

On Effective Decidability of the Homeomorphism Problem for Non-Compact Surfaces

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ABSTRACT. Here is investigated a class of non-compact surfaces without boundary for which the homeomorphism problem is decidable in an effective way. These surfaces, called equational, are characterized by three equivalent properties of global regularity concerning triangulations, infinite connected sums and ideal boundary.

Here we investigate effective computability of the homeomorphism problem for triangulable non-compact surfaces.

It is a well known fact that compact surfaces without boundary are classified by two invariants : genus and orientability (see e.g. [Mas67]). Since these data are computable from any triangulation, we have an algorithm to solve the homeomorphism problem in the compact case. In the case of non-compact surfaces without boundary, the classification theorem states that two such surfaces are homeomorphic if and only if there exists a homeomorphism between their ideal boundaries which conserves genus and orientability properties (Kerékjártó, Richards [Ric63]). However, this result does not directly lead to an effective classification method. The purpose of this paper is to give an *algorithm* which effectively decides, from triangulations, whether two such surfaces are homeomorphic.

The method is based on the work of Pierce [Pie72] which investigates a special kind of zero-dimensional compact metric spaces called *finite type* spaces. Pierce showed that the existence of homeomorphisms between such spaces can be reduced to the existence of isomorphisms between finite algebras. We restrict the study to non-compact surfaces without boundary whose ideal boundary is a space of finite type in the sense of Pierce.

A non-compact surface without boundary can be defined in several ways : by a triangulation, by an infinite connected sum (Corollary 1.5) or by its ideal boundary. Expressed at these various levels, the above restriction appears to be a natural *regularity* condition :

- It is equivalent with the property of having triangulations which are equational infinite hypergraphs. This means that they have triangulations which are defined in a constructive way by a special type of inductions on hypergraphs (see [Cou90]

or definition below ¹). Such a surface will be said to be *equational*. Let us note that the notion of equational hypergraph extends those of context-free graph which first appeared in [MS85].

- Treelike structures of zero-dimensional compact spaces (see Theorem 1.4 below) will allow us to define *the treelike connected sum* i.e. the connected sum of infinitely many compact surfaces performed along a tree. As a generalization of the fact that every compact surface without boundary is a connected sum of spheres, tori and projective planes, it will be showed that every non-compact surface without boundary is a treelike connected sum of spheres, tori and projective planes. This result is a new point of view of Richards's results [Ric63].

We will see that equational surfaces are the non-compact surfaces obtained from treelike connected sum on *regular* trees i.e. trees with only finitely many subtrees up to isomorphism (see Appendix). The notion of regular tree generalizes to trees the property of ultimate periodicity for infinite sequences. This fact expresses the property of being equational as a kind of global regularity.

- The work of Pierce [Pie72] generalizes the theorem stating that the Cantor space is the only perfect, zero dimensional, compact, metric space. Following Pierce, a zero dimensional compact metric space X is said to be of *finite type* if the powerset of X , provided with the usual boolean operations ($\cup, \cap, \cdot^c, \emptyset$), and also the closure and the topological derivative has a finite sub-algebra. Under some technical hypothesis, Pierce has showed that such a sub-algebra characterizes the homeomorphism class of X .

Equational surfaces are those whose ideal boundaries are spaces of finite type. For example, punctured surfaces (i.e. compact surfaces with only a finite number of removed points) are equational.

Here is how our algorithm goes. First, the instances of the algorithm are equational surfaces. Such a surface is given by an induction scheme defining an equational triangulation. From it, we compute a regular tree defining the surface by treelike connected sum; this tree is encoded by a special kind of automaton called a Moore machine (cf Appendix). By a method inspired by Head [Hea86], we then compute a finite Pierce algebra characterizing the homeomorphism class of the ideal boundary. Eventually, we have to test the existence of isomorphisms between finite algebras, which can be made in an effective way.

Let us remark that effective computability of the homeomorphism problem for non-compact surfaces does not make sense in the general case since the cardinality of the set of homeomorphism classes of non-compact surfaces without boundary is at least 2^{\aleph_0} (Corollary 4.3).

1. The Classification of Non-Compact Surfaces.

1.1. Kerékjártó-Richards Theorem.

A *region* of a topological space X is a connected open subset of X . A *closed region* is the closure of a region. We say that a subset is *bounded* if its closure is compact. Let S denote a non-compact surface without boundary. Let us consider

¹The reader could see that these induction schemes actually are equations in a certain magma associated to the notion of graph ([Cou89, Cour96]).

nested sequences $P_1 \supset \dots \supset P_n \supset \dots$ of unbounded regions of S such that : the boundary of P_n is compact for all n and for any bounded subset A of S , $A \cap P_n = \emptyset$ for n sufficiently large. Two such sequences $\{P_i\}_{i \geq 0}$ and $\{P'_i\}_{i \geq 0}$ are said to be equivalent if and only if for all n there exists n' such that $P'_{n'} \subset P_n$ and vice versa. An equivalent class of such sequences is called an *end* of S . The set of ends of S is denoted by $B(S)$. Let U denote a region of S with bounded boundary and let U^* denote the set of ends represented by some sequences $\{P_i\}_{i \geq 0}$ such that $P_n \subset U$ for n large enough. Let \mathcal{B} be the set of all such U^* ; $B(S)$ endowed with the topological structure of basis \mathcal{B} is called the *ideal boundary* of S . Fundamental properties of the ideal boundary can be found for instance in [AS60].

We now turn to the genus and orientability notions in the non-compact case, which can be found in [Ric63, Mas67]. S is said to be of *genus* $k \in \mathbb{N}$ if there exists a bounded region A of genus k such that $S \setminus A$ is planar. If such an integer k does not exist, S is said to be of *infinite genus*. In the same way, S is said to be *finitely non orientable* if S is not orientable and there exists a bounded region A such that $S \setminus A$ is orientable. If such a region does not exist, S is said to be *infinitely non orientable*. A finitely non orientable surface is said to be of *even type* of non orientability (respectively *odd type*) if any large enough bounded region is of even genus (respectively odd genus).

Let p^* be an end of S represented by a sequence $\{P_i\}_{i \geq 0}$; p^* is said to be *planar* (respectively *orientable*) if P_n is planar (respectively orientable) for n sufficiently large. Let $B'(S)$ (respectively $B''(S)$) denote the set of non planar (respectively non orientable) ends. $B(S)$, $B'(S)$ and $B''(S)$ are compact (see [AS60]).

THEOREM 1.1 (Kerékjártó-Richards [Ric63]).

Let S_1, S_2 be two non-compact surfaces without boundary of the same orientability and genus type in the sense of the previous definitions. Then S_1 and S_2 are homeomorphic if and only if there exists a homeomorphism

$$f : (B(S_1), B'(S_1), B''(S_1)) \rightarrow (B(S_2), B'(S_2), B''(S_2))$$

See [Ric63] for a proof.

1.2. The Treelike Structure of Non-Compact Surfaces Without Boundary.

The ideal boundary is a zero-dimensional compact metric space. Theorem 1.4 below shows that such a space has a treelike nature. For instance, we can look at the Cantor space as the set of infinite paths of a complete binary tree. This fact leads us to consider a kind of treelike global structure on a non-compact surface without boundary.

First, by defining the *treelike connected sum*, we shall define precisely what we mean by “treelike global structure”. Then, we show that every non-compact surface without boundary is a treelike connected sum (Corollary 1.5).

In an obvious way, we can associate to a tree t a one dimensional simplicial complex whose vertices are the nodes of t and whose 1-simplexes make links between fathers and sons ². Therefore, a tree can be seen as a topological space. Now, imagine the tree thickened slightly; let us consider the surface S of the solid

²For notations, definitions and basics about trees, the reader is referred to the appendix at the end of this article.

obtained in this way. Then $B(S)$ is homeomorphic to the space of infinite paths in t (cf Appendix). Figure 1 shows an example, where t is the complete binary tree. The surface is homeomorphic to a sphere minus a Cantor set.

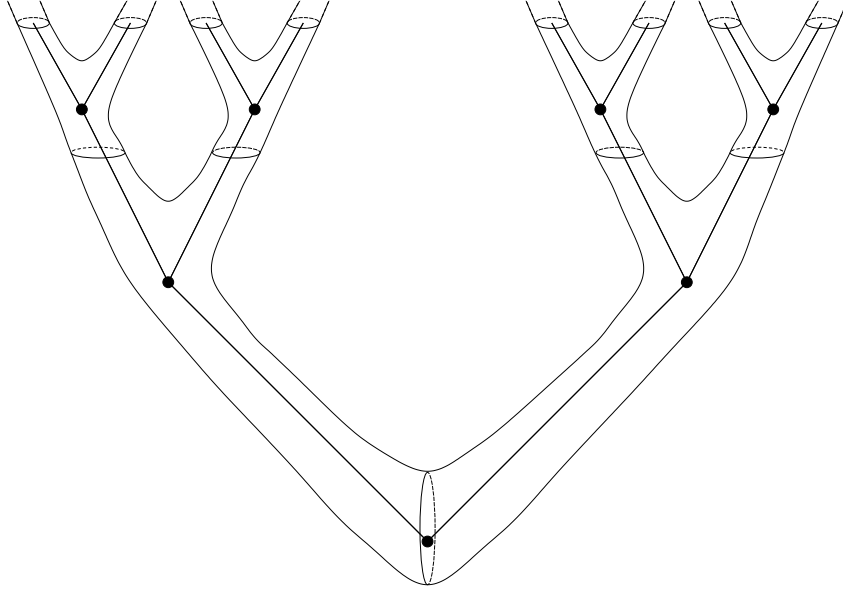


FIGURE 1. The “thickened” binary tree can be obtained by treelike connected sum on a binary tree whose nodes are labelled by spheres.

