

# Automata, algebras and logic for languages of labeled birooted trees



(overlapping tile languages)

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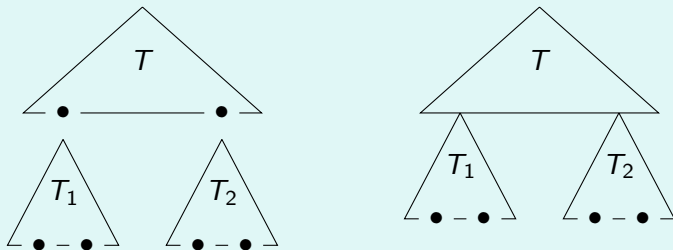
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# Plan of the talk

- 1 Labeled birooted trees
- 2 Languages
- 3 Automata :  $\text{MSO}^\uparrow$
- 4 Algebra :  $\text{BOOL}(\text{MSO}^\uparrow)$
- 5 Conclusion

# Algebras for finite trees : the functional approach

## Leaf to tree substitution

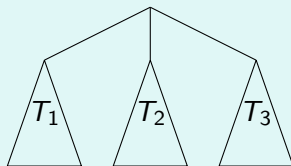
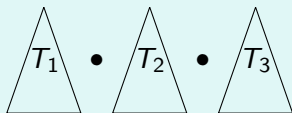


## Remark

Infinitely many sorts/unbounded arity

# Algebras for finite trees : the forest approach

## Forest concatenation and conversion to trees

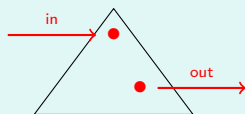


## Remark

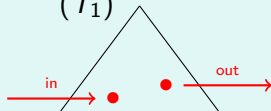
Two sorted algebra

# Algebras for finite trees : the (overlapping) tile approach

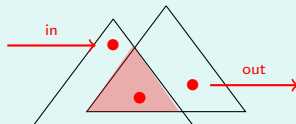
## Tree enriched with “input” and “output” roots



$(T_1)$



$(T_2)$



$(T_1 \cdot T_2)$

Product = synchronization + fusion

## Remark

Partial product (made complete with a zero). A modeling facility in computational music [1, 2]

## Question

**Automata**, **algebras** and **logic** for languages of overlapping tiles ?

# 1. Labeled birooted trees

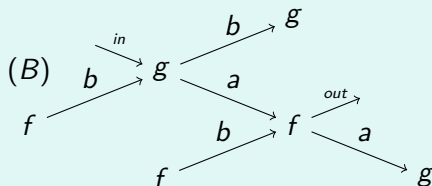
Musing around Scheiblich-Munn presentation of the Free inverse monoids

## Labeled Munn's trees (or birooted $F$ -trees)

Let  $F$  be a finite vertex alphabet, let  $A$  be a finite edge alphabet  $A$ .

### Birooted $F$ -tree

A finite **directed tree-shaped graph** with  **$F$ -labeled vertices** and  **$A$ -labeled edges**, both forward and backward **deterministic**,

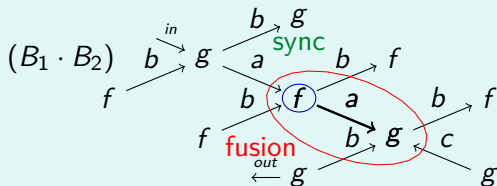
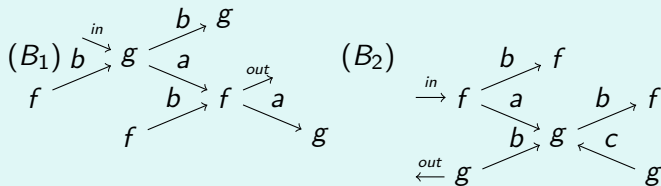


with distinguished **input root** and **output root**.



# Birooted $F$ -trees product

Product = sync + fusion

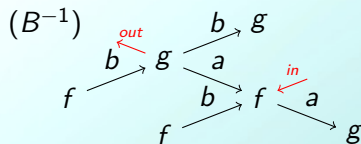
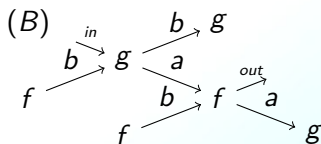


or the *undefined birooted tree* 0 when fusion fails.

