

A Toolkit for First Order Extensions of Monadic Games

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Abstract. In 1974 R. Fagin proved that properties of structures which are in NP are exactly the same as those expressible by existential second order sentences, that is sentences of the form: there exist \vec{P} such that φ , where \vec{P} is a tuple of relation symbols. and φ is a first order formula. Fagin was also the first to study monadic NP: the class of properties expressible by existential second order sentences where all the quantified relations are unary.

In [AFS00] Ajtai, Fagin and Stockmeyer introduce closed monadic NP: the class of properties which can be expressed by a kind of monadic second order existential formula, where the second order quantifiers can interleave with first order quantifiers. In order to prove that such alternation of quantifiers gives substantial additional expressive power they construct graph properties \mathcal{P}_1 and \mathcal{P}_2 : \mathcal{P}_1 is expressible by a sentence with the quantifier prefix in the class $(\exists\forall)^* \exists^*(\exists\forall)^*{}^1$ but not by a boolean combination of sentences from monadic NP (i.e with the prefix of the form $\exists^*(\exists\forall)^*$) and \mathcal{P}_2 is expressible by a sentence $\exists^*(\exists\forall)^* \exists^*(\exists\forall)^*$ but not by a Boolean combination of sentences of the form $(\exists\forall)^* \exists^*(\exists\forall)^*$. A natural question arises here whether the hierarchy inside closed monadic NP, defined by the number of blocks of second order existential quantifiers, is strict.

In this paper we present a technology for proving some non expressibility results for monadic second order logic. As a corollary we get a new, easy, proof of the two results from [AFS00] mentioned above. With our technology we can also make a first small step towards an answer to the hierarchy question by showing that the hierarchy inside closed monadic NP does not collapse on a first order level. The monadic complexity of properties definable in Kozen's mu-calculus is also considered as our technology also applies to the mu-calculus itself.

1 Introduction

1.1 Previous works

In 1974 R. Fagin proved that the properties of structures which are in \mathcal{NP} are exactly the same as those expressible by existential second order sentences, known also as Σ_1^1

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¹ In this paper we use the symbols \exists, \forall for the first order quantifiers and \exists, \forall for the monadic second order quantifiers

sentences, i.e. sentences of the form: *there exist relations \vec{P} such that φ* , where \vec{P} is a tuple of relation symbols (possibly of high arity) and φ is a first order formula.

Fagin was also the first to study *monadic NP*: the class of properties expressible by existential second order sentences where all quantified relations are unary. The reason for studying this class was the belief that it could serve as a training ground for attacking the “real problems” like whether NP equals co-NP. It is not hard to show ([F75]) that monadic NP is different from monadic co-NP. A much stronger result has even been proved by Matz and Thomas ([MT97]). They show that the monadic hierarchy, the natural monadic counterpart of the polynomial hierarchy, is strict (a property is in the k -th level of the monadic hierarchy if it is expressible by a sentence of monadic second order logic where all the second order quantifiers are at the beginning and there are at most $k - 1$ alternations between second order existential and second order universal quantifiers).

An important part of research in the area of monadic NP is devoted to the possibility of expressing different variations of graph connectivity. Already Fagin’s proof that monadic NP is different from monadic co-NP is based on the fact that connectivity of undirected graphs is not expressible by a sentence in monadic Σ_1^1 , while non-connectivity obviously is. Then de Rougemont [dR87] and Schwentick [S95] proved that connectivity is not in monadic NP even in the presence of various built-in relations.

However, as observed by Kanellakis, the property of reachability (for undirected graphs) is in monadic NP (reachability is the problem if, for a given graph and two distinguished nodes s and t , there is a path from s to t in this graph). It follows that connectivity, although not in monadic NP, is expressible by a formula of the form $\forall x \forall y \exists \vec{P} \varphi$. This observation leads to the study of *closed monadic NP*, the class of properties expressible by a sentence with quantifier prefix of the form $(\exists^*(\exists \forall)^*)^*$, and of the *closed monadic hierarchy*, the class of properties expressible by a sentence with quantifier prefix of the form $((\exists^*(\exists \forall)^*)^*(\forall^*(\exists \forall)^*)^*)^*$.