The following construction extends the idea of thickened trees. Let us consider the set \mathcal{T} of infinite deterministic trees of finite degree whose nodes are labelled by compact surfaces without boundary, i.e. for such a tree t and for $w \in \text{Dom}(t)$, $t(w)$ is compact surface without boundary. We assume that the surfaces labelling any two distinct nodes of t are disjoint. One associates to each tree $t \in \mathcal{T}$ a non-compact surface $\Sigma(t)$ obtained by gluing together the surfaces associated to adjacent nodes of t . The construction of $\Sigma(t)$ is as following :

Let A be the alphabet of t . For all $w \in \text{Dom}(t) \setminus \{\varepsilon\}$, let $f_w : D \rightarrow t(w)$ be an embedding of the unit disk D into the compact surface $t(w)$. For all $w \in \text{Dom}(t)$ and $a \in A$ such that $w.a \in \text{Dom}(t)$, let $s_w^a : D \rightarrow t(w)$ also be an embedding. One assumes that all these embeddings have pairwise disjoint images.

Now, let $\bar{t}(w)$ denote the surface obtained after cutting out $f_w(\overset{\circ}{D})$ and $s_w^a(\overset{\circ}{D})$ where $w.a \in \text{Dom}(t)$.

$$\text{Let } K = \bigcup_{w \in \text{Dom}(t)} \bar{t}(w)$$

and let \mathcal{R} be the equivalence relation on K generated by the pairs (x, x') such that there exists $z \in S^1 = \text{Fr}(D)$, $w \in \text{Dom}(t)$ and $a \in A$ such that $w.a \in \text{Dom}(t)$, $f_{w.a}(z) = x$ and $s_w^a(z) = x'$.

We define $\Sigma(t) = K/\mathcal{R}$.

We say that $\Sigma(t)$ is obtained by *treelike connected sum* on t . One can verify that the homeomorphism class of $\Sigma(t)$ depends neither on the choice of the f_w nor on the choice of the s_w^a .

The following result describes the encoding of the ends of $\Sigma(t)$ by the infinite paths of t . It can be readily checked.

For each path $\alpha = (w_n)_{n \geq 0} \in \text{Path}(t)$, let $e(\alpha)$ be the end of $\Sigma(t)$ defined by the subsets P_n made of all the pieces coming from the nodes of t_{w_n} , the subtree of t of root w_n .

LEMMA 1.2. *Let*

- $\text{Path}'(t)$ be the set of paths $\alpha = (w_n)_n$ such that $\forall N \in \mathbb{N} \exists n > N$ such that $t(w_n)$ is not planar,
- $\text{Path}''(t)$ be the set of paths $\alpha = (w_n)_n$ such that $\forall N \in \mathbb{N} \exists n > N$ such that $t(w_n)$ is non orientable.

Then $e : (\text{Path}(t), \text{Path}'(t), \text{Path}''(t)) \rightarrow (B(\Sigma(t)), B'(\Sigma(t)), B''(\Sigma(t)))$ is a homeomorphism.

By Theorem 2.48 of [HY61], the ideal boundary is a metrisable topological space. Moreover it is compact and zero dimensional [AS60].

THEOREM 1.3 ([HY61, Theorem 2.94]).

Let X be a compact, zero dimensional metric space. Then there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n, \dots$ of finite coverings, each \mathcal{U}_n being a collection of disjoint sets of diameter less than $1/n$ which are both open and closed and \mathcal{U}_{n+1} being a refinement of \mathcal{U}_n for each n .

We can find the following theorem in [HY61, Theorem 2.95] in terms of inverse limits. The proof is rewritten here in the tree framework. It shows how to encode the points of a compact zero-dimensional metric space by the infinite paths of a tree, which are objects of combinatorial nature.

THEOREM 1.4.

Under the hypothesis of the previous theorem, X is homeomorphic to the space of infinite paths of a tree.

PROOF.

Let $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n, \dots$ be a sequence of finite covering given by Theorem 1.3. Let $U_{n,1}, U_{n,2}, \dots, U_{n,k_n}$ denote the elements of \mathcal{U}_n (subsets of X). Let us consider the tree t whose nodes are the $U_{n,k}$ and whose edges are defined by relations $U_{n+1,i} \subset U_{n,i'}$ which are interpreted by stating that $U_{n,i'}$ is the father of $U_{n+1,i}$. More precisely, we consider the tree t without label whose domain $\text{Dom}(t)$ is a set of finite sequences of positive integers :

$$\text{Dom}(t) = \{l_1 l_2 \dots l_n \mid U_{n,l_n} \subset U_{n-1,l_{n-1}} \subset \dots \subset U_{1,l_1}\}$$

There is an obvious one-to-one correspondence denoted by σ between the elements of $\text{Dom}(t)$ and the $U_{n,k}$ associating $l_1 l_2 \dots l_n$ to U_{n,l_n} .

Now, we can associate to any point $x \in X$ the path $p(x) = (w_n)_{n \geq 0} \in \text{Path}(t)$ defined by : $\forall n \in \mathbb{N}$, $x \in \sigma(w_n)$ or, equivalently, $\{x\} = \bigcap_{n \geq 0} \sigma(w_n)$. It is easy to see that p is a one-to-one mapping from X onto $\text{Path}(t)$. Since the $U_{n,k}$ form a basis for the topology of X , one can also verify that p is bicontinuous. \square

PROPOSITION 1.1 (I. Richards [Ric63]).

Let X be a zero dimensional compact metric space and let $X'' \subset X'$ be two closed subsets of X . Then, there exists a non-compact surface S without boundary such that $B(S) \simeq X$, $B'(S) \simeq X'$ and $B''(S) \simeq X''$.

PROOF. We actually use the ideas of the proof of Richards, but here we seek a surface obtained by treelike connected sum.

Let us consider a tree t as given in the proof of Theorem 1.4, i.e. obtained from a sequence of finite coverings of X . This tree is of finite but possibly unbounded degree.

Here we construct a treelike connected sum over t . The labels of t are defined in the following way :

Let $w \in \text{Dom}(t)$,

- if $\sigma(w) \cap X'' \neq \emptyset$ then $t(x)$ is a projective plane,
- if $\sigma(w) \cap X' \neq \emptyset$ and $\sigma(w) \cap X'' = \emptyset$, $t(x)$ is a torus,
- if $\sigma(w) \cap X' = \emptyset$, $t(x)$ is a sphere.

Keeping the notations of Theorem 1.4 and Lemma 1.2, one can check that $e \circ p : (X, X', X'') \rightarrow (B(\Sigma(t)), B'(\Sigma(t)), B''(\Sigma(t)))$ is an homeomorphism. \square

COROLLARY 1.5.

Every non-compact surface without boundary is a treelike connected sum of spheres, tori and projective planes.

2. Equational Surfaces.

2.1. Definition.

This section is devoted to the introduction of the data structure (equational hypergraph) which encodes the infinite triangulations which we investigate. This allows us to deal with non-compact surfaces in a constructive way.

The main reference about equational hypergraphs is [Cou90].

DEFINITION 2.1 (Hypergraph).

Let A be a finite alphabet. Let n be an integer. A n -hypergraph over A is a tuple $G = (V_G, E_G, \text{lab}_G, \text{vert}_G, \text{src}_G)$ where :

- V_G is a set whose elements are called vertices.
- E_G is a set whose elements are called hyperedges, $V_G \cap E_G = \emptyset$.
- $\text{lab}_G : E_G \rightarrow A$ defines a label for each hyperedge.
- $\text{vert}_G : E_G \rightarrow V_G^*$ associates a finite sequence of vertices to each hyperedge (V_G^* denotes the set of finite sequences of elements of V_G). For all $e \in E_G$, the number of vertices appearing in $\text{vert}_G(e)$ is called the arity of e . Formally, $\text{vert}_G(e)$ can be the empty sequence. But for the purpose of encoding triangulations, this will not append.
- $\text{src}_G \in V_G^n$ is a tuple of n distinguished vertices called the sources of G .

This definition extends the notion of pointed oriented graph. Edges are replaced by hyperedges which are allowed to connect more than 2 vertices in a certain order; this is encoded by vert_G . The root is replaced by an ordered finite set of vertices called sources which is encoded by src_G . Note that only hyperedges are labelled.

Let $G = (V_G, E_G, \text{lab}_G, \text{vert}_G, \text{src}_G)$ be an n -hypergraph over A . A *sub-hypergraph* of G is a k -hypergraph $K = (V_K, E_K, \text{lab}_K, \text{vert}_K, \text{src}_K)$ such that $V_K \subset V_G$, $E_K \subset E_G$, $\text{lab}_K = \text{lab}_G|_{E_K}$, for all $e \in E_K$, $\text{vert}_K(e)$ is equal to $\text{vert}_G(e)$ after deletion of vertices which do not belong to V_K and the sources of K are those of G which are elements of V_K in the same ordering. The *hypergraph complement* of K in G , denoted by $G \setminus K$, is the sub-hypergraph of G defined by $V_{G \setminus K} = (V_G \setminus V_K) \cup \text{vert}_G(E_G \setminus E_K)$ and $E_{G \setminus K} = E_G \setminus E_K$ where $\text{vert}_G(E_G \setminus E_K)$ denotes the set of vertices which belong to some hyperedge of $E_G \setminus E_K$. Two vertices $x, y \in V_G$ are said to be *connected* if there is a sequence $x = x_0, x_1, \dots, x_k = y$ of vertices such that for all $i \in [0, k - 1]$, there exists $e_i \in E_G$ such that x_i and x_{i+1} both appear in $\text{vert}_G(e_i)$. G is said to be *connected* if every pair of vertices is connected.

