

FONCTIONS PARKING ET POLYOMINOS

COLLABORATEURS

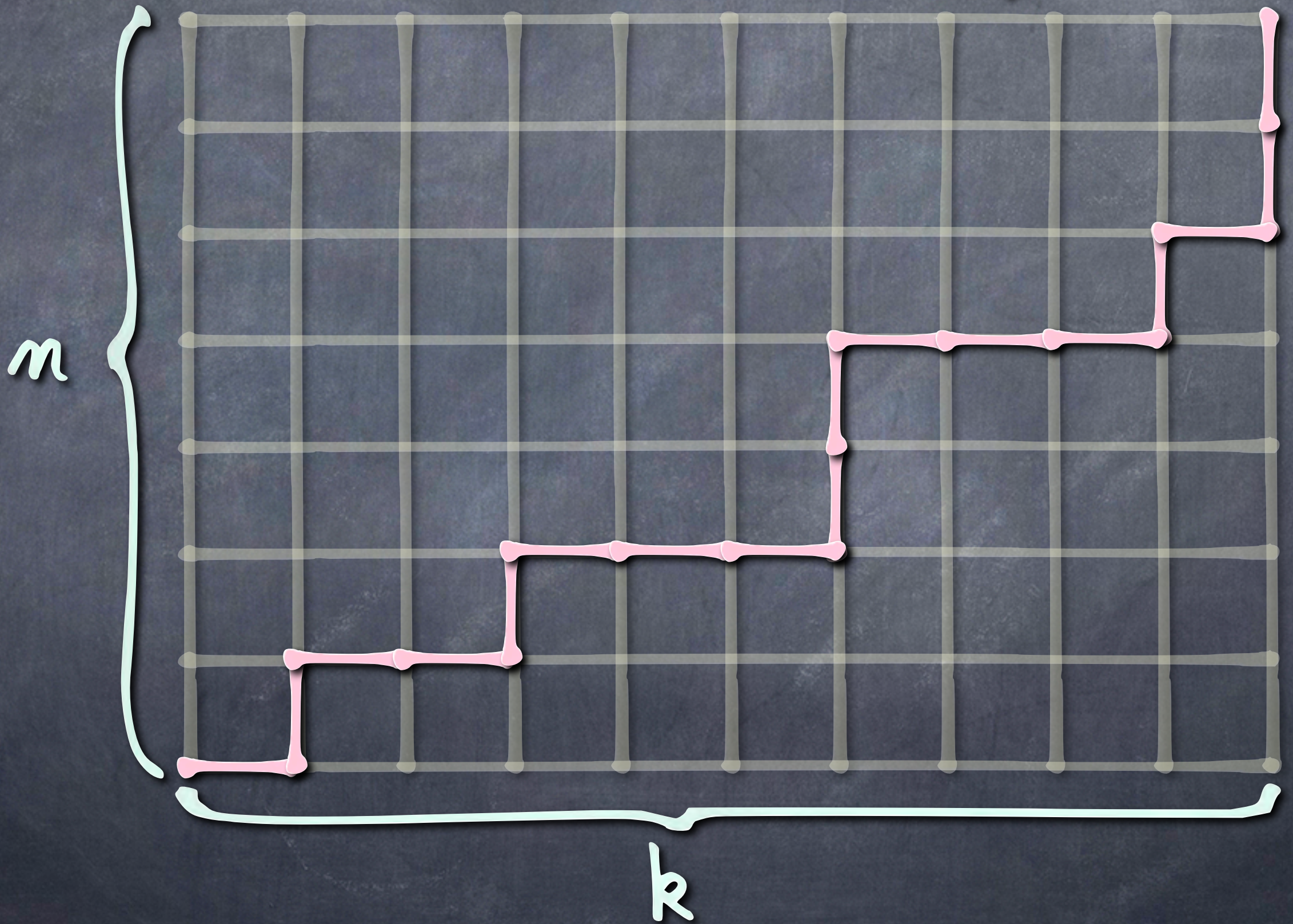


ADRIANO



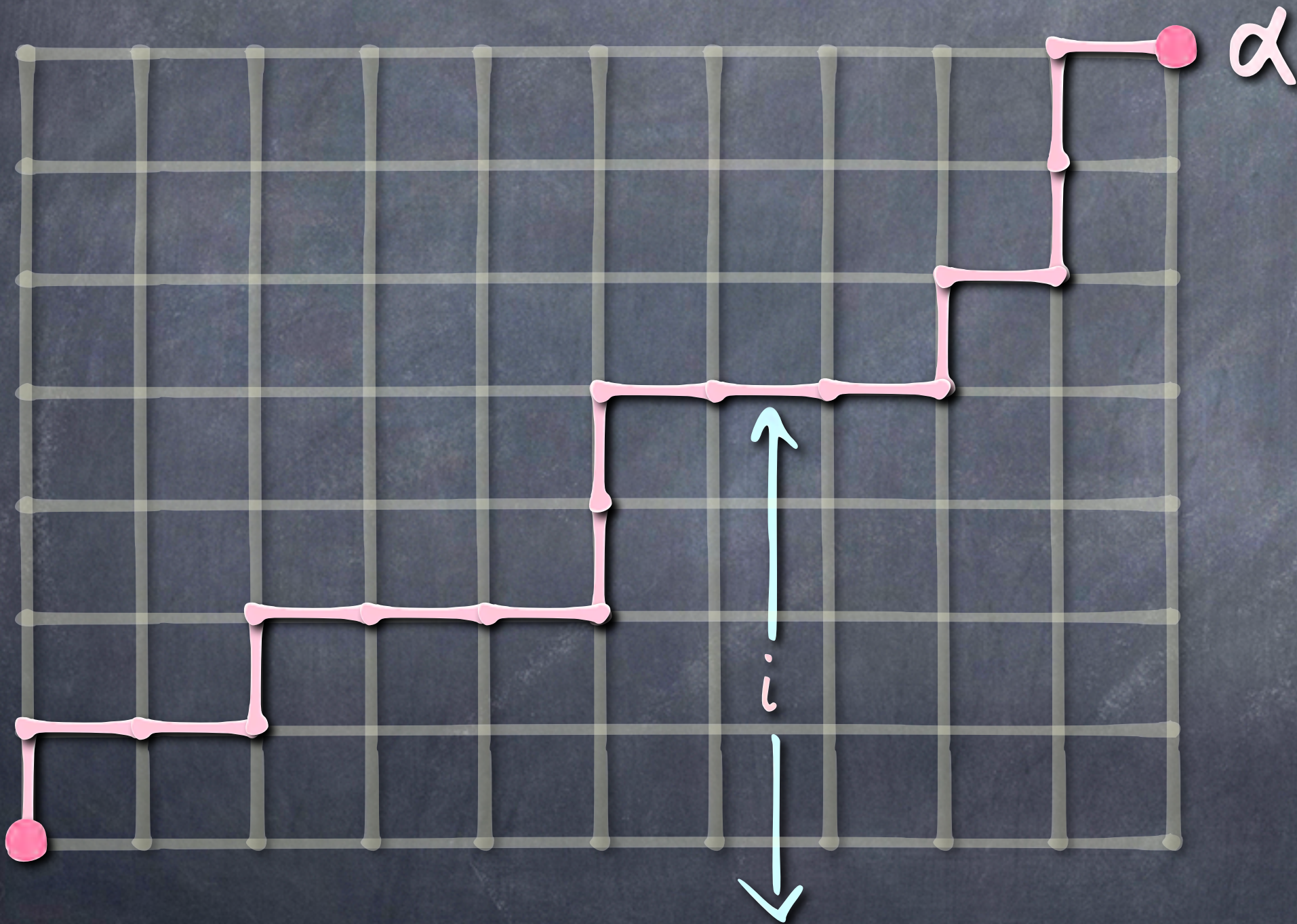
JEAN-CHRISTOPHE

$$\# \mathcal{P}_{k,m} = \binom{m+k}{k}$$



CHEMINS ET MONÔMES

$$x_\alpha := \prod_i x_i$$