In [AFS00] and [AFS98] Ajtai, Fagin and Stockmeyer argue that closed monadic NP is even a more interesting object of study than monadic NP: it is still a subclass of NP (and also the k -th level of closed monadic hierarchy is still a subclass of the k -th level of polynomial hierarchy), it is defined by a simple syntax and it is closed under first order quantifications. In order to prove that such alternation of quantifiers gives substantial additional expressive power they construct graph properties \mathcal{P}_1 and \mathcal{P}_2 such that \mathcal{P}_1 is expressible by a sentence with the quantifier prefix in the class $(\exists \forall)^* \exists^*(\exists \forall)^*$, but not by a Boolean combination of sentences from monadic NP (i.e with the prefix of the form $\exists^*(\exists \forall)^*$) and \mathcal{P}_2 is expressible by a sentence $\exists^*(\exists \forall)^* \exists(\exists \forall)^*$ but not by a Boolean combination of sentences of the form $(\exists \forall)^* \exists^*(\exists \forall)^*$. The non expressibility results for \mathcal{P}_1 and \mathcal{P}_2 in [AFS00] are by no means easy and constitute the main technical contribution of this long paper. As the authors write: *Our most difficult result is the fact that there is an undirected graph property that is in closed monadic NP but not in the first order/Boolean closure of monadic NP. In the game corresponding to the first order/Boolean closure of monadic NP, played over graphs G_0 and G_1 , the spoiler not only gets to choose which of G_0 and G_1 he wishes to color, but he does not have to make his selection until after a number of pebbling moves had been played. Thus, not only are we faced with the situation where the spoiler gets to choose which structure*

to color, but apparently also for the first time, we are being forced to consider a game where there are pebbling rounds both before and after the coloring round.

There are many natural open questions in the area, most of them stated in [AFS00]: is the hierarchy inside closed monadic NP strict? We mean here the hierarchy defined by the number of blocks of second order existential quantifiers, alternating with first order quantifiers. Is there any property in the monadic hierarchy (or, equivalently, in the closed monadic hierarchy) which is not in closed monadic NP? Is the closed monadic hierarchy strict? These questions seem to be quite hard: so far we do not know any property in the (closed) monadic hierarchy which would not be expressible by a sentence with quantifier prefix $\exists^*(\forall\exists)^*\exists^*(\forall\exists)^*$.

1.2 Our contribution

In this paper we present an inductive and compositional technology for proving some non expressibility results for monadic second order logic. In particular, our technology gives an alternative simple solution to all the technical problems described in the citation from [AFS00] above. But unlike the construction in [AFS00], which is specific for first order/Boolean closure of monadic NP, our technology is universal: it deals with first order/Boolean closure of most monadic classes.

To be more precise, we show how to construct, for any given property \mathcal{S} not expressible by a sentence with quantifier prefix in some non trivial² class W , two properties $bool(\mathcal{S})$ and $reach(\mathcal{S})$ which are not much harder than \mathcal{S} and such that (1) property $bool(\mathcal{S})$ cannot be expressed by boolean combination of sentences with quantifier prefix in W and (2) property $reach(\mathcal{S})$ cannot be expressed by a sentence with quantifier prefix vw where $v \in (\exists + \forall)^*$ is a block of first order quantifiers and $w \in W$. Saying that $bool(\mathcal{S})$ and $reach(\mathcal{S})$ are *not much harder than* \mathcal{S} we mean that if \mathcal{S} is expressible by a sentence with quantifier prefix in some class V then $bool(\mathcal{S})$ is expressible by a sentence with the prefix of the form $\exists\exists v$ where $v \in V$ and $reach(\mathcal{S})$ is expressible by a sentence with the prefix of the form $\exists\forall v$ where $v \in V$. The non expressibility proof for $reach$ generalizes the second author's proof of the fact that directed reachability is not expressible by a sentence with the prefix of the form $(\forall\exists)^*\exists^*(\forall\exists)^*$ [M99].

Our lower bounds are proved in the language of Ehrenfeucht-Fraïssé games. To show that, for example, $reach(\mathcal{S})$ cannot be expressed by a sentence with a prefix of the form $\forall\exists w$ where $w \in W$ we assume as (inductive) hypothesis that there are two structures $P \in \mathcal{S}$ and $R \notin \mathcal{S}$ such that Duplicator has a winning strategy in the game (corresponding to the prefix w) on (P, R) . Then we show how to apply some graph composition methods to get, from P and R , new structures $P_1 \in reach(\mathcal{S})$ and $R_1 \notin reach(\mathcal{S})$ such that Duplicator has a winning strategy in the game (corresponding to the new prefix $\forall\exists w$) on (P_1, R_1) . But since we know nothing about P and R our knowledge about P_1 and R_1 is quite limited, so the strategy for Duplicator uses as a black box the unknown Duplicator's strategy in a game on (P, R) .

