

A note on the satisfiability problem in fixpoint calculi (Draft Version)

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Abstract. In this paper, we study the satisfiability problem in fixpoint calculi. A notion of disjunctive formulas is defined to characterize a class of fixpoint calculi in which the satisfiability problem is simple: provided the underlying lattice of interpretation is compact in some sense, satisfiability is decidable in linear time. Then we generalize this approach to a broader class of fixpoint calculi giving some simple condition for a calculus to be semantically equivalent to a disjunctive one.

This approach generalizes important results in fixpoint calculi and automata theory. In particular it induces an abstract characterization of the notion of finite non deterministic automata, i.e. automata without universal branching or, following the terminology of Muller and Schupp, non alternating automata.

Introduction

In this paper, we investigate the satisfiability problem in fixpoint calculi. More precisely, given a complete lattice M equipped with a set of monotonic increasing functions, we want to know if there exists an algorithm that, given any positive fixpoint expression (or fixpoint formula) α built over these functions, decide if the value α_M of the formula α in the lattice M is distinct from the bottom element (in this case we say α is satisfiable in M) or not.

Such a problem generalizes many other problems. For instance, it has been shown recently that fixpoint calculi are tightly related with automata theory [1]; fixpoint calculi induces a notion of automaton which generalizes most usual notions of finite automaton. It turns out that solving the satisfiability problem for fixpoint calculi also solves the emptiness problem for languages recognizable by means of many kind of (today or future) notions of finite automaton.

Before giving an overview of our paper, let us review some simple facts on the satisfiability problem.

First, with the disjunction operator \vee , the satisfiability problem is straightforward since a disjunction of two formula $\alpha \vee \beta$ is satisfiable if and only if one of the disjunct α or β is satisfiable.

With the conjunction operator \wedge , the satisfiability problem becomes difficult in general. In classical language theory for instance, given two regular expression α_1 and α_2 (which are some normal forms of fixpoint expression built without conjunctions) deciding if the language $L(\alpha_1 \vee \alpha_2) = L(\alpha_1) \cup L(\alpha_2)$ is non empty is immediate. Deciding if the language $L(\alpha_1 \wedge \alpha_2) = L(\alpha_1) \cap L(\alpha_2)$ is non empty is more difficult. It requires some (implicit or explicit) basic automata theory.

Notice that it may also happen that the satisfiability problem is, even with the conjunction operator, still easy. For instance, over languages of binary trees, given two formulas α_1 and α_2 over trees, one may define, as in [6], the formula $a(\alpha_1, \alpha_2)$ which denote the language of all trees which root are labeled by a and which left and right sons respectively belong to the languages $L(\alpha_1)$ and $L(\alpha_2)$. Obviously, an implicit conjunction is hidden in this operator. But the satisfiability problem is still easy since the satisfiability of the formula $a(\alpha_1, \alpha_2)$ is equivalent to the satisfiability of both the formula α_1 and the formula α_2 .

From the previous examples one is tempted to say that an operator f has a simple behavior w.r.t. the satisfiability problem when checking the satisfiability of some formula $f(\alpha_1, \dots, \alpha_n)$ is equivalent with checking a boolean combination (which depends on f) of the satisfiability of the arguments $\alpha_1, \dots, \alpha_n$. In the absence of fixpoint, such a condition obviously leads to formulas for which satisfiability is easy.

In the present paper, we essentially show that when all operators satisfy the previous property, provided the underlying lattice is compact in some sense, the satisfiability problem for arbitrary fixpoint formulas is also easy. At first sight such a case seems quite restrictive (e.g. no explicit conjunction may occur in formulas). But we also show, applying a formerly proved reduction theorem [1], that, under some simple conditions, fixpoint calculi are reducible to this case. In fact, the decision procedure we describe in this paper can be applied to many fixpoint calculi: from Park's μ -calculus over finite and infinite words [7] to the fixpoint calculi over tree-shaped structures defined in [10], passing by Kozen's modal μ -calculus [3] and Niwiński fixpoint calculus over binary trees [6].

The paper is organized as follows. In the first part we recall some basic definitions about fixpoint formulas and their interpretations in complete lattices called, in this occasion, fixpoint algebras.

In the second part, we define a topological structure in fixpoint algebras which leads us to the notion of compact algebras.

In the third part, the notion of disjunctive functions (functions with a simple behavior w.r.t. satisfiability) is defined. Basic properties of these functions are also given. This notion of disjunctiveness generalizes a similar notion presented in [2] for Kozen's modal μ -calculus.