$$x_2^2 x_3^3 x_5^3 x_6 x_8$$

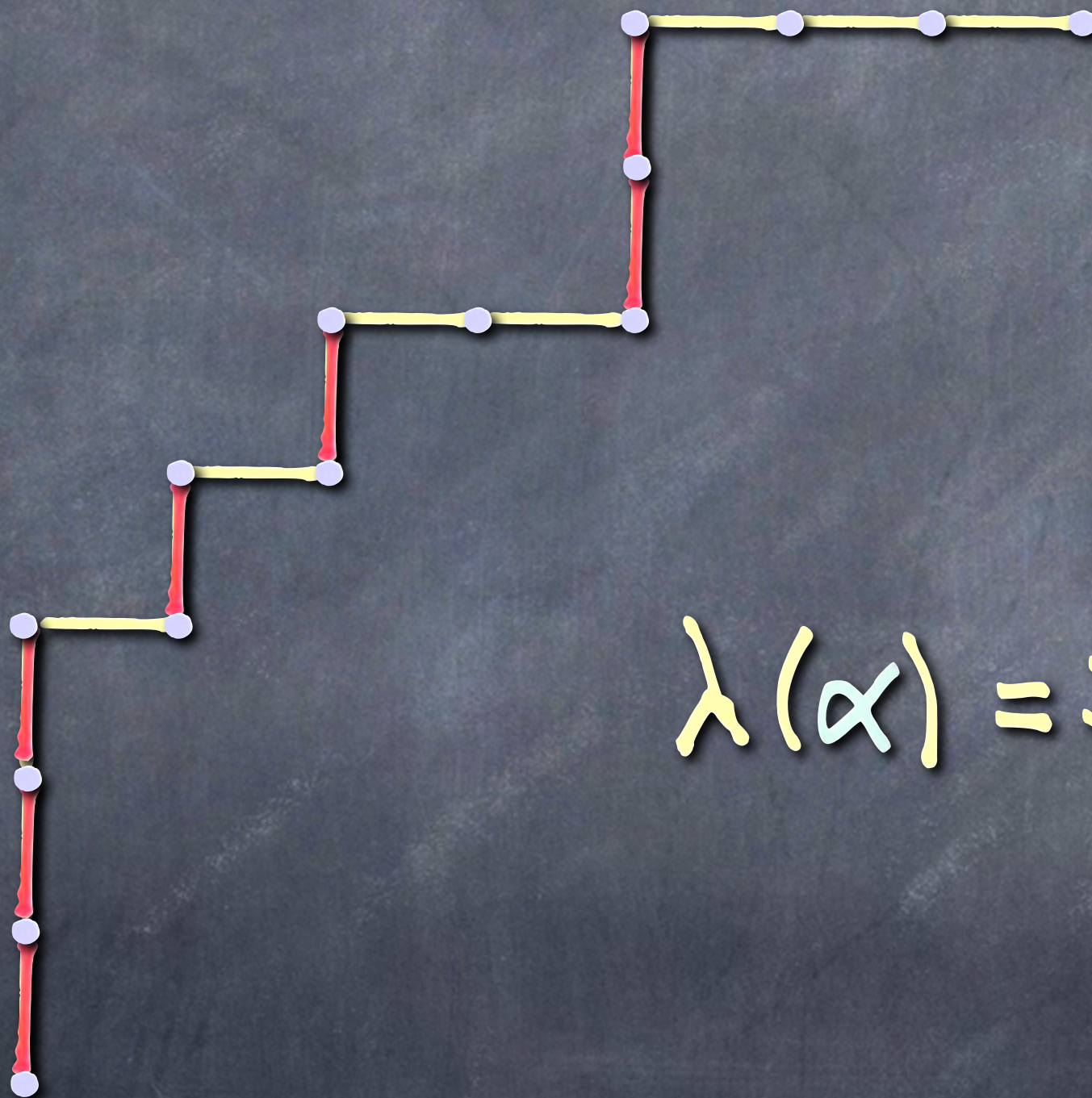
FONCTIONS
HOMOGÈNES
COMPLÈTES

FONCTION GÉNÉRATRICE
DES CHEMINS

$$h_k(x_1, x_2, \dots, x_{m+1}) = \sum_{\alpha \in \mathcal{L}_{k,m}} x_\alpha$$

$$h_k(\underbrace{1, 1, \dots, 1}_{m+1}) = \binom{m+k}{k}$$