Graphs are usually interpreted as 1-dimensional simplicial complexes. We use hypergraphs to encode simplicial complexes of any dimension. Basics about simplicial complexes can be found for instance in [HY61, Chap. 5].

DEFINITION 2.2 (Simplicial complex associated to a hypergraph).

We associate to a hypergraph G an abstract simplicial complex S_G defined as following :

- The set of vertices of S_G is V_G .
- Each hyperedge $e \in E_G$ gives rise to a simplex s_e which is the set of vertices appearing in $\text{vert}_G(e)$. We also add to S_G all the subsets of s_e in order to satisfy the heredity condition.

The topological space defined by S_G , i.e. the geometric realisation of S_G (cf [HY61]) is denoted by $|G|$ and is called the geometric carrier of G .

Our definition of hypergraph is more complicated than the usual one, which can be found for instance in [Ber73]. In fact, the ordering of vertices around a hyperedge i.e. the ordering of $\text{vert}_G(e)$, the labels and the sources are not needed when encoding simplicial complexes. However, this set of data is essential for the construction of an equational hypergraph. Let us note that this definition is particularly well adapted to software development.

DEFINITION 2.3 (System of hypergraph equations).

A system of hypergraph equations is a set of n equations

$$L = \langle u_1 = H_1, \dots, u_n = H_n \rangle$$

where $U = \{u_1, \dots, u_n\}$ is the set of unknowns; H_i is a finite n_i -hypergraph over the alphabet $\{\#\} \cup U$ where n_i is such that every hyperedge in the system labelled by u_i is of arity n_i . Hyperedges labelled by $\#$ are called terminal and the others are called nonterminal.

Figure 2 shows an example of such a system.

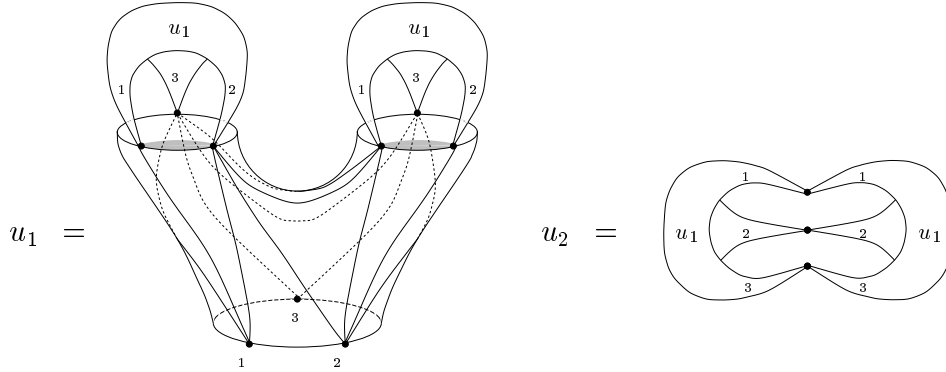


FIGURE 2. Example of system of hypergraph equations.

Here we represent a system of two hypergraph equations $\langle u_1 = H_1, u_2 = H_2 \rangle$. H_1 is represented on the left hand side. It has 9 vertices represented as points, 11 terminal hyperedges of arity 3 and 2 nonterminal ones of arity 3 labelled by u_1 . The labels of the terminal hyperedges are omitted. Its sources are the three vertices at the bottom of the drawing. The numbers associated to them define their ordering. The two nonterminal hyperedges, both labelled by u_1 are represented on the top as kind of parachutes. Their legs indicate which vertices they connect. The numbers written in them describe the ordering of these vertices. This ordering is essential as it encodes a gluing process. Finally, terminal hyperedges form a triangulation of a pair of pants. As they will be eventually interpreted as simplexes, the orderings of their vertices are not indicated in the drawing. H_2 is represented on the right hand side. It has 3 vertices, 2 nonterminal hyperedges each of arity 3 and labelled by u_1 and no terminal hyperedge. It has no sources. The canonical solution associated to u_2 has only terminal hyperedges of arity 3. If we interpret them as 2-simplexes, then we get a triangulation of the surface represented in Figure 1 i.e. the sphere minus a Cantor set. This is an equational surface.

Here we keep the notations of Definition 2.3. A solution of L is a tuple (G_1, \dots, G_n) of n hypergraphs such that :

- G_i is a n_i -hypergraph i.e. it has n_i sources. G_i is the solution associated to u_i .
- For all $i \in [1, n]$, (G_1, \dots, G_n) satisfies the equation $u_i = H_i$. This means that if for all $j \in [1, n]$, we glue G_j in H_i in place of each hyperedge labelled by u_j , then we get a hypergraph isomorphic to G_i . By gluing G_j in place of a hyperedge e labelled by u_j , we mean the following operation : let G_j^e be an isomorphic copy of G_j . For all $k \in [1, n_j]$, we glue the k th source of G_j^e on the k th vertex of e . Finally we delete e .

Note that the semantics of the terminal hyperedges and the nonterminal ones are different. The former encode simplexes and the latter encode gluing operations.

Solution of such systems are far from being unique. However we will see a way to construct a canonical one.

Let e_{i1}, \dots, e_{im_i} denote the nonterminal hyperedges of H_i . This notation fixes an (arbitrary) ordering on the set of nonterminal hyperedges of H_i : we say that e_{ij} is the j th nonterminal hyperedge of H_i . Let $l_{ij} \in [1, n]$ be such that e_{ij} is labelled by $u_{l_{ij}}$. Let $m = \max_i m_i$.

Let T_i denote the deterministic tree labelled by $[1, n]$ over the alphabet $[1, m]$ defined inductively by the two following properties :

- The root of T_i is labelled by i .
- Let μ be a node of T_i labelled by $i' \in [1, n]$; then μ has $m_{i'}$ sons and if μ_j denotes the j th one, then $T_i(\mu_j) = l_{i'j}$.

T_i is called the i th *syntactic tree* associated to L . We remark that T_i is a regular tree i.e. T_i has only finitely many subtrees up to isomorphism (see Appendix). Indeed, T_i has at most n subtrees up to isomorphism.

We consider for every node μ of T_i , a hypergraph \overline{H}_μ which is a copy of $H_{T_i(\mu)}$ *without its nonterminal hyperedges*. We assume that the \overline{H}_μ are pairwise disjoint. We will glue the \overline{H}_μ to each other following the edges of T_i . Let \mathcal{R} be the equivalence relation on $\bigcup_\mu V_{\overline{H}_\mu}$ generated by the pairs (x_1, x_2) such that :

- if μ_1, μ_2 are the nodes of T_i such that $x_1 \in V_{\overline{H}_{\mu_1}}$ and $x_2 \in V_{\overline{H}_{\mu_2}}$, then μ_1 is the father of μ_2 .
- Let j be such that μ_2 is the j th son of μ_1 . There exists an integer k such that x_1 was the k th vertex of the j th nonterminal hyperedge of \overline{H}_{μ_1} and x_2 is the k th source of \overline{H}_{μ_2} .

The hypergraph obtained from $\bigcup_\mu \overline{H}_\mu$ after identification of \mathcal{R} -equivalent vertices is called the i th *canonical solution* of L . Its sources are, by definition, the sources of \overline{H}_ε where ε is the root of T_i .

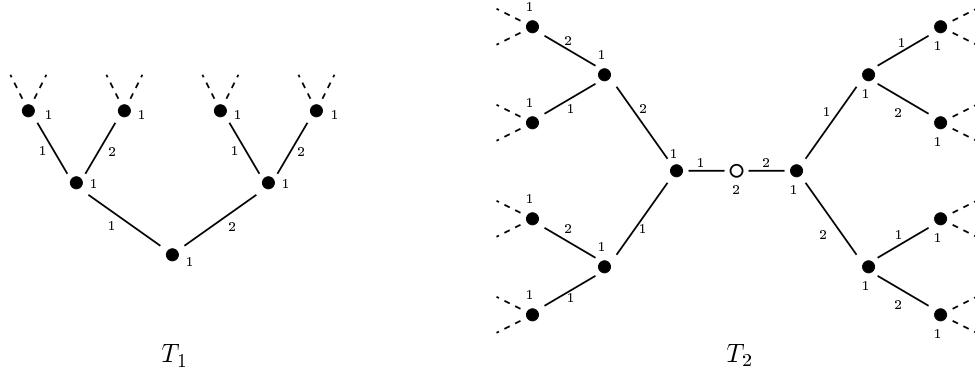


FIGURE 3. Syntactic trees associated to the system of hypergraph equations described in Figure 2.

DEFINITION 2.4 (Equational hypergraphs).

A hypergraph is said to be equational if it is a canonical solution of a system of hypergraph equations.

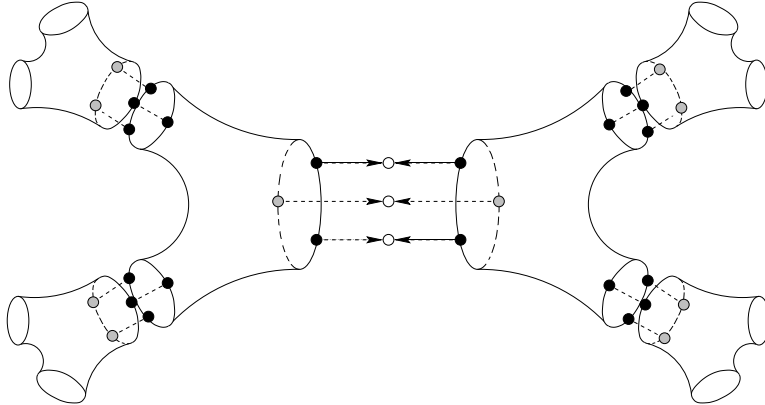


FIGURE 4. Construction of the canonical solution associated to u_2 for the system represented in Figure 2. Here are represented some \overline{H}_μ . To simplify the drawing, terminal hyperedges have been omitted. Note that \overline{H}_ε (in the center of the drawing) is made of only 3 vertices (represented as white points). The gluings follow the dashed lines.