In the fourth part, we prove the main theorem of our paper. Namely, over disjunctive formulas, i.e. formulas built only with disjunctive functions, provided the underlying fixpoint algebra is compact, the satisfiability problem is decidable in time linear in the size of formulas. In some sense, the restriction to disjunctive

functions ensures that no universal branching will occur in the underlying automata. While arbitrary fixpoint formulas generalize Muller and Schupp alternating automaton [4], disjunctive formulas generalize non deterministic Rabin automaton [8].

In the last part, we recall the reduction theorem proved in [1] (a generalization of Muller and Schupp simulation theorem [5]) and show how it enables us to reduce the satisfiability problem in many fixpoint calculi to the satisfiability problem on disjunctive formulas.

As a conclusion, we examined the compactness hypothesis which may be difficult to check. We also examined how far we are from solving a related question: obtaining complete and finite axiomatization of truth in fixpoint calculi as it has already been done for Kozen's modal μ -calculus [9].

1 Preliminaries

We call functional signature any set Σ of function symbols equipped with an arity function $\rho : \Sigma \rightarrow \mathbb{N}$ over these symbols.

1.1. Definition (Fixpoint algebras). Given a signature Σ a *fixpoint algebra* over Σ is a complete lattice $\langle M, \vee_M, \wedge_M \rangle$ with the bottom and top element denoted by \perp_M and \top_M together with, for any symbol $f \in \Sigma$, a monotonic increasing function $f_M : M^{\rho(f)} \rightarrow M$ called the *interpretation of f in M* .

In the sequel, to be consistent with symbol names, we always assume that for any fixpoint algebra M , any symbol among \perp , \top , \wedge or \vee which appears in Σ is respectively interpreted in M as \perp_M , \top_M , \wedge_M or \vee_M . Also, for any set $E \subseteq M$ we denote by $\bigvee_M E$ (resp. $\bigwedge_M E$) the least upper bound (resp. the greatest lower bound) of the set E .

1.2. Example. Any complete boolean algebra is a fixpoint algebra over signature $\{\vee, \wedge\}$.

1.3. Definition (Fixpoint formulas). Given a signature Σ and a set of variable symbol \mathcal{X} disjoint from Σ the set $\Sigma_\mu(\mathcal{X})$ of *fixpoint formulas*, simply called formulas in the sequel, is inductively defined by the following rule:

1. X is a formula for any variable $X \in \mathcal{X}$,
2. $f(\alpha_1, \dots, \alpha_{\rho(f)})$ is a formula for any $f \in \Sigma$ and any formula $\alpha_1, \dots, \alpha_{\rho(f)}$,
3. $\nu X.\alpha$ and $\mu X.\alpha$ are formulas for any $X \in \mathcal{X}$ and any formula α .

Given a formula $\alpha \in \Sigma_\mu(\mathcal{X})$, we say an occurrence of variable X is *bound* in α when it occurs in a subformula of the form $\sigma X.\alpha_1$ for $\sigma = \mu$ or ν . Unbounded occurrences of variable X in α are called *free*. A formula α is a *closed formula* when no variable occurs free. The set of closed formulas is denoted by Σ_μ . We also denote by $\Sigma(\mathcal{X})$ the set of formulas built without fixpoint construction, i.e. formulas with no sub-formulas of the form $\sigma X.\alpha_1$ for $\sigma = \mu$ or ν .

Notation: Given a set of variable $\{X_1, \dots, X_n\}$, we denote by $\alpha(X_1, \dots, X_n)$ any formula $\alpha \in \Sigma_\mu(\mathcal{X})$ such that any variable occurring free in α belongs to the set $\{X_1, \dots, X_n\}$.

1.4. Definition (Formulas semantics). Given a fixpoint algebra M , given a valuation of variables $V : \mathcal{X} \rightarrow M$, any formula α is interpreted as an element $\llbracket \alpha \rrbracket_V^M$ of M inductively defined by :

1. $\llbracket X \rrbracket_V^M = V(X)$,
2. $\llbracket f(\alpha_1, \dots, \alpha_{\rho(f)}) \rrbracket_V^M = f_M(\llbracket \alpha_1 \rrbracket_V^M, \dots, \llbracket \alpha_{\rho(f)} \rrbracket_V^M)$,
3. $\llbracket \nu X. \alpha \rrbracket_V^M = \bigvee \left\{ e \in M : e \leq \llbracket \alpha \rrbracket_{V[e/X]}^M \right\}$,
4. $\llbracket \mu X. \alpha \rrbracket_V^M = \bigwedge \left\{ e \in M : e \geq \llbracket \alpha \rrbracket_{V[e/X]}^M \right\}$,

where $V[e/X]$ denotes the valuation defined for any variable $Y \in \mathcal{X}$ by

$$V[e/X](Y) = \begin{cases} e & \text{when } X = Y, \\ V(Y) & \text{otherwise.} \end{cases}$$