# An inverse semigroup

## Semigroup/Monoid

The set  $\mathcal{B}(F)$  of birooted  $F$ -tree is a semigroup.  
 When  $F$  is a singleton, it is isomorphic to  $FIM(A)$ .  
 Otherwise, it is extended with a unit 1.



## Lemma (Inverses)

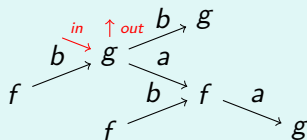
For every  $B \in \mathcal{B}(F)$  there exists a **unique**  $B^{-1} \in \mathcal{B}(F)$  such that

$$B \cdot B^{-1} \cdot B = B \text{ and } B^{-1} \cdot B \cdot B^{-1} = B^{-1}$$

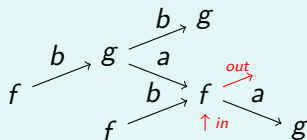
# Left and right projections (and Green's relations)

Elements  $B \cdot B^{-1}$  and  $B^{-1} \cdot B$  are idempotents

$$B^R = B \cdot B^{-1}$$



and  $B^L = B^{-1} \cdot B$

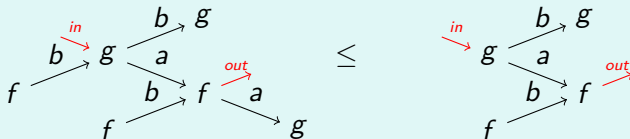


with  $B^R = C^R$  iff  $B\mathcal{R}C$  (resp.  $B^L = C^L$  iff  $B\mathcal{L}C$ ).

# The natural order

## Definition (Natural Order)

For every  $B$  and  $C$ , let  $B \leq C$  when  $B = B^R C$  (equivalently  $B = C B^L$ ).



## Lemma

The natural order is *stable* under product and the idempotents are the *subunits*  $U(\mathcal{B}(F)) = \{B \in \mathcal{B}(F) : B \leq 1\}$ .

## 2. Languages

From REC to MSO

# Languages of birooted $F$ -trees

## Operations on languages

Given  $M$  and  $N \subseteq \mathcal{B}(F) - 0$ :

- **sum** :  $M + N = M \cup N$ ,
- **product** :  $M \cdot N = \{B \cdot C : B \in M, C \in N, B \cdot C \neq 0\}$ ,
- **star** :  $M^* = \bigcup_{n \in \mathbb{N}} M^n$  with  $M^0 = \{1\}$  and  $M^{n+1} = M \cdot M^n$ ,
- **inverse** :  $M^{-1} = \{B^{-1} : B \in M\}$ ,
- **idempotent projection** :  $M^E = \{B : B \in M, B \cdot B = B\}$ ,
- **left and right projection** :  $M^L = \{B^{-1} \cdot B : B \in M\}$  and  $M^R = \{B \cdot B^{-1} : B \in M\}$ ,
- **others** ...

with upward or downward closed variants.

## First facts

### Theorem (Robustness)

*The class of languages of birooted trees definable in MSO is closed under all operators including upward and downward closure w.r.t. the natural order.*

### Lemma (Hierarchy)

$$REC \subset REG \subset 1-REG \subseteq \dots \subseteq k-REG \subseteq \dots \subseteq MSO$$

*where  $k$  in  $k-REG$  is the maximum allowed nested-depth of context projection operators.*

### Question

Automata/algebras for a rich class of languages of birooted trees ?

$$REC \subset ? \subset MSO$$

### 3. Automata : $\text{MSO}^\uparrow$

Non deterministic automata for languages of birooted  $\mathcal{F}$ -trees  
Walking automata are considered elsewhere [3].



## Birooted $F$ -tree automata

Automaton :  $\mathcal{A} = \langle Q, \delta, \Delta, W \rangle$

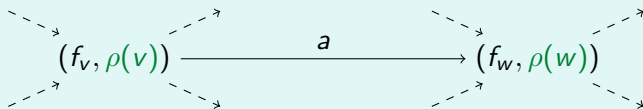
with a (finite) set of states  $Q$ , a **state table**  $\delta : F \rightarrow \mathcal{P}(Q)$ , a **transition table**  $\Delta : A \rightarrow \mathcal{P}(Q \times Q)$  and an **acceptance condition**  $W \subseteq Q \times Q$ .

