

Graph polynomials and relation with physics

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J. Noncomm. Geom. **4** (2010)

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- Tutte polynomial - some definitions
- QFT and Feynman integrals; parametric representation
- Relation Tutte polynomial - parametric representation
- Graph linear maps and differential equations
- Proof of the universality of the Tutte polynomial
- Perspectives

Points of interest (among others) for the Tutte polynomial

- property of universality
- relation to the chromatic polynomial (which counts the number of distinct ways to color a graph) - specification of the Tutte polynomial
- relations with physics (statistical physics models, quantum field theory)
 - **Combinatorial Physics**
- *etc.*

Graph theory - some definitions

loop - edge which starts and ends on the same vertex (*tadpole edge*)

bridge - an edge whose removal increases by 1 the number of connected components of the graph (*IPR edge*)

regular edge - edge which is neither a bridge nor a loop

spanning tree - connected subgraph with no cycle, connecting all vertices

2 natural operations for an edge e in a graph G :

① deletion $\rightarrow G - e$

② contraction $\rightarrow G/e$

\hookrightarrow associated to these operations - the Tutte polynomial

Tutte polynomial

(W. T. Tutte, *Graph Theory*, '84, H. H. Crapo, *Aequationes Mathematicae*, '69)

a 1st definition - deletion/contraction:

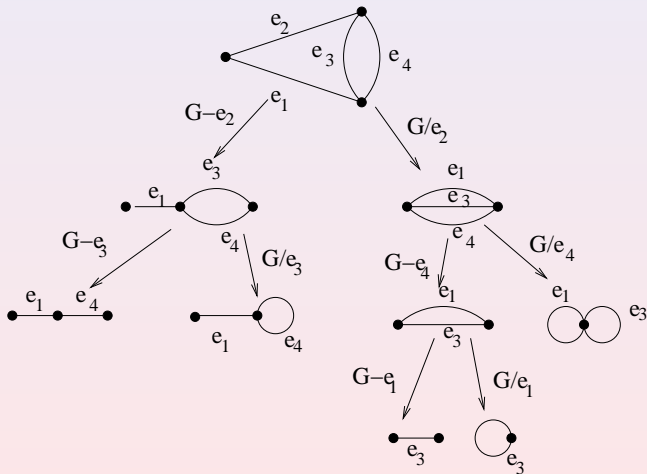
e - regular line

$$T(G; x, y) := T(G/e; x, y) + T(G - e; x, y)$$

→ terminal forms - m bridges and n loops

$$T(G; x, y) := x^m y^n.$$

Exemple



$$T(G; x, y) = x^2 + xy + x + y + y^2.$$

Rank, nullity and the Tutte polynomial

$A \subseteq E$ - a subgraph of the graph G

$$r(A) := |V(G)| - k(A),$$

$r(A)$ - the **rank** of the subgraph A

$V(G)$ - number of vertices of the graph G

$k(A)$ - number of connected components of A

$$n(A) := |A| - r(A),$$

$n(A)$ - nullity of the subgraph A

2nd definition of the Tutte polynomial - **sum over subgraphs**:

$$T_G(x, y) := \sum_A (x - 1)^{r(E) - r(A)} (y - 1)^{n(A)}.$$

the two definitions are equivalent

Multivariate Tutte polynomial

(A. Sokal, *London Math. Soc. Lecture Note Ser.*, 2005)

w_e , $e = 1, \dots, |E|$ (different variable for each edge)
 $|E|$ - the total number of edges

1st definition - deletion/contraction:

$$Z(G; q, \mathbf{w}) := Z(G/e; q, \mathbf{w}) + \beta_e Z(G - e; q, \mathbf{w}),$$

e - not necessary regular

→ terminal forms with v vertices and without edges

$$Z_G(q, \mathbf{w}) := q^v.$$

Multivariate Tutte polynomial - 2nd definition

2nd definition - sum over subgraphs:

$$Z(G; q, \mathbf{w}) := \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} w_e.$$

the two definitions are equivalent

The polynomial $Z(G; q, \mathbf{w})$ is directly related to the *Potts model* in
statistical physics

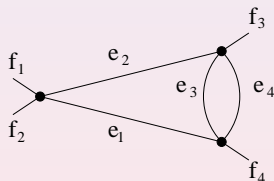
- combinatorial physics

Quantum field theory (QFT)