Given \mathcal{C} a class of structures, we say that a formula α_1 is *refinement* of formula α_2 over class \mathcal{C} , which is noted $\alpha_1 \preceq_{\mathcal{C}} \alpha_2$ when, for any fixpoint algebra $M \in \mathcal{C}$, any valuation of variable $V : \mathcal{X} \rightarrow M$,

$$\llbracket \alpha_1 \rrbracket_V^M \leq_M \llbracket \alpha_2 \rrbracket_V^M$$

This relation is a preorder and we note $\simeq_{\mathcal{C}}$ the induced equivalence. When $\alpha_1 \simeq_{\mathcal{C}} \alpha_2$ (i.e. both $\alpha_1 \preceq_{\mathcal{C}} \alpha_2$ and $\alpha_2 \preceq_{\mathcal{C}} \alpha_1$) we say formulas α_1 and α_2 are *semantically equivalent w.r.t. \mathcal{C}* . When this equality holds for arbitrary fixpoint algebra and arbitrary valuation the subscript will be omitted.

In particular, $\mu X.X \simeq \perp$ and $\nu X.X \simeq \top$. In the sequel we will always assume, without increase of expressive power, that *both constant symbols \top and \perp belong to Σ* .

1.5. Example. Given an alphabet $A = \{a_1, \dots, a_n\}$, given a signature $\Sigma_1 = \{\perp, \top, \wedge, \vee, S_1\}$ with $\rho(S_1) = n$, we define the fixpoint algebra of *languages of infinite words* (ω -languages for short) on the alphabet A as $M = \langle \mathcal{P}(A^\omega), \cup, \cap \rangle$ with, for any $L_1, \dots, L_n \in \mathcal{P}(A^\omega)$,

$$S_{1M}(L_1, \dots, L_n) = \bigcup_{i \in [1, n]} a_i.L_i$$

where $a.L = \{a.w : w \in L\}$. In [7], the signature is slightly different since Park uses mappings of the form $L \mapsto a_i.L$. These mappings can be built from symbols of Σ_1 as mappings $X \mapsto S_{1M}(t_1, \dots, t_n)$ with $t_j = L$ when $j = i$ and $t_j = \perp_M$ otherwise, i.e. the interpretation in M of a term noted $a_i.X$ of the form $S_1(\perp, \dots, \perp, X, \perp, \dots, \perp)$. In this algebra, one can check that formula $\alpha = \nu X(\mu Y(b(X) \vee a(Y)))$ denotes, the set of all infinite words on alphabet

$\{a, b\}$ with infinitely many b . An equivalent regular expression for this language is $(a^*b)^\omega$.

The algebra of *languages of finite and/or infinite words* is defined from signature Σ_1 extended with a new constant symbol ϵ interpreted as the empty word.

In formulas semantics, valuations gives meaning to free variables. Abstracting from this meaning, interpretations of formulas become functions. The following definition formalized this intuition.

1.6. Definition (Functional interpretations of formulas). Given a fixpoint algebra M , any formula $\alpha(X_1, \dots, X_n)$, we define the *functional interpretation* of $\alpha(X_1, \dots, X_n)$ in M , denoted by $\alpha_M(X_1, \dots, X_n)$, as the function from M^n to M defined by :

$$(e_1, \dots, e_n) \mapsto \alpha_M(e_1, \dots, e_n) = \llbracket \alpha \rrbracket_{V[e_1/X_1] \dots [e_n/X_n]}^M$$

In the sequel, in the notation $\alpha_M(X_1, \dots, X_n)$, the sequence of variables shall be omitted when it causes no ambiguity.

Remark: One can check, by induction on the syntactic complexity of formulas, that the function $\alpha_M(X_1, \dots, X_n)$ is a monotonic increasing function in any of its argument. The Knaster-Tarski's Lemma applies showing that the functional interpretation in M of the formula $\beta = \sigma X_i. \alpha(X_1, \dots, X_n)$ is the least fixpoint (when $\sigma = \mu$) or the greatest fixpoint (when $\sigma = \nu$) of the mapping (over functions from M^{n-1} to M) defined by

$$F \mapsto \alpha_M(e_1, \dots, e_{i-1}, F(e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n), e_{i+1}, \dots, e_n)$$

In the sequel, we shall use both notations

$$(\sigma X_i. \alpha)_M(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$$

or

$$\sigma X_i. \alpha_M(X_1, \dots, X_{i-1}, X_i, X_{i+1}, \dots, X_n)$$

to denote such a fixpoint.

