

Efficient Experimental Mathematics for Lattice Path Combinatorics

Alin Bostan



Based on joint works with
M. Bousquet-Mélou, F. Chyzak, M. van Hoeij, M. Kauers,
I. Kurkova, S. Melczer, L. Pech, K. Raschel, B. Salvy

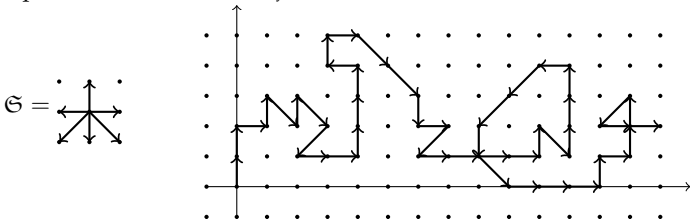
Journées de combinatoire de Bordeaux
LaBRI, Feb. 5, 2015

Enumerative combinatorics of lattice walks

- ▷ Nearest-neighbor walks in the quarter plane = walks in \mathbb{N}^2 starting at $(0,0)$ and using steps in a prefixed subset \mathfrak{S} of

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- ▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

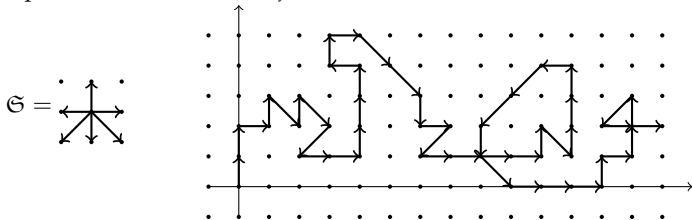


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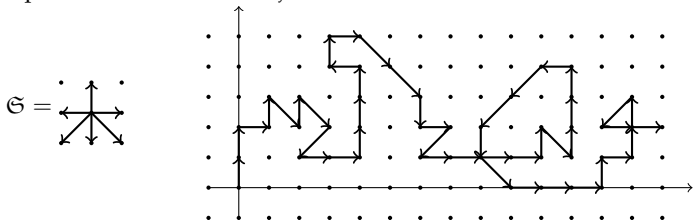
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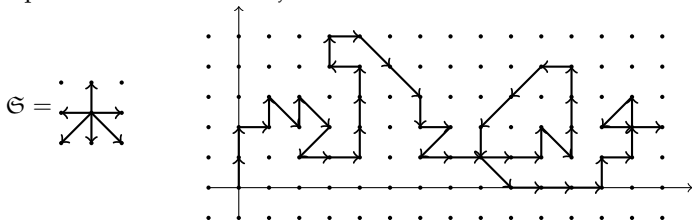
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- ▷ $f_n = \sum_{i,j \geq 0} f_{n;i,j}$ = number of total walks with length n .

▷ Complete generating series:

$$F(t; x, y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]].$$

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- ▷ Special, combinatorially meaningful specializations:

- $F(t; 0, 0)$ counts excursions;
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Combinatorial questions: Given \mathfrak{S} , what can be said about $F(t; x, y)$, resp. $f_{n,i,j}$, and their variants?

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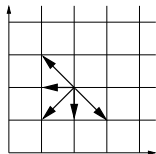
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Our goal: Use computer algebra to give computational answers.

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:

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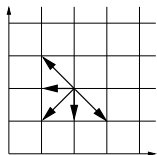
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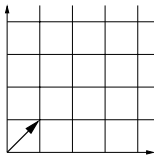
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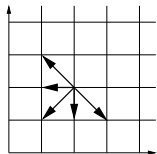
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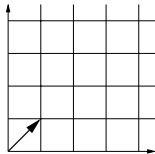
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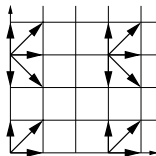
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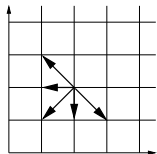
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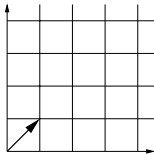
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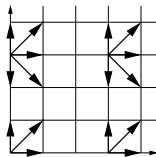
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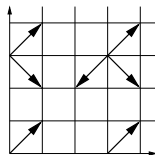
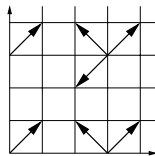
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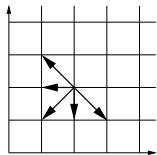
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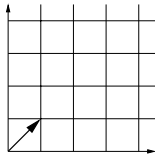
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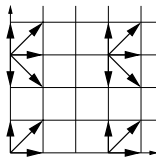
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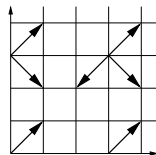
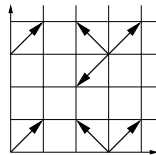
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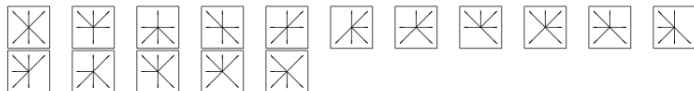
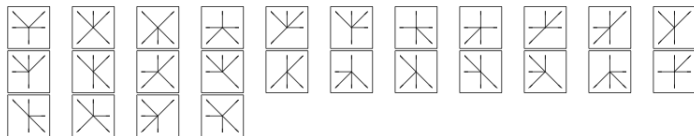
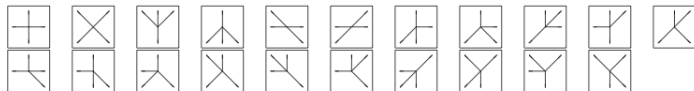
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One is left with [79 interesting distinct models](#).