Here we only are interested in equational hypergraphs defining triangulations of surfaces in the sense of Definition 2.2 :

DEFINITION 2.5 (Equational surfaces).

A surface is said to be equational if it has a triangulation defined by an equational hypergraph. Such a triangulation is said to be equational.

EXAMPLE 2.6. Every compact surface is equational.

EXAMPLE 2.7. Every surface obtained from a compact surface without boundary by removing a finite number of points is equational.

An equational surface is defined by a system of hypergraph equations, which is a finite set of data. Thus, the effective computability of the homeomorphism problem for this kind of surface makes sense.

Corollary 1.5 shows that every non-compact surface without boundary has a triangulation of bounded tree-width (the notion of tree-width appeared in [RS84]). On the other hand, equational graphs are the graphs of bounded tree-width satisfying a certain regularity property which extends to graphs the regularity property for trees (see [Cou90, Theorem 6.5]). From this point of view, the introduction of equational triangulations appears to be natural.

Note that surfaces are locally compact spaces. So, the only hypergraphs in which we are interested are of finite degree (i.e. there is a finite number of hyperedges around a vertex). On the other hand, equational graphs of finite degree actually are context-free graphs in the sense of [MS85].

2.2. Global Regularity of Equational Surfaces.

We need the notion of *barycentric subdivision* (see for instance [HY61]).

LEMMA 2.8.

Let T be a simplicial complex defined by a system L of hypergraph equations. Then we can effectively compute a system L' defining the barycentric subdivision T' of T .

Each terminal hyperedge has to be subdivided. \square

Let $L = \langle u_i = H_i \rangle_{1 \leq i \leq n}$ be a system of hypergraph equations. Let $|\overline{H}_i|$ denote the topological space associated to the simplicial complex obtained from H_i after deletion of nonterminal hyperedges.

LEMMA 2.9.

Let S be an equational surface, then S is homeomorphic to the geometric carrier of the last canonical solution of a system $L = \langle u_i = H_i \rangle_{1 \leq i \leq n}$ such that for all $i \in [1, n]$, $|\overline{H}_i|$ is a bordered compact surface.

PROOF.

Let $L = \langle u_i = H_i \rangle_{1 \leq i \leq n}$ be a system of hypergraph equations defining a triangulation of S . Let G_1, \dots, G_n be the canonical solutions of L . Let us assume that the triangulation of S is encoded by G_n .

Step 1 : Even if it means adding new equations, we can assume that the vertices of each nonterminal hyperedge are pairwise distinct. Indeed, for instance, let e be a nonterminal hyperedge labelled by u_i whose second vertex is equal to the first. Then we add a new unknown whose associated right member is obtained from H_i by gluing the first source with the second. Details are left to the reader.

Step 2 : We can transform L in such a way that there is no source in any nonterminal hyperedges of any H_i . We define the *order 1 development* of H_i , denoted by $D^1(H_i)$, to be the hypergraph obtained after the replacement of each nonterminal hyperedge by the corresponding H_j . Then, let $D^n(H_i)$ be the order 1 development of $D^{n-1}(H_i)$. Recall that G_i is of finite degree. Let us consider a right member H_i of L and a source s of H_i . The number of triangles i.e. terminal 3-hyperedges, and more generally the number of terminal hyperedges of G_i which contains s is finite. So, there exists a minimal integer k_s such that they are all in $D^{k_s}(H_i)$. Now, suppose that s still is in a nonterminal hyperedge e of $D^{k_s}(H_i)$. Let u_j be the label of e . Then G_j contains a source connected to no triangle; it is useless and we can delete it. The system after transformation is still denoted by L (H_i becomes $D^{k_i}(H_i)$ where $k_i = \max k_s$, maximum over all the sources of H_i).

Step 3 : We can transform L such that any two distinct nonterminal hyperedges have no common vertex. Assuming the result obtained in the previous step, we only have to replace each H_i by its order 1 development.

Step 4 : We can transform L such that each vertex of H_i has a neighbourhood homeomorphic to a disk or a half disk in $|\overline{H}_i|$. Let us consider a right member H_i of L and a vertex x having no neighbourhood homeomorphic to a disk or a half disk.

If x does not belong to any nonterminal hyperedge and is not a source, then all the terminal hyperedges in G_i containing x already are in H_i . Therefore, x has a neighbourhood homeomorphic to a disk in $|\overline{H}_i|$ which is a contradiction.

If x is a source, then x has to be glued on a vertex which is not a source and which has no neighbourhood homeomorphic to a disk or a half disk. This vertex fits in the next case.

Suppose that x belongs to a nonterminal hyperedge. Let us replace L by its double barycentric subdivision. By step 3, there is a unique right member $H_{i'}$ which have to be pasted on x . Let s' be the source of $H_{i'}$ which has to be glued on x . Let us consider the triangles which contain s' . We shall delete them from $H_{i'}$, and paste them around x in H_i and in every right member containing a nonterminal hyperedge labelled by $u_{i'}$. After this, x will have a neighbourhood homeomorphic to a disk in $|\overline{H}_i|$; and we will delete s' from $H_{i'}$.

Let T' be such a triangle; it has been created during the double barycentric subdivision. The other vertices of T' , which are denoted by a and b , have been also created during this process. After deletion of T' , a and b become sources; note that they could already have been sources. And they have neighbourhoods homeomorphic to a half disk in $|\overline{H}_{i'} \setminus T'|$ (cf. Figure 5). The proof is by induction on the number of vertices having no neighbourhood homeomorphic to a disk or a half disk.

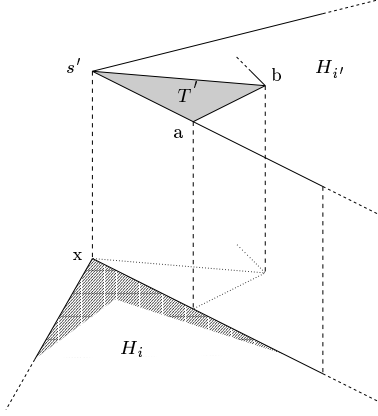


FIGURE 5

Step 5 : It remains to transform L such that each $|\overline{H}_i|$ becomes connected. Since S is assumed to be connected, we can assume that each G_i is an infinite connected hypergraph, even if it means separating it into several unknowns. Under this hypothesis, H_i is a connected hypergraph. Let us assume that $|\overline{H}_i|$ is not connected i.e. connectivity of H_i is achieved using nonterminal hyperedges. Since $|G_i|$ is connected, after enough developments of H_i , all the sources belong to the same component of $|\overline{H}_i|$ whose underlying hypergraph is denoted by C_i . Let K_{i1}, \dots, K_{ij} be the connected components of $H_i \setminus C_i$. We will replace H_i by C_i in the system and add new unknowns w_{i1}, \dots, w_{ij} and new equations of the form $w_{ij} = K_{ij}$. Let $v_1^{ij}, \dots, v_k^{ij}$ be the vertices of the frontier between K_{ij} and C_i i.e. $V_{K_{ij}} \cap V_{C_i}$ (k depends on i and j). We define the sources of K_{ij} to be $v_1^{ij}, \dots, v_k^{ij}$ in that order i.e. $\text{src}_{K_{ij}} = (v_1^{ij}, \dots, v_k^{ij})$. For each i, j , we add to C_i a new nonterminal hyperedge e_{ij} with $\text{lab}(e_{ij}) = w_{ij}$ and $\text{vert}(e_{ij}) = v_1^{ij} \dots v_k^{ij}$. We get a hypergraph still denoted by C_i . The new hyperedge e_{ij} encodes the gluing of K_{ij} on C_i . Let $\text{src}_{C_i} = \text{src}_{H_i}$.

Then, we replace the equation $u_i = H_i$ by $u_i = C_i$ and add to L the new equations $w_{ij} = K_{ij}$. K_{ij} is a connected hypergraph; however $|\overline{K}_{ij}|$ maybe not connected. But all its nonterminal hyperedges are labelled on $\{u_1, \dots, u_n\}$. Thus, we replace any nonterminal hyperedge of K_{ij} by its corresponding right member which is now a C_i . $|\overline{K}_{ij}|$ becomes then a connected.

Even if the number of equation increased, the number of right members with non connected geometric carrier decreased strictly. So, by induction, the system becomes as we desire. \square

THEOREM 2.10.

A non-compact surface without boundary is equational if and only if it is a treelike connected sum on a regular tree.

PROOF.

(i) (\Leftarrow) Let S be a surface defined by treelike connected sum on a regular tree t . Recall that t is labelled on the set of compact surfaces. Let $\mathcal{A} = \langle Q, \delta, A, q_0, \tau \rangle$ be the Moore machine³ associated to t . For all $q \in Q$, let H_q be a finite hypergraph encoding a triangulation of $\tau(q)$. For all $q \neq q_0$, let us cut out a disk in H_q , the vertices of which will be the sources of H_q . For each transition $\delta(q, a) = q'$, let us cut out a disk in H_q and paste instead of it, a nonterminal hyperedge whose label is $u_{q'}$. Eventually, we consider the system $L = \langle u_q = H_q \rangle_{q \in Q}$. The precise definitions of the sources and the hyperedges of H_q can easily be written out.