FORME D'UN CHEMIN α



$$\lambda(\alpha) = 3211$$

$P_{k,m}$: POLYOMINOS
PARALLÉLOGRAMMES

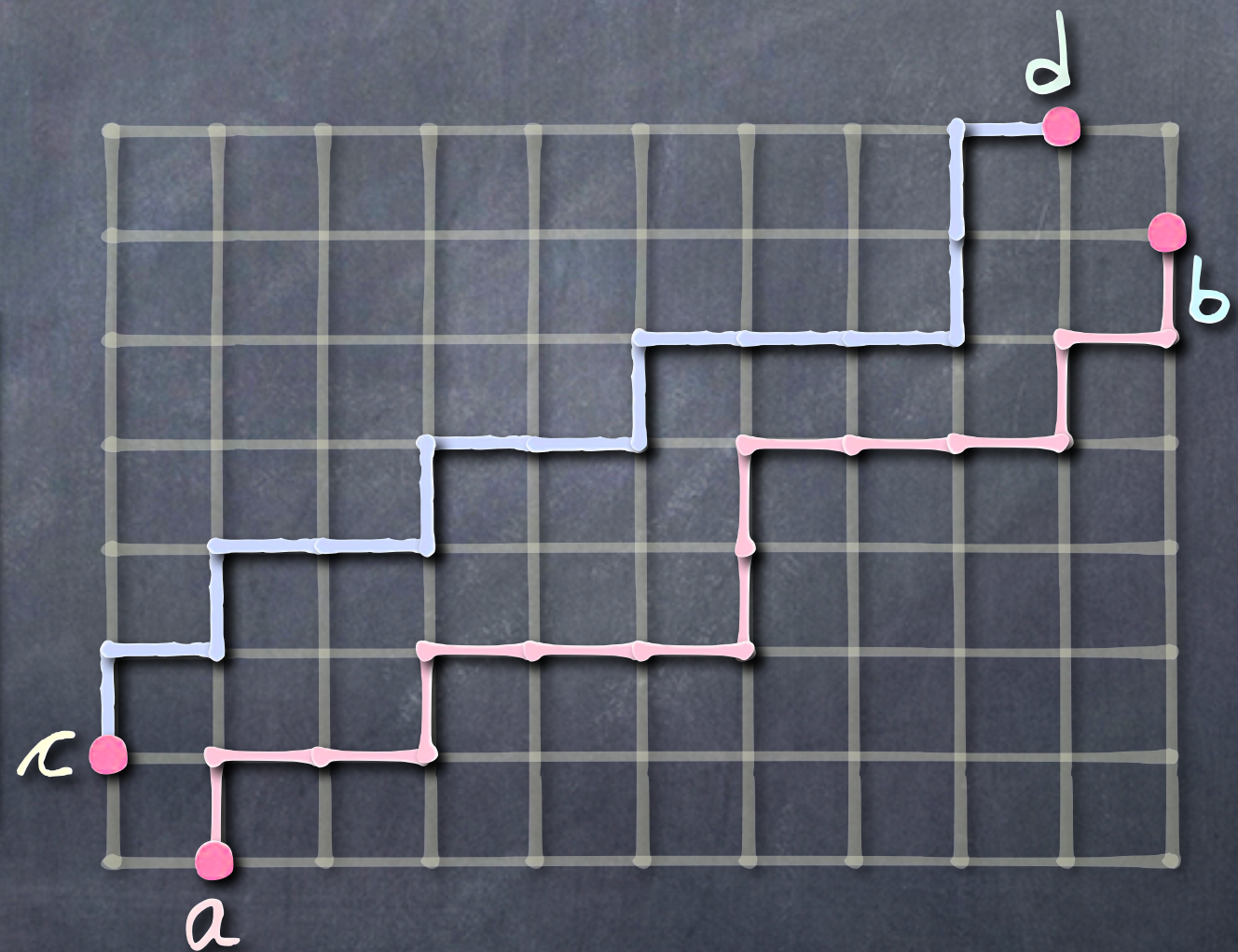




LINDSTRÖM - GESSEL - VIENNOT

$$\det \begin{pmatrix} \binom{m+k}{k} & \binom{m+k}{k+1} \\ \binom{m+k}{k-1} & \binom{m+k}{k} \end{pmatrix}$$