The 79 models



Non-singular



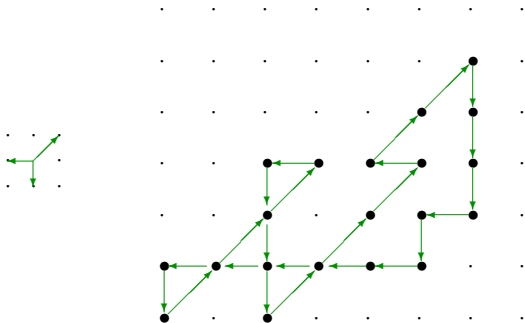
Singular

Two important models: **Kreweras** and **Gessel** walks

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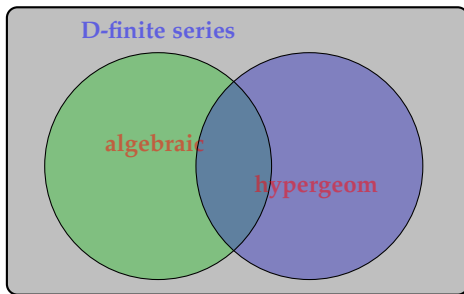


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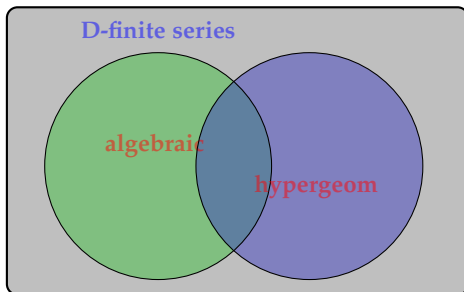


Example: A **Kreweras** excursion.

Important classes of univariate power series

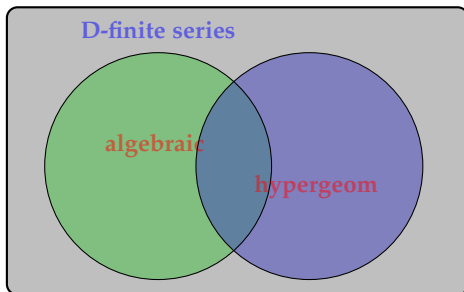


Important classes of univariate power series



Algebraic: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$, i.e., $P(t, S(t)) = 0$.

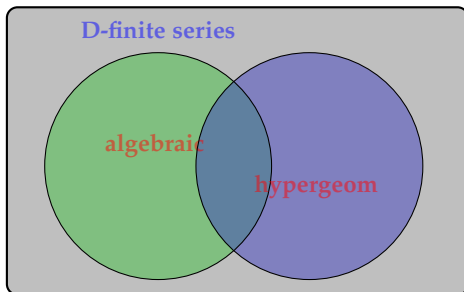
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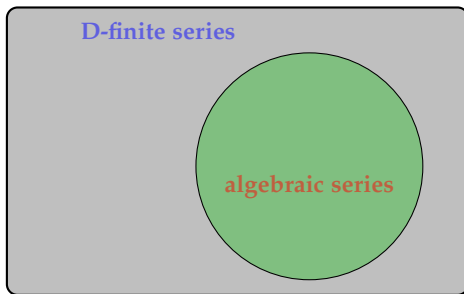


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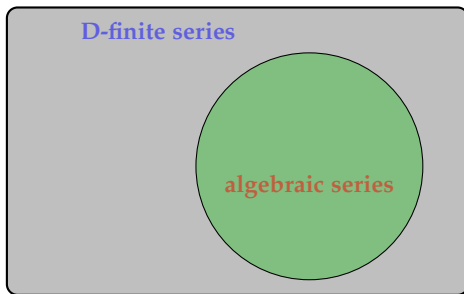
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Hypergeometric: $S(t) = \sum_{n=0}^{\infty} s_n t^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g.,

$${}_2F_1\left(\begin{matrix} a & b \\ c \end{matrix} \middle| t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1) \cdots (a+n-1).$$



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Main results (I): algebraicity of Gessel walks

Theorem [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t; 0, 0) = {}_3F_2\left(\begin{matrix} 1/3 & 2/3 & 1 \\ 3/2 & 2 \end{matrix} \middle| 27t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

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▷ Fresh news: recent human proof [B., Kurkova & Raschel 2014].

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▷ **Guess'n'Prove** method, using **Hermite-Padé approximants**.

Main results (II): Explicit form for $G(t; x, y)$

Theorem [B., Kauers & van Hoeij 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \dots$ be a root of

$$(V - 1)(1 + 3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \dots$ be a root of

$$\begin{aligned} &x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2 \\ &- xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0, \end{aligned}$$

let $W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \dots$ be a root of

$$y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.$$

Then $G(t; x, y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

▷ Computer-driven discovery and proof; no human proof yet.

