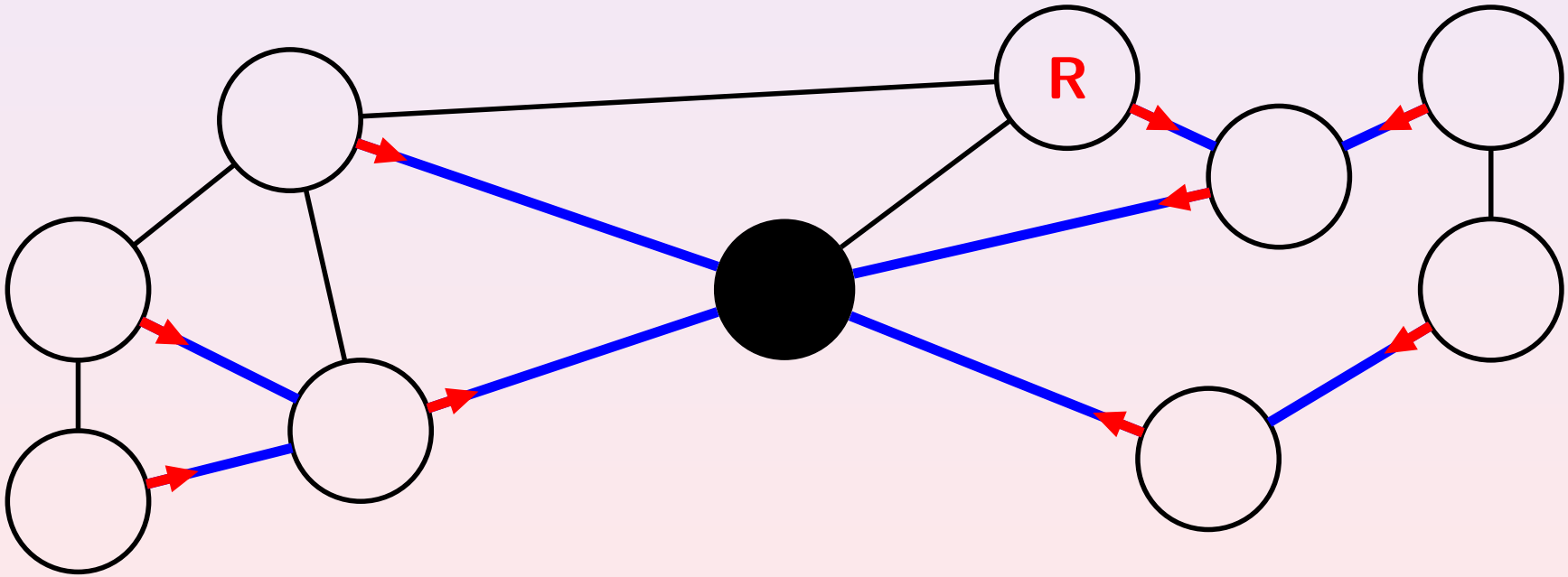

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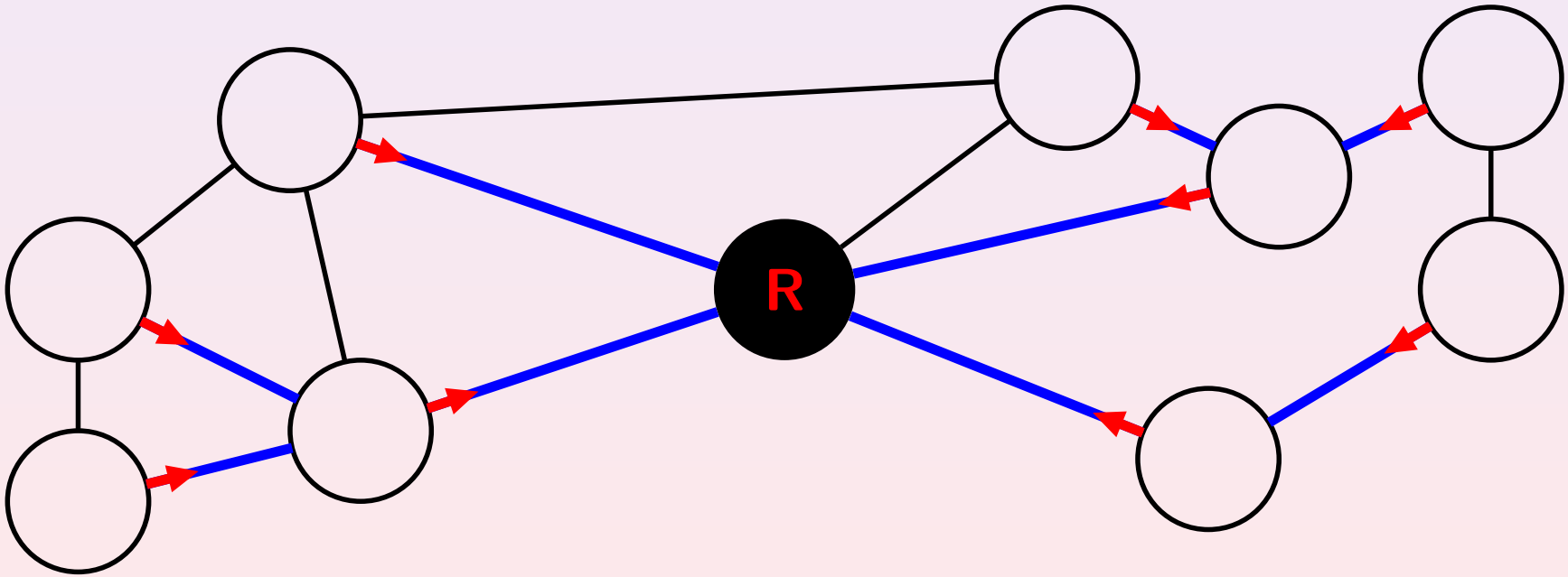

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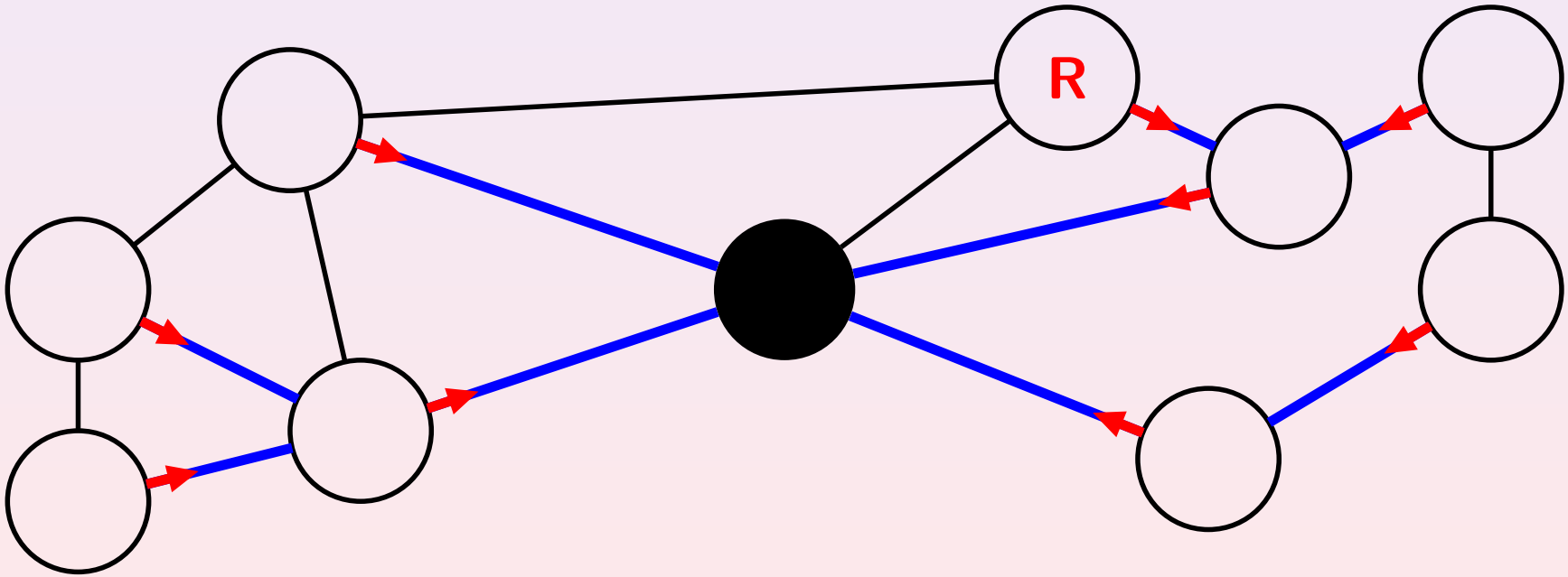