With our technology we can make the first small step answering the hierarchy questions. To be more precise, we show that the hierarchy inside closed monadic NP does

² see definition below

not collapse on any first order level. Since we do not need to care if the w (the prefix which does not express \mathcal{S}) contains, or not, universal second order quantifiers a variety of results of this kind can also be proved with our technology about the structure of closed monadic hierarchy.

A new, very easy, proof of the results from [AFS00] is just a corollary of our method.

It also appears that - with minor modifications - the above inductive constructions can also be applied inside Kozen's mu-calculus [Ko83]. This constitutes a first small step towards trying to understand, over finite models, the (descriptive) complexity (in terms of patterns of FO and/or monadic quantifiers' prefix) of properties definable in the mu-calculus.

2 Technical Part

2.1 Structures

All the structures we consider in this paper are finite graphs (directed or not). The signature of the structures may also contain some additional unary relations ("colors") and constants (s and t).

2.2 Games

Definition 1. 1. A pattern of a monadic game (or just pattern) is any word over the alphabet $\{\forall, \exists, \forall, \exists, \oplus\}$.

2. If w is a pattern then the pattern \bar{w} (dual to w) is inductively defined as $\forall\bar{v}, \exists\bar{v}, \forall\bar{v}, \exists\bar{v}$ or $\oplus\bar{v}$ if w equals $\exists v, \forall v, \exists v, \forall v$ or $\oplus v$ respectively. The dual of the empty word is the empty word.

\forall and \exists still keep the meaning of universal and existential first order quantifiers, while \forall and \exists are universal and existential monadic second order (set) quantifiers. As you will soon see \oplus should be understood as a sort of boolean closure of a game. We will use the abbreviation FO for the regular expression $(\forall + \exists)$.

Definition 2. Let P and R be two relational structures over the same signature. Let w be some pattern. An Ehrenfeucht-Fraïssé game with pattern w over (P, R) is then the following game between 2 players, called Spoiler and Duplicator:

1. If w is the empty word then the game is over and Duplicator wins if the substructures induced in P and in R by all the constants in the signature are isomorphic. Spoiler wins if they are not isomorphic.
2. If w is nonempty then:
 - (a) If $w = \exists v$ ($w = \forall v$) for some v then a new constant symbol c is added to the signature, Spoiler chooses the interpretation of c in P (R resp.) and then Duplicator chooses the interpretation of c in R (P resp.). Then they play the game with pattern v on the enriched structures.

- (b) If $w = \exists v (w = \forall v)$ for some v then a new unary relation symbol C is added to the signature, Spoiler chooses the interpretation of C in P (R resp.) and then Duplicator chooses the interpretation of C in R (P resp.) Then they play the game with pattern v on the enriched structures.
- (c) If $w = \oplus v$ for some v then Spoiler can decide if he prefers to continue with the game with pattern v or rather with \bar{v} . Then they play the game with the pattern chosen by Spoiler.

The part of the game described by item (a) is called a first order round, or pebbling round. The part described by item (b) is a second order round, or coloring round.

Definition 3. We say that a property (i.e a class of structures) \mathcal{S} is expressible by a pattern w if for each two structures $P \in \mathcal{S}$ and $R \notin \mathcal{S}$ Spoiler has a winning strategy in the game with pattern w on (P, R) . If W is a set of patterns then we say that \mathcal{S} is expressible in W if there exists a $w \in W$ such that \mathcal{S} is expressible by w .

The following theorem illustrates the links between games and logics. We skip its proof as well known (see for example [EF] and [AFS00]):

- Theorem 1.**
1. Monadic NP is exactly the class of properties expressible by $\exists^* FO^*$;
 2. The boolean closure of monadic NP is exactly the class of properties expressible by $\oplus \exists^* FO^*$;
 3. The first order closure of monadic NP is exactly the class of properties expressible by $FO^* \oplus \exists^* FO^*$;
 4. $2k$ -th level of the monadic hierarchy is exactly the class of properties expressible by $(\exists^* \forall^*)^k FO^*$;
 5. $2k$ -th level of the closed monadic hierarchy is exactly the class of properties expressible by $(FO^* \exists^* \forall^*)^k FO^*$;
 6. Closed monadic NP is exactly the class of properties expressible by $(FO^* \exists^*)^*$;

The last theorem motivates:

Definition 4. A non trivial class of game patterns (or just class) is a set of game patterns denoted by a regular expression without union over the alphabet $\{\oplus, \exists, \forall, FO\}$, which ends with FO^* and contains at least one \forall^* or \exists^*

In the sequel, all classes of game patterns we consider are non trivial.