Main results (II): Explicit form for $G(t; x, y)$

Theorem [B., Kauers & van Hoeij 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \dots$ be a root of

$$(V - 1)(1 + 3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \dots$ be a root of

$$\begin{aligned} &x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2 \\ &- xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0, \end{aligned}$$

let $W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \dots$ be a root of

$$y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.$$

Then $G(t; x, y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t-xy} - \frac{1}{tx(y+1)}.$$

- ▷ Computer-driven discovery and proof; no human proof yet.
- ▷ Proof uses **guessed minimal polynomials** for $G(t; x, 0)$ & $G(t; 0, y)$

Main results (III): Conjectured D-Finite $F(t; 1, 1)$ [B. & Kauers 2009]

	OEIS	\mathfrak{S}	Pol size	ODE size		OEIS	\mathfrak{S}	Pol size	ODE size
1	A005566		—	3, 4	13	A151275		—	5, 24
2	A018224		—	3, 5	14	A151314		—	5, 24
3	A151312		—	3, 8	15	A151255		—	4, 16
4	A151331		—	3, 6	16	A151287		—	5, 19
5	A151266		—	5, 16	17	A001006		2, 2	2, 3
6	A151307		—	5, 20	18	A129400		2, 2	2, 3
7	A151291		—	5, 15	19	A005558		—	3, 5
8	A151326		—	5, 18					
9	A151302		—	5, 24	20	A151265		6, 8	4, 9
10	A151329		—	5, 24	21	A151278		6, 8	4, 12
11	A151261		—	4, 15	22	A151323		4, 4	2, 3
12	A151297		—	5, 18	23	A060900		8, 9	3, 5

Equation sizes = {order, degree}@{algeq, diffeq}

- ▷ Computerized discovery by enumeration + Hermite–Padé
- ▷ 1–22: Confirmed by human proofs in [Bousquet-Mélou & Mishna 2010]
- ▷ 23: Confirmed by a human proof in [B., Kurkova & Raschel 2014]

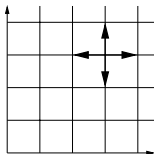
Main results (III): Conjectured D-Finite $F(t; 1, 1)$ [B. & Kauers 2009]

	OEIS	\mathfrak{G}	alg	asympt		OEIS	\mathfrak{G}	alg	asympt
1	A005566		N	$\frac{4}{\pi} \frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi} \frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi} \frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi} \frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi} \frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi} \frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}} \frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi} \frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}} \frac{6^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
9	A151302		N	$\frac{1}{3} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3} \sqrt{\frac{7}{3\pi}} \frac{7^n}{n^{1/2}}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)} \frac{6^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)} \frac{4^n}{n^{2/3}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi} \frac{(2B)^n}{n^2}$					

$$A = 1 + \sqrt{2}, B = 1 + \sqrt{3}, C = 1 + \sqrt{6}, \lambda = 7 + 3\sqrt{6}, \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$$

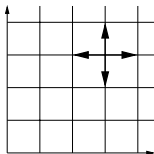
► Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.

The group of a walk: an example



The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$

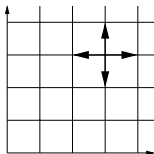
The group of a walk: an example



The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \quad \phi(x, y) = \left(\frac{1}{x}, y\right),$$

The group of a walk: an example



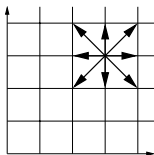
The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \quad \phi(x, y) = \left(\frac{1}{x}, y\right),$$

and thus under any element of the group

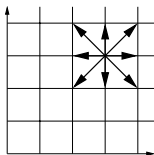
$$\langle \psi, \phi \rangle = \left\{ (x, y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

The group of a walk: the general case



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j) \in \mathfrak{S}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$

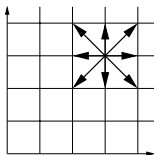
The group of a walk: the general case



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j) \in \mathfrak{S}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$ is left invariant under

$$\psi(x, y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right),$$

The group of a walk: the general case



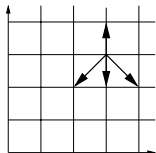
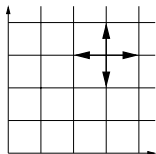
The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j) \in \mathfrak{S}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$ is left invariant under

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and thus under any element of the group

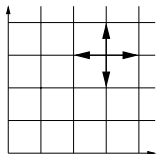
$$\mathcal{G}_{\mathfrak{S}} := \langle \psi, \phi \rangle.$$

Examples of groups

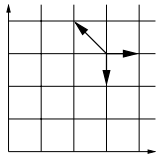


Order 4,

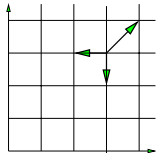
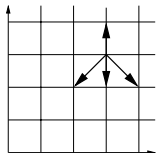
Examples of groups



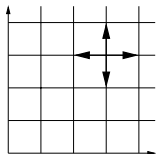
Order 4,



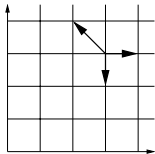
order 6,



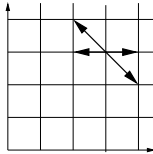
Examples of groups



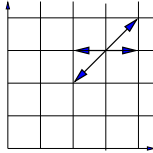
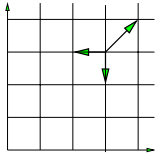
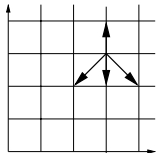
Order 4,