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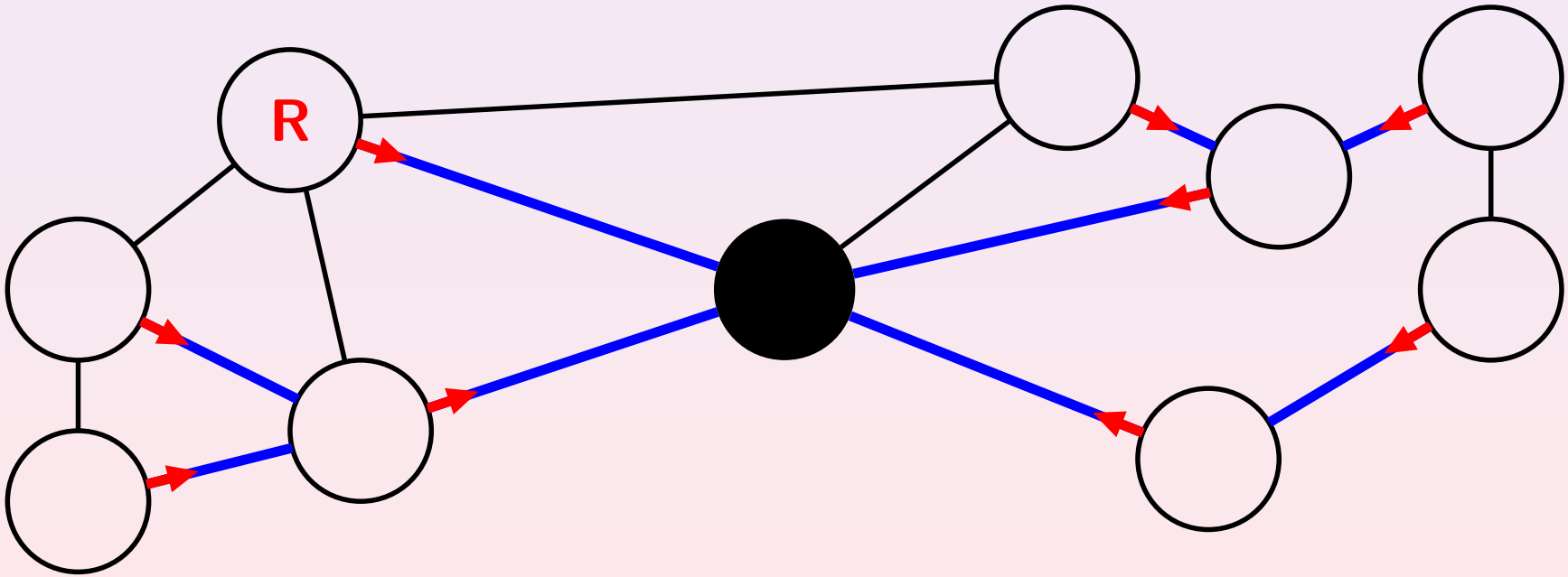

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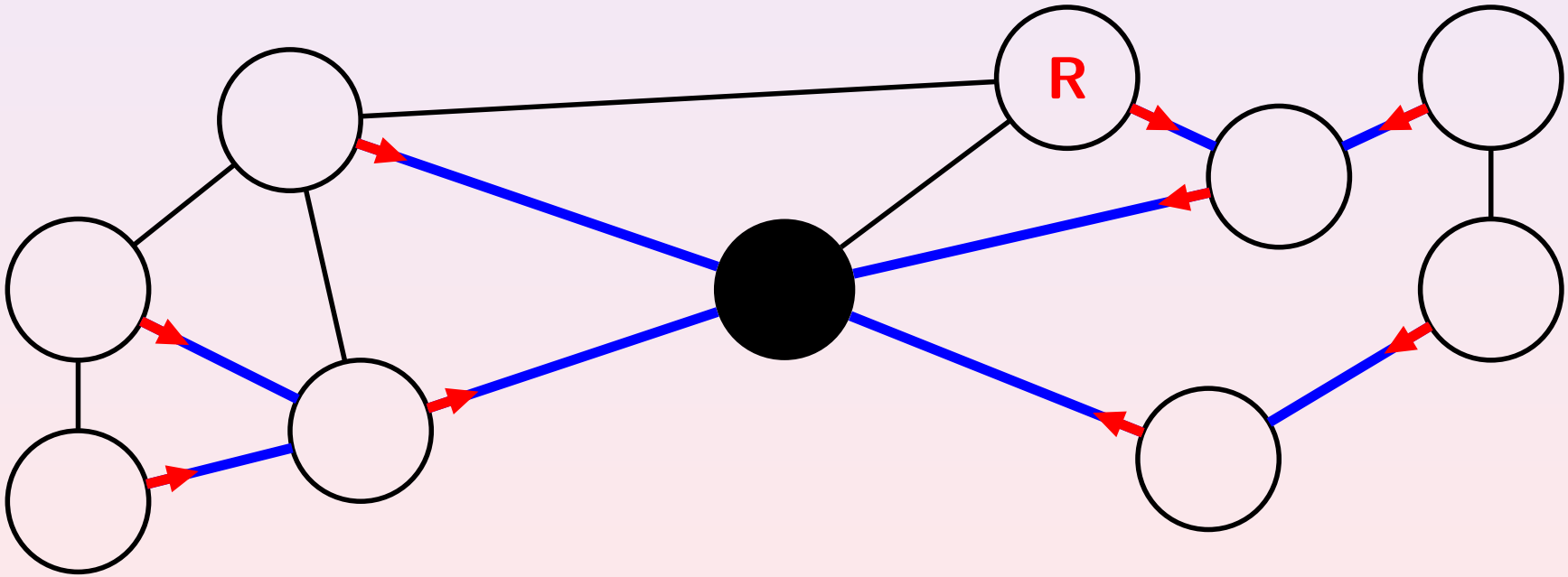

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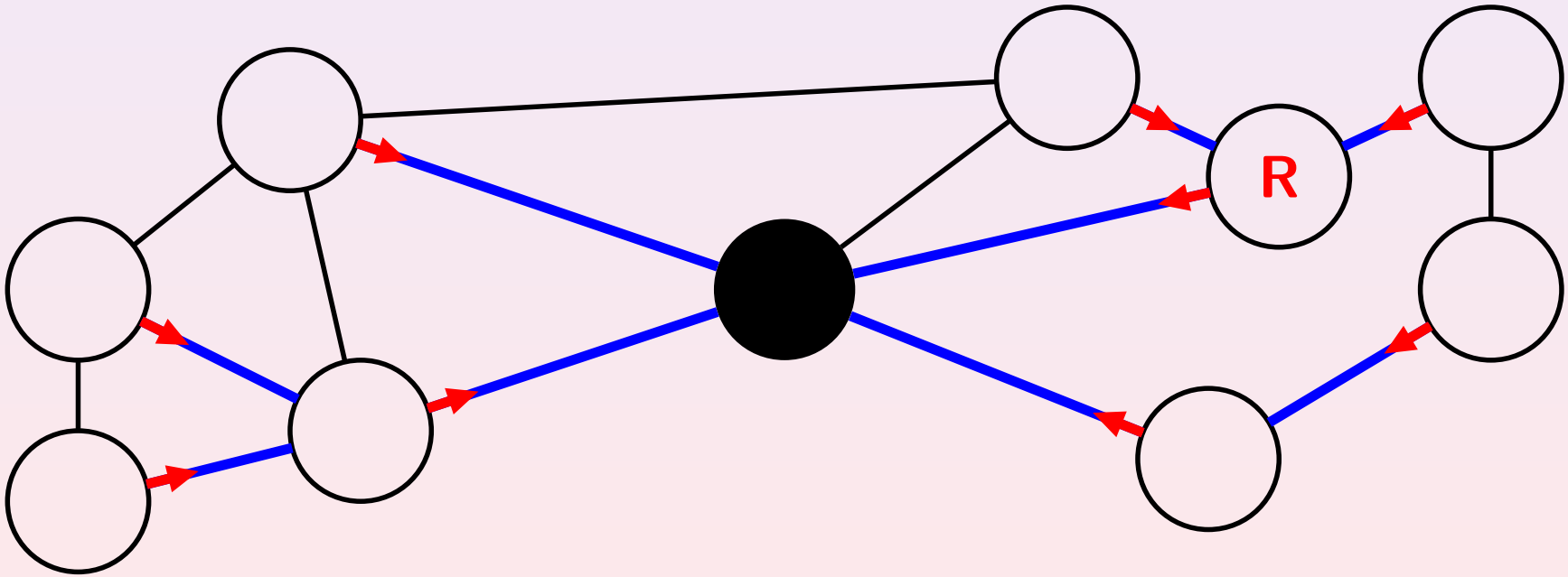