(ii) (\Rightarrow) Here we use the method of canonical exhaustion [AS60].

For this we need the notion of *regular neighbourhood* which first appeared in [Whi39]; see e.g. [Hem76] for a modern approach. Let K be a simplicial complex, σ a simplex of K and τ a $(\dim \sigma - 1)$ -dimensional face of σ belonging to no other simplex of K . The simplicial complex $K \setminus \{\sigma, \tau\}$ is said to be obtained from K by *elementary collapsing*. If L is a complex obtained from K after a finite sequence of elementary collapsings, we say that K *collapses* to L (which is denoted by $K \searrow L$). Note that $|L|$ is a deformation retract of $|K|$. Let $n > 0$ be a non-null integer. Let T be a simplicial complex whose geometric carrier $|T|$ is a n -dimensional manifold and let L be a subcomplex of T . We call *regular neighbourhood* of $|L|$ in $|T|$ any simplicial n -dimensional submanifold of $|T|$ of the form $|K|$ where K is a subcomplex of T and $K \searrow L$. Generally, $|K|$ is not a neighbourhood of $|L|$ in the usual sense.

LEMMA 2.11.

Let T and L be as above. Let T'' and L'' denote respectively the double barycentric subdivisions of T and L , Let $N(L'', T'')$ be the union of all the simplexes or faces of simplexes of T'' meeting a simplex of L'' . Then $|N(L'', T'')|$ is a regular neighbourhood of L'' in T'' .

See [Whi39, Hem76] for a proof.

Let us turn to the proof of (\Rightarrow). Let S be an equational surface defined by a system $L = \langle u_i = H_i \rangle_{1 \leq i \leq n}$. Here we keep the notations of the construction of equational hypergraphs. By lemma 2.9, we can assume that all the $|\overline{H}_i|$ are bordered surfaces.

³See Appendix for definition of Moore machines and their connection with regular trees.

Step 1 : Let us consider F_i the sub-hypergraph of H_i made of the sources of H_i and their 2-hyperedges (even if it means adding new hyperedges, we can consider that any two vertices connected by an hyperedge are connected by a hyperedge of arity 2). Since $|\overline{H}_i|$ is a bordered compact surface, $|F_i|$ is made of pairwise disjoint simple closed curves of the boundary. We will transform L such that each $|F_i|$ will be made of only one simple closed curve.

Let i_0 be an integer such that F_{i_0} contains at least two simple closed curves C_1 and C_2 ; let α be a simplicial path in H_{i_0} connecting C_1 and C_2 (cf Figure 6). Let K be a regular neighbourhood of $C_1 \cup \alpha \cup C_2$ in H_{i_0} . K is homeomorphic to a pair of pants; C_1 and C_2 are two components of its boundary; let C denote the third. Let us consider H_{i_1}, \dots, H_{i_k} the right members of L containing some nonterminal hyperedges labelled by u_{i_0} . We will delete K from H_{i_0} and paste it on the H_{i_j} for $j = 1, \dots, k$.

Let us consider the sub-hypergraph H'_{i_0} obtained from $H_{i_0} \setminus K$ after the following modification of the sources : the sources not contained in the interior of K remain sources of H'_{i_0} and the vertices of C become new sources, replacing the sources of C_1 and C_2 . It remains to paste K on H_{i_j} for $j = 1, \dots, k$. Let e be an hyperedge of H_{i_j} labelled by u_{i_0} . Let v_1, v_2, \dots, v_n be the vertices connected by e i.e. $\text{vert}_{H_{i_j}}(e) = v_1 v_2 \dots v_n$. Even if it means renumbering the vertices of e and the sources of H_{i_0} , we can assume that there exists m such that C_1 and C_2 have to be pasted on v_m, v_{m+1}, \dots, v_n . After deleting e , we paste K on these vertices and add a new hyperedge e' whose vertices are $v_1, v_2, \dots, v_{m-1}, v'_m, \dots, v'_{n'}$ where $v'_m, \dots, v'_{n'}$ are the vertices of C in the same ordering than the one defined by $\text{src}_{H'_{i_0}}$; e' is labelled by a new unknown u'_{i_0} . After doing this operation on each hyperedge labelled by u_{i_0} , the equation $u'_{i_0} = H'_{i_0}$ shall replace $u_{i_0} = H_{i_0}$ in L . Eventually the number of components of $|F_i|$ has decreased; the proof is by induction.

Let us remark that we already have a treelike connected sum. However, we will show in the following step how to get a treelike connected sum of spheres, tori and projective planes.

Step 2 : We will construct an automaton encoding the regular tree defining

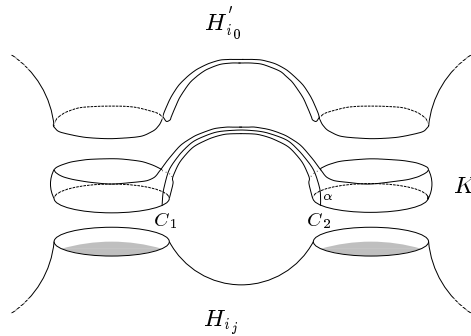


FIGURE 6. Canonical exhaustion

the treelike connected sum. For all $i \in [1, n]$, $|\overline{H}_i|$ is a bordered compact surface; hence it is a sphere or a connected sum of tori and projective planes : $|\overline{H}_i| = P_{i_1} \# P_{i_2} \# \dots \# P_{i_{p_i}}$ where P_{i_p} is a torus or a projective plane for all $p \in [1, p_i]$ and $P_{i_{p_i}}$ is a sphere with holes. Note that this decomposition is computable from H_i .

To construct the automaton, the idea is to associate to each H_i a finite graph (actually a chain) whose vertices are labelled by spheres, tori or projective planes. Then we connect these graphs to each other according to the system L in order to obtain the underlying graph of a deterministic finite automaton which encodes the regular tree associated to the treelike connected sum. This automaton is provided with an output over the set of three elements containing the sphere, the projective plane and the torus.

Rigorously, Let e_{i1}, \dots, e_{im_i} denote the nonterminal hyperedges of H_i . Let $l_{ij} \in [1, n]$ be such that e_{ij} is labelled by $u_{l_{ij}}$. Let $m = \max_i m_i$.

$\mathcal{A} = \langle Q, \delta, A, q_0, \tau \rangle$ is defined by :

- $A = [1, m]$.
- $Q = \{P_{ip} : 1 \leq i \leq n, 1 \leq p \leq p_i\}$.
- For $1 \leq p < p_i$, let $\delta(P_{ip}, 1) = P_{ip+1}$.
For all $i \in [1, n]$ and all $j \in [1, m_i]$, $\delta(P_{ip_i}, j) = P_{l_{ij}1}$.
- $q_0 = P_{n1}$.
- $\tau(P_{ip})$ is obtained from P_{ip} by pasting disks on its holes.

We can verify that \mathcal{A} actually define a regular treelike connected sum defining S . \square

3. Decidability.

3.1. Pierce's Theorem.

In this part, X denotes a zero dimensional compact metric space. If $A \subset X$, the closure of A is denoted by \bar{A} and its topological derivative is denoted by A' .

DEFINITION 3.1.

The powerset of X , which is denoted by $\mathcal{P}(X)$, provided with the usual boolean operations (\cup, \cap, \cdot^c), the 0-arity operation \emptyset , the closure and the topological derivative is called an algebra. The smallest sub-algebra of $\mathcal{P}(X)$, denoted by $\mathcal{U}(X)$, is called the topological boolean algebra of X . If it is finite then X is said to be of finite type.

Let \mathcal{U} be a sub-algebra of $\mathcal{P}(X)$. An *atom* of \mathcal{U} is an element $a \in \mathcal{U}$ such that for all $b \in \mathcal{U}$, $b \subset a$ implies $b = a$ or $b = \emptyset$.

PROPOSITION 3.1 ([Pie72]).

Let \mathcal{U} be a non trivial finite sub-algebra of $\mathcal{P}(X)$. Suppose that \mathcal{U} is generated by its closed elements. Then there exists a finite sequence $\emptyset = A_0 \subset A_1 \subset \dots \subset A_{n-1} \subset A_n = X$ of closed elements of \mathcal{U} such that for $i \in [0, n-1]$, $a_i = A_{i+1} \setminus A_i$ is an atom of \mathcal{U} .

See [Pie72, Cor. 3.18] for proof.

Here we study the form of the a_i :

First, $a_0 = A_1$ is a closed atom. Thus it is compact and we have $a'_0 = \emptyset$ or a_0 .

- If $a'_0 = \emptyset$ i.e. each point of a_0 is isolated, then, since it is compact, a_0 is a finite set.
- If $a'_0 = a_0$ then a_0 is homeomorphic to the Cantor set (cf [HY61, chap 2.15]).

Now let us look at a_i for $i > 0$. If a_i is closed, then we are in the previous case. Thus, let us assume that $\overline{a_i} \neq a_i$; in particular, a_i is not compact. Then we have $\overline{a_i} = a_{j_1} \cup a_{j_2} \cup \dots \cup a_{j_l}$ where $j_k \leq i$.

- If $a'_i = \emptyset$ then a_i is thus a discrete countable set.
- If $a'_i = a_i$, one can show that a_i is the disjoint union of infinitely (but countably) many Cantor sets (see [Pie72, thm 2.1]).

THEOREM 3.2 ([Pie72]).