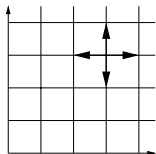
order 6,



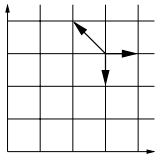
order 8,



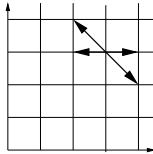
Examples of groups



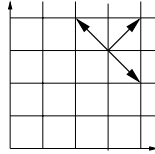
Order 4,



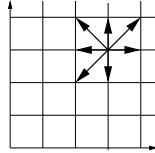
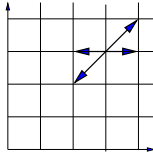
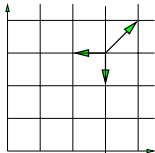
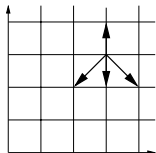
order 6,



order 8,

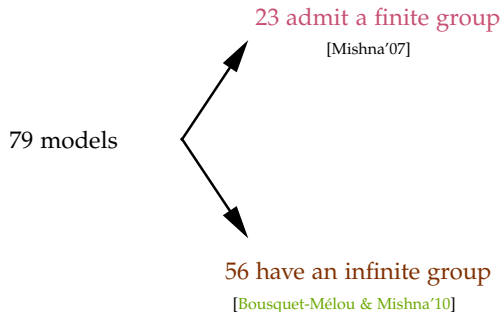


order ∞ .

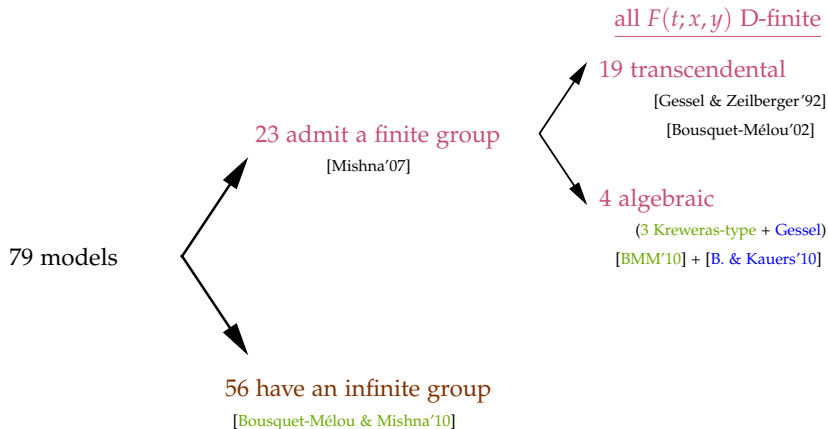


79 models

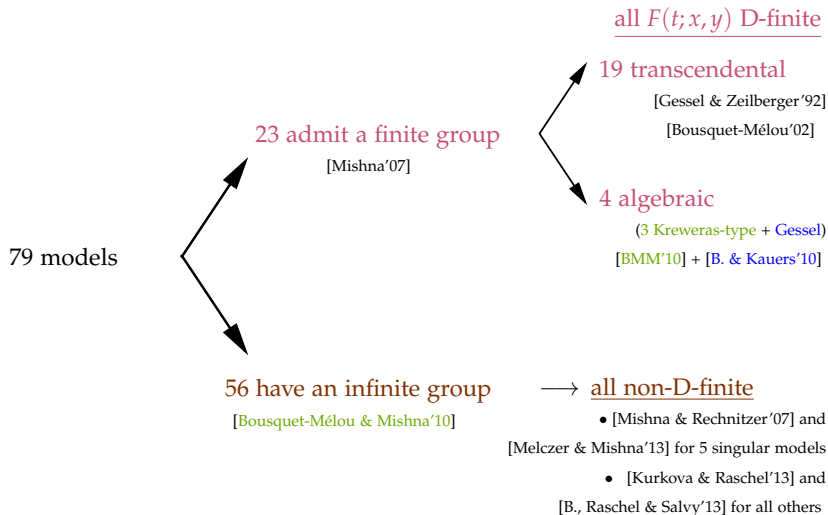
The 79 models: finite and infinite groups



The 79 models: finite and infinite groups

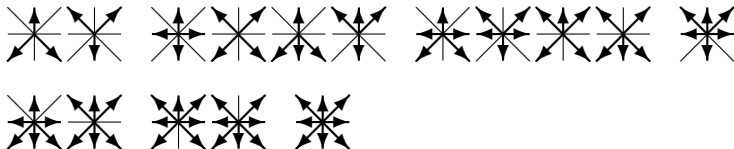


The 79 models: finite and infinite groups



The 23 models with a finite group

(i) 16 with a *vertical symmetry*, and group isomorphic to D_2



(ii) 5 with a *diagonal or anti-diagonal symmetry*, and group isomorphic to D_3

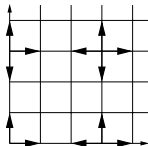


(iii) 2 with group isomorphic to D_4



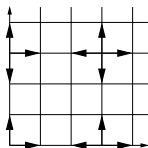
(i): vertical symmetry; (ii)+(iii): zero drift $\sum_{s \in \mathfrak{S}} s$

In red, models with algebraic generating functions



$J = -\frac{1}{t} + \sum_{(i,j) \in \mathfrak{S}} x^i y^j = x + \frac{1}{x} + y + \frac{1}{y} - \frac{1}{t}$ is **invariant** under the change of (x, y) into, respectively:

$$\left(\frac{1}{x}, y\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(x, \frac{1}{y}\right).$$

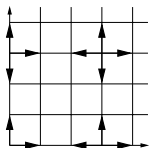


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$$\left(\frac{1}{x}, y\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(x, \frac{1}{y}\right).$$

“Kernel equation”:

$$J(t; x, y)xytF(t; x, y) = txF(t; x, 0) + tyF(t; 0, y) - xy$$

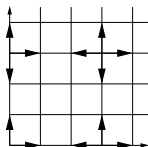


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$$\left(\frac{1}{x}, y\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(x, \frac{1}{y}\right).$$