POLYOMINOS PARALLELOGRAMMES

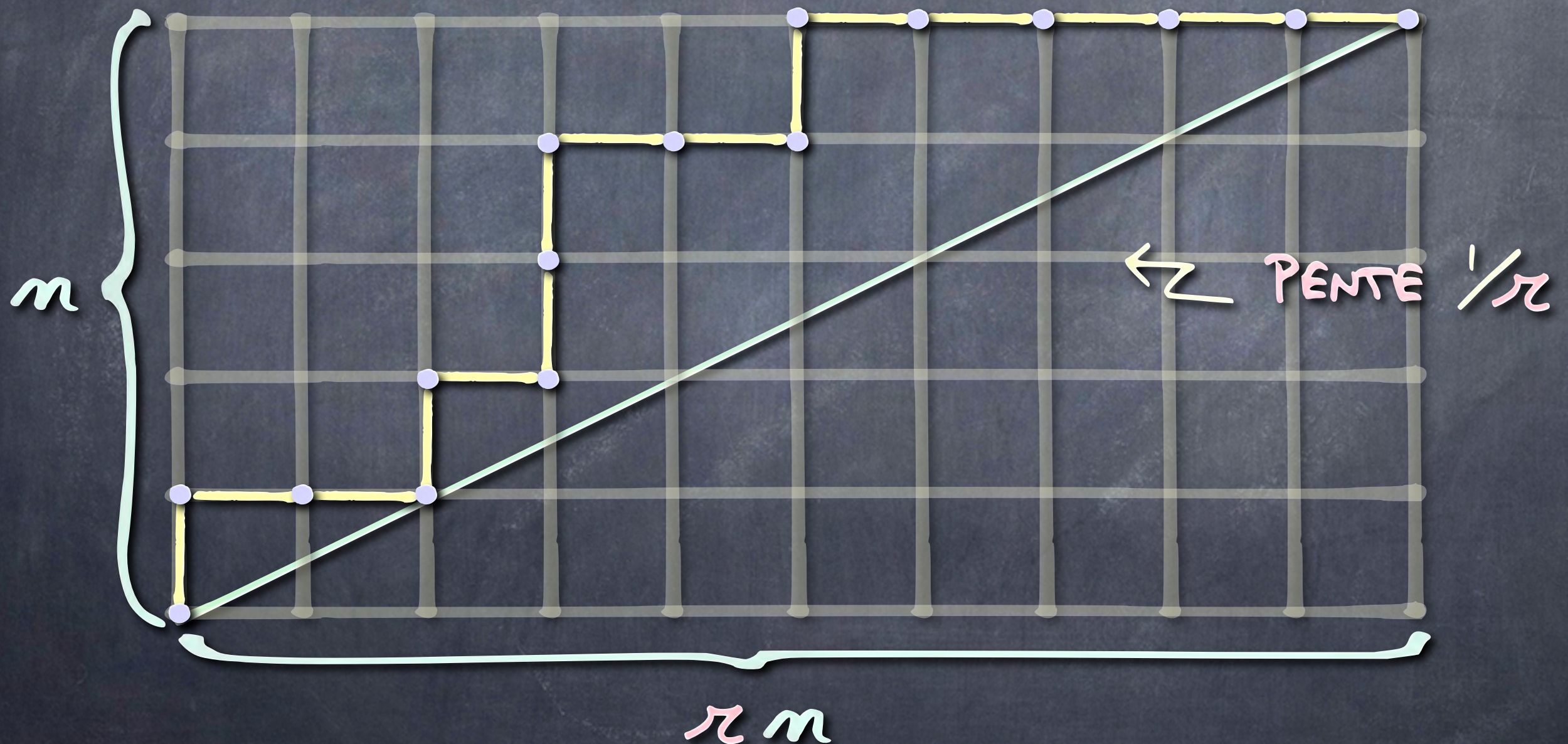


POLYOMINOS PARALLÉLOGRAMMES

$$\# \mathcal{P}_{k+1, n+1} = \frac{1}{n+k+2} \binom{n+k}{k} \binom{n+k+2}{k+1}$$

CHEMINS DE π -DYCK

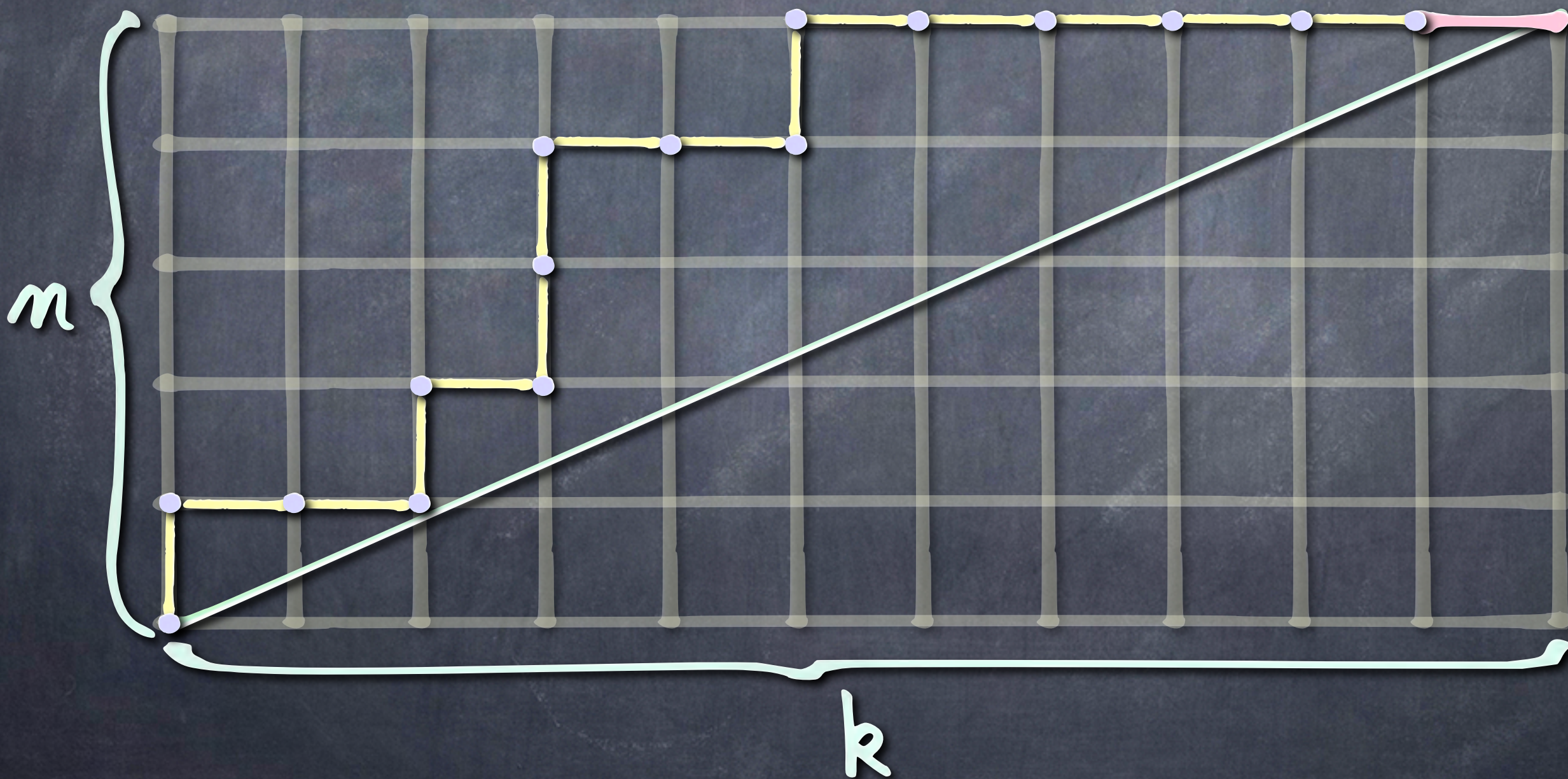
$$\frac{1}{(\pi m + 1)} \binom{(\pi + 1)m}{m}$$