2.3 Graph operations

The techniques we are going to present are inductive and compositional. *Inductive* means here that we will assume as a hypothesis that there is a property expressible by some class of patterns W_1 but not by W and then, under this hypothesis, we will prove that there is a property expressible in the class $V_1 W_1$ but not in the class $V W$ where V_1 and V will be some (short) prefixes. The word *compositional* means here that the pair of structures (P_{VW}, R_{VW}) (on which Duplicator has a winning strategy in a VW game) will be directly constructed from the pair of structures (P_W, R_W) (on

which Duplicator has a winning strategy in a W game). For this construction we do not need to know anything about the original structures.

In the sequel, we will assume that all our structures are connected and that the signature contains a constant s (for *source*). This is possible thanks to the following natural definition and obvious lemma:

Definition 5. Let S be a property of structures (with the signature without constant s). Then $\text{cone}(S)$ is the property of structures (with the same signature, enriched with constant s): For every x distinct from s there is an edge from s to x and the substructure induced by all the vertices distinct from s has the property S .

Lemma 1. If S is expressible by w then $\text{cone}(S)$ also is. If S is not expressible by w then there is a pair of connected structures (P, R) (see Definition 6 below) such that P has the property $\text{cone}(S)$, R does not, and Duplicator has a winning strategy in the w game on (P, R) . ■

Now we introduce some notations for graph operations. As we just mentioned we assume that all the graphs we are dealing with are connected and have some distinguished node s . Some of them will also have another distinguished node t (for *target*).

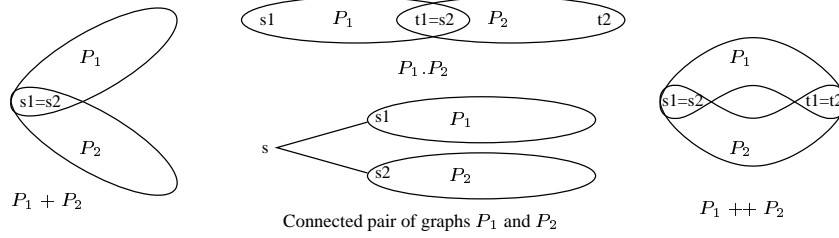


Fig. 1. Some graph operations.

- Definition 6.**
1. Let U denote the graph containing just two vertices, s and t , and one edge $E(s, t)$.
 2. If \mathcal{A} is a set of graphs, then $\Sigma_{P \in \mathcal{A}}^s P$ ($\Sigma_{P \in \mathcal{A}}^{st} P$) is the union of all graphs in \mathcal{A} with all the s vertices identified (resp. and all the t vertices identified). We will use also the notation $\Sigma_c^s P$ ($\Sigma_c^{st} P$) if \mathcal{A} contains just c copies of the same structure P . If there are only two elements, say P and R in \mathcal{A} , then we write $P+R$ (or $P++R$) instead of $\Sigma_{P \in \mathcal{A}}^s P$ (or $\Sigma_{P \in \mathcal{A}}^{st} P$).
 3. If P is a graph with constants s and t then $P.R$ (or PR for short) is the graph being a union of P and R with t of P identified with s of R (so that s of the new graph is the s of P and the t of the new graph is the t of R if it exists).
 4. If \mathcal{A} is a set of graphs then the graph $\Sigma_{P \in \mathcal{A}}^s (UP)$ will be called a connected set of graphs. If there are just two elements in \mathcal{A} then we will call it a connected pair of graphs.

2.4 Some simple lemmas about games

Let us start with an obvious lemma, which would remain true even without the assumption that the relations introduced during the second order rounds are unary:

Lemma 2. *If the graphs P and R are isomorphic then Duplicator has a winning strategy in the w game on (P, R) whatever w is. ■*

The following Lemmas 3-5 are not much harder to prove than Lemma 2 but the assumption that games are monadic is crucial here:

Lemma 3. *If Duplicator has winning strategies in w games on (P_1, R_1) and on (P_2, R_2) then he also has winning strategies in w games on (P_1+P_2, R_1+R_2) , on (P_1++P_2, R_1++R_2) and on (P_1P_2, R_1R_2) . ■*