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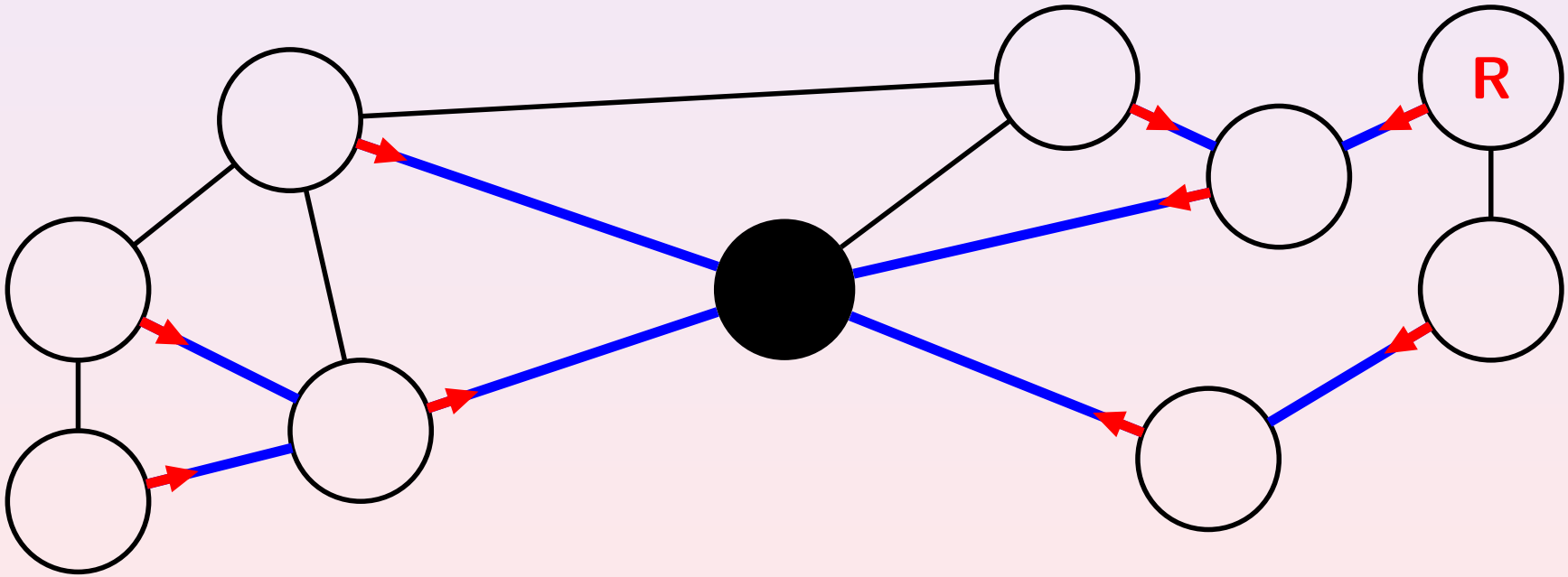

Naive upper bounds

Trivial observations

- Maximum: $M_{up}(T) \leq \lceil \log \Delta \rceil$, $\Delta =$ maximum degree
- Sum: $S_{up}(T) \leq \sum_v \lceil \log \deg(v) \rceil \longrightarrow O(n \log \Delta)$ bits

The bound on $M_{up}(T)$ is tight.

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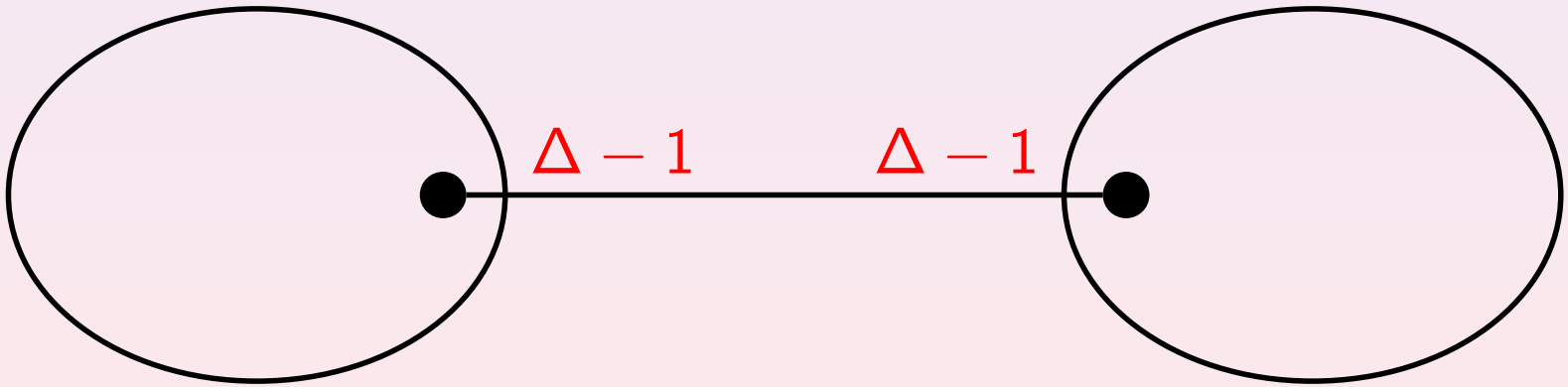

Optimal algorithm

Remark

There is a polynomial time algorithm that given a graph G and a port numbering, construct a spanning tree T for G minimizing $S_{up}(T)$.

Use a Directed Minimum Spanning Tree algorithm.

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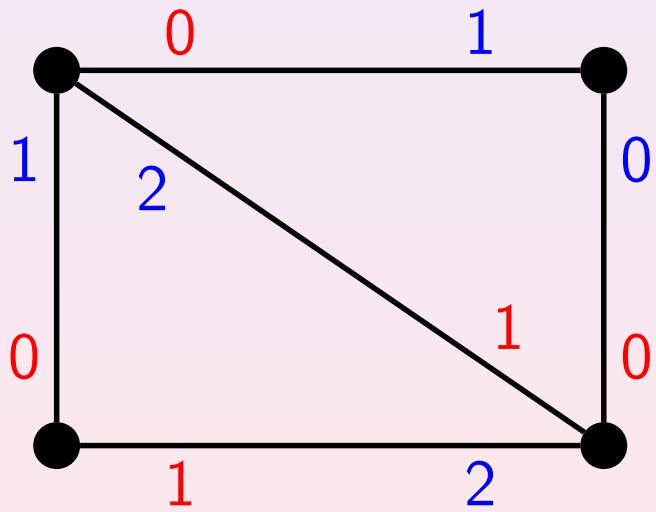

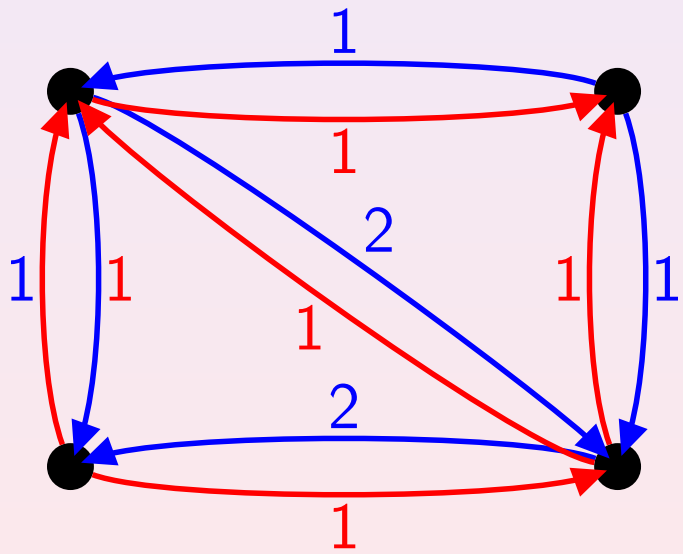

Use a Directed Minimum Spanning Tree algorithm.