CHEMINS DE k, m -DYCK

k ET m
RELATIVEMENT
PREMIERS

$$\frac{1}{k+m} \binom{k+m}{m}$$



DERIVATION OF A NEW FORMULA FOR THE NUMBER OF
MINIMAL LATTICE PATHS FROM $(0, 0)$ TO (km, kn) HAVING
JUST t CONTACTS WITH THE LINE $my = nx$ AND HAVING
NO POINTS ABOVE THIS LINE; AND A PROOF OF GROSS-
MAN'S FORMULA FOR THE NUMBER OF PATHS WHICH
MAY TOUCH BUT DO NOT RISE ABOVE THIS LINE

By M. T. L. BIZLEY, F.I.A., F.S.S., F.I.S.

WHITWORTH⁽¹⁾ deals in Chapter v of *Choice and Chance* with the problem of finding the number of minimal lattice paths from $(0, 0)$ to (k, k) which do not cross the line $y = x$. By a lattice path is meant a path joining two points with integral coefficients by a line composed of horizontal and vertical steps of unit length. A minimal lattice path from $(0, 0)$ to (x, y) , say, is a lattice path where the total number of steps is $(x + y)$; in other words, all the steps are onwards. In what follows minimal lattice paths only will be considered, and the words 'minimal lattice' will be omitted.

Although Whitworth deals only with the case where the boundary line (i.e. the line which the path must not cross) is $y = x$, the more general case of a boundary $\alpha y = x$ has been solved provided α is a positive integer^{(2), (3)}. The number of paths from $(0, 0)$ to $(\alpha l, l)$ which may touch but never rise above $\alpha y = x$ is $\frac{1}{l\alpha + 1} \binom{l\alpha + l}{l}$.

Grossman⁽⁴⁾ announced without proof in 1950 a formula for the number of paths from $(0, 0)$ to (km, kn) which may touch but never rise above the line $my = nx$, where k is a positive integer and m and n are coprime positive integers; thus (km, kn) is any point having positive integral coefficients. Grossman's formula is

$$\sum F_1^{k_1} F_2^{k_2} \dots / k_1! k_2! \dots,$$

where

$$F_j = \frac{1}{j(m+n)} \binom{jm+jn}{jm},$$

J. INST. ACTUARIES, 77 (1952)

FORMULE DE GROSSMAN

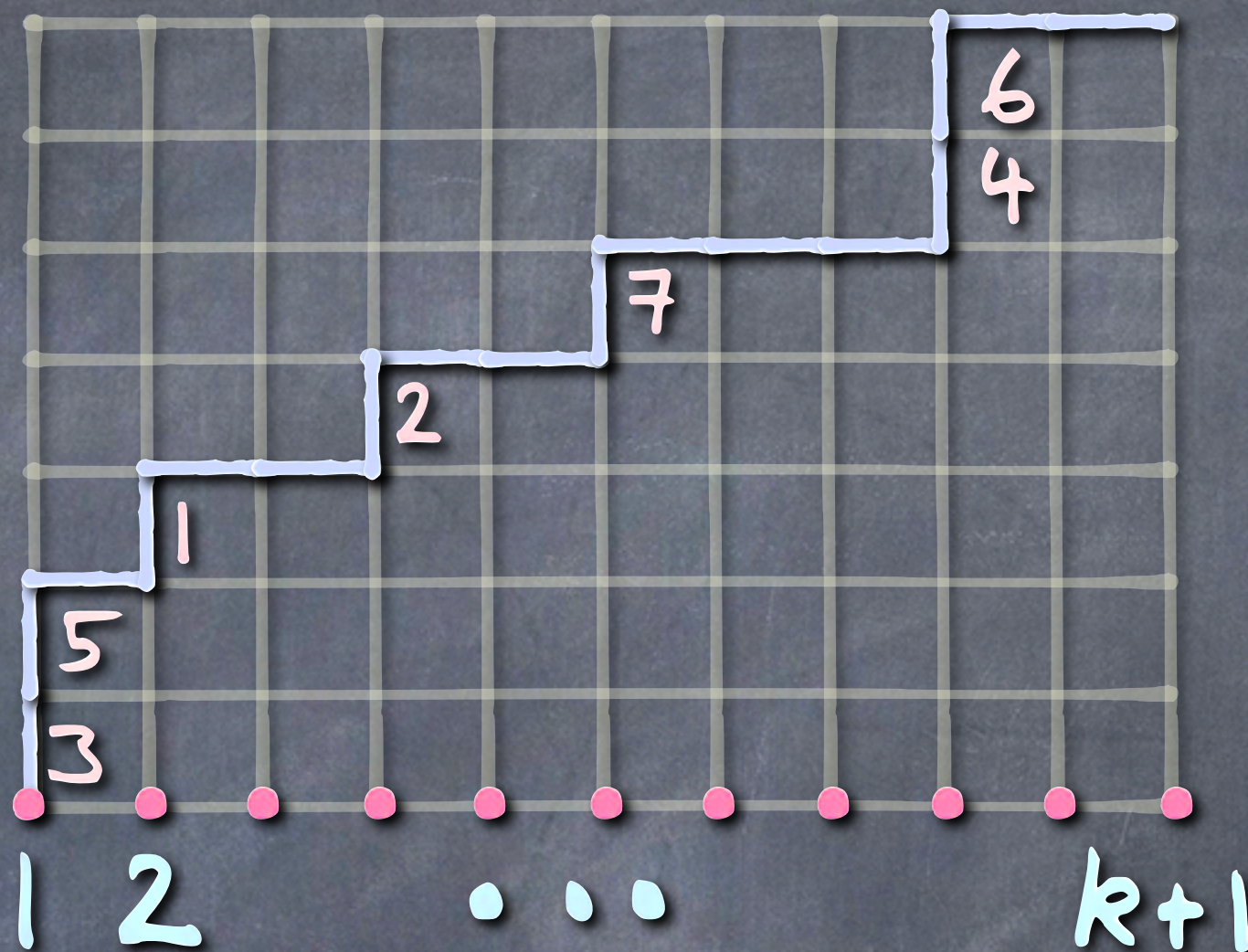
k ET m
RELATIVEMENT
PREMIERS

$$\#DYCK_{dk, dm} = h_d \mid p_j \leftarrow \frac{1}{k+m} \binom{j(k+m)}{jm}$$

$$\#DYCK_{2k, 2m} = \frac{1}{2} \left(\frac{1}{k+m} \binom{k+m}{m} \right)^2 + \frac{1}{2(k+m)} \binom{2k+2m}{2m}$$