The following proposition is a fundamental tool to study formulas semantics.

1.7. Proposition (Transfinite approximation). *For any fixpoint algebras M , there exists an ordinal τ_M such that for any formula α :*

$$\nu X. \alpha \simeq_M \nu^{\tau_M} X. \alpha \text{ and } \mu X. \alpha \simeq_M \mu^{\tau_M} X. \alpha$$

with the semantics of $\nu^\tau. \alpha$ and $\mu^\tau. \alpha$ inductively defined by :

1. $\llbracket \nu^0 X. t \rrbracket_V^M = \top_M$ and $\llbracket \mu^0 X. t \rrbracket_V^M = \perp_M$,
2. $\llbracket \nu^{\tau+1} X. t \rrbracket_V^M = \llbracket t[\nu^\tau X. T/X] \rrbracket_V^M$ and $\llbracket \mu^{\tau+1} X. t \rrbracket_V^M = \llbracket t[\mu^\tau X. T/X] \rrbracket_V^M$,
3. and, for any limit ordinal τ , $\llbracket \nu^\tau X. t \rrbracket_V^M = \bigwedge_{\tau_1 < \tau} \llbracket \nu^{\tau_1} X. t \rrbracket_V^M$
and $\llbracket \mu^\tau X. t \rrbracket_V^M = \bigvee_{\tau_1 < \tau} \llbracket \mu^{\tau_1} X. t \rrbracket_V^M$.

2 Compact algebras

Before introducing the notion of disjunctive functions which is a central notion in our approach to the satisfiability problem, we need first to introduce some topological-like notions over fixpoint algebras.

2.1. Definition. Given a fixpoint algebra M over signature Σ , the set C_M of *closed elements* of M is define as the smallest set such that:

1. \perp_M and $\top_M \in C_M$,
2. for any function symbol $f \in \Sigma$, any closed element $e_1, \dots, e_{\rho(f)} \in C_M$, $f(e_1, \dots, e_{\rho(f)})$ is a closed element,
3. for any set $E \subseteq C_M$ of closed element, $\bigwedge_M E$ is a closed element,

Remark: In the case M is a continuous lattice (when \vee is continuous w.r.t. to \wedge) with $\vee \in \Sigma$ the set of closed elements defined above induces a topology in the usual sense over M , i.e. the least upper bound of any finite subset of set C_M belong to set C_M and the greatest lower bound of any finite or infinite subset of set C_M belong as well to set C_M .

The following proposition (from which comes our definition) shall be useful in the sequel.

2.2. Proposition. *Given a fixpoint algebra M , for any formula $\alpha(X_1, \dots, X_n)$ built without least fixpoint construction, the function $\alpha_M(X_1, \dots, X_n)$ is a closed function, i.e. for any closed element $e_1, \dots, e_n \in M$, $\alpha_M(e_1, \dots, e_n)$ is a closed element.*

Proof: A straightforward induction on the syntactic complexity of formula α , applying transfinite approximation of greatest fixpoint. \square

2.3. Definition (Compact algebra). We say a fixpoint algebra M is *compact* when, for any set $E \subseteq C_M$ of closed points, $\bigwedge E = \perp$ if and only if $\bigwedge F = \perp$ for some finite subset F of set E .

2.4. Example. The fixpoint algebra of languages of finite words *is not* compact as shown, for instance, by the language defined by

$$\bigwedge_{i \in \mathbb{N}} a^i \cdot \top$$

which is empty. In opposite, the fixpoint algebra of languages of finite and infinite words is a compact algebra.

3 Disjunctive algebras

3.1. Definition (Boolean projection). Given a fixpoint algebra M over signature Σ , the *boolean projection* \mathcal{X}_M over M is defined as the mapping

$$\mathcal{X}_M : M \rightarrow \{\top, \perp\}$$

$$x \mapsto \mathcal{X}_M(x) = \begin{cases} \perp & \text{when } x = \perp, \\ \top & \text{otherwise.} \end{cases}$$

Remark: Extending in a straightforward way the usual (model theoretical) notion of satisfiability, a closed formula $\alpha \in \Sigma_\mu$ is *satisfiable* in an algebra M when $\mathcal{X}_M(\alpha_M) = \top$.