Run of  $\mathcal{A}$  on  $B : \rho : \text{dom}(B) \rightarrow Q$

with, for every  $v \in \text{dom}(B)$

- **state coherence** :  $\rho(v) \in \delta(f_v)$ ,

and for every for  $w \in \text{dom}(B)$  with  $v \xrightarrow{a} w$  for some  $a \in A$ :

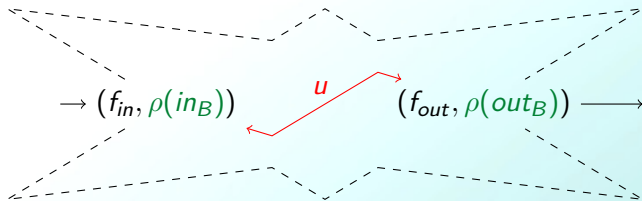


- **transition coherence** :  $(\rho(v), \rho(w)) \in \Delta(a)$ .

## Tile automata languages

### Accepting runs

Run  $\rho : \text{dom}(B) \rightarrow Q$  of  $\mathcal{A}$  on  $B$  is **accepting** when  $(\rho(\text{in}_B), \rho(\text{out}_B)) \in W$ .



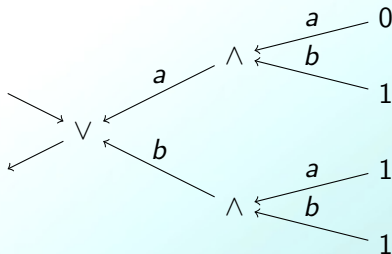
### Recognized language $L(\mathcal{A})$

The language  $L(\mathcal{A})$  is defined as the set of (non zero) birooted  $F$ -tree  $B \in \mathcal{B}(F)$  such that **there is an accepting run** of  $\mathcal{A}$  on  $B$ .

## An example: boolean birooted trees

### Fact

One can define (via a bottom tree automaton) the birooted  $F$ -tree automaton for (some closure of) the language of *idempotent birooted boolean trees* that “evaluate” to true.



### Remark

Birooted tree automata languages belongs to  $\text{Mon}\Sigma_1$ .

# Expressiveness

## Remark

Every language  $L$  recognized by an automaton is upward closed in the natural order.

## Theorem ( $\text{MSO}^\uparrow$ )

*A language of birooted  $F$ -trees  $L \subseteq \mathcal{B}(F)$  is recognized by a finite state tile automaton if and only if  $L$  is definable in  $\text{MSO}$  and upward closed.*

## Proof.

Slightly technical encoding of automata for mono-rooted tree (with distinguished vertex output) into birooted tree automata.  $\square$

## Corollary

*This class of languages is closed under union, intersection, (upward closed) product, (upward closed) star, projection. . .*

## 4. Algebra : $\text{BOOL}(\text{MSO}^\uparrow)$

Ordered monoids with local units and premorphisms

## Towards an associated algebra

### Induced premorphism

Let  $\mathcal{A} = \langle Q, \delta, \Delta, W \rangle$  be an automaton and let

$$\varphi : \mathcal{B}(F) \rightarrow \mathcal{P}(Q \times Q)$$

be the mapping defined by  $\varphi(0) = \emptyset$ ,  $\varphi(1) = I_Q$  and, for every non trivial  $B = \langle t, u \rangle$ ,  $\varphi(B)$  is the set of pair  $(\rho(1), \rho(u))$  for runs  $\rho$  of  $\mathcal{A}$  over  $B$ .

### Lemma

*The mapping  $\varphi$  recognizes  $L(\mathcal{A})$  in the sense that  $L(\mathcal{A}) = \varphi^{-1}(\varphi(L(\mathcal{A})))$ .*

### Question

What can we say about  $\varphi^{-1}(X)$  for  $X \subseteq \mathcal{P}(Q \times Q)$  ?

## Towards an associated algebra: objects

### Lemma

*The sub-monoid  $M_\varphi$  induced by  $\varphi(\mathcal{B}(F))$  ordered by inclusion is a partially ordered monoid (with stable order).*

### Lemma

*The sub-monoid  $M_\varphi$  is closed under “left” and “right” projection defined, for every  $X \subseteq Q \times Q$  by*

$$X^L = \{(q, q) \in Q \times Q : \exists p, (p, q) \in X\}$$

*and*

$$X^R = \{(p, p) \in Q \times Q : \exists q, (p, q) \in X\}$$

*with the (characterizing) properties*

$$X^L = \min\{Y \leq 1 : X \cdot Y = X\} \text{ and } X^R = \min\{Y \leq 1 : Y \cdot X = X\}$$

