Decomposing graphs into highly irregular subgraphs

A connected graph \( G \) is highly irregular if every two of its adjacent vertices have distinct degrees.

An improper \( k \)-edge-colouring of \( G \) is locally irregular if each of the \( k \) subgraphs it induces is highly irregular.

The irregular chromatic index of \( G \), denoted \( \chi_{irr}(G) \), is the least number of colours used by a locally irregular edge-colouring of \( G \).

This parameter is undefined for paths and cycles with odd length, and a family of tree-like graphs with maximum degree 3 \([3]\).

We conjecture that all colourable graphs have irregular chromatic index at most 3.

This conjecture is verified for numerous classes of graphs \([1]\), including trees, complete graphs, \( d \)-regular graphs with \( d \geq 10^7 \), ...

From the algorithmic points of view, determining the irregular chromatic index of any tree can be done in linear time \([2]\).

If the input has colour, say, green by any locally irregular spreading gadget, then the two outputs have colour green.

We can check whether two outputs of two literal gadgets have the same colour.

Construction of the reduced graph \( G_F \)

Each clause \( C \) from a formula \( F \) is associated with one clause gadget \( G_C(C) \) of \( G_F \), and similarly for each literal \( x \) and one literal gadget \( G_x(x) \).

The clause and literal gadgets are connected in accordance with the structure of \( F \); i.e. if \( C = (x_1 \lor x_2 \lor x_3) \) is a clause of \( F \), then the first output of \( G_C(C) \) is used to construct \( G_F(x_1) \), the second one for \( G_F(x_2) \), and the third one for \( G_F(x_3) \).

The outputs of two literal gadgets associated with a literal and its negation are connected.

In any locally irregular 2-edge-colouring \( \phi_{irr} \) of \( G_F \):

- Each clause gadget "sends" colour green to exactly one literal gadget it is connected with.
- All of the inputs of a literal gadget have the same colour.
- The outputs of two literal gadgets associated with a literal and its negation have distinct colours.

From \( \phi_{irr} \), we can deduce a truth assignment satisfying \( F \), and vice-versa.

References