3.2. Definition (Disjunctive function). We say a function $F : M^n \rightarrow M$ is *disjunctive* when:

1. if $n = 0$ then $F = c_M$ for some constant symbol in Σ ,
2. if $n \neq 0$ then there exists a boolean function $\mathcal{X}_M(F)$, called the *boolean projection* of function F , such that, for any *closed elements* $e_1, \dots, e_n \in C_M$,

$$\mathcal{X}_M(F(e_1, \dots, e_n)) = \mathcal{X}_M(F)(\mathcal{X}_M(e_1), \dots, \mathcal{X}_M(e_n))$$

Remark: In condition 2 above, function $\mathcal{X}_M(F)$ is uniquely determined, for any *boolean elements* $e_1, \dots, e_n \in \{\top, \perp\}$, by

$$\mathcal{X}_M(F)(e_1, \dots, e_n) = \mathcal{X}_M(F(e_1, \dots, e_n))$$

3.3. Example. The joint function \vee is disjunctive with $\mathcal{X}_M(\vee) = \vee$. In general the meet function \wedge is not disjunctive. For instance, given $M = \mathcal{P}(\{a, b\})$ one has $\mathcal{X}_M(\{a\} \cap \{b\}) = \perp$ while $\mathcal{X}_M(\{a\}) \wedge \mathcal{X}_M(\{b\}) = \top$.

3.4. Proposition. *The set of disjunctive function is closed under function composition. The boolean projection of a composition of disjunctive functions is the composition of their boolean projections.*

Proof: By definition \mathcal{X}_M is “functorial” over disjunctive functions! □

3.5. Proposition. *The boolean projection is invariant under iteration of disjunctive functions, i.e. for any disjunctive function $F(X_1, \dots, X_n)$, any $i \in [1, n]$*

$$\mathcal{X}_M(F(X_1, \dots, X_n)) = \mathcal{X}_M(F(X_1, \dots, X_{i-1}, F(X_1, \dots, X_n), X_{i+1}, \dots, X_n))$$

Proof: Obvious since for any *boolean function* F , any $i \in [1, n]$, $F(X_1, \dots, X_n) = F(X_1, \dots, X_{i-1}, F(x_1, \dots, X_n), X_{i+1}, \dots, X_n)$. □

3.6. Definition. Given a fixpoint algebra M , we say that a formula $\alpha \in \Sigma_\mu(\mathcal{X})$ is a *disjunctive formula* over M when, for any function symbol f occurring in α , the function f_M is disjunctive. By extension, we say that the fixpoint algebra M is a *disjunctive algebra* when the interpretation f_M of any symbol $f \in \Sigma$, is a disjunctive function.

Notation: In the sequel, for any formula α , we denote by $\mathcal{X}_M(\alpha)(X_1, \dots, X_n)$, when it does exist, the boolean projection of the functional interpretation of formula $\alpha(X_1, \dots, X_n)$. In particular, for any function symbol $f \in \Sigma$ such that the function f_M is disjunctive, we denote by $\mathcal{X}_M(f)$ the boolean projection of f_M .

As far as complexity is concerned, we also assume that given any algebra M , any function symbol $f \in \Sigma$ such that the function f_M is disjunctive, *the function $\mathcal{X}_M(f)$ is given in such a way it is computable in time linear in the number of its argument.* Since $\mathcal{X}_M(f)$ is a boolean function this is always possible.

4 Satisfiability in compact and disjunctive algebras

In the absence of fixpoint construction, disjunctiveness is a straightforward sufficient condition for the satisfiability problem to be simple. More precisely:

4.1. Proposition. *When M is a disjunctive fixpoint algebra, the satisfiability of any closed formula α built without fixpoint is decidable in time linear in the size of α .*

Proof: By induction on formula α . If α is a constant symbol then $\mathcal{X}_M(\alpha)$ is computable in constant time. Otherwise α is of the form $f(\alpha_1, \dots, \alpha_{\rho(f)})$ with

$$\mathcal{X}_M(\alpha) = \mathcal{X}_M(f)(\mathcal{X}_M(\alpha_1), \dots, \mathcal{X}_M(\alpha_{\rho(f)}))$$

hence induction hypothesis applies giving us the result. \square

4.2. Definition (Lower and upper projection). Given a formula $\alpha \in \Sigma_\mu(\mathcal{X})$, we define the *lower projection* and the *upper projection* of formula α as the formula $\downarrow\alpha$ and the formula $\uparrow\alpha$ inductively defined by the following rules:

1. $\downarrow X = X$ and $\uparrow X = X$ for any variable $X \in \mathcal{X}$,
2. $\downarrow f(\alpha_1, \dots, \alpha_{\rho(f)}) = f(\downarrow\alpha_1, \dots, \downarrow\alpha_{\rho(f)})$
and $\uparrow f(\alpha_1, \dots, \alpha_{\rho(f)}) = f(\uparrow\alpha_1, \dots, \uparrow\alpha_{\rho(f)})$,
3. $\downarrow \mu X. \alpha = \downarrow\alpha[\perp/X]$ (resp. $\uparrow \mu X. \alpha = \mu X. \uparrow\alpha$),
4. $\downarrow \nu X. \alpha = \nu X. \downarrow\alpha$ (resp. $\uparrow \nu X. \alpha = \uparrow\alpha[\top/X]$),

Remark: For any formula $\alpha \in \Sigma_\mu(\mathcal{X})$, $\downarrow\alpha \preceq \alpha \preceq \uparrow\alpha$ and $\downarrow\uparrow\alpha \simeq \downarrow\alpha$.

4.3. Theorem (Disjunctiveness). *For any fixpoint formula α , for any disjunctive and compact fixpoint algebra M , the functional interpretation α_M of the formula α is disjunctive with $\mathcal{X}_M(\alpha) = \mathcal{X}_M(\uparrow\alpha)$.*

Proof: The proof goes in two steps. We first show this result holds for formula built without least fixpoint construction and then we extend it to the general case. In the sequel, let M be a disjunctive and compact fixpoint algebra.

4.4. Lemma. *For any formula $\alpha(X_1, \dots, X_n)$ built without least fixpoint construction, function $\alpha_M(X_1, \dots, X_n)$ is disjunctive with $\mathcal{X}_M(\alpha)(X_1, \dots, X_n) = \mathcal{X}_M(\uparrow\alpha)(X_1, \dots, X_n)$.*

Proof: By induction on the structure of formula $\alpha(X_1, \dots, X_n)$.

1. When formula α is a variable X nothing has to be done since $\uparrow X = X$.
2. When formula α is of the form $f(\alpha_1, \dots, \alpha_{\rho(f)})$, by induction hypothesis, for any $i \in [1, \rho(f)]$, the function α_{iM} is disjunctive with $\mathcal{X}_M(\alpha_i) = \mathcal{X}_M(\uparrow \alpha_i)$. Hence, for f_M is disjunctive, by composition of disjunctive function (see proposition 3.5), α_M is disjunctive with

$$\begin{aligned} \mathcal{X}_M(\alpha)(X_1, \dots, X_n) \\ = \mathcal{X}_M(f)(\mathcal{X}_M(\uparrow \alpha_1)(X_1, \dots, X_n), \dots, \mathcal{X}_M(\uparrow \alpha_{\rho(f)})(X_1, \dots, X_n)) \end{aligned}$$

3. When formula α is of the form $\nu X.\beta$, we prove by induction on ordinals that, for any ordinal $\tau > 1$, $(\nu^\tau X.\beta)_M$ is disjunctive with

$$\mathcal{X}_M(\nu^\tau X.\beta)(X_1, \dots, X_n) = \mathcal{X}_M(\beta[\top/X])(X_1, \dots, X_n)$$

which, applying Proposition 1.7 and the induction hypothesis on $\beta[\top/X]$ enable us to conclude.

The induction on ordinals goes as follows.

- (a) When $\tau = 1$, $\nu^1 X.\beta \simeq_M \beta[\top/X]$, hence the result.
- (b) When τ is a successor ordinal, i.e. $\tau = \tau_1 + 1$, we have $\nu^{\tau_1+1} X.\beta \simeq \beta[\nu^{\tau_1} X.\beta/X]$. Applying the general induction hypothesis, we know the function $\beta_M(X, X_1, \dots, X_n)$ is disjunctive hence, applying proposition 3.5, $\mathcal{X}_M(\beta)(X, X_1, \dots, X_n) = \mathcal{X}_M(\beta[\beta/X])(X, X_1, \dots, X_n)$. Then, by composition of disjunctive function, i.e. with X instantiated into the formula $(\nu^{\tau_1} X.\beta)(e_1, \dots, e_n)$ on which the induction hypothesis applies, we conclude the induction step.
- (c) When τ is a limit ordinal, for any closed elements $e_1, \dots, e_n \in C_M$,

$$\alpha_M(e_1, \dots, e_n) = \bigwedge_{t_1 < \tau} (\nu^{t_1} X.\beta_M)_M(e_1, \dots, e_n)$$

In this case the compactness hypothesis applies (for all elements of the form $(\nu^{\tau_1} X.\beta)_M(e_1, \dots, e_n)$ are closed) hence $\mathcal{X}_M(\alpha_M(e_1, \dots, e_n)) = (\nu^{\tau_1} X.\beta)_M(e_1, \dots, e_n)$ for some $\tau_1 < \tau$ and the induction hypothesis applies.