“Kernel equation”:

$$\begin{aligned} J(t; x, y)xytF(t; x, y) &= txF(t; x, 0) + tyF(t; 0, y) - xy \\ -J(t; x, y)\frac{1}{x}ytF(t; \frac{1}{x}, y) &= -t\frac{1}{x}F(t; \frac{1}{x}, 0) - tyF(t; 0, y) + \frac{1}{x}y \end{aligned}$$

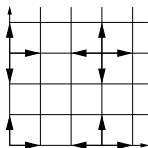


$J = -\frac{1}{t} + \sum_{(i,j) \in \mathfrak{S}} x^i y^j = x + \frac{1}{x} + y + \frac{1}{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

$$\left(\frac{1}{x}, y\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(x, \frac{1}{y}\right).$$

“Kernel equation”:

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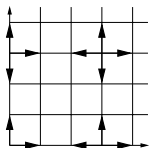


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“Kernel equation”:

$$\begin{aligned} J(t; x, y)xytF(t; x, y) &= txF(t; x, 0) + tyF(t; 0, y) - xy \\ - J(t; x, y)\frac{1}{x}y tF(t; \frac{1}{x}, y) &= -t\frac{1}{x}F(t; \frac{1}{x}, 0) - tyF(t; 0, y) + \frac{1}{x}y \\ J(t; x, y)\frac{1}{x}\frac{1}{y}tF(t; \frac{1}{x}, \frac{1}{y}) &= t\frac{1}{x}F(t; \frac{1}{x}, 0) + t\frac{1}{y}F(t; 0, \frac{1}{y}) - \frac{1}{x}\frac{1}{y} \\ - J(t; x, y)x\frac{1}{y}tF(t; x, \frac{1}{y}) &= -txF(t; x, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y}) + x\frac{1}{y} \end{aligned}$$



$J = -\frac{1}{t} + \sum_{(i,j) \in \mathcal{G}} x^i y^j = x + \frac{1}{x} + y + \frac{1}{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

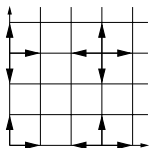
$$\left(\frac{1}{x}, y\right), \left(x, \frac{1}{y}\right), \left(x, \frac{1}{y}\right).$$

“Kernel equation”:

$$\begin{aligned} J(t; x, y)xytF(t; x, y) &= txF(t; x, 0) + tyF(t; 0, y) - xy \\ -J(t; x, y)\frac{1}{x}y tF(t; \frac{1}{x}, y) &= -t\frac{1}{x}F(t; \frac{1}{x}, 0) - tyF(t; 0, y) + \frac{1}{x}y \\ J(t; x, y)\frac{1}{x}\frac{1}{y}tF(t; \frac{1}{x}, \frac{1}{y}) &= t\frac{1}{x}F(t; \frac{1}{x}, 0) + t\frac{1}{y}F(t; 0, \frac{1}{y}) - \frac{1}{x}\frac{1}{y} \\ -J(t; x, y)x\frac{1}{y}tF(t; x, \frac{1}{y}) &= -txF(t; x, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y}) + x\frac{1}{y} \end{aligned}$$

Summing up yields:

$$\sum_{\theta \in \mathcal{G}} (-1)^\theta \theta(xytF(t; x, y)) = \frac{-xy + \frac{1}{x}y - \frac{1}{x}\frac{1}{y} + x\frac{1}{y}}{J(t; x, y)}$$



$J = -\frac{1}{t} + \sum_{(i,j) \in \mathfrak{S}} x^i y^j = x + \frac{1}{x} + y + \frac{1}{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

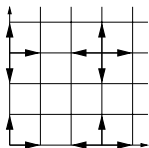
$$\left(\frac{1}{x}, y\right), \left(x, \frac{1}{y}\right), \left(x, \frac{1}{y}\right).$$

“Kernel equation”:

$$\begin{aligned} J(t; x, y)xytF(t; x, y) &= txF(t; x, 0) + tyF(t; 0, y) - xy \\ -J(t; x, y)\frac{1}{x}y tF(t; \frac{1}{x}, y) &= -t\frac{1}{x}F(t; \frac{1}{x}, 0) - tyF(t; 0, y) + \frac{1}{x}y \\ J(t; x, y)\frac{1}{x}\frac{1}{y}tF(t; \frac{1}{x}, \frac{1}{y}) &= t\frac{1}{x}F(t; \frac{1}{x}, 0) + t\frac{1}{y}F(t; 0, \frac{1}{y}) - \frac{1}{x}\frac{1}{y} \\ -J(t; x, y)x\frac{1}{y}tF(t; x, \frac{1}{y}) &= -txF(t; x, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y}) + x\frac{1}{y} \end{aligned}$$

Summing up yields:

$$[x^>][y^>] \sum_{\theta \in \mathfrak{G}} (-1)^\theta \theta(xytF(t; x, y)) = [x^>][y^>] \frac{-xy + \frac{1}{x}y - \frac{1}{x}\frac{1}{y} + x\frac{1}{y}}{J(t; x, y)}$$



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▷ Implies D-finiteness of $F(t; x, y)$, but does not yield easily an ODE for it

Main results (IV): explicit expressions for the 19 D-finite transcendental models

Theorem [B., Chyzak, van Hoeij, Kauers & Pech 2014]

Let \mathfrak{S} be one of the 19 models with finite group $\mathcal{G}_{\mathfrak{S}}$, and such that the generating series $F = F_{\mathfrak{S}}(t; x, y)$ is not algebraic.