Lemma 4. *For every structure P and pattern w there exists a number n such that provided $m \geq n$ then Duplicator has winning strategies in the w games on $(\Sigma_m^s P, \Sigma_{m+1}^s P)$ and $(\Sigma_m^{st} P, \Sigma_{m+1}^{st} P)$*

Proof. Induction on the structure of w . Use the fact that for a structure P of some fixed size there are only finitely many colorings of it, so if we have enough copies some colorings must repeat many times. ■

Lemma 5. *Let P be a connected pair of structures P_1 and P_2 and let R be a connected pair of structures R_1 and R_2 . Suppose for some (non trivial³) class V there exists $v \in V$ such that Spoiler has a winning strategy on the v games on (P_1, R_1) and on (P_1, R_2) . Then there exists $w \in \exists V$ such that Spoiler has a winning strategy in the w games on (P, R) .*

Proof. The strategy of Spoiler is to take as his first constant the source of P_1 in P . Duplicator must answer either with the source of R_1 or of R_2 , and so he must make a commitment on which of the two structures is going to play the role of P_1 in R now. The cases are symmetric, so let us assume he decides on R_1 . Then Spoiler uses his strategy for the v game on (P_1, R_1) to win the game. Actually, Spoiler must force Duplicator to move only inside the structures P_1 and R_1 . This can be achieved with one more coloring round (at any time in the v game) subsequently playing a w -game for some $w \in V$ since V is non trivial. The next remark makes this observation more precise. ■

Remark 6 *After the first round, when Spoiler picks the source of P_1 and Duplicator answers by the source of R_1 , Spoiler must force Duplicator to restrict the moves of the remaining game only to the structures P_1 and R_1 . In other words, Spoiler needs to be sure that each time he picks a constant inside P_1 (R_1) Duplicator actually answers with a constant inside R_1 (P_1). This can be secured with the use of an additional coloring round: Spoiler paints P_1 (or R_1 , he is as happy with a \exists round as with a \forall one) with some color leaving the rest of P unpainted. Duplicator must answer by painting R_1 (P_1) with this color, leaving the rest of R unpainted. Otherwise, this will be detected by Spoiler with the use of the final first order rounds. Notice that the additional coloring round can take place at any moment of the game, and so that the strategy is available for Spoiler for some $\exists V$ game since V is a nontrivial class of patterns.*

³ see Definition 4

2.5 A tool for the boolean closure

Let \mathcal{S} be any property. Then, a connected pair of structures $UP+UR$ will be called $\mathcal{S}\mathcal{S}$ if both the structures P and R belong to \mathcal{S} , $\bar{\mathcal{S}}\mathcal{S}$ if exactly one of them belongs to \mathcal{S} and $\bar{\mathcal{S}}\bar{\mathcal{S}}$ otherwise.

Definition 7. For a property \mathcal{S} define $\text{bool}(\mathcal{S})$ as the property: the structure is a connected set of connected pairs of structures, and at least one of those pairs is $\mathcal{S}\mathcal{S}$.

Lemma 7. Suppose a property \mathcal{S} is not expressible in class W , but both \mathcal{S} and its complement $\bar{\mathcal{S}}$ are expressible in some other class V . Then $\text{bool}(\mathcal{S})$ is not expressible in $\oplus W$ but is expressible in $\exists\exists V$.

Proof. Let us first show that there exists $w \in V$ such that, provided $P \in \text{bool}(\mathcal{S})$ and $R \notin \text{bool}(\mathcal{S})$, Spoiler has a winning strategy in the $\exists\exists w$ game on (P, R) . This will prove that property $\text{bool}(\mathcal{S})$ is expressible by $\exists\exists V$.

First observe that if R is not a connected set of pairs then either the vertices of R at distance less than 2 from s do not form a tree, or there is a vertex at distance 2 from s whose degree is not 3, or R is not connected, or there is a vertex x at distance 2 from s such that the structure resulting from removing x (and all the three adjacent edges) from R has less than 3 connected components. In each of those cases Spoiler can win some game in $\exists V$ for every nontrivial V .

If R is a connected set of pairs then in his first move Spoiler takes as his constant the source of some $\bar{\mathcal{S}}\mathcal{S}$ pair in P . Duplicator must answer by showing a source of some pair in R . There are two cases: either Duplicator shows a source of some $\mathcal{S}\mathcal{S}$ pair in R or a source of some $\bar{\mathcal{S}}\bar{\mathcal{S}}$ pair in R . In each of the two cases we may think that one pair of structures has been selected in P and one in R . Spoiler can restrict the game to the two selected pairs (see Remark 6). Then we use Lemma 5 to finish the proof.