Let X, Y be two zero dimensional compact metric spaces of finite type. Let \mathcal{A} and \mathcal{B} be finite sub-algebras of $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ respectively. Suppose that \mathcal{A} and \mathcal{B} are generated by their closed elements. Let $\rho : \mathcal{A} \rightarrow \mathcal{B}$ be an isomorphism such that for all atoms $p \in \mathcal{A}$ of finite cardinality, $\text{Card}(\rho(p)) = \text{Card}(p)$. Then there exists a homeomorphism $\varphi : X \rightarrow Y$ such that $\varphi(C) = \rho(C)$ for all $C \in \mathcal{A}$.

3.2. Computation of Pierce Algebras.

Let $\mathcal{A} = \langle Q, \delta, A, q_0, \tau \rangle$ be the deterministic finite automaton as introduced in the proof of Theorem 2.10. Let $\mathcal{L}^\omega(\mathcal{A})$ denote the language of \mathcal{A}^ω which is recognized by \mathcal{A} i.e. the set of infinite words of $\mathcal{L}^\omega(\mathcal{A})$ which can be associated to the trace of an infinite path in \mathcal{A} (see Appendix).

Our aim in this part is to compute the Pierce algebra of $\mathcal{L}^\omega(\mathcal{A})$ from the structure of \mathcal{A} . The idea, which is due to Head [Hea86], is to associate to each state q of \mathcal{A} , the language of infinite words going infinitely many times through q :

$$\text{In}(q) = \{u \in \mathcal{L}^\omega(\mathcal{A}) \mid u \text{ encodes a path in } \mathcal{A} \text{ going infinitely many times through } q \}$$

Before going further, we have to avoid some pathological situations. Let us delete one at a time states $q \neq q_0$ such that $\text{In}(q) = \emptyset$; such states make no difference. After deleting q , the labels of the transitions are taken in A^+ instead of in A : a pair of transitions of the form $\delta(q_1, a) = q, \delta(q, a') = q_2$ is replaced by $\delta(q_1, a.a') = q_2$. The automaton obtained after doing this is still denoted by \mathcal{A} .

We associate to each subset $Y \subset Q$, the language of $\mathcal{L}^\omega(\mathcal{A})$ defined by $\text{In}(Y) = \bigcup_{q \in Y} \text{In}(q)$. We would like this map to preserve boolean operations; it is obvious that $\text{In}(\emptyset) = \emptyset, \text{In}(Q) = \mathcal{L}^\omega(\mathcal{A})$ and $\text{In}(Y_1 \cup Y_2) = \text{In}(Y_1) \cup \text{In}(Y_2)$; but we only have $\text{In}(Y_1 \cap Y_2) \subseteq \text{In}(Y_1) \cap \text{In}(Y_2)$. We thus introduce the following definition :

DEFINITION 3.3.

$Y \subset Q$ is said to be coherent if for all mutually accessible states $p, q \in Q$, we have $p, q \in Y$ or $p, q \notin Y$.

A coherent subset is a union of strongly connected components of \mathcal{A} .

For all coherent sets Y_1 and Y_2 , $\text{In}(Y_1 \cap Y_2) = \text{In}(Y_1) \cap \text{In}(Y_2)$ and also $\text{In}(Y_1^c) = \text{In}(Y_1)^c$.

We have to understand how In encodes topological operations. So, we introduce the following operations :

- $Y^K = \{q \in Q \mid Y \text{ is accessible from } q\}$
- $Y^D = \{q \in Y \mid \text{there are at least two distinct infinite words of } \mathcal{L}^\omega(\mathcal{A}) \text{ beginning at } q \text{ and going only through states of } Y \}$

One can verify that for every coherent set Y , $\text{In}(Y^K) = \overline{\text{In}(Y)}$ and $\text{In}(Y^D) = \text{In}(Y)'$ (cf [Hea86]). Note that \cdot^K and \cdot^D are computable in an effective way. The set of coherent subsets of Q (denoted by \mathcal{C}) is a sub-algebra of $(\mathcal{P}(Q), \cup, \cap, \cdot^C, \emptyset, \cdot^K, \cdot^D)$.

PROPOSITION 3.2 ([Hea86]).

$\text{In} : (\mathcal{C}, \cup, \cap, \cdot^C, \emptyset, \cdot^K, \cdot^D) \rightarrow (\mathcal{P}(\mathcal{L}^\omega(\mathcal{A})), \cup, \cap, \cdot^C, \emptyset, \bar{\cdot}, \cdot')$ is a morphism.

Since In is injective, $(\mathcal{C}, \cup, \cap, \cdot^C, \emptyset, \cdot^K, \cdot^D)$ is isomorphic to a sub-algebra of $\mathcal{P}(\mathcal{L}^\omega(\mathcal{A}))$ of finite cardinality. Of course, this sub-algebra is computable from \mathcal{A} .

3.3. The Homeomorphism Problem for Equational Surfaces.

Let L be a system of hypergraph equations defining a triangulation of an equational surface S . The proofs of Lemma 2.9 and Theorem 2.10 lead to an algorithm for computing an automaton $\mathcal{A} = \langle Q, \delta, A, q_0, \tau \rangle$ defining a regular tree t such that $\Sigma(t)$ is homeomorphic to S . The previous part shows how to compute from \mathcal{A} some finite sub-algebras of $\mathcal{P}(B(S))$. We will conclude that the homeomorphism problem for equational surfaces is decidable.

Let N (respectively O) denote the smallest coherent subset of Q (recall that Q is the set of states of \mathcal{A}) containing all the states labelled by the projective plane (respectively the torus). Let e denote the homeomorphism between $\text{Path}(t)$ and $B(S)$ defined in Lemma 1.2. $\mathcal{L}^\omega(\mathcal{A})$ and $\text{Path}(t)$ are canonically homeomorphic (cf Appendix). So we can consider e as a map from $\mathcal{L}^\omega(\mathcal{A})$ to $B(S)$. We then have easily the following facts :

- $e(\text{In}(N)) = B''(\Sigma(t))$
- $e(\text{In}(N \cup O)) = B'(\Sigma(t))$

The following theorem states that the global regularity studied in the previous parts is equivalent to the property of having a finite type ideal boundary.

THEOREM 3.4.

A surface S is equational if and only if the sub-algebra of $\mathcal{P}(B(S))$ generated by $\{B'(S), B''(S)\}$ is of finite cardinality. Moreover, this sub-algebra is computable from a system of hypergraph equations defining S and vice versa.

PROOF.

(i) We compute the (finite) sub-algebra of \mathcal{C} which is generated by N and O . Recall that \mathcal{C} denote the set of coherent subsets of states of \mathcal{A} . By Proposition 3.2, it is isomorphic to the desired sub-algebra of $\mathcal{P}(B(S))$.

(ii) Let \mathcal{U} be the sub algebra of $\mathcal{P}(B(S))$ generated by $\{B'(S), B''(S)\}$. Let us consider a decomposition as given by Proposition 3.1 :

$$\emptyset = A_0 \subset A_1 \subset \dots \subset A_{n-1} \subset A_n = B(S)$$

where $a_i = A_{i+1} \setminus A_i$ is an atom of \mathcal{U} for all $i \in [0, n-1]$. The proof is by induction on n , the length of this decomposition, with the following induction hypothesis :
 IH_n : We can compute a Moore machine \mathcal{A} whose states are labelled by spheres, tori or projective planes such that $(\mathcal{L}^\omega(\mathcal{A}), \text{In}(N \cup O), \text{In}(N))$ is homeomorphic to $(B(S), B'(S), B''(S))$.

IH_0 : let us assume that $\mathcal{U} = \{\emptyset, B(S)\}$.

If $B(S)$ is perfect, it is homeomorphic to the Cantor set. \mathcal{A} is then made of two

states denoted by q_0 and q with two distinct transitions whose origins and targets are q itself and one connecting q_0 to q (cf Figure 7). The details of the alphabet of \mathcal{A} are left to the reader.

If all the points of $B(S)$ are isolated, then it is finite since it is compact. Let $c \in \mathbb{N}$ be its cardinality. \mathcal{A} is made of $c + 1$ states q_0, \dots, q_c , c edges of origin q_0 and targets q_1, \dots, q_c , and for $j = 1..c$, an edge of origin and target q_j (cf Figure 7).

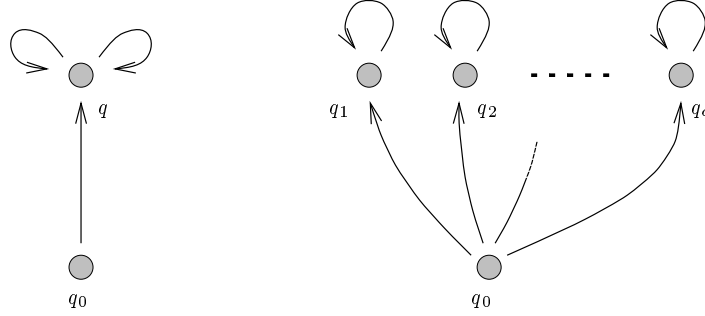


FIGURE 7

Let us assume now that IH_n is true.