QFT - graph theory and combinatorics

Φ^4 model - 4-valent vertices

$\Phi(x)$ - a field, $x \in \mathbb{R}^4$ (the space-time)



propagator (associated to each edge of the graph):

$$C(p_\ell, m) = \frac{1}{p_\ell^2 + m^2}, \quad p_\ell \in \mathbb{R}^4, \quad i = 1, \dots, |E|, \quad m \in \mathbb{R} \text{ the mass}$$

→ part of the integrands of some *Feynman integral* \mathcal{A}_G

Parametric representation of the Feynman integrals

introduction of some parameters a :

$$\frac{1}{p_\ell^2 + m^2} = \int_0^\infty da_\ell e^{-a(p_\ell^2 + m^2)}, \quad \forall \ell = 1, \dots, E$$

→ Gaussian integration over internal momenta p_i

$$\implies \mathcal{A}(G; p_{\text{ext}}) = \int_0^\infty \frac{e^{-V(G; p_{\text{ext}}, \mathbf{a})/U(G; \mathbf{a})}}{U(G; \mathbf{a})^2} \prod_{\ell=1}^E (e^{-m^2 a_\ell} da_\ell)$$

U, V - polynomials in the parameters a

Kirchhoff-Symanzik polynomials

$$U(G, \mathbf{a}) = \sum_{\mathcal{T}} \prod_{\ell \notin \mathcal{T}} a_\ell,$$

\mathcal{T} - spanning tree of the graph

Theorem

$$U(G; \mathbf{a}) = a_e U(G - e; \mathbf{a}) + U(G/e; \mathbf{a})$$

terminal forms (graph formed only of tadpoles)

$$U(G; \mathbf{a}) = \prod_{e \text{ tadpole}} a_e.$$



T. Krajewski, V. Rivasseau, A. T., Z. Wang, *J. Noncomm. Geom.* (2010)

Proof - Grassmannian development of Pfaffians

Grassmann (anti-commuting) variables:

$$\chi_i \chi_j = -\chi_j \chi_i,$$

$$\implies \chi_i^2 = 0$$

Grassmann integration:

$$\int d\chi = 0 \quad \text{and} \quad \int \chi d\chi = 1.$$

powerful tool to manipulate:

① **determinants (and Pfaffians):**

$$\det M = \int \prod_{i=1}^n d\bar{\psi}_i d\psi_i e^{-\sum_{i,j=1}^n \bar{\psi}_i M_{ij} \psi_j}$$

② **minors:** $\det M_{ij} = \int \prod_{i=1}^n d\bar{\psi}_i d\psi_i (\psi_i \bar{\psi}_j) e^{-\sum_{i,j=1}^n \bar{\psi}_i M_{ij} \psi_j}$

Relation with the multivariate Tutte polynomial - the polynomial U_G satisfies the deletion/contraction relation

The Kirchoff-Symanzik polynomial can be obtained as a limit of the multivariate Tutte polynomial:

$$U(G; \mathbf{a}) = \left[\left(\prod_{e \in E} w_e \right) \lim_{q' \rightarrow 0} \frac{1}{(q')^{p(G)}} \lim_{q \rightarrow 0} q^{-k(G)} Z(G; q, q' \mathbf{w}) \right]_{w_e^{-1} = a_e}.$$

where $p(G) := |V(G)| - k(G)$.



A. T., *Sém. Loth. Comb.* (2012)

Some algebra - what is a Hopf algebra?

A **bialgebra** over a field \mathbb{K} is a \mathbb{K} -linear space endowed with an unital associative algebra and a counital, coassociative coalgebra structure

- a **product** m (assembles) - an algebra structure
- a **coproduct** Δ (disassembles) - a coalgebra structure

such that some compatibility conditions are satisfied.

A **Hopf algebra** \mathcal{H} over a field \mathbb{K} is a bialgebra over \mathbb{K} equipped with an **antipode** map $S : \mathcal{H} \rightarrow \mathcal{H}$.

Examples of combinatorial Hopf algebras:

Sym, QSym, FQSym etc. - [selection-deletion rule](#)

J.-C. Aval, A. Boussicault, ...