□

4.5. Lemma. *The functional interpretation of any formula $\alpha(X_1, \dots, X_n)$ is disjunctive with $\mathcal{X}_M(\alpha)(X_1, \dots, X_n) = \mathcal{X}_M(\downarrow \alpha)(X_1, \dots, X_n)$.*

Proof: Again, the proof goes by induction on the structure of α . Lemma 4.4 ensures that formula $\downarrow \alpha$ (built without least fixpoint construction) is disjunctive.

1. When formula α is a variable X or when it is of the form $f(\alpha_1, \dots, \alpha_{\rho(f)})$ the proof is essentially the same as above to relate α and $\downarrow \alpha$.

2. When formula α is of the form $\mu X.\beta$, applying Proposition 1.7, for any element $e_1, \dots, e_n \in M$, $\alpha_M(e_1, \dots, e_n) = \perp_M$ iff $(\beta[\perp/X])_M(e_1, \dots, e_n) = \perp_M$ hence the result, applying the induction hypothesis on the formula $\beta[\perp/X]$.
3. When formula α is of the form $\nu X.\beta$, for any $e_1, \dots, e_n \in C_M$, we have $\mathcal{X}_M(\downarrow \alpha_M(e_1, \dots, e_n)) \leq \mathcal{X}_M(\alpha_M(e_1, \dots, e_n))$ since $\downarrow \alpha \preceq \alpha$. We prove the converse by transfinite induction. More precisely, we shows that, for any ordinal τ , any $e_1, \dots, e_n \in C_M$, whenever $(\nu^\tau X.\downarrow \beta)_M(e_1, \dots, e_n) = \perp$ then $(\nu^\tau X.\beta)_M(e_1, \dots, e_n) = \perp$. The induction on ordinals goes as follows.
 - (a) When $\tau = 0$ nothing has to be done.
 - (b) When τ is a successor ordinal, i.e. $\tau = \tau_1 + 1$, we have

$$\begin{aligned} \mathcal{X}_M((\nu^{\tau_1+1} X.\beta)_M(e_1, \dots, e_n)) \\ = \mathcal{X}_M(\beta)(\mathcal{X}_M((\nu^{\tau_1} X.\beta)_M(e_1, \dots, e_n)), \mathcal{X}_M(e_1), \dots, \mathcal{X}_M(e_n)) \end{aligned}$$

hence the result applying the general induction hypothesis on formula $\beta(X, X_1, \dots, X_n)$ (with $\mathcal{X}_M(\beta)$ a monotonic increasing function) and the induction hypothesis for ordinal τ_1 .

- (c) When τ is a limit ordinal, we have

$$\mathcal{X}_M((\nu^\tau X.\downarrow \beta)(e_1, \dots, e_n)) = \mathcal{X}_M\left(\bigwedge_{\tau_1 < \tau} (\nu^{\tau_1} X.\downarrow \beta_M(e_1, \dots, e_n))\right) = \perp$$

Again compactness applies hence $(\nu^{\tau_1} X.\downarrow \beta)_M(X, e_1, \dots, e_n) = \perp$ for some ordinal $\tau_1 < \tau$. Then, applying the induction hypothesis on τ_1 , i.e. $(\nu^{\tau_1} X.\beta)_M(e_1, \dots, e_n) = \perp$, we have $(\nu^\tau X.\beta)_M(e_1, \dots, e_n) = \perp$ which conclude the proof.

□

Proof (Theorem 4.3 continued): Hence for any formula α , α_M is disjunctive with $\mathcal{X}_M(\alpha) = \mathcal{X}_M(\downarrow \alpha)$ (applying Lemma 4.5) hence $\mathcal{X}_M(\alpha) = \mathcal{X}_M(\uparrow \alpha)$ (applying Lemma 4.4 on formula $\downarrow \alpha$). □

Remark: Compactness is essential in the previous proofs. However it is only used over closed elements which are definable by means of transfinite approximations of greatest fixpoint. It may be the case that this hypothesis can be weakened without falsifying the previous theorem.