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Example (King walks in the quarter plane, A025595)

$$F_{\begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \\ \leftarrow \\ \swarrow \\ \downarrow \end{array}}(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\frac{3}{2}, \frac{3}{2} \mid \frac{16x(1+x)}{(1+4x)^2}\right) dx$$
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \dots$$

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- ▷ Computer-driven discovery and proof; no human proof yet.
- ▷ Proof uses **creative telescoping**, **ODE factorization**, **ODE solving**.

Theorem [B., Rachel & Salvy 2013]

Let \mathfrak{G} be one of the 51 non-singular models with infinite group $\mathcal{G}_{\mathfrak{G}}$.
Then $F_{\mathfrak{G}}(t; 0, 0)$, and in particular $F_{\mathfrak{G}}(t; x, y)$, are non-D-finite.

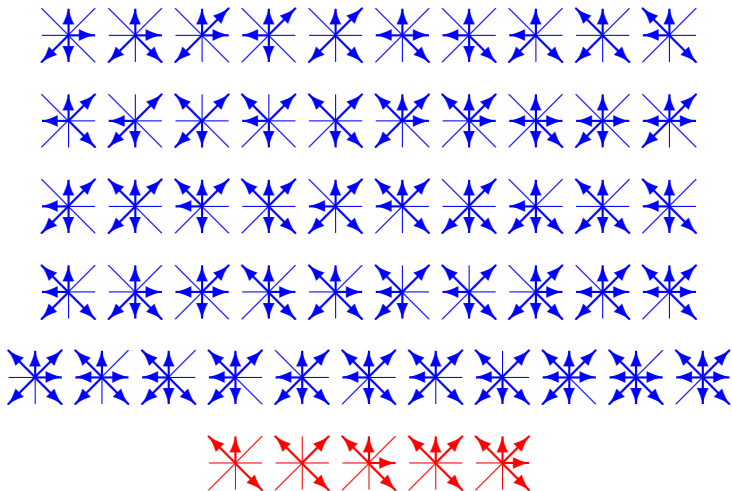
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- ▷ *Algorithmic proof.* Uses Gröbner basis computations, polynomial factorization, cyclotomy testing.
- ▷ Based on *two ingredients*: asymptotics + irrationality.

- ▷ [Kurkova & Raschel 2013] Human proof that $F_{\mathfrak{G}}(t;x,y)$ is non-D-finite.
- ▷ No human proof yet for $F_{\mathfrak{G}}(t;0,0)$ non-D-finite.

The 56 models with infinite group



In **blue**, non-singular models, solved by [B., Raschel & Salvy 2013]

In **red**, singular models, solved by [Melczer & Mishna 2013]

Example: the scarecrows

[B., Raschel & Salvy 2013]: $F_{\mathfrak{S}}(t;0,0)$ is not D-finite for the models



For the 1st and the 3rd, the excursions sequence $[t^n] F_{\mathfrak{S}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \dots$$

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\dots$

The *irrationality* of α prevents $F_{\mathfrak{S}}(t;0,0)$ from being D-finite.

The Main Theorem Let \mathfrak{S} be one of the 74 non-singular models. The following assertions are equivalent:

- (1) The full generating series $F_{\mathfrak{S}}(t; x, y)$ is D-finite
- (2) the excursions generating series $F_{\mathfrak{S}}(t; 0, 0)$ is D-finite
- (3) the excursions seq. $[t^n] F_{\mathfrak{S}}(t; 0, 0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$
- (4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^* \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)
- (5) the step set \mathfrak{S} has either an axial symmetry, or zero drift and cardinal different from 5.

Summary – classification of 2D non-singular walks

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Moreover, under (1)–(5), $F_{\mathfrak{S}}(t; x, y)$ is **algebraic** if and only if the step set \mathfrak{S} has **positive covariance** $\sum_{(i,j) \in \mathfrak{S}} ij - \sum_{(i,j) \in \mathfrak{S}} i \cdot \sum_{(i,j) \in \mathfrak{S}} j > 0$.

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In this case, $F_{\mathfrak{S}}(t; x, y)$ is expressible using **nested radicals**. If not, $F_{\mathfrak{S}}(t; x, y)$ is expressible using **iterated integrals of ${}_2F_1$ expressions**.

- (1) for proving **algebraicity** / D-finiteness
 - (1a) **Guess'n'Prove**
 - (1b) Diagonals of rational functions

- (2) for proving non-D-finiteness
 - (2a) Infinite number of singularities, or lacunary
 - (2b) Asymptotics

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(1b) Diagonals of rational functions

Hermite-Padé approximants
Creative telescoping

(2) for proving non-D-finiteness

(2a) Infinite number of singularities, or lacunary

(2b) Asymptotics

▷ All methods are *algorithmic*.

Guess'n'Prove

Experimental mathematics –~~Guess'n'Prove~~– approach:

(S1) *Generate data*

(S2) *Conjecture*

(S3) *Prove*

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guess candidates for minimal polynomials of $F_{\mathfrak{G}}(t; x, 0)$ and $F_{\mathfrak{G}}(t; 0, y)$,
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+ **Efficient Computer Algebra**

Step (S1): high order series expansions

$f_{\mathfrak{S}}(n; i, j)$ satisfies the recurrence with constant coefficients

$$f_{\mathfrak{S}}(n+1; i, j) = \sum_{(u,v) \in \mathfrak{S}} f_{\mathfrak{S}}(n; i-u, j-v) \quad \text{for } n, i, j \geq 0$$

+ init. cond. $f_{\mathfrak{S}}(0; i, j) = \delta_{0,ij}$ and $f_{\mathfrak{S}}(n; -1, j) = f_{\mathfrak{S}}(n; i, -1) = 0$.