Optimal algorithm

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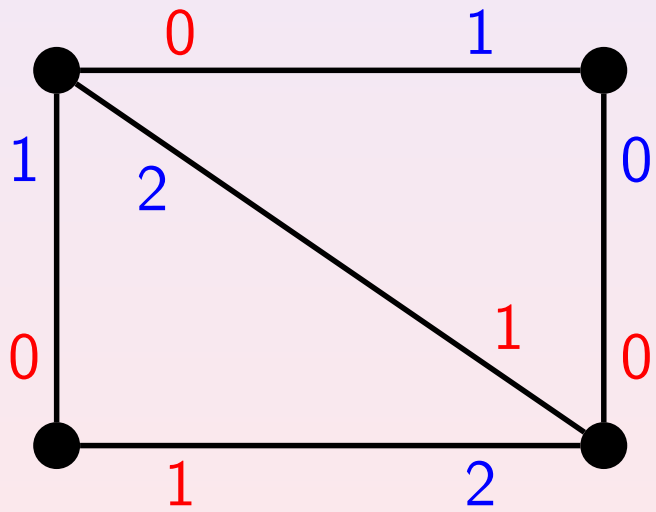

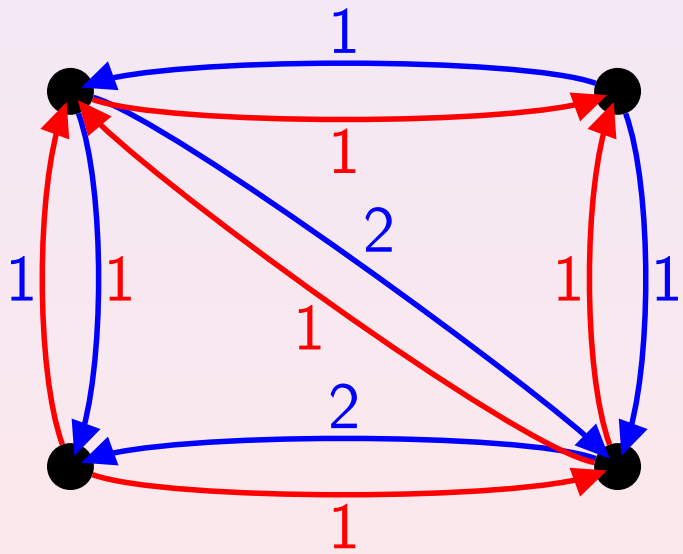

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Our results (1)

Theorem 1

In complete graphs with arbitrary labeling:

- There exists T such that $S_{up}(T) = O(n)$

Theorem 2

In arbitrary graphs with symmetric port assignments:

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Outline

Algorithm for complete graphs

Main idea

- Start with isolated vertices, the initial trees.
- Merge trees to finally obtain a spanning tree.

Description of phase k

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1. Identify the collection of small trees for the phase: $\mathcal{T}_{\text{small}}(k) = \{T \mid \text{size}(T) < 2^k\}$.
2. For each tree $T \in \mathcal{T}_{\text{small}}(k)$, select the edge $e(T)$ of minimum weight going from the root outside T.
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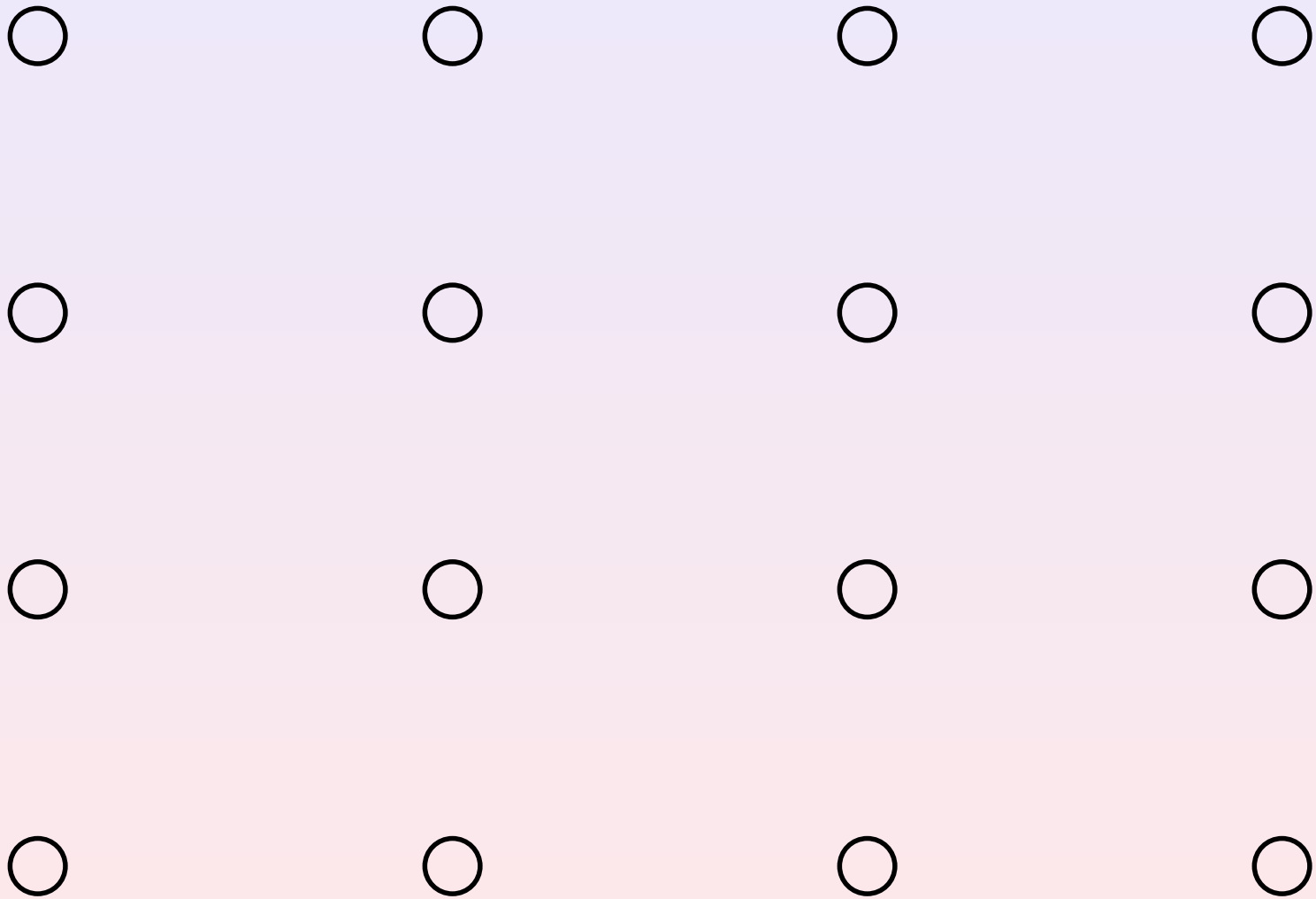

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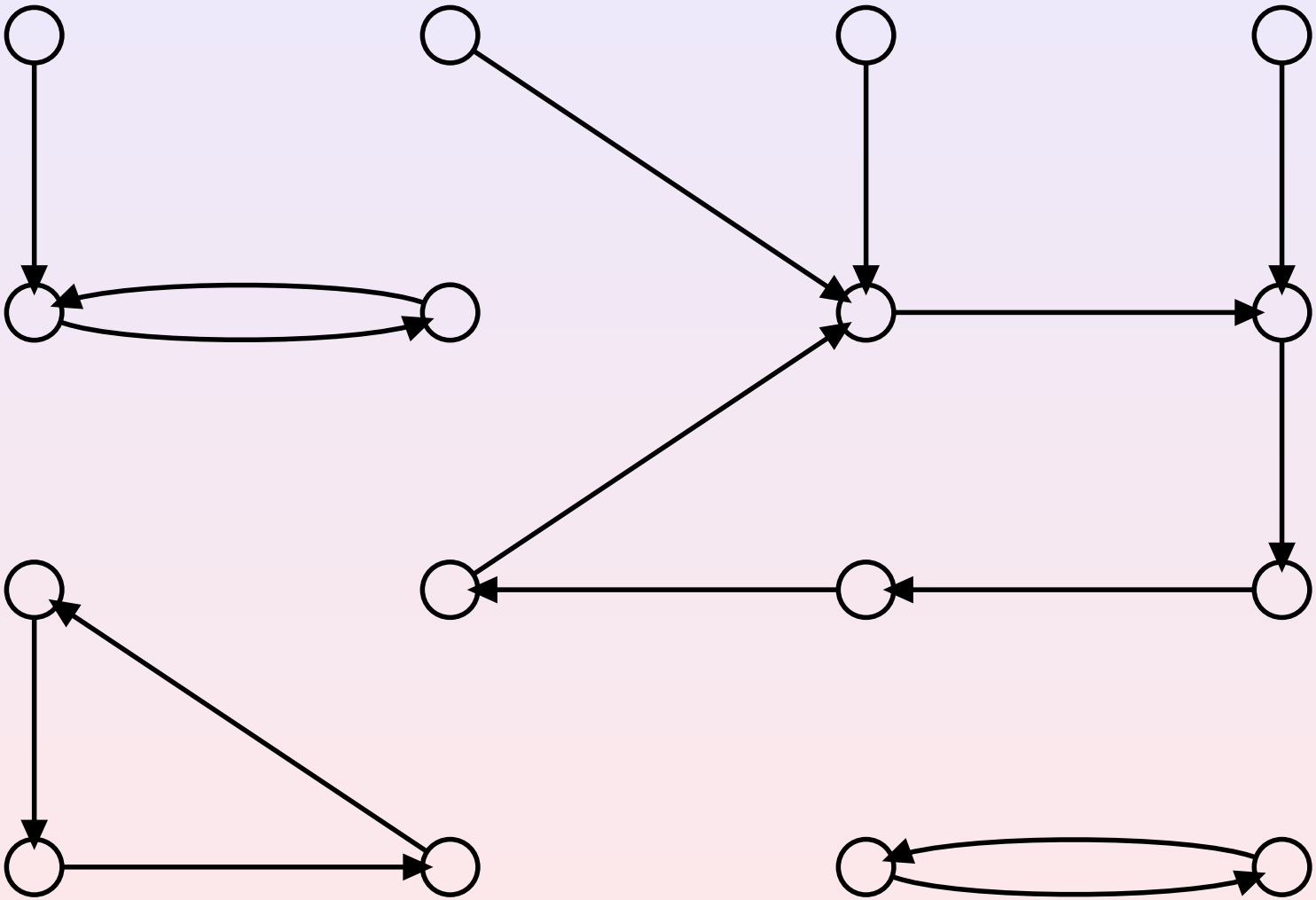