A Hopf algebra of graphs



W. Schmitt, *J. Pure Applied Alg.* (1994)

Product: disjoint reunion of graphs

Coproduct:

$$\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$$

$$\Delta(G) := \sum_{A \subseteq E} A \otimes G/A.$$


same type of structure as the Connes-Kreimer Hopf algebra encoding the combinatorics of renormalization in QFT




A. Connes and D. Kreimer, *Commun. Math. Phys.* (2000)

2 liner maps

$$\delta_{\text{loop}}, \delta_{\text{bridge}} : \mathcal{H} \rightarrow \mathbb{K}$$

$$\delta_{\text{loop}}(G) := \begin{cases} 1_{\mathbb{K}} & \text{if } G = \text{loop}, \\ 0_{\mathbb{K}} & \text{otherwise.} \end{cases}$$


$$\delta_{\text{bridge}}(G) := \begin{cases} 1_{\mathbb{K}} & \text{if } G = \text{bridge}, \\ 0_{\mathbb{K}} & \text{otherwise.} \end{cases}$$


- From an algebraic point of view, δ_{loop} and δ_{bridge} are infinitesimal Hopf algebra characters
- δ_{loop} and δ_{bridge} are related to one-edge graphs

Convolution product:

$$f * g = m \circ (f \otimes g) \circ \Delta,$$

Let $\alpha : \mathcal{H} \rightarrow \mathbb{K}$

$$\begin{aligned} \alpha(G; x, y, s) &:= \exp_* s \{ \delta_{\text{bridge}} + (y - 1) \delta_{\text{loop}} \} \\ &\quad * \exp_* s \{ (x - 1) \delta_{\text{bridge}} + \delta_{\text{loop}} \} (G). \end{aligned}$$

(the non-trivial part of the coproduct is nilpotent)

From an algebraic point of view, α is a Hopf algebra character

Relation between the map α and the Tutte polynomial

Lemma

$$\exp_* \{ a\delta_{\text{bridge}} + b\delta_{\text{loop}} \} (G) = a^{r(G)} b^{n(G)}.$$

Proposition

$$\alpha(G; x, y, s) = s^{|E|} T(G; x, y).$$

The map α - differential equation solution

Proposition

The map α is the solution of the differential equation:

$$\frac{d\alpha}{ds}(G) = (x\alpha * \delta_{\text{bridge}} + y\delta_{\text{loop}} * \alpha + [\delta_{\text{bridge}}, \alpha]_* - [\delta_{\text{loop}}, \alpha]_*)(G),$$

where $[f, g]_* := f * g - g * f$.

Differential equation of the same type as
the Renormalisation Group equation in QFT

We take a four-variable graph polynomial $Q(G; x, y, a, b)$ which has the following properties:

- a multiplicative law on disjoint union and one-point joints
- if e is a bridge, then

$$Q(G; x, y, a, b) = xQ(G - e; x, y, a, b), \quad (1)$$

- if e is a loop, then

$$Q(G; x, y, a, b) = yQ(G/e; x, y, a, b), \quad (2)$$

- if e is neither a loop nor a bridge, then

$$Q(G; x, y, a, b) = aQ_{G-e; x, y, a, b} + bQ(G/e; x, y, a, b). \quad (3)$$

a Tutte-Grothendieck invariant

$$\beta(G; x, y, a, b, s) := s^{|E|} Q(G; x, y, a, b).$$

From an algebraic point of view, β is a Hopf algebra character

Proposition

The map β satisfies the following differential equation:

$$\frac{d\beta}{ds}(G) = (x\beta * \delta_{\text{bridge}} + y\delta_{\text{loop}} * \beta + b[\delta_{\text{bridge}}, \beta]_* - a[\delta_{\text{loop}}, \beta]_*)(G).$$

Universality property - recipe theorem

Theorem

If one has a four-variable graph polynomial $Q(G; x, y, a, b)$ satisfying a multiplicative law on disjoint reunions and one-point joints, and conditions (1) - (3), then one has:

$$Q(G; x, y, a, b) = a^{n(G)} b^{r(G)} T(G; \frac{x}{b}, \frac{y}{a}).$$

Proof: differential equation change of variable

Any Tutte-Grothendieck invariant must be some evaluation of the Tutte polynomial

↔ Universality proof using differential equations
(the usual proofs use involved edge induction arguments)



A. T., invited contribution *CRC Handbook "The Tutte polynomial"*
(Editors: J. Ellis-Monaghan & I. Moffatt)

Other results & perspectives for future work

- relation between the Bollobás-Riordan polynomial (or the topological Tutte polynomial) and the Kirchoff-Symanzik polynomial of non-commutative QFT

↪ non-trivial dependence on the **genus**



T. Krajewski, V. Rivasseau, A. T., Z. Wang, *J. Noncomm. Geom.* (2010)

- proof of the universality of the Tutte polynomial for matroids



G. Duchamp, N. Hoang-Nghia, T. Krajewski, A. T., *Adv. Appl. Math.* (2014)

- proof of the universality of the Bollobás-Riordan polynomial



T. Krajewski, I. Moffatt, A. T., submitted (2015)

- relation with polynomials of graphs on pseudo-surfaces



T. Krajewski, I. Moffatt, A. T., work in progress

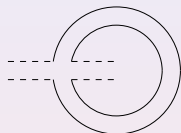
- same approach for Sym, QSym, FQSym *etc.*;
- unification under a common framework
- unification of the Bollobás-Riordan definitions; topological generalization of the graph (and matroid) Courtiel approach