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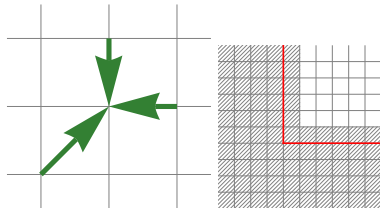
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Example: for the **Kreweras walks**,

$$\begin{aligned} k(n+1; i, j) &= k(n; i+1, j) \\ &+ k(n; i, j+1) \\ &+ k(n; i-1, j-1) \end{aligned}$$



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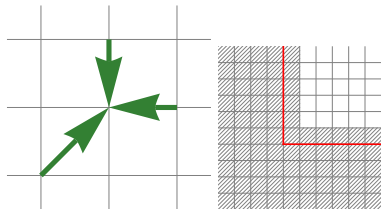
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▷ Recurrence is used to compute $F_{\mathfrak{S}}(t; x, y) \bmod t^N$ for large N .

$$\begin{aligned} K(t; x, y) &= 1 + xyt + (x^2y^2 + y + x)t^2 + (x^3y^3 + 2xy^2 + 2x^2y + 2)t^3 \\ &\quad + (x^4y^4 + 3x^2y^3 + 3x^3y^2 + 2y^2 + 6xy + 2x^2)t^4 \\ &\quad + (x^5y^5 + 4x^3y^4 + 4x^4y^3 + 5xy^3 + 12x^2y^2 + 5x^3y + 8y + 8x)t^5 + \dots \end{aligned}$$

Step (S2): guessing equations for $F_{\mathfrak{S}}(t; x, y)$, a first idea

In terms of generating series, the recurrence on $k(n; i, j)$ reads

$$\begin{aligned} & (xy - (x + y + x^2y^2)t)K(t; x, y) \\ & = xy - xt K(t; x, 0) - yt K(t; 0, y) \end{aligned} \quad (\text{KerEq})$$

▷ A similar kernel equation holds for $F_{\mathfrak{S}}(t; x, y)$, for any \mathfrak{S} -walk.

Corollary. $F_{\mathfrak{S}}(t; x, y)$ is algebraic (resp. D-finite) if and only if $F_{\mathfrak{S}}(t; x, 0)$ and $F_{\mathfrak{S}}(t; 0, y)$ are both algebraic (resp. D-finite).

▷ **Crucial** simplification: equations for $G(t; x, y)$ are **huge** ($\approx 30\text{Gb}$)

Step (S2): guessing equations for $F_{\mathfrak{S}}(t; x, 0)$ & $F_{\mathfrak{S}}(t; 0, y)$

Task 1: Given the first N terms of $S = F_{\mathfrak{S}}(t; x, 0) \in \mathbb{Q}[x][[t]]$, search for a *differential equation* satisfied by S at precision N :

$$c_r(x, t) \cdot \frac{\partial^r S}{\partial t^r} + \cdots + c_1(x, t) \cdot \frac{\partial S}{\partial t} + c_0(x, t) \cdot S = 0 \pmod{t^N}.$$

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- Both tasks amount to **linear algebra** in size N over $\mathbb{Q}(x)$.
- In practice, we use **modular** Hermite-Padé approximation (**Beckermann-Labahn** algorithm) combined with (rational) **evaluation-interpolation** and **rational number reconstruction**.
- Fast (FFT-based) arithmetic in $\mathbb{F}_p[t]$ and $\mathbb{F}_p[t]\langle \frac{t}{\partial t} \rangle$.

Step (S2): guessing equations for $G(t; x, 0)$ and $G(t; 0, y)$

Using $N = 1200$ terms of $G(t; x, y)$, we guessed candidates

- $\mathcal{P}_{x,0}$ in $\mathbb{Z}[x, t, T]$ of degree $(32, 43, 24)$, coefficients of 21 digits
- $\mathcal{P}_{0,y}$ in $\mathbb{Z}[y, t, T]$ of degree $(40, 44, 24)$, coefficients of 23 digits

such that

$$\mathcal{P}_{x,0}(x, t, G(t; x, 0)) = \mathcal{P}_{0,y}(y, t, G(t; 0, y)) = 0 \pmod{t^{1200}}.$$

[†]of order 11, and bidegree $(96, 78)$ for $G(t; x, 0)$, and $(68, 28)$ for $G(t; 0, y)$

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▷ Guessing $\mathcal{P}_{x,0}$ by *undetermined coefficients* would require solving a dense linear system of size $\approx 100\,000$, and ≈ 1000 digits entries!

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3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite, and so is (r_n) :

$$(n+2)(3n+5)r_{n+1} - 4(6n+5)(2n+1)r_n = 0, \quad r_0 = 1$$

\Rightarrow solution $r_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n = g_n$, thus $g(t) = r(t)$ is algebraic.

Step (S3): rigorous proof for Kreweras walks



1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \dots$ in the kernel equation

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shows that $U = K(t; x, 0)$ satisfies the *reduced kernel equation*

$$0 = x \cdot y_0 - x \cdot t \cdot U(t, x) - y_0 \cdot t \cdot U(t, y_0)$$

(RKerEq)

Step (S3): rigorous proof for *Kreweras walks*



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4. **Resultant computations** + verification of initial terms
 $\implies U = H(t, x)$ also satisfies (RKerEq).
5. *Uniqueness:* $H(t, x) = K(t; x, 0) \implies K(t; x, 0)$ is algebraic!