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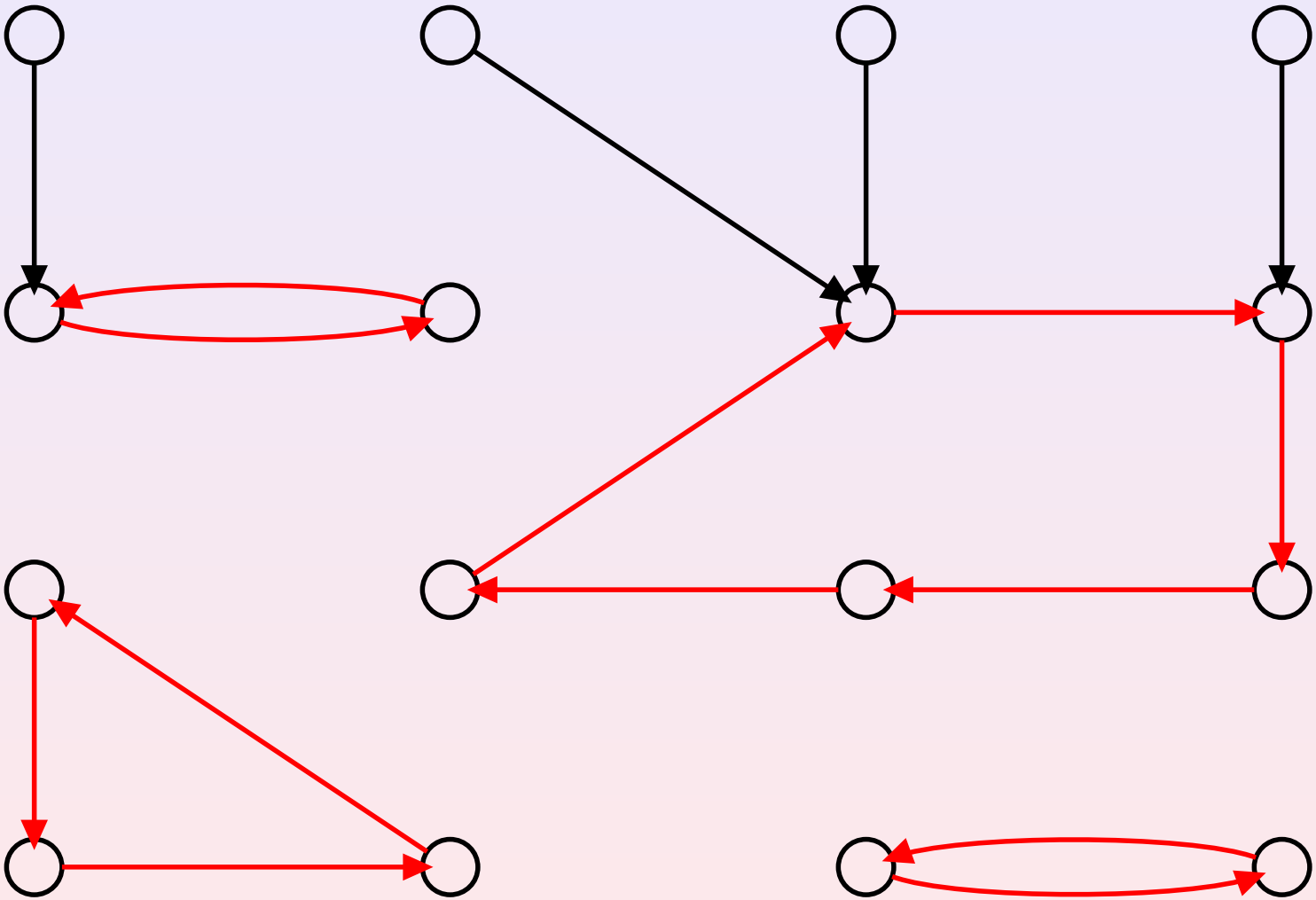

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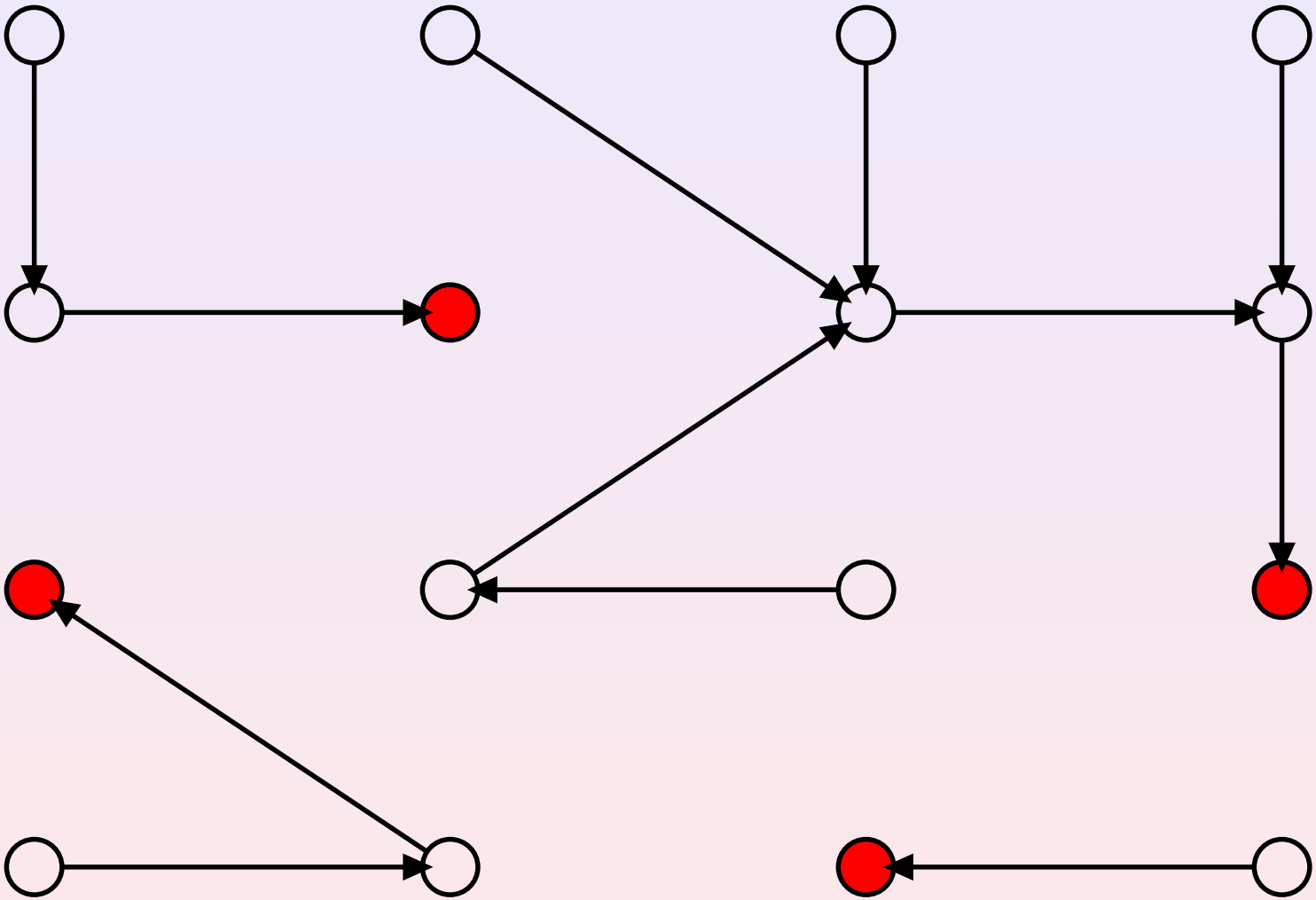