Now we will show that whatever a pattern $\oplus w$ is, where $w \in W$, there exist two structures $P \in \text{bool}(\mathcal{S})$ and $R \notin \text{bool}(\mathcal{S})$ such that Duplicator has a winning strategy in the $\oplus w$ game on (P, R) . Let (P_1, R_1) be such a pair of structures that $P_1 \in \mathcal{S}$, $R_1 \notin \mathcal{S}$ and Duplicator has a winning strategy in the w game on (P_1, R_1) . Let c be some huge constant. Let $R = \Sigma_c^s (U(UP_1+UP_1)+U(UR_1+UR_1))$. So R is a connected set of $2c$ connected pairs, c of them are $\bar{\mathcal{S}}\bar{\mathcal{S}}$ and c are $\mathcal{S}\mathcal{S}$. Obviously, $R \notin \text{bool}(\mathcal{S})$. Let $P = R+U(UP_1+UR_1)$ be R with one more pair, a $\mathcal{S}\bar{\mathcal{S}}$ one, so that $P \in \text{bool}(\mathcal{S})$.

Now, if Spoiler in his first move decides to play the game w on P and R then remark that P is $Q_1+Q_2+Q_3$ where $Q_1 = \Sigma_c^s (U(UP_1+UP_1))$, $Q_2 = \Sigma_c^s (U(UR_1+UR_1))$ and $Q_3 = U(UR_1+UP_1)$ while R is $Q_4+Q_5+Q_6$ where $Q_4 = \Sigma_c^s (U(UP_1+UP_1))$, $Q_5 = \Sigma_{c-1}^s (U(UR_1+UR_1))$ and $Q_6 = U(UR_1+UR_1)$. We know that Duplicator has a winning strategies in w games on (Q_1, Q_4) (by Lemma 2), on (Q_2, Q_5) (by Lemma 4) and on (Q_3, Q_6) (by Lemma 3, since he has a winning strategy in a w game on (P_1, R_1)). So, again by Lemma 3 he has a winning strategy in w game on (P, R) .

If Spoiler decides in his first round to continue with \bar{w} rather than w then take Q_1, Q_2, Q_3 as before but $Q_4 = \Sigma_{c-1}^s (U(UP_1+UP_1))$, $Q_5 = \Sigma_c^s (U(UR_1+UR_1))$ $Q_6 = U(UP_1+UP_1)$ and use the same reasoning, using the fact that Duplicator has a winning strategy in the \bar{w} game on (R_1, P_1) . ■

2.6 A tool for first order quantifiers

Now the signature of our structures will contain additional unary relation symbol G (for *gate*). For a given structure P , and for two its vertices x, y , such that $G(y)$ holds let $P_{x,y}$ be the structure consisting of the connected component of $P - \{x\}$, containing y as its source. $P - \{x\}$ is here understood to be the structure resulting from P after removing x and all its adjacent edges. So $P_{x,y}$ could be read as "the structure you enter from x crossing the gate y " (see Figure 2).

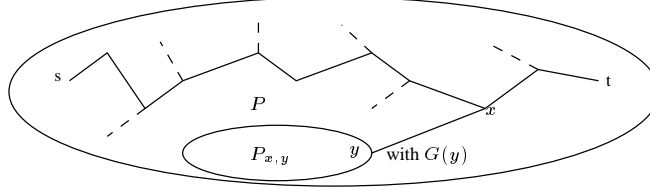


Fig. 2. $P_{x,y}$ is the structure you enter from x crossing the gate y .

Definition 8. Let \mathcal{S} be some property of structures. Then $reach(\mathcal{S})$ will be the following property (of a structure P): there is a path from s to t such that for every x on this path it holds that (i) $x \notin G$ and (ii) for every y such that $E(x, y)$ and $G(y)$ the structure $P_{x,y}$ has the property \mathcal{S} .

By a *path from s to t* we mean a subset H of the set of vertices of the structure such that $s, t \in H$, each of s and t has exactly one adjacent vertex in H and each element of H which is neither s nor t has exactly 2 adjacent vertices in H . The fact that H is a path is expressible by FO^* .

Lemma 8. 1. Suppose a property \mathcal{S} is not expressible in some class W . Then $reach(\mathcal{S})$ is not expressible in FO^*W ;
 2. Suppose a property \mathcal{S} is expressible in some class W . Then $reach(\mathcal{S})$ is expressible in the class $\exists \forall W$.