## Towards an associated algebra: arrows

### Lemma

The mapping  $\varphi : \mathcal{B}^1(F) \rightarrow M_\varphi$  is a **premorph**, i.e. for every  $B$  and  $C \in \mathcal{B}(F)$ :

- *monotonic*: if  $B \leq C$  then  $\varphi(B) \leq \varphi(C)$
- *sub-multiplicative*:  $\varphi(B \cdot C) \leq \varphi(B) \cdot \varphi(C)$

### Lemma (Adequacy)

For every  $B \in \mathcal{B}^1(F)$ ,

$$\varphi(X^L) = (\varphi(X))^L \text{ and } \varphi(X^R) = (\varphi(X))^R$$

and, for every  $B$  and  $C \in \mathcal{B}^1(F)$

if  $B \cdot C$  is disjoint then  $\varphi(B \cdot C) = \varphi(B) \cdot \varphi(C)$



# Effectivity

## Lemma (Effectivity)

*As  $\mathcal{B}^1(F)$  is finitely generated by disjoint product and left and right projection,  $\varphi(B)$  is effectively computable in linear time.*

## Definition (QREC)

A language  $L \subseteq \mathcal{B}(F)$  is QREC when there is an **adequate premorphism**

$$\varphi : \mathcal{B}^1(F) \rightarrow S$$

into an **finite adequately ordered monoid**  $S$  such that

$$L = \varphi^{-1}(\varphi(L))$$

# Expressiveness

## Theorem

A language  $L \subseteq \mathcal{B}(F)$  is QREC if and only if  $L$  is a finite boolean combination of languages recog. by finite state automata, i.e.

$$\text{QREC} = \text{BOOL}(\text{MSO}^{\uparrow})$$

## proof.

(QREC)  $\Rightarrow$  MSO : simulate disjoint decomposition.

(QREC + MSO)  $\Rightarrow$  BOOL(MSO<sup>↑</sup>) : let  $\varphi : \mathcal{B}(F) \rightarrow S$  an adequate premorphism with finite  $S$ . For every  $x \in S$  we have:

$$\varphi^{-1}(x) = \underbrace{\left( \bigcup_{y \in x^{\uparrow}} \varphi^{-1}(y) \right)}_{\text{MSO}^{\uparrow}} \cap \underbrace{\left( \bigcup_{y \in x^{\downarrow}} \varphi^{-1}(y) \right)}_{\text{MSO}^{\downarrow}}$$

# Expressiveness

proof (continued).

BOOL(MSO<sup>↑</sup>)  $\Rightarrow$  QREC : since MSO<sup>↑</sup>  $\subseteq$  QREC and QREC is closed under complement, it suffices to prove closure under union.

Let  $\varphi_1 : \mathcal{B}(F) \rightarrow S_1$  and  $\varphi_2 : \mathcal{B}(F) \rightarrow S_2$  two adequate premorphisms with finite  $S_1$  and  $S_2$ .

Then  $S_1 \times S_2$  is adequately ordered and

$$\varphi = \langle \varphi_1, \varphi_2 \rangle : \mathcal{B}(F) \rightarrow S_1 \times S_2$$

recognizes the union of any languages  $L_1$  and  $L_2$  resp. recognized by  $\varphi_1$  and  $\varphi_2$ .



## Example

### Fact

*The language of birooted boolean idempotent trees that evaluate to true is quasi-recognized by a finite aperiodic adequately ordered monoid.*

Thanks to Pin's semigroup software for analyzing the property of the recognizer.

### Remark

Strictness of  $QREC \subset MSO$  is witnessed by  $L = \{a^{2n}a^{-2n} : n \in \mathbb{N}\}$  that is MSO but not QREC.

## 5. Conclusion

## A work in progress

- Canonical automata/syntactic monoids ?  
Ehresmann ordered monoids ?
- Algebraic construction for (closed) product, stars ?
- Language classification ?
- Extension to infinite/continuous structures ?

# The underlying bigger picture

Based on Kellendonk models proposed for quasi-crystal structure description:

## Inverse semigroup theory *for* computer science

- System programming tools and concepts [4] ?
- Robust mathematical framework shaped towards computer system modeling [5] ?
- Experiments in reactive music systems [2] and, beyond, reactive systems ?

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