4.6. Corollary. *[Satisfiability] In particular, for any compact and disjunctive fixpoint algebra M , for any closed fixpoint formula $\alpha \in \Sigma_\mu$, the satisfiability of formula α on M is computable in time linear in the size of formula α .*

Proof: The proof of the theorem above shows that $\mathcal{X}_M(\alpha) = \mathcal{X}_M(\uparrow \alpha)$ hence it can be computed by induction applying the following rules:

1. $\mathcal{X}_M(\mu X.\alpha_1) = \mathcal{X}_M(\alpha_1[\perp/X])$,
2. $\mathcal{X}_M(\nu X.\alpha_1) = \mathcal{X}_M(\alpha_1[\top/X])$,
3. $\mathcal{X}_M(f(\alpha_1, \dots, \alpha_{\rho(f)})) = \mathcal{X}_M(f)(\mathcal{X}_M(\alpha_1), \dots, \mathcal{X}_M(\alpha_{\rho(f)}))$,

which, under the assumption that every function $\mathcal{X}_M(f)(x_1, \dots, x_{\rho(f)})$ is presented in such a way it can be computed in time $\rho(f)$, gives us the result. \square

Remark: Obviously, the satisfiability problem for disjunctive fixpoint formulas is easier than the emptiness problem for automata. However, since there is an exponential blow-up translating an automaton (with parity condition) into a fixpoint formula [1], this is not a surprise.

5 Reduction to the disjunctive case

An immediate weakness of the decision procedure obtained above is that it does not apply to formulas where the conjunction \wedge occurs since it is not a disjunctive operator. The following result, obtained in [1], remedy to this fact.

5.1. Definition. Given a fixpoint algebra M , we say that the conjunction \wedge commutes with Σ on M when, for any finite multiset $\{f_i\}_{i \in [1, n]}$ of functional symbols of $\Sigma - \{\wedge\}$ there exists a function G built with symbols of $\Sigma - \{\wedge\}$ such that an equation of the form:

$$\bigwedge_{i \in [1, n]} \{f_i(x_i)\} = G(\bigwedge \mathbf{y}_1, \dots, \bigwedge \mathbf{y}_m)$$

holds on \mathcal{C} , where :

1. x_i s are vectors of distinct variables of the appropriate length,
2. y_j s are vectors of distinct variables taken among those appearing in x_i s,
3. $\bigwedge y_j$ s denote the g.l.b. applied to the set of all variables occurring in y_j .

5.2. Example. In the algebra of languages of infinite words, \wedge commutes with signature Σ_1 . For Kozen's modal μ -calculus things are less simple since one needs to change the signature. This case from which the present approach was inspired, has been extensively described in [2].

The following theorem is proved in [1].

5.3. Theorem (Reduction). *When the conjunction \wedge commutes with Σ on M any closed fixpoint formula α is equivalent over M to a formula $\hat{\alpha}$ built without the symbol \wedge .*

Proof: Going back to the proof presented in [1], one can easily check the size of $\hat{\alpha}$ is less than a triple exponent of the size of formula α . \square

It follows:

5.4. Corollary. *In any compact fixpoint algebra M over a signature Σ such that:*

1. *the conjunction \wedge commutes with Σ on M ,*
2. *for any symbol $f \in \Sigma - \{\wedge\}$, the function f_M is disjunctive,*

the satisfiability problem is decidable in triple exponential time over the size of fixpoint formulas.

Proof: Since \wedge commutes with Σ on M , applying Theorem 5.3, any closed formula α is, over the fixpoint algebra M , equivalent to a formula $\hat{\alpha}$ over the signature $\Sigma - \{\wedge\}$. Then, Corollary 4.6 applies, giving us the result. \square

Conclusion

In this paper, we have defined a general notion of disjunctive formulas for which, in compact algebra, the satisfiability problem is easy; the reduction theorem gives sufficient condition for the satisfiability problem to be reducible to this case. Although such a decision procedure applies to many fixpoint calculi, the compactness condition can be difficult to check. We miss here some simple and general conditions for compactness¹ to hold in a fixpoint algebra.

One interesting case to look at seems to be fixpoint calculi which arise from monadic fixpoint logic, i.e. where fixpoint formulas are interpreted over lattices of sets of elementary equivalent structures with interpretations of function symbols being definable in First-Order Logic. At the moment it is not clear whether compactness, that holds for First-Order Logic, will help or not.

The satisfiability problem is also, inherently, related with the problem of finding complete finite axiomatization of formulas' validity. For Kozen's modal μ -calculus [3], such an axiomatization has already been found and proved complete [9]. There, an important intermediate step was to consider the notion of disjunctive (modal) formulas for which both satisfiability and provability are easy. Since we generalize, in the present approach, the notion of disjunctive formulas, we may expect to obtain completeness results soon.

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¹ or a slightly weaker notion as noticed at the end of the proof of Theorem 4.3

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