Proof. 1. First of all we will show that if \mathcal{S} is not expressible in W , then also $reach(\mathcal{S})$ is not expressible in W . For a given $w \in W$ there are structures P and R such that $P \in \mathcal{S}$, $R \notin \mathcal{S}$ and Duplicator has a winning strategy in the w game on (P, R) . Consider a structure T whose only elements are s, t, x, y , whose edges are $E(s, x), E(x, t), E(x, y)$ and for which $G(y)$ holds. Let P_0 be the union of T and P , with y of T identified with s of P . The s and t of P_0 are s and t of T . Let R_0 be the structure constructed in the same way from T and R . Then obviously $P_0 \in reach(\mathcal{S})$, $R_0 \notin reach(\mathcal{S})$ and Duplicator has a winning strategy in the w game on (P_0, R_0) . Notice that both P_0 and R_0 have the following property :

(*) (property of structure Q) if x is reachable from s or from t by a path disjoint from G and if y is such that $G(y)$ and $E(x, y)$ then $Q_{x,y}$ contains neither s of Q nor t of Q .

Now let P and R be structures, both satisfying (*) and such that $P \in reach(\mathcal{S})$, $R \notin reach(\mathcal{S})$ and Duplicator has a winning strategy in a w game on (P, R) . In order to prove our claim it is enough (by induction) to construct structures (P_1, R_1) both satisfying (*) and such that $P_1 \in reach(\mathcal{S})$, $R_1 \notin reach(\mathcal{S})$ and Duplicator has a winning strategy in a $\forall\exists w$ game on (P_1, R_1) . Let n be a huge enough constant. Define: $R_1 = (\Sigma_n^{st}(PR)) ++ (\Sigma_n^{st}(RP))$ and $P_1 = R_1 ++ PP$. Obviously $P_1 \in reach(\mathcal{S})$ and $R_1 \notin reach(\mathcal{S})$ hold. Now will show a winning strategy for Duplicator in a $\forall\exists w$ game on (P_1, R_1) . In his first round Spoiler selects some constant in R_1 . Duplicator answers with the same constant in P_1 (this is possible since R_1 can be viewed as a subset of P_1). Now notice that after this first round R_1 can be seen as

$$RP ++ PR ++ (\Sigma_{n-1}^{st}(PR)) ++ (\Sigma_{n-1}^{st}(RP))$$

and P_1 as

$$RP ++ PR ++ (\Sigma_{n-1}^{st}(PR)) ++ (\Sigma_{n-1}^{st}(RP)) ++ PP$$

where the constant selected in the first round is in the first $RP ++ PR$, both in R_1 and in P_1 . By Lemma 2 and Lemma 3 it is now enough to show that Duplicator has a winning strategy in the remaining $\exists w$ game on (P_2, R_2) where

$$P_2 = \Sigma_{n-1}^{st}(PR) ++ (\Sigma_{n-1}^{st}(RP)) ++ PP$$

and

$$R_2 = \Sigma_{n-1}^{st}(PR) ++ (\Sigma_{n-1}^{st}(RP))$$

Let Spoiler select some constant in P_2 .

If Spoiler selects a constant in $\Sigma_{n-1}^{st}(PR) ++ (\Sigma_{n-1}^{st}(RP))$ then Duplicator answers with the same constant in R_2 and then wins easily. The only interesting case is when Spoiler selects his constant in PP . Suppose it is selected in the first P (the other case is symmetric). Then Duplicator answers by selecting the same constant in the P of some PR in R_2 . Notice that $P_2 = Q_1 ++ Q_2 ++ (\Sigma_{n-1}^{st}(RP))$ and $R_2 = Q_3 ++ Q_4 ++ (\Sigma_{n-1}^{st}(RP))$, where $Q_1 = PP$, $Q_2 = \Sigma_{n-1}^{st}(PR)$, $Q_3 = PR$ and $Q_4 = \Sigma_{n-2}^{st}(PR)$, and where some constant is already fixed in the first P of Q_1 and in the P of Q_3 . Now the w game remains to be played. But since Duplicator has a winning strategy in the w game on (P, R) he also has (by Lemmas 2 and 3) a winning strategy in a w game on (Q_1, Q_3) . By Lemma 4 he has a winning strategy in a w game on (Q_2, Q_4) and so, again by Lemma 3 we get a winning strategy for Duplicator in the $\exists w$ game on (P_2, R_2) .