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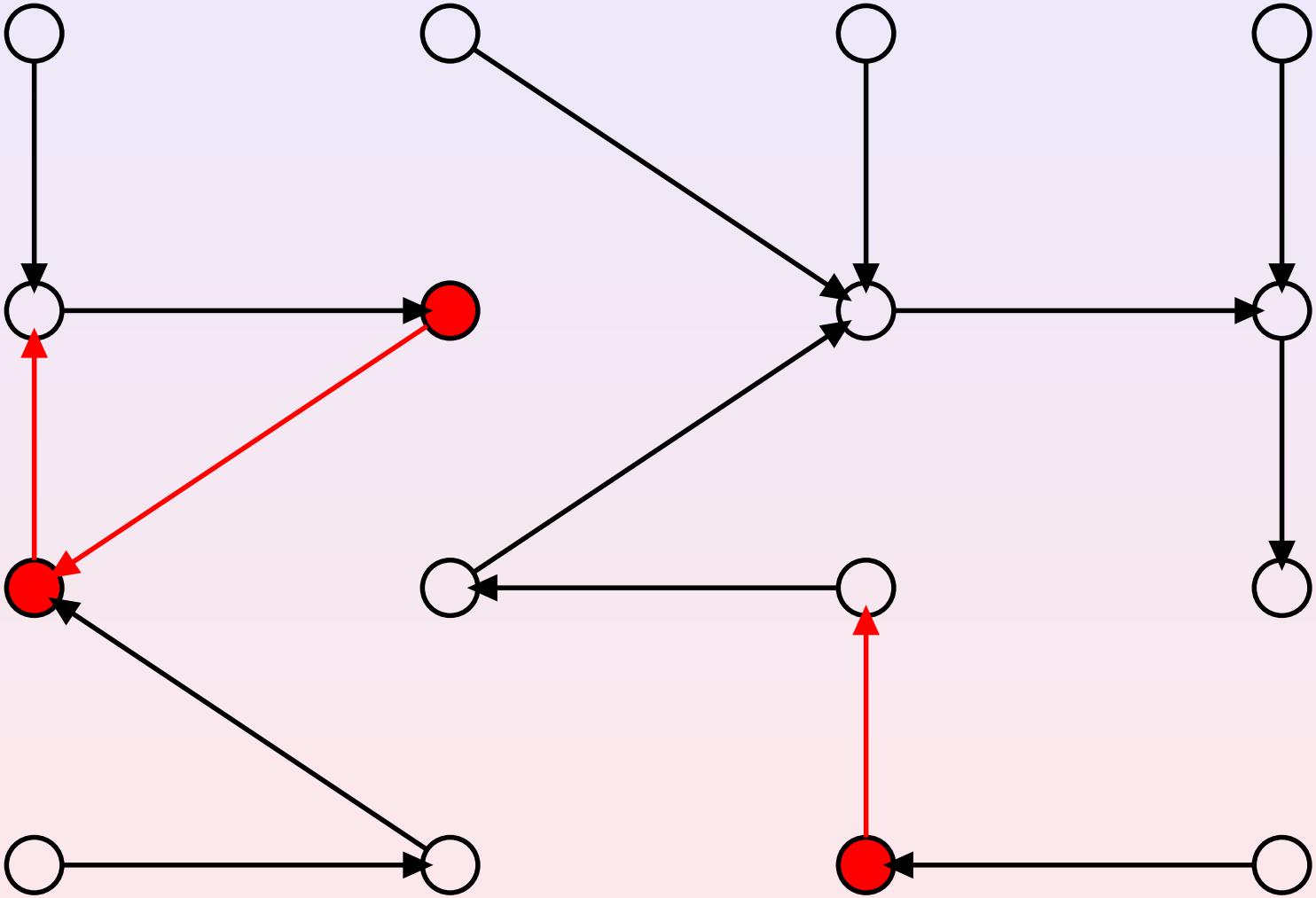

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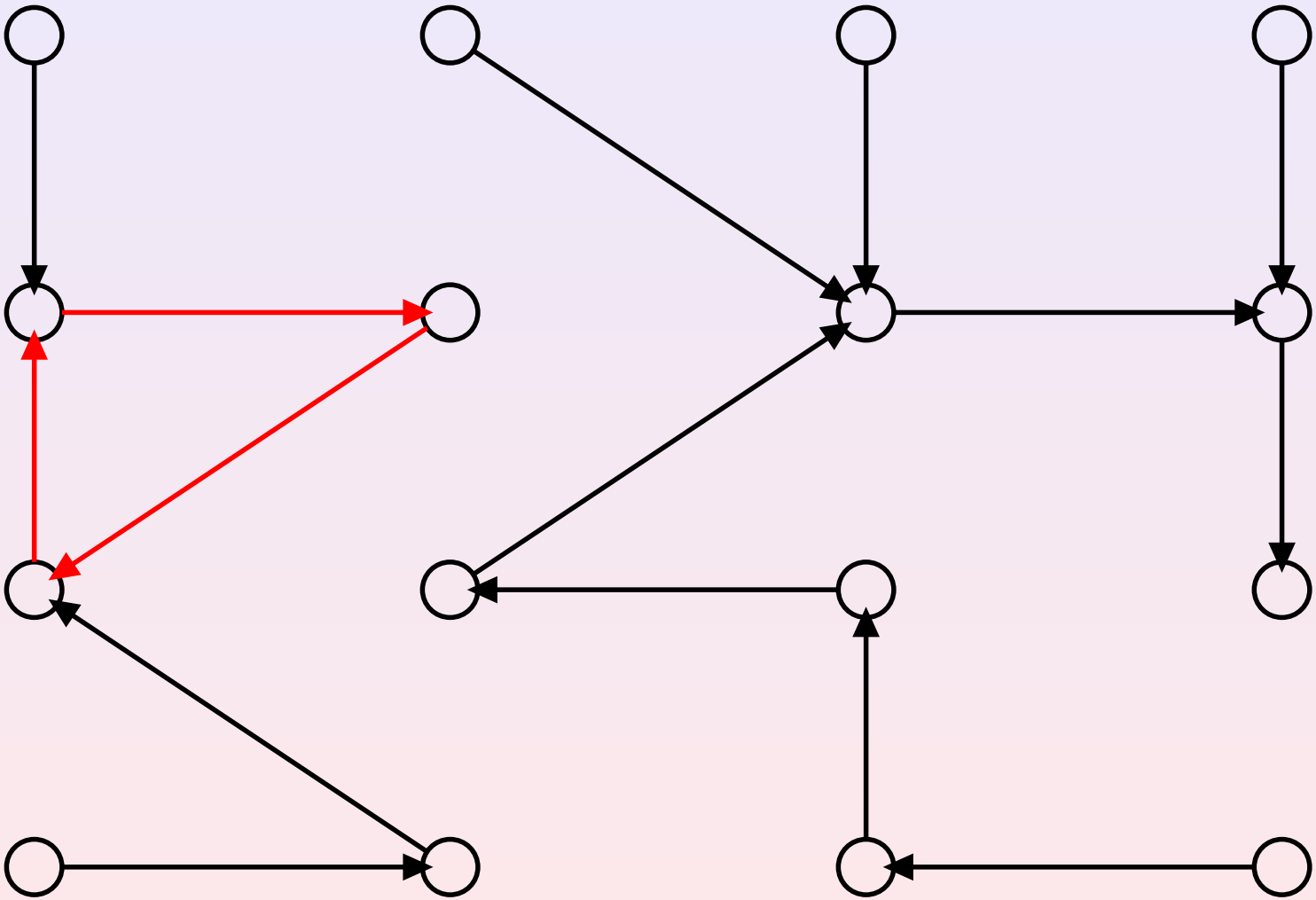

