

THE MONADIC COMPLEXITY OF THE
PROPOSITIONAL FIXPOINT CALCULUS :
KNOWN RESULTS
AND OPEN PROBLEMS

DAVID JANIN

LaBRI - Bordeaux

APRIL 2000

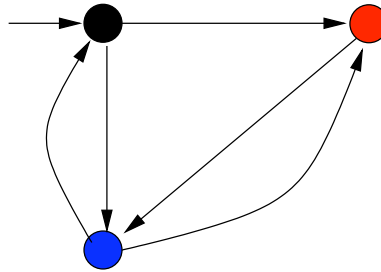
Overview of the talk

1. the mu-calculus
 - 1.1. definition : models, formulas
 - 1.2. example : reachability and alternating reachability,
 - 1.3. expressive power : over (deterministic) trees, over graphs
2. complexity
 - 2.1. best known algorithm complexity for model checking
 - 2.2. monadic complexity over trees, over finite graphs,
 - 2.3. open problems
3. more on hard formulas
 - 3.1. parity games
 - 3.2. classical results
 - 3.3. relationship with the mu-calculus
 - 3.4. alternating reachability : the return !

DEFINITION, EXAMPLES AND EXPRESSIVENESS

Models

Models are graphs with a source : $G = \langle S_G, s_G, R_G \rangle$ with some distinguished sets of vertices P_G, Q_G, \dots that induce colors, e.g. $black = P_G \cap Q_G$, $blue = P_G \cap \bar{Q}_G, \dots$



A graph is called **deterministic** when no common successors are identically colored.

Propositional fixpoint calculus

Syntax :

- atomic formulas X , $\neg X$, P or $\neg P$,
- $\alpha_1 \wedge \alpha_2$, $\alpha_1 \vee \alpha_2$,
- $\diamond\alpha$ and $\square\alpha$,
- $\mu X.\alpha$ and $\nu X.\alpha$ ¹,

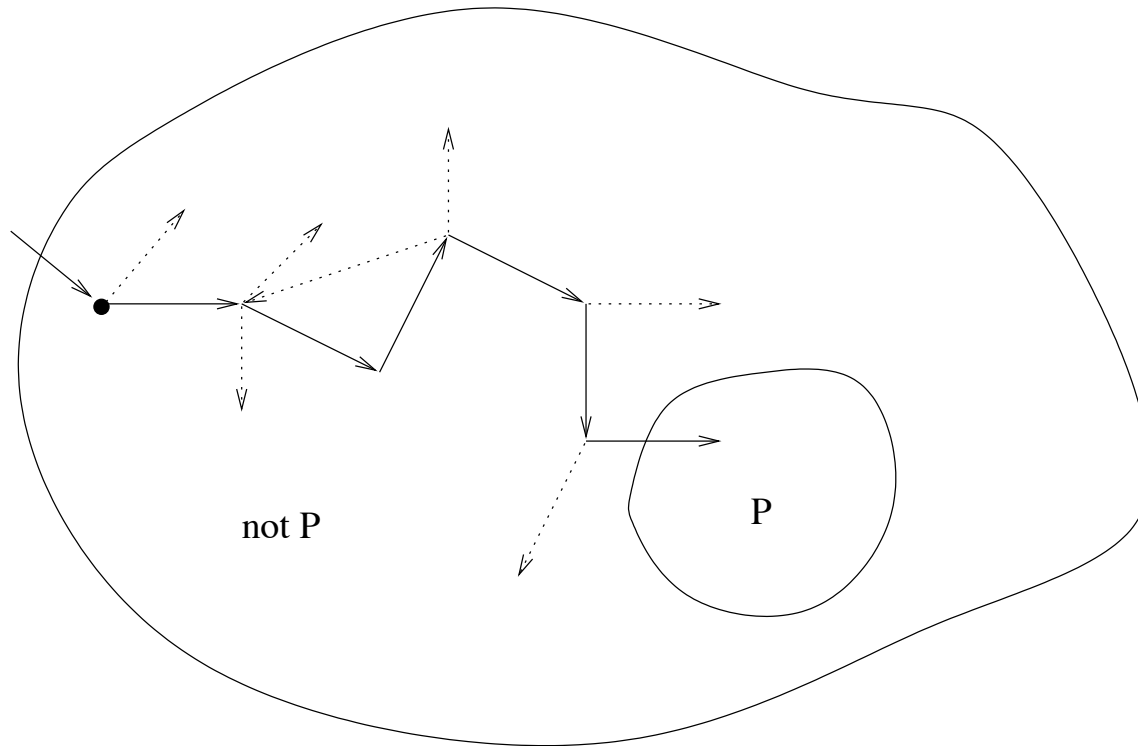
Semantics : Each mu-calculus formula α is interpreted as a unary predicate $\llbracket\alpha\rrbracket(z)$ defined (in MSOL) by :