By Proposition 1.1, $(A_{n-1}, A_{n-1} \cap B'(S), A_{n-1} \cap B''(S))$ can be considered as the ideal boundary of a surface. So, by induction, we can construct an automaton $\mathcal{A}' = \langle Q', \delta', A', q'_0, \tau' \rangle$ associated to it. If $\bar{a}_{n-1} = a_{n-1}$, the construction is as above. Thus, let us assume that $\bar{a}_{n-1} \cap A_{n-1} \neq \emptyset$. Let $C' = \text{In}^{-1}(\bar{a}_{n-1} \cap A_{n-1})$; it is a coherent set of Q' . Let C'_1, \dots, C'_s denotes the strongly connected components of C' (one can verify that each component is actually made of only one state). Let q be a new state, with one or two transitions of origin and target q itself depending on whether $a'_{n-1} = \emptyset$ or $a'_{n-1} = a_{n-1}$. Recall that if $a'_{n-1} = \emptyset$, a_{n-1} is made of isolated points and if $a'_{n-1} = a_{n-1}$, it is made of Cantor sets. We then add s new transitions connecting C'_1, \dots, C'_s to q . Let us consider a word u in $\text{In}(C_i)$. While it reads u , \mathcal{A} runs through C_i and at any time it can move into q . Thus u is the limit of a sequence of $\text{In}(q)$.

As \mathcal{U} is generated by $\{B'(S), B''(S)\}$, all its atoms are included in $B'(S)$ (respectively $B''(S)$) or disjoint from it. Outputs of \mathcal{A} can be easily defined according to that. \square

COROLLARY 3.5.

The homeomorphism problem for non-compact equational surfaces without boundary is decidable.

The above discussion allows us to construct an algorithm deciding the homeomorphism problem for surfaces which are of infinite genus. In the finite genus case, one can compute in an effective way the genus and the non orientability type. Thus, it is possible to construct a general algorithm. Details are left to the reader.

4. Examples of Non Equational Surfaces.

Here we give a method of construction of zero-dimensional compact metric spaces which are not of finite type. In particular, this construction gives examples of

non equational surfaces. We will also show that the set of homeomorphism classes of non-compact surfaces is not countable. This fact gives another proof of the existence of non equational surfaces.

We consider, in the unit interval $I = [0, 1]$ of \mathbb{R} , the Cantor set defined by

$$C = \left\{ \sum_{n \geq 1} \frac{x_n}{3^n} : (x_n)_{n \geq 1} \in \{0, 2\}^{\mathbb{N}} \right\}$$

and the set of midpoints of the intervals complementary to the Cantor set :

$$K = \left\{ \sum_{n=0}^N \frac{x_n}{3^n} + \frac{1}{2 \times 3^N} : x_0 = 0 \text{ and } x_1 \dots x_N \in \{0, 2\} \right\}.$$

Let $K_1 = C \cup K$, endowed with the topology induced by I . We also set $K_0 = C$. We will construct by induction a sequence $(K_n)_{n \geq 0}$ of zero dimensional subspaces of I . K_0 and K_1 are already defined, let us assume that K_0, K_1, \dots, K_{n-1} are already constructed.

For each isolated point x of K_{n-1} , let us pick two points $a < b$ such that $[a, b] \cap K_{n-1} = \{x\}$ and $x = \frac{a+b}{2}$. We consider now for all $x \in K_{n-1}$ the sequence $(y_{k,x})_{k \geq 1}$ defined by $y_{k,x} = x + (-1)^k \frac{b-a}{2^k}$. Let $K_n = K_{n-1} \cup \{y_{k,x} \mid x \text{ isolated in } K_{n-1} \text{ and } k \geq 1\}$. The following result can be readily checked.

LEMMA 4.1. $K'_n = K_{n-1}$ for all $n \geq 1$.

Let us consider for $n \geq 1$, D_n the image of K_n by the one-to-one increasing affine function denoted by φ_n mapping $[0, 1]$ onto $[\frac{1}{n} - \frac{1}{2n(n+1)}, \frac{1}{n} + \frac{1}{2n(n+1)}]$.

Let Φ be the map from $\{0, 1\}^{\mathbb{N}}$ into the set of zero dimensional compact metric spaces defined by :

For $b = (b_n)_{n \geq 0} \in \{0, 1\}^{\mathbb{N}}$,

$$\Phi(b) = \{0\} \cup \bigcup \{D_{n+1} : n \geq 0 \text{ and } b_n = 1\}$$

If \mathbb{I} denotes the integer sequence defined by $\mathbb{I}_n = 1$ for all $n \geq 0$. We have $\Phi(\mathbb{I})'' \subsetneq \Phi(\mathbb{I})'$ but $\Phi(\mathbb{I})''$ is homeomorphic to $\Phi(\mathbb{I})'$. Thus the sequence of topological derivatives of \mathbb{I} is infinite which implies that $\Phi(\mathbb{I})$ is not of finite type. This allows us to construct a non equational non-compact surface (Proposition 1.1, Theorem 3.4).

We define the map Λ which associates a sequence of $\{0, 1\}^{\mathbb{N}}$ to each zero dimensional compact metric space X as following :

$\Lambda(X)$ is constructed in two steps :

- Let us consider the sequence $(X_n)_{n \geq 0}$ of topological spaces defined using X by the following induction scheme :

$$\begin{cases} X_0 = X, \\ X_{n+1} = \overline{X'_n \setminus X''_n}. \end{cases}$$

- This sequence allows us to define $\Lambda(X) = (b_n)_{n \geq 0}$ with

$$b_n = \begin{cases} 1 & \text{if } X'_n \text{ is not homeomorphic to } X_{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Of course, if X and Y are homeomorphic then $\Lambda(X) = \Lambda(Y)$.

PROPOSITION 4.1.

For all $b = (b_n)_n \in \{0, 1\}^{\mathbb{N}}$ such that for all $N \geq 0$, there exists $n \geq N$ such that $b_n = 1$, we have $b = \Lambda(\Phi(b))$.

PROOF.

Let δ be the shifting operator on sequences defined by :

let $b = (b_n)_n \in \{0, 1\}^{\mathbb{N}}$, for all $n \geq 0$, $\delta(b)_n = b_{n+1}$.

Then $\overline{\Phi(b)' \setminus \Phi(b)''}$ is homeomorphic to $\Phi(\delta(b))$.

Indeed :

$$\begin{aligned} \Phi(b)' &= \{0\} \cup \bigcup \{D'_{n+1} : n \geq 0 \text{ and } b_n = 1\} \\ &= \{0\} \cup \bigcup \{\varphi_{n+1}(K_{n+1})' : n \geq 0 \text{ and } b_n = 1\} \\ &= \{0\} \cup \bigcup \{\varphi_{n+1}(K'_{n+1}) : n \geq 0 \text{ and } b_n = 1\}. \end{aligned}$$

Note that $0 \in \Phi(b)'$ because b is not ultimately always equal to 0.

According to lemma 4.1, we have :

$$\Phi(b)' = \{0\} \cup \bigcup \{\varphi_{n+1}(K_n) : n \geq 0 \text{ and } b_n = 1\}$$

$$\text{and } \Phi(b)'' = \{0\} \cup \bigcup \{\varphi_{n+1}(K_{n-1}) : n \geq 0 \text{ and } b_n = 1\}$$

where K_{-1} denotes the Cantor set.

$$\text{Thus } \Phi(b)' \setminus \Phi(b)'' = \bigcup \{\varphi_{n+1}(K_n \setminus K_{n-1}) : n \geq 0 \text{ and } b_n = 1\}$$

$$\text{and } \overline{\Phi(b)' \setminus \Phi(b)''} = \{0\} \cup \bigcup \{\varphi_{n+1}(\overline{K_n \setminus K_{n-1}}) : n \geq 0 \text{ and } b_n = 1\}.$$

It is then easy to see that

$$\overline{K_n \setminus K_{n-1}} = \begin{cases} K_n & \text{if } n > 0, \\ \emptyset & \text{if } n = 0 \text{ i.e. } K_n = K_0 = K_{-1} = C. \end{cases}$$

Thus we get

$$\begin{aligned} \overline{\Phi(b)' \setminus \Phi(b)''} &= \{0\} \cup \bigcup \{\varphi_{n+1}(K_n) : n \geq 1 \text{ and } b_n = 1\} \\ &= \{0\} \cup \bigcup \{\varphi_{n+2}(K_{n+1}) : n \geq 0 \text{ and } b_{n+1} = 1\}. \end{aligned}$$

The last space is homeomorphic to

$$\Phi(\delta(b)) = \{0\} \cup \bigcup \{\varphi_{n+1}(K_{n+1}) : n \geq 0 \text{ and } b_{n+1} = 1\}$$

Indeed, let us consider for all n , the one-to-one mapping

$$\psi_n : \varphi_{n+2}([0, 1]) \longrightarrow \varphi_{n+1}([0, 1]) \text{ defined by } \psi_n = \varphi_{n+1} \varphi_{n+2}^{-1}.$$

$$\text{Let } \psi = \bigcup \{\psi_n : n \geq 0\} : \bigcup \{\varphi_{n+2}([0, 1]) : n \geq 0\} \longrightarrow \bigcup \{\varphi_{n+1}([0, 1]) : n \geq 0\}.$$

Then we have $\Phi(\delta(b)) = \psi(\overline{\Phi(b)' \setminus \Phi(b)''})$. Recall that $\Lambda(\Phi(b))$ is defined from the following sequence of spaces :

$$\begin{cases} X_0 = \Phi(b) \\ X_{k+1} = \overline{X'_k \setminus X''_k} \end{cases}$$

By the above discussion we can see that for all $k \geq 0$:

$$X_k = \psi^{-k}(\Phi(\delta^k(b))).$$

Thus $\Lambda(\Phi(b))_k = 0$ if and only if $\Phi(\delta^k(b))'$ is homeomorphic to $\Phi(\delta^{k+1}(b))$.

$$\begin{aligned} \Phi(\delta^k(b))' &= \{0\} \cup \bigcup \{\varphi_{n+1}(K_{n+1})' : n \geq 0 \text{ and } b_{n+k} = 1\} \\ (4.1) \qquad &= \{0\} \cup \bigcup \{\varphi_{n+1}(K_n) : n \geq 0 \text{ and } b_{n+k} = 1\}. \end{aligned}$$

And

$$\begin{aligned} \Phi(\delta^{k+1}(b)) &= \{0\} \cup \bigcup \{\varphi_{n+1}(K_{n+1}) : n \geq 0 \text{ and } b_{n+k+1} = 1\} \\ &= \{0\} \cup \bigcup \{\varphi_n(K_n) : n \geq 1 \text{ and } b_{n+k} = 1\} \end{aligned}$$

Suppose that, in Formula 4.1, $b_k = 1$. Then $\Phi(\delta^k(b))'$ contains a Cantor set i.e. $\varphi_1(K_0)$ which is not the closure of a sequence of isolated points. There are accumulation points which are not limits of sequences of isolated points. This is not true for $\Phi(\delta^{k+1}(b))$ and so, these spaces cannot be homeomorphic.