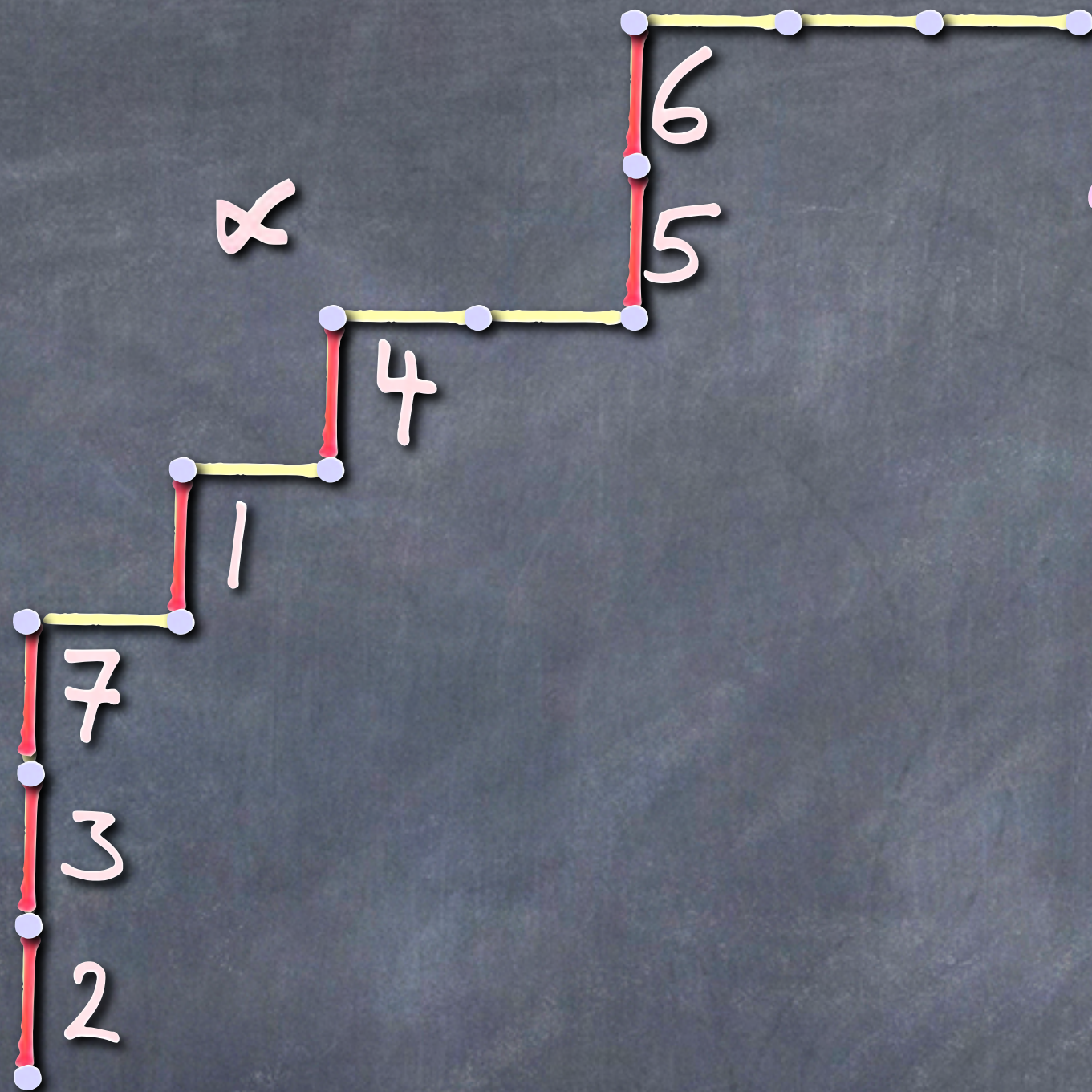
ÉTIQUETAGES

$$\# L_{k,m} = (k+1)^m$$



$$f: \{1, 2, \dots, m\} \longrightarrow \{1, 2, \dots, k+1\}$$

$f^{-1}(i) =$ ENSEMBLE DES
ÉTIQUETTES SUR
LES PAS NORDS



ACTION
 S_m

$$\binom{m}{\lambda(\alpha)} = \# \text{ D'ÉTIQUETAGES DE } x$$

$$(k+1)^n = \sum_{\alpha \in \mathcal{P}_{k,m}} \binom{n}{\lambda(\alpha)}$$

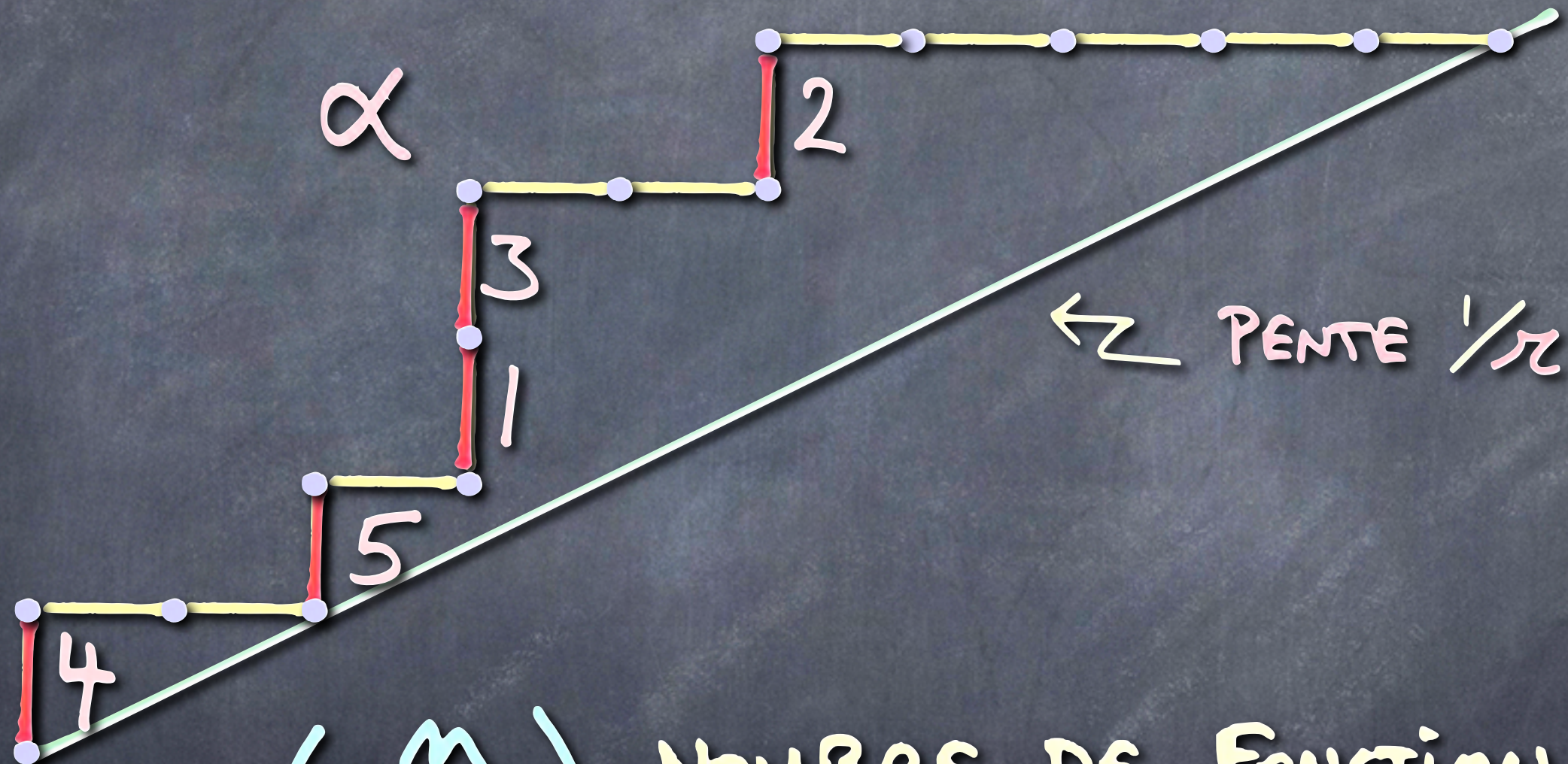
CARACTÉRISTIQUE
DE
FROBENIUS