$$\begin{array}{ll}
 \llbracket X \rrbracket = X(z) & \llbracket \diamond\alpha \rrbracket = \exists x R(z, x) \wedge \llbracket \alpha \rrbracket[x/z] \\
 \llbracket P \rrbracket = P(z) & \llbracket \square\alpha \rrbracket = \forall x R(z, x) \Rightarrow \llbracket \alpha \rrbracket[x/z] \\
 \llbracket \alpha_1 \wedge \alpha_2 \rrbracket = \llbracket \alpha_1 \rrbracket \wedge \llbracket \alpha_2 \rrbracket & \llbracket \mu X.\alpha \rrbracket = \forall X (X \subseteq \llbracket \alpha \rrbracket) \Rightarrow X(z) \\
 \llbracket \alpha_1 \vee \alpha_2 \rrbracket = \llbracket \alpha_1 \rrbracket \vee \llbracket \alpha_2 \rrbracket & \llbracket \nu X.\alpha \rrbracket = \exists X (X \subseteq \llbracket \alpha \rrbracket) \wedge X(z) \\
 \llbracket \neg\alpha \rrbracket = \neg \llbracket \alpha \rrbracket &
 \end{array}$$

where $X \subseteq \llbracket \alpha \rrbracket$ stands for $\forall z X(z) \Rightarrow \llbracket \alpha \rrbracket(z)$.

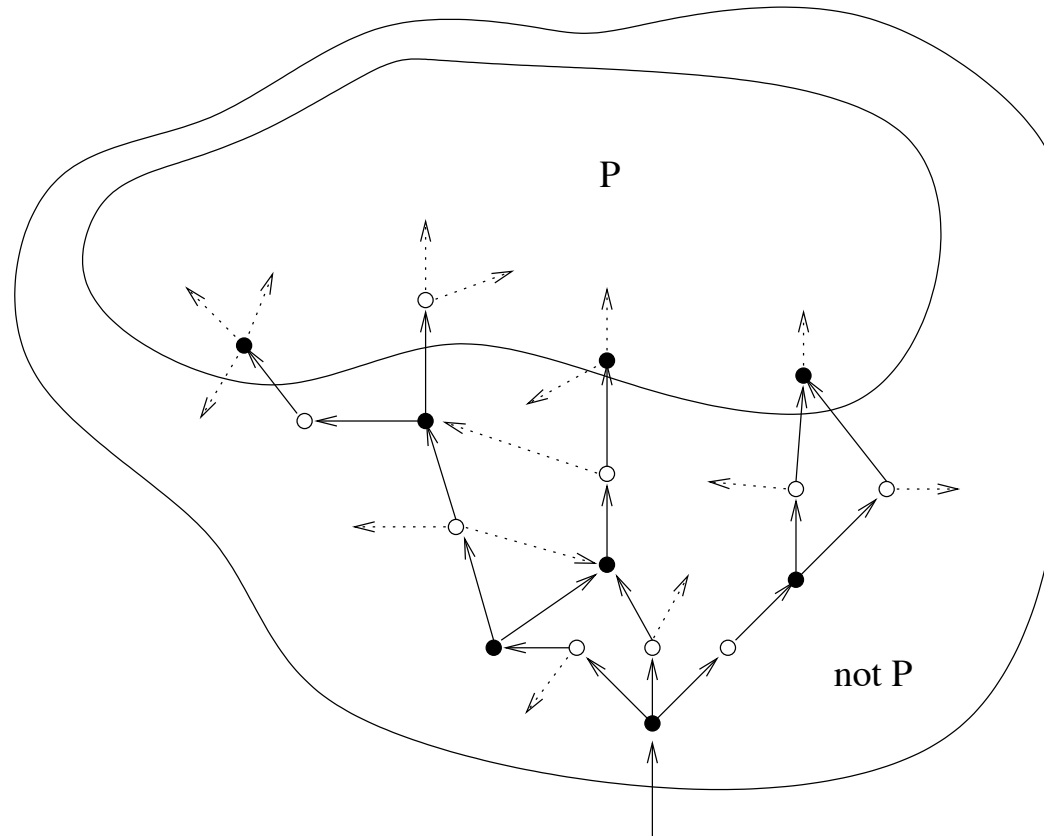
¹provided X is positive in α

$$\text{reach} = \mu X.(P \vee \diamond X)$$



Computed as the least fixpoint of the mapping $X \mapsto (P \vee \diamond X)$,
 $G \models \llbracket \text{reach} \rrbracket(s_G)$ iff there is a path from s_G to a vertex in P .

$$\text{altr} = \mu X.(P \vee (Q \wedge \diamond X) \vee (\neg Q \wedge \square X))$$



$G \models [\text{altr}](s_G)$ iff from s_G player Q has a winning strategy (against player \bar{Q}) to reach a vertex of P .

Expressiveness (extending Rabin's 1969 results on S2S)

Theorem (Niwiński 1986) Over (deterministic) trees : mu-calculus = *MSOL*.

Theorem (J - Walukiewicz 1996) Over graphs :
mu-calculus = bisimulation invariant fragment of *MSOL*

ALGORITHMIC COMPLEXITY

A complexity measure : the fixpoint alternation depth

Alternation depth of least and greatest fixpoint :

Let $M_0 = N_0$ be the set of mu-formulae with no fixpoint.