2. Suppose $P \in reach(\mathcal{S})$ and $R \notin reach(\mathcal{S})$. Spoiler, in his first move fixes a path in P , as in the definition of $reach(\mathcal{S})$. Duplicator answers selecting a set in R . If the set selected by Duplicator is not a path from s to t then Spoiler only needs some fixed number of first order rounds to win. If it is such a path then there must be some x on the path, and some y such that $E(x, y)$, $G(y)$ hold in R and $R_{x,y} \notin \mathcal{S}$. Now Spoiler uses his two first order universal rounds to fix those x and y . Duplicator answers with some two points z, t in P such that $E(z, t)$ and $G(t)$ hold in P . But, since $P \in reach(\mathcal{S})$ it turns out that $P_{z,t} \in \mathcal{S}$, so Spoiler can use rounds of the remaining w game to secure a win (a trick from Remark 6 will be needed here to restrict the w game to $P_{x,y}, R_{z,t}$).

■

Remark 9 *The role of predicate G is not crucial for the construction above. It could be replaced by a graph gadget if the reader wishes to see \mathcal{P}_2 being a property of undirected uncolored graphs.*

Another way to avoid the unary relation G (as suggested by Larry Stockmeyer) is to define $\text{reach}(S)$ as: there is a path from s to t such that for every x on this path and every y such that $E(x, y)$ and y is not on this path, the structure $P_{x,y}$ has the property S .

2.7 Corollaries

As the first application of our toolkit we reprove the results from [AFS00]:

Theorem 2. *There exists property \mathcal{P}_1 expressible in $FO^* \exists^* FO^*$ but not in $\oplus \exists^* FO^*$. There exists property \mathcal{P}_2 expressible in $\exists FO^* \exists^* FO^*$ but not in $FO^* \oplus \exists^* FO^*$.*

Proof. Let $Cted$ be the property of connectivity. It is well known that $Cted$ is not expressible in $\exists^* FO^*$ but both $Cted$ and its complement are expressible in $\forall \exists^* FO^*$. now take $\mathcal{P}_1 = \text{bool}(\text{cone}(Cted))$ and $\mathcal{P}_2 = \text{reach}(\text{bool}(\text{cone}(Cted)))$. Use Lemmas 7 and 8 to finish the proof. ■

A new result we can prove is that even if the hierarchy inside closed monadic NP collapses, it does not collapse on a first order level:

Theorem 3. *If there is a property expressible in FO^*W but not in W , where $W = (\exists^* FO^*)^k$ then there is a property expressible in $\exists FO^*W$ but not in FO^*W .*

Proof. This follows immediately from Lemma 8 ■

Several similar results can be proved for the closed monadic hierarchy or reproved for the monadic hierarchy (see [MT97] and [Ma99] sections 4.4 and 4.5).

It is interesting to remark that the inductive constructions presented here are also definable (with minor and insignificant variations) inside Kozen's propositional μ -calculus [Ko83].

More precisely, given some unary predicates S , one may define in the μ -calculus the new predicates that depend on S : $\text{Bool}(S) = \diamond(\diamond S \wedge \diamond \neg S)$ and $\text{Reach}(S) = \mu X.(\Box(G \Rightarrow S) \wedge (\diamond X \vee T))$ which almost denote the same constructions (here the “target” constant t is replaced by the set of “possible targets” T and the “source” constant s is the implicit free FO variable in any mu-calculus formula).

From Lemmas 7 and 8 (which extend to these definitions inside the mu-calculus) and the fact that (the mu-calculus version of) directed reachability: $\text{dreach} = \mu X.(\diamond X \vee T)$ is not expressible in $\exists^* FO^*$ while both dreach and its complement are expressible in $\exists \exists^* FO^*$, one has :

Corollary 1. *There are properties \mathcal{R}_1 and \mathcal{R}_2 definable in monadic μ -calculus such that \mathcal{R}_1 is expressible in $FO^* \exists FO^* \exists^* FO^*$ but not in $\oplus \exists^* FO^*$ and \mathcal{R}_2 is expressible in $\exists FO^* \exists FO^* \exists^* FO^*$ but not in $FO^* \oplus \exists^* FO^*$.*

Proof. Take $\mathcal{R}_1 = \text{Bool}(\text{dreach})$ and $\mathcal{R}_2 = \text{Reach}(\text{dreach})$ and apply Lemmas 7 and 8 to finish the proof. ■

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