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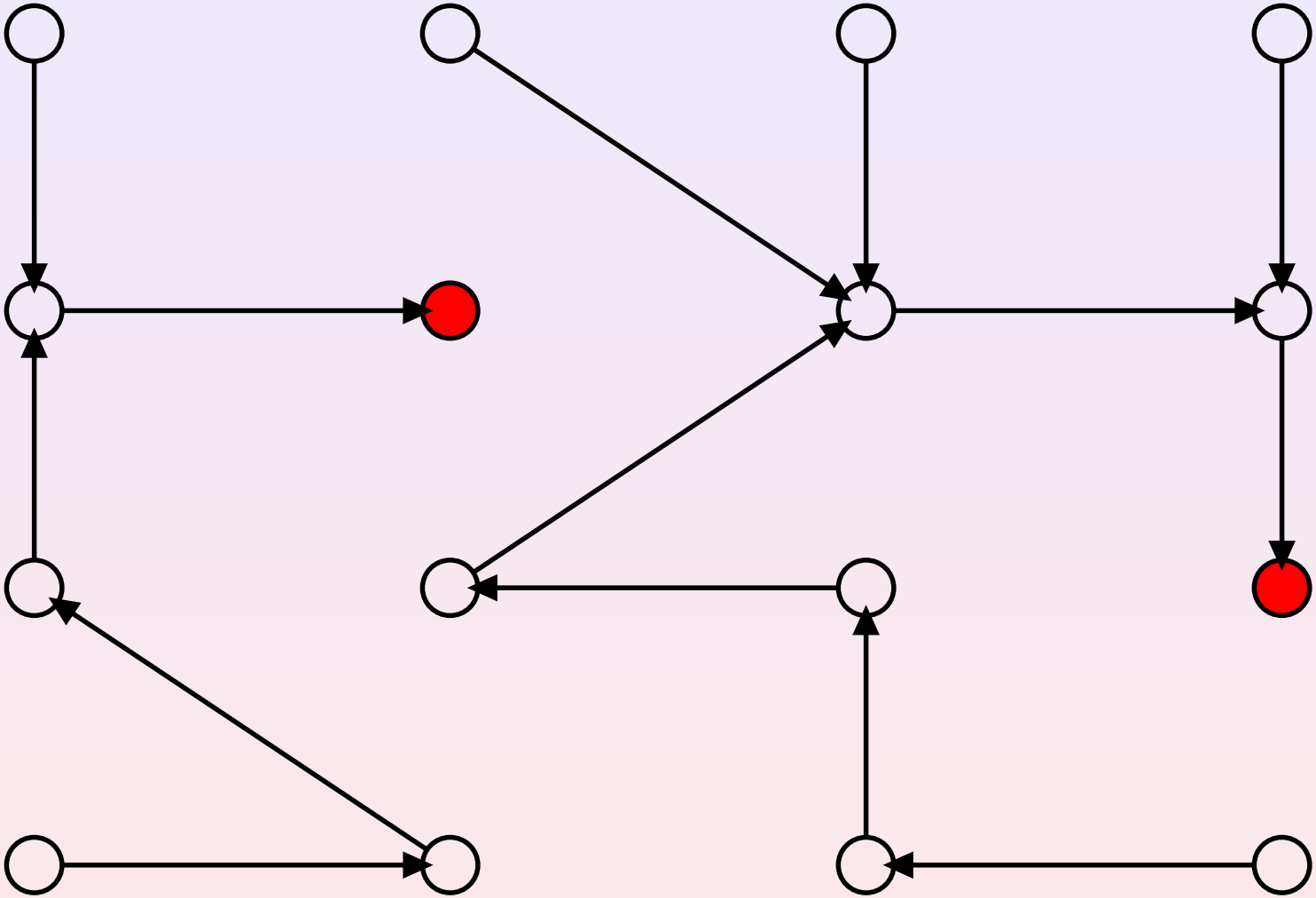

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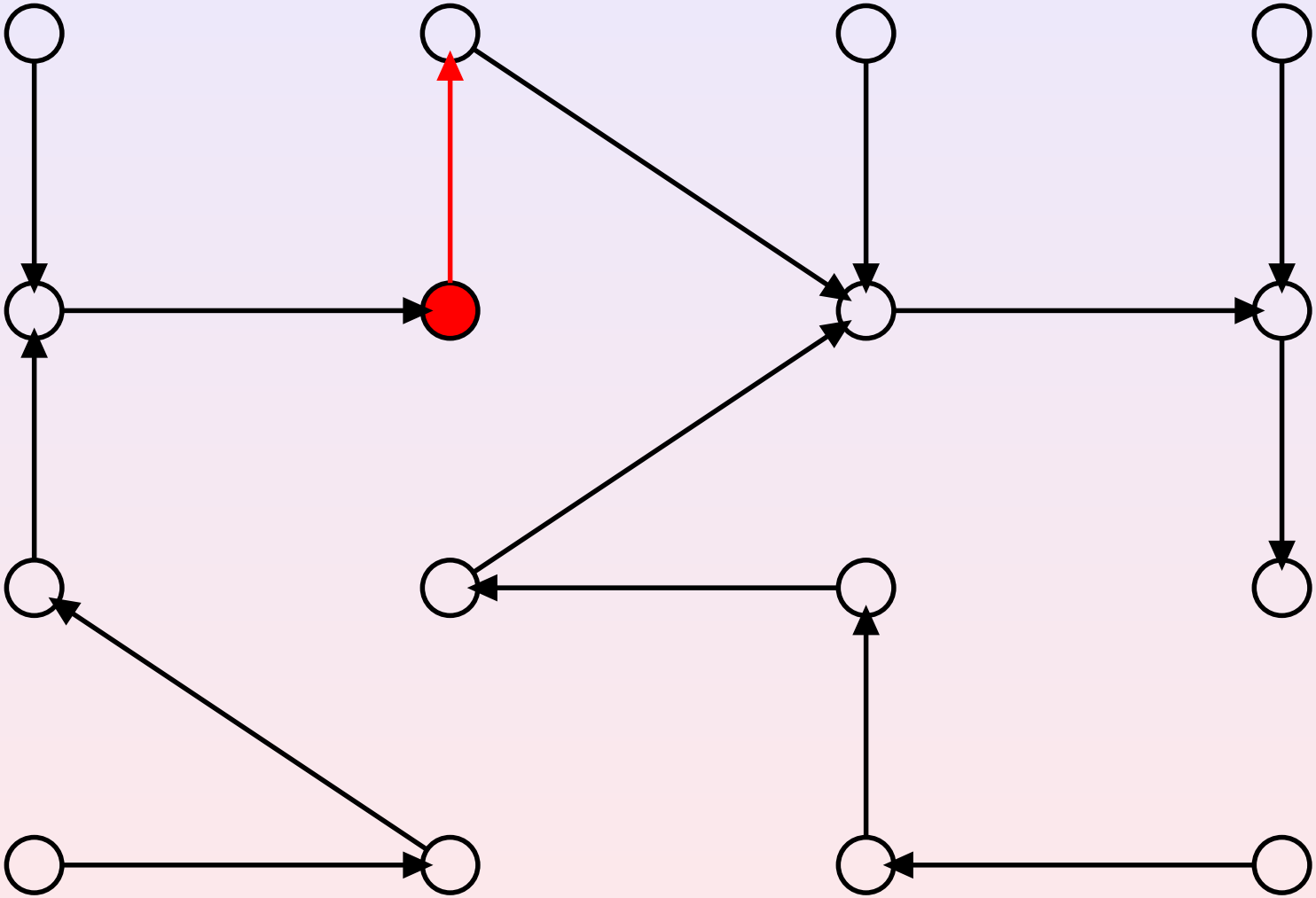