For each d , let M_{d+1} (resp. N_{d+1}) be the closure of N_d (resp. M_d) by : conjunction, disjunction, modalities and least fixpoint $\mu X.\alpha$ (resp. greatest fixpoint $\nu X.\alpha$).

Algorithmic complexity (best known result)

Computing $G \models \llbracket \alpha \rrbracket (s_G)$ for (finite) graph G and $\alpha \in M_d \cup N_d$ can be made in $O(|\alpha \times G|^{\lceil d/2 \rceil})$.

Better model-checking algorithm ?

Remark All known model checking algorithm stick more or less to the computation of least and greatest fixpoint alternatively...

and

Theorem (Bradfield 1996, Arnold 1999) The fixpoint alt. depth hierarchy is strict, i.e. $M_d \subset M_{d+1} \neq M_d$ ($0 \leq d$)

DESCRIPTIVE COMPLEXITY

Over arbitrary graphs

Theorem (from Street-Emerson 89)

$$\text{mu-calculus} \subseteq \text{NP} \cap \text{co-NP}$$

Theorem (with Emerson-Jutla 91)

$$M_d \subseteq (\text{monadic}) \Pi_d \quad (d \geq 0)$$

$$N_d \subseteq (\text{monadic}) \Sigma_d \quad (d \geq 0)$$

Open problem : mu-calculus \subseteq (monadic) Π_k for some k ?

Over (deterministic) trees

Theorem (Rabin 1969) $MSOL = \Sigma_2(\leq_R) \neq \Sigma_1(\leq_R)$

Theorem (J-Lenzi 1999) $MSOL = \Sigma_3(R) \neq \Sigma_2(R)$

Theorem (Lenzi 2000) $N_2 = \Sigma_2(R)$

where N_2 also captures the class of languages of trees definable by (so called) **special tree automaton**.

Over finite graphs

Theorem (from Emerson and Jutla 91)

In the deterministic case : mu-calculus \subseteq (monadic) Σ_3 .

Remark : the monadic complexity of the mu-calculus collapses whenever quantification over edges is available “for free”...

In general : two open problems

1. do mu-calculus formulae witness the infiniteness of the monadic quantifier alternation hierarchy (although mu-calculus properties sound unrelated with Matz and Thomas witnesses) ?
2. do mu-calculus formulae witness some (yet unknown) infiniteness of the (first order closed) monadic quantifier alternation hierarchy...?

DIFFICULT PROPERTIES

Parity game

A **parity game** is defined by :

1. a set of positions $P = P_O \uplus P_E$,
2. a move relation $R \subseteq P \times P$,
3. an index function $\Omega : P \rightarrow \mathbb{N}$ with $|\pi(P)| < +\infty$.

A **play** between player O and player E is a maximal sequence of positions $\pi = (p_0, p_1, p_2, \dots, p_i, p_{i+1}, \dots)$ such that, from each position p_i , next position p_{i+1} must be chosen by player O when $p_i \in P_O$ or by player E when $p_i \in P_E$ with the constraint $(p_i, p_{i+1}) \in R$

Wining conditions :

When π is finite, the player that cannot move is the loser.

When π is infinite, given $Inf(\pi) = \{k : |\{i : k = \Omega(p_i)\}| = +\infty\}$

$$\begin{cases} \text{player } E \text{ wins when } \min(Inf(\pi)) \text{ is } \textit{even}, \\ \text{player } O \text{ wins when } \min(Inf(\pi)) \text{ is } \textit{odd}. \end{cases}$$

Standard results on parity games

Theorem 1 (Mc Naughton) Parity game are determined
(i.e. from each position, one of the player has a wining strategy).

Theorem 2 (Mc Naughton) Winning strategies are memoryless
(i.e. from a given position of his, the winning player can always make the same move)

Theorem 3 (Street-Emerson 89)² With $\Omega(P) = [0, d]$

$$W_O^d = \mu X_0 \nu X_1 \cdots \sigma X_d \bigvee_{k \in [0, d]} \Omega^{-1}(k) \wedge [(P_O \wedge \diamond X_k) \vee (P_E \wedge \square X_k)]$$

$$W_E^d = \nu X_0 \mu X_1 \cdots \sigma' X_d \bigvee_{k \in [0, d]} \Omega^{-1}(k) \wedge [(P_O \wedge \square X_k) \vee (P_E \wedge \diamond X_k)]$$

²and these formulas witness the infiniteness of the fixpoint hierarchy

Difficult mu-formulas

Theorem (from Street - Emerson 89) Computing $G \models \llbracket \alpha \rrbracket (s_G)$ for some $\alpha \in M_d \cup N_d$ is equivalent to solve a parity game (FO-definable over G) with indices $[0, d]$ of size $|G| \times |\alpha|$.

Corollary For each d the maximal computational/descriptive complexity of N_d (resp. M_d) equals the computational/descriptive complexity of W_E^d (resp. W_O^d).

Particular case : alternating reachability property is, up to non significant details, property W_O^0 .

Conjecture : $W_O^0 \in \text{monadic co-NP} \setminus \text{closed monadic NP}$.