Now if $b_k = 0$, we get

$$\Phi(\delta^k(b))' = \{0\} \cup \bigcup \{\varphi_{n+1}(K_n) : n \geq 1 \text{ and } b_{n+k} = 1\},$$

which is homeomorphic to

$$\Phi(\delta^{k+1}(b)) = \{0\} \cup \bigcup \{\varphi_n(K_n) : n \geq 1 \text{ and } b_{n+k} = 1\}$$

Finally $\Lambda(\Phi(b))_k = 0$ if and only if $b_k = 0$. □

COROLLARY 4.2.

Φ is injective in the sense that $\Phi(b)$ and $\Phi(b')$ are homeomorphic if and only if $b = b'$.

By Proposition 1.1, we can associate to each $\Phi(b)$ a planar non-compact surface. Since the set of infinite sequence is not countable, we get

COROLLARY 4.3.

The set of homeomorphism classes of non-compact planar surfaces has at least cardinality 2^{\aleph_0} .

This proves again by a cardinality argument that there exist many non equational surfaces.

5. Perspectives

Since the notion of equational triangulation can be defined in the general framework of polyhedra, several generalisations could be investigated :

1. Homeomorphism problem for equational non-compact *bordered* surfaces. Note that there is no generalisation of Kerékjártó-Richards Theorem in this case.
2. Equational n -manifolds. For instance, the Whitehead manifold Wh_3 has an equational triangulation.
3. Constructive ways to define hyperbolic structures on non-compact manifolds.

6. Appendix : Basic Definitions in Formal Language Theory

Here we give basic definitions in formal language theory dealing with the notion of tree (see [HU79, Cou83] for more details).

Let A be an at most countable set (called an *alphabet* when it is finite) and let A^* (respectively A^ω) denote the set of words i.e. finite sequences (respectively infinite words i.e. infinite sequences) over A and ε the empty word. Let $A^+ = A^* \setminus \varepsilon$. Let $w_0, w_1 \in A^*$. The concatenation of w_0 with w_1 is denoted by $w_0.w_1$ and $|w_0|$ denotes the length of w_0 . We say that w_0 is a prefix of w_1 if there exists $w \in A^*$ such that $w_1 = w_0.w$.

DEFINITION 6.1 (Labelled Deterministic Tree).

Let S be a set. A deterministic tree over A labelled by S is a map $t : \text{Dom}(t) \rightarrow S$ where $\text{Dom}(t) \subset A^*$ such that :

- $\text{Dom}(t)$ is prefix-closed i.e. every prefix of an element of $\text{Dom}(t)$ is also in $\text{Dom}(t)$. The elements of $\text{Dom}(t)$ are called the nodes of t ; ε is called the root of t . For all node $w_1 \neq \varepsilon$, there exists a unique $w_0 \in \text{Dom}(t)$ and $a \in A$ such that $w_1 = w_0.a$. We say that w_0 is the father of w_1 .
- Every node of t is of finite degree i.e. every node has a finite number of sons.

Let $w_0 \in \text{Dom}(t)$, the *subtree* t_{w_0} of t of root w_0 is the map which associates the label $t(w_0.w)$ to each w such that $w_0.w \in \text{Dom}(t)$.

An *infinite path* of t is an infinite sequence $(w_n)_{n \geq 0}$ of nodes of t such that $w_0 = \varepsilon$ is the root and for all $n \geq 0$, w_n is the father of w_{n+1} . Let $\text{Path}(t)$ denote the set of infinite paths of t . Note that infinite paths are encoded by words of A^ω . This allows us to consider on $\text{Path}(t)$ the *topology* inherited from the product topology of A^ω . For any node x of t , let P_x denotes the set of infinite paths going through x . Then the topology of $\text{Path}(t)$ is generated by $\{P_x \mid x \text{ node of } t\}$.

Let t_1 and t_2 be two trees over A_1 and A_2 labelled by S_1 and S_2 respectively. We say that t_1 and t_2 are *isomorphic* if there exists two bijections $\sigma_A : A_1 \rightarrow A_2$ and $\sigma_S : S_1 \rightarrow S_2$ such that, if $\sigma_A^* : A_1^* \rightarrow A_2^*$ denotes the canonical extension of σ_A , then $\sigma_A^*|_{\text{Dom}(t_1)}$ is a bijection between $\text{Dom}(t_1)$ and $\text{Dom}(t_2)$ and $t_2 = \sigma_S \circ t_1 \circ \sigma_A^{*-1}$.

DEFINITION 6.2 (Regular trees).

A tree will be said to be *regular* if it has only finite number of subtrees up to isomorphism.

Now, we will see that regular trees are in correspondence with a special kind of automata called Moore machines [HU79, chap 2.7].

DEFINITION 6.3 (Moore Machine).

A *deterministic finite automaton with output over S* or (*deterministic*) *Moore machine* is a tuple $\mathcal{A} = \langle Q, \delta, A, q_0, \tau \rangle$ where A is an alphabet, Q is a finite set called the set of states; $\delta : Q \times A \rightarrow Q$ is the transition map; $q_0 \in Q$ is called the initial state; $\tau : Q \rightarrow S$ assigns an output to each state. Sometimes, $\tau(q)$ is called the label of q .

Let \mathcal{A} be an automaton, we consider $\mathcal{L}(\mathcal{A}) \subset A^*$, the language over A accepted by \mathcal{A} . In order to define it, we have to extend δ . Let us consider the map $\hat{\delta} : \text{Dom}(\hat{\delta}) \subset Q \times A^* \rightarrow Q$ defined by the following inductive rules :

- for all $q \in Q$, $\hat{\delta}(q, \varepsilon) = q$,

- for all $q \in Q$, $w \in A^*$, $a \in A$ such that $\hat{\delta}(q, w)$ is defined : $\hat{\delta}(q, w.a) = \delta(\hat{\delta}(q, w), a)$.

A state q is said to be *reachable* if there exists a word $w \in A^*$ such that $q = \hat{\delta}(q_0, w)$ is defined. Unreachable states play no role in the behaviour of the automaton; so, we will assume that all the states are reachable.

Let $\mathcal{L}(\mathcal{A}) = \{w \in A^* \mid \hat{\delta}(q_0, w) \text{ is defined}\}$. Such a machine can also be used to deal with languages of infinite words in some constructive way. An infinite word is recognized by \mathcal{A} if all its prefixes are recognized :

$\mathcal{L}^\omega(\mathcal{A}) = \{u = a_0 a_1 \dots a_k \dots \in A^\omega \mid \forall k \geq 0 : \hat{\delta}(q_0, a_0 a_1 \dots a_k) \text{ is defined}\}$. For an overview on automata on infinite words, see for instance [**Tho90**].

Here we turn to the correspondence between regular trees and Moore machines. Let t be a regular tree over an alphabet A labelled by S . Let $t_0 = t, \dots, t_n$ be the isomorphism classes of its subtrees. S is finite because its only elements which are labels of nodes of t are the $t_i(\varepsilon)$ for $i = 0 \dots n$. We associate to t the Moore machine $\mathcal{A}_t = \langle Q, \delta, A, q_0, \tau \rangle$ defined by $Q = \{t_0, \dots, t_n\}$; $\delta(t_i, a) = t_j$ if and only if $(t_i)_a = t_j$; $q_0 = t_0$; for all i , $\tau(t_i) = t_i(\varepsilon)$.

We obtain the converse by performing a kind of development of \mathcal{A}_t : Let $\mathcal{A} = \langle Q, \delta, A, q_0, \tau \rangle$ deterministic finite automaton labelled over a set S . Let $\mathcal{L}(\mathcal{A})$ be the language over A accepted by \mathcal{A} . Let us define the A -tree $t_{\mathcal{A}}$ associated to \mathcal{A} by :

- $\text{Dom}(t_{\mathcal{A}}) = \mathcal{L}(\mathcal{A}) \subset A^*$.
- For all $w \in \text{Dom}(t_{\mathcal{A}})$, $t_{\mathcal{A}}(w) = \tau(\hat{\delta}(q_0, w))$.

The reader can easily check that $t_{\mathcal{A}}$ is a regular tree and $t = t_{\mathcal{A}_t}$. Note that for a regular tree t , $\mathcal{L}^\omega(\mathcal{A}_t)$ and $\text{Path}(t)$ are canonically homeomorphic.

7. Acknowledgement

The author is greatly indebted to P. Narbel and G. Sénizergues for suggesting the problem, pointing out several references and for helpful conversations. The author wants to thank D. Epstein for his careful reading and many essential comments.

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