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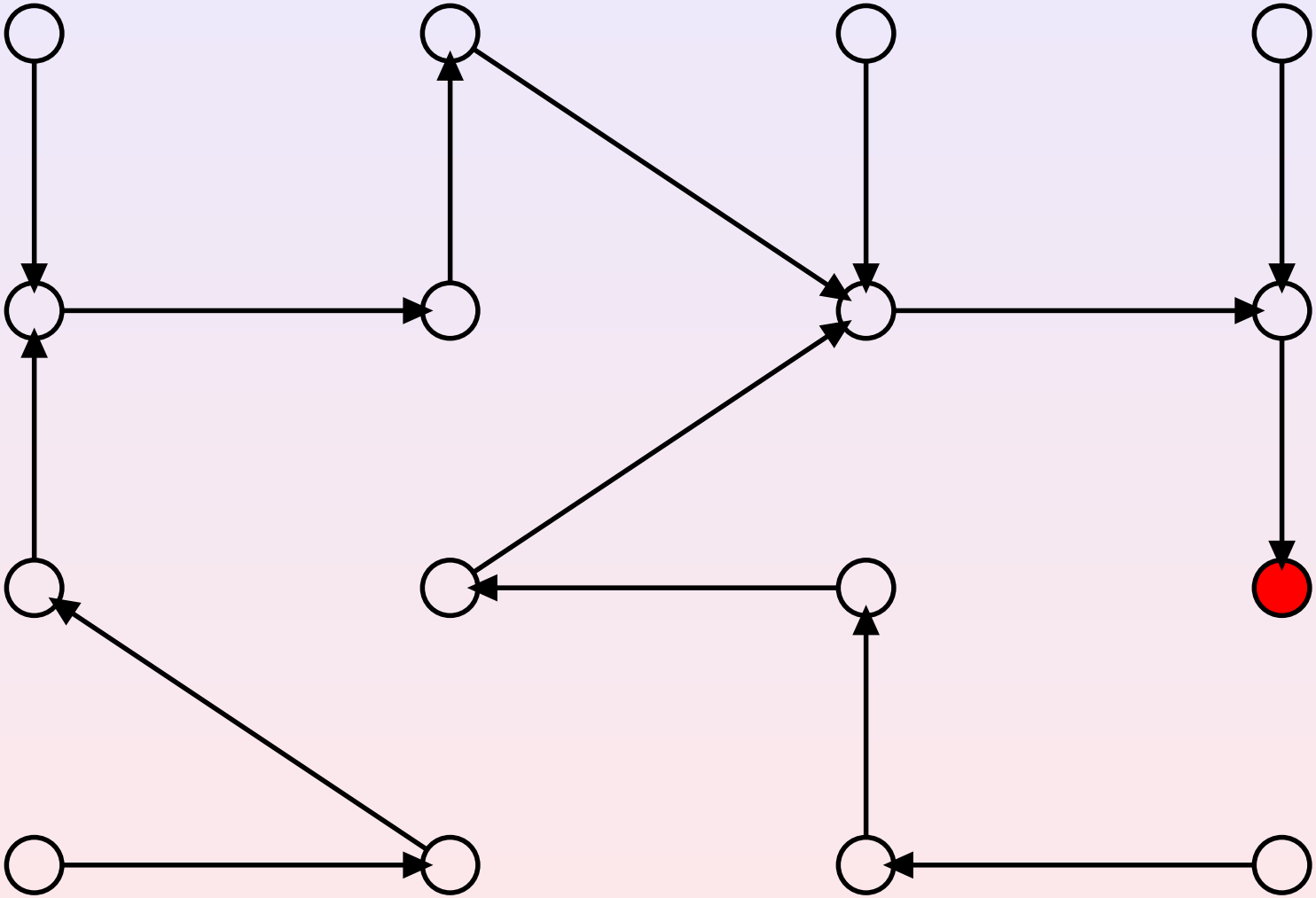

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Claim

1. The trees of a given phase form a partition of $V(G)$;
2. At the beginning of phase 1, all trees are of size $1 \geq 2^0$;
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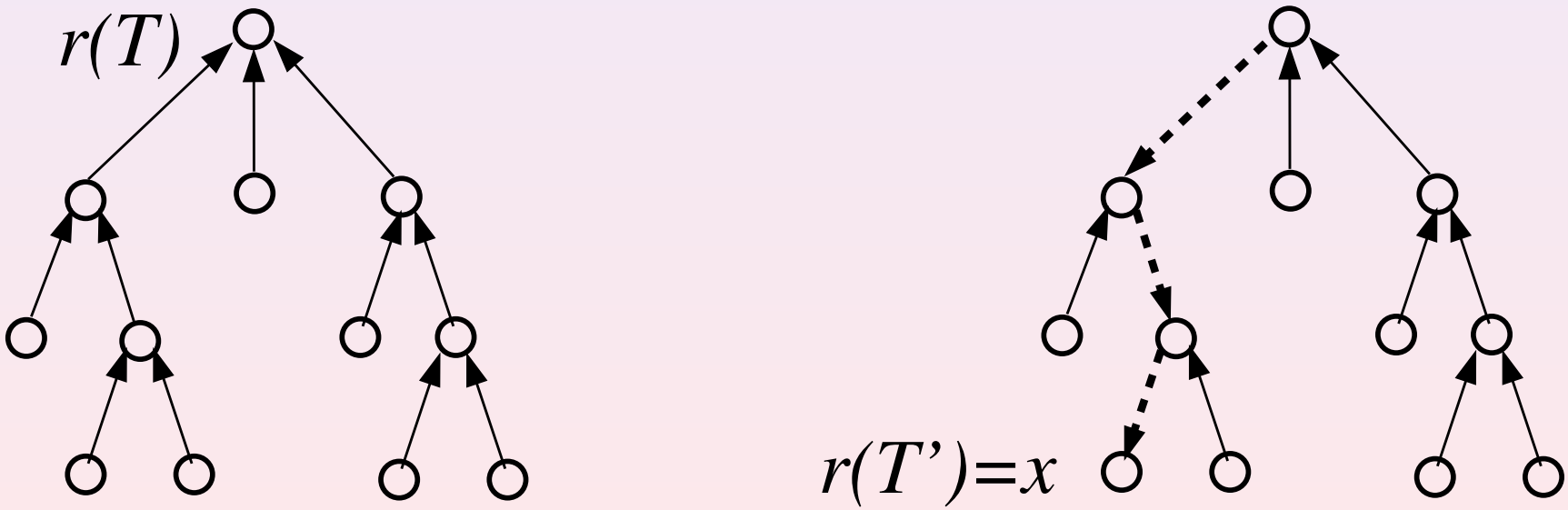

Outline

Arbitrary graphs with arbitrary labeling

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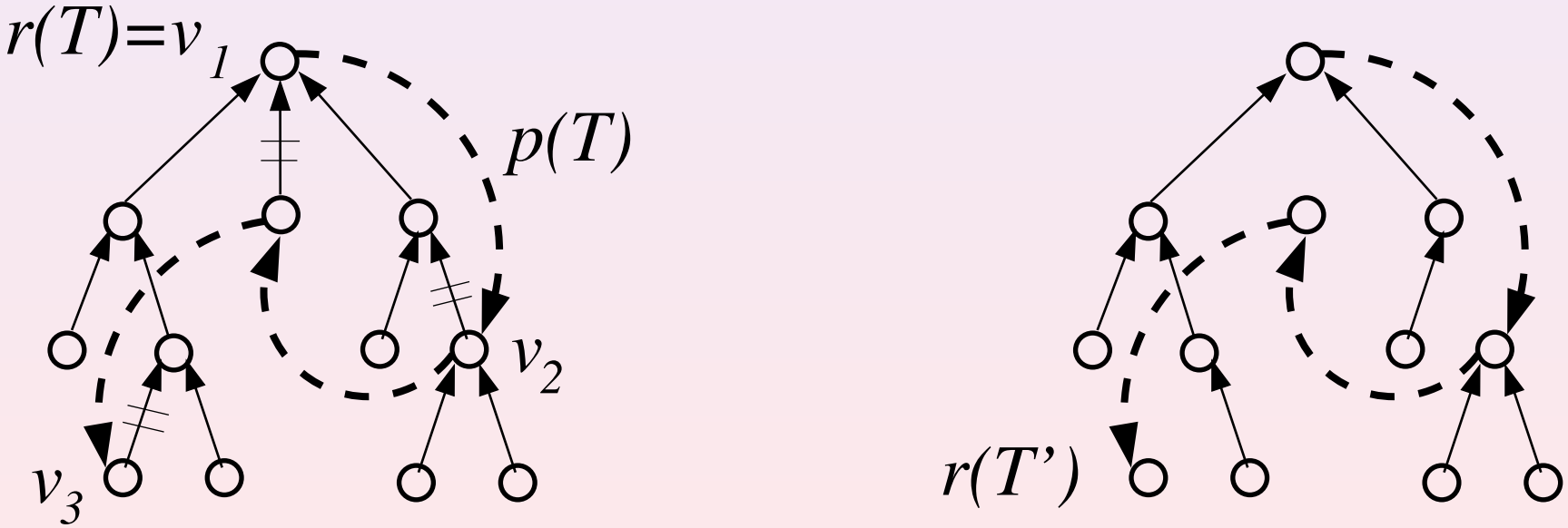

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Open problem

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- Upper bounds
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Close the gap.

For further information

An extended version of this paper will appear in the Proceedings of IWDC 2005.

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