Algebraicity of **Kreweras walks**: our Maple proof in a nutshell

```
[bostan@inria ~]$ maple
  \\~/|      Maple 19 (APPLE UNIVERSAL OSX)
._|\\|   |/|_.. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2014
 \ MAPLE / All rights reserved. Maple is a trademark of
 <____ ____> Waterloo Maple Inc.
 |           Type ? for help.

# HIGH ORDER EXPANSION (S1)
> st,bu:=time(),kernelopts(bytesused):
> f:=proc(n,i,j)
option remember;
  if i<0 or j<0 or n<0 then 0
  elif n=0 then if i=0 and j=0 then 1 else 0 fi
  else f(n-1,i-1,j-1)+f(n-1,i,j+1)+f(n-1,i+1,j) fi
end:
> S:=series(add(add(f(k,i,0)*x^i,i=0..k)*t^k,k=0..80),t,80):

# GUESSING (S2)
> libname:=".",libname:gfun:-version();
                                     3.62
> gfun:-seriestoalgeq(S,Fx(t)):
> P:=collect(numer(subs(Fx(t)=T,%[1])),T):

# RIGOROUS PROOF (S3)
> ker := (T,t,x) -> (x+T+x^2*T^2)*t-x*T:
> pol := unapply(P,T,t,x):
> p1 := resultant(pol(z-T,t,x),ker(t*z,t,x),z):
> p2 := subs(T=x*T,resultant(numer(pol(T/z,t,z)),ker(z,t,x),z)):
> normal(primpart(p1,T)/primpart(p2,T));
                                     1

# time (in sec) and memory consumption (in Mb)
> trunc(time()-st),trunc((kernelopts(bytesused)-bu)/1000^2);
                                     7, 617
```

Step (S3): rigorous proof for Gessel walks

Same philosophy, but several complications:

- stepset diagonal symmetry is lost: $G(t; x, y) \neq G(t; y, x)$;
- $G(t; 0, 0)$ occurs in (KerEq);
- equations are $\approx 5\,000$ times bigger.

→ replace equation (RKerEq) by a *system* of 2 reduced kernel equations.

→ fast algorithms needed (e.g., [B., Flajolet, Salvy & Schost 2006] for computations with algebraic series).



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Fast computation of special resultants

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- ☺ 2D classification of $F(t;0,0)$ and $F(t;x,y)$ is **fully completed**
- ☺ **robust algorithmic** methods, based on computer algebra algorithms:
 - **Guess'n'Prove**
 - **Creative Telescoping** for integration of rational functions
- ☺ Brute-force and/or use of naive algorithms = **hopeless**.
E.g. size of algebraic equations for $G(t;x,y) \approx 30\text{Gb}$.
- ☺ Remarkable properties **discovered experimentally**. E.g.:
all algebraic models have **solvable Galois groups**

$$G(t;1,1) = -\frac{3}{6t} + \frac{\sqrt{3}}{6t} \sqrt{U(t) + \sqrt{\frac{16t(2t+3)+2}{(1-4t)^2U(t)} - U(t)^2 + 3}}$$

$$\text{where } U(t) = \sqrt{1 + 4t^{1/3}(4t+1)^{1/3}/(4t-1)^{4/3}}.$$

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- ☺ **lack of “purely human” proofs** for many results. E.g.:
 ${}_2F_1$ s expressions for $F(t;1,1)$; asymptotics of f_n
- ☺ **still missing a unified proof** of: **finite group** \leftrightarrow **D-finite**
- ☺ **open: is $F(t;1,1)$ non-D-finite** in the 51 non-singular models with infinite group?

▷ **3D walks** [B., Bousquet-Mélou, Kauers & Melczer 2014]

Many distinct models: 11 074 225 (instead of 79)

- We focus on the 20 804 models with at most 6 steps.
- [B. & Kauers, 2009] conjectured 35 D-finite. **Now proved.**
- 116 **new** D-finite models: **guessed**, then **proved**.
- **New** phenomenon (empirically discovered, no proof yet): 19 models (e.g. **3D Kreweras**) with **finite group**, but possibly **non-D-finite** GF (!?)

▷ **Longer 2D steps** [B., Bousquet-Mélou & Melczer 2014]

- 680 models with one large step, 643 **proved non D-finite**, 32 of 37 have differential equations **guessed**.
- 5910 models with two large steps, 5754 **proved non D-finite**, 69 of 156 have differential equations **guessed**.

- *Automatic classification of restricted lattice walks*, with M. Kauers. Proc. FPSAC, 2009.
- *The complete generating function for Gessel walks is algebraic*, with M. Kauers. Proc. Amer. Math. Soc., 2010.
- *Explicit formula for the generating series of diagonal 3D Rook paths*, with F. Chyzak, M. van Hoeij and L. Pech. Séminaire Lotharingien de Combinatoire, 2011.
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- *A human proof of Gessel's lattice path conjecture*, with I. Kurkova, K. Raschel, 2015.
- *Explicit Differentiably Finite Generating Functions of Walks with Small Steps in the Quarter Plane*, with F. Chyzak, M. Kauers, M. van Hoeij and L. Pech, 2015.

Thanks for your attention!