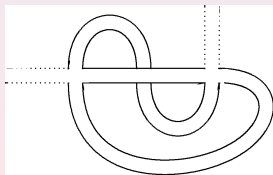
J. Courtiel, thèse LABRI (oct. 2014), J. Courtiel, arXiv:1412.2081[math.CO]

Je vous remercie pour votre
attention !

Generalization: ribbon graphs



$$bc = 2$$



$$bc = 1$$

bc - number of connected components of the graph's boundary
(if the graph is connected, bc - the number of faces)

Bollobás-Riordan polynomial R_G

(B. Bollobás and O. Riordan, *Proc. London Math. Soc.*, **83** 2001, *Math. Ann.*, **323** (2002)

J. Ellis-Monaghan and C. Merino, arXiv:0803.3079[math.CO], 0806.4699[math.CO])

↪ generalization of the Tutte polynomial for ribbon graphs

$$R_G(x, y, z) = \sum_{H \subset G} (x-1)^{r(G)-r(H)} y^{n(H)} z^{k(H)-bc(H)+n(H)}.$$

the additional variable z keeps track of the additional topological information (bc or the graph genus g)

↪ some generalizations:

(S. Chumotov, *J. Combinatorial Theory* **99** (2009), F. Vignes-Tourneret, *Discrete Mathematics* **309** (2009)

Deletion/contraction for the Bollobás-Riordan polynomial

$R_G(x, y, z) = R_{G/e}(x, y, z) + R_{G-e}(x, y, z)$, e semi-regular edge
terminal forms (graphs \mathcal{R} with 1 vertex):

$$k(\mathcal{R}) = V(\mathcal{R}) = k(H) = V(H) = 1 \rightarrow R(x, y, z) = R(y, z)$$

$$R_{\mathcal{R}}(y, z) = \sum_{H \subset \mathcal{R}} y^{E(H)} z^{2g(H)}.$$

Multivariate Bollobás-Riordan polynomial

generalization of the Bollobás-Riordan polynomial analogous to the generalization of the Tutte polynomial

$$Z_G(x, \{\beta_e\}, z) = \sum_{H \subset G} x^{k(H)} \left(\prod_{e \in H} \beta_e \right) z^{bc(H)}.$$

↔ satisfies the deletion/contraction relation

Noncommutative quantum field theory (NCQFT)

NCQFTs - ribbon graphs



Parametric representation for a noncommutative Φ^4 model

$$\mathcal{A}_G^*(p) = \int_0^\infty \frac{e^{-V^*(p,\alpha)/U^*(\alpha)}}{U^*(\alpha)^{\frac{D}{2}}} \prod_{\ell=1}^L (e^{-m^2 \alpha_\ell} d\alpha_\ell)$$

Theorem:

$$U^* = \left(\frac{\theta}{2}\right)^{bc-1+2g} \sum_{\mathcal{T}^*} \prod_{\ell \notin \mathcal{T}^*} 2 \frac{\alpha_\ell}{\theta}$$

θ - noncommutativity parameter

\mathcal{T}^* - \star -trees (non-trivial generalization of the notion of trees)

Relation to the multivariate Bollobás-Riordan polynomial

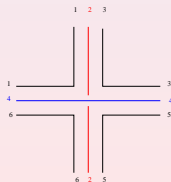
$$U_G^*(\{\alpha_e\}) = \alpha_e U_{G-e}^* + U_{G/e}^*.$$

for the sake of completeness ...

$$U_G^*(\alpha, \theta) = (\theta/2)^{E-V+1} \left(\prod_{e \in E} \alpha_e \right) \times \lim_{w \rightarrow 0} w^{-1} Z_G \left(\frac{\theta}{2\alpha_e}, 1, w \right).$$

relation between combinatorics and QFTs

- other type of topological polynomials related to other QFT models (T. Krajewski *et. al.*, arXiv:0912.5438) - no deletion/contraction property
- generalization to tensor models (appearing in recent approaches for a theory of quantum gravity)



(R. Gurău, *Annales H. Poincaré* **11** (2010), J. Ben Geloun *et. al.*, *Class. Quant. Grav.* **27** (2010))