SÉRIE
DE
POLYA

$$L_{k,m}(z) = \sum_{\alpha \in \mathcal{P}_{k,m}} h_{\lambda(\alpha)}(z)$$

$$= \sum_{\mu \vdash n} S_{\mu}(\underbrace{1, 1, \dots, 1}_{k+1}) S_{\mu}(z)$$

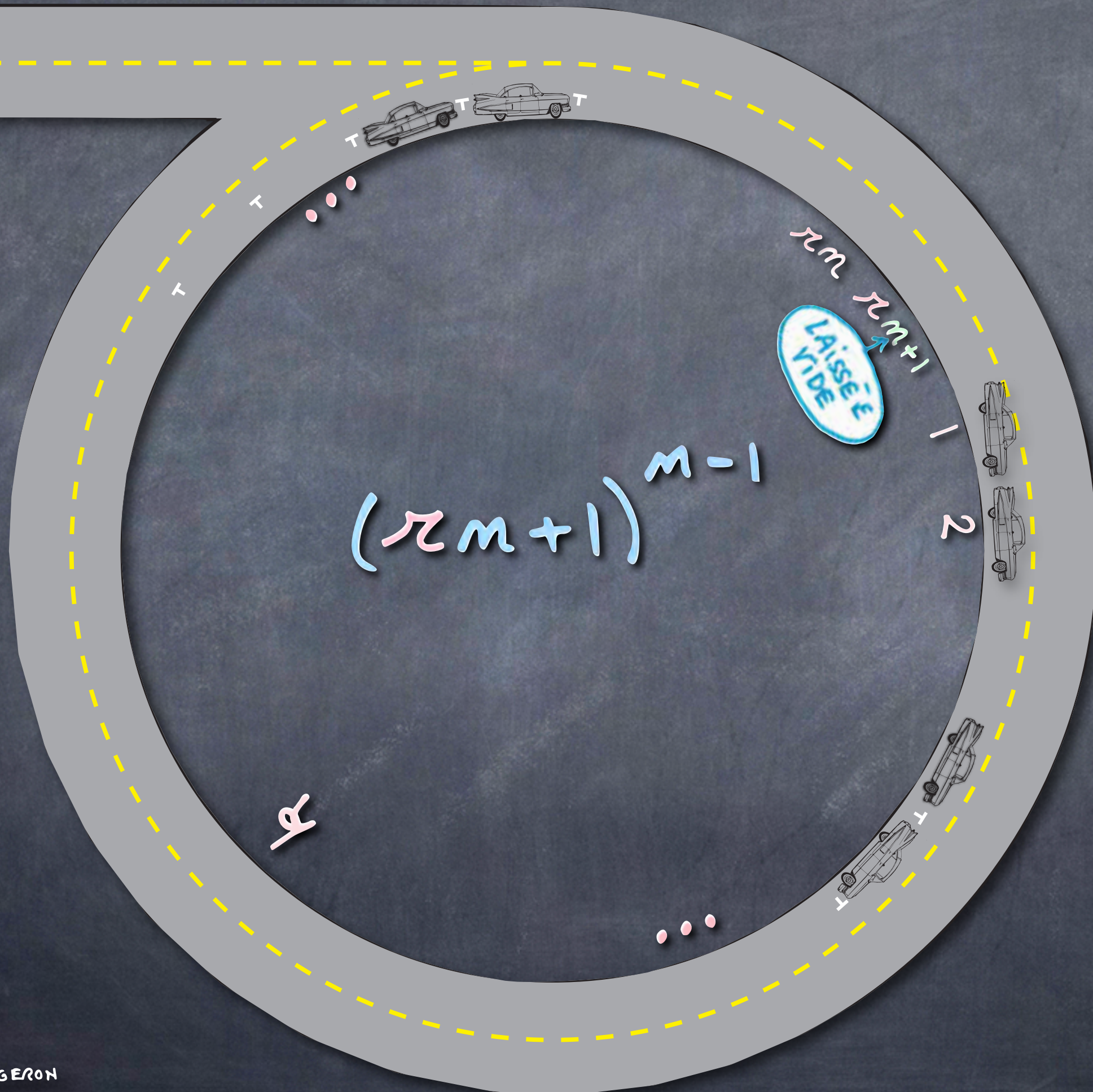
π -functions PARKING



← PENTE $1/\pi$

$\binom{n}{\lambda(\alpha)}$

NOMBRE DE FONCTION
PARKING DE FORME α



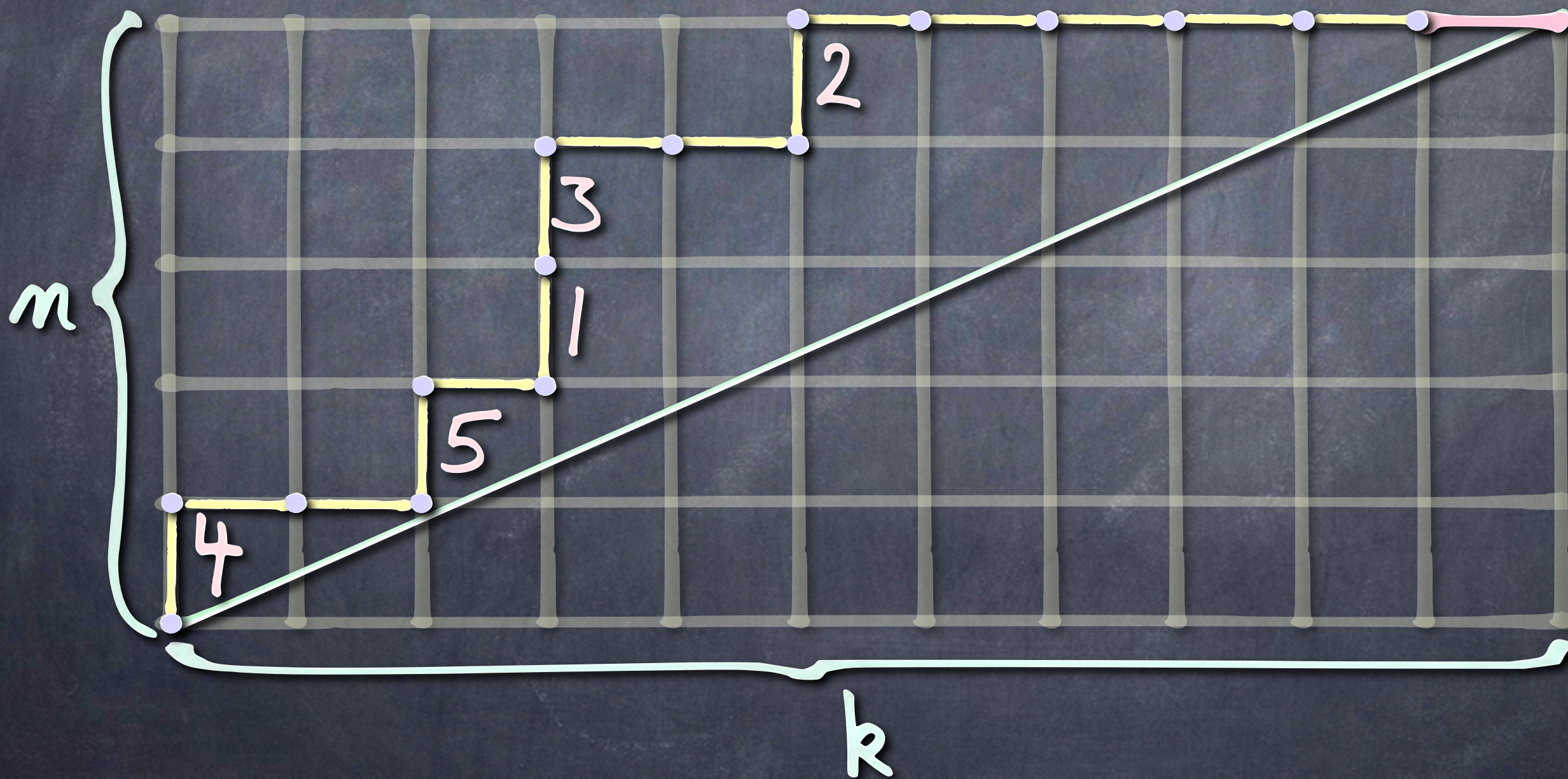
π -functions PARKING

$$(\pi n + 1)^{n-1} = \sum_{\substack{\beta \\ \pi\text{-DYCK}}} \binom{n}{\lambda(\beta)}$$

k ET m
RELATIVEMENT
PREMIERS

k, m -FUNCTIONS PARKING

$$k^{m-1}$$



FORMULE 'A LA GROSSMAN

$$\sum_{d=0}^{\infty} \# \text{PART}_{d_k, d_m} \frac{x^{d_m}}{(d_m)!} = e^{\sum_{j \in I} (j_k)^{j_m-1} \frac{x^{j_m}}{(j_m)!}}$$

FROBENIUS

k, m -FUNCTIONS PARKING

$$\text{PARK}_{k,m}(z) := \sum_{\alpha \in \text{DYCK}_{k,m}} h_{\lambda(\alpha)}$$

k ET m
RELATIVEMENT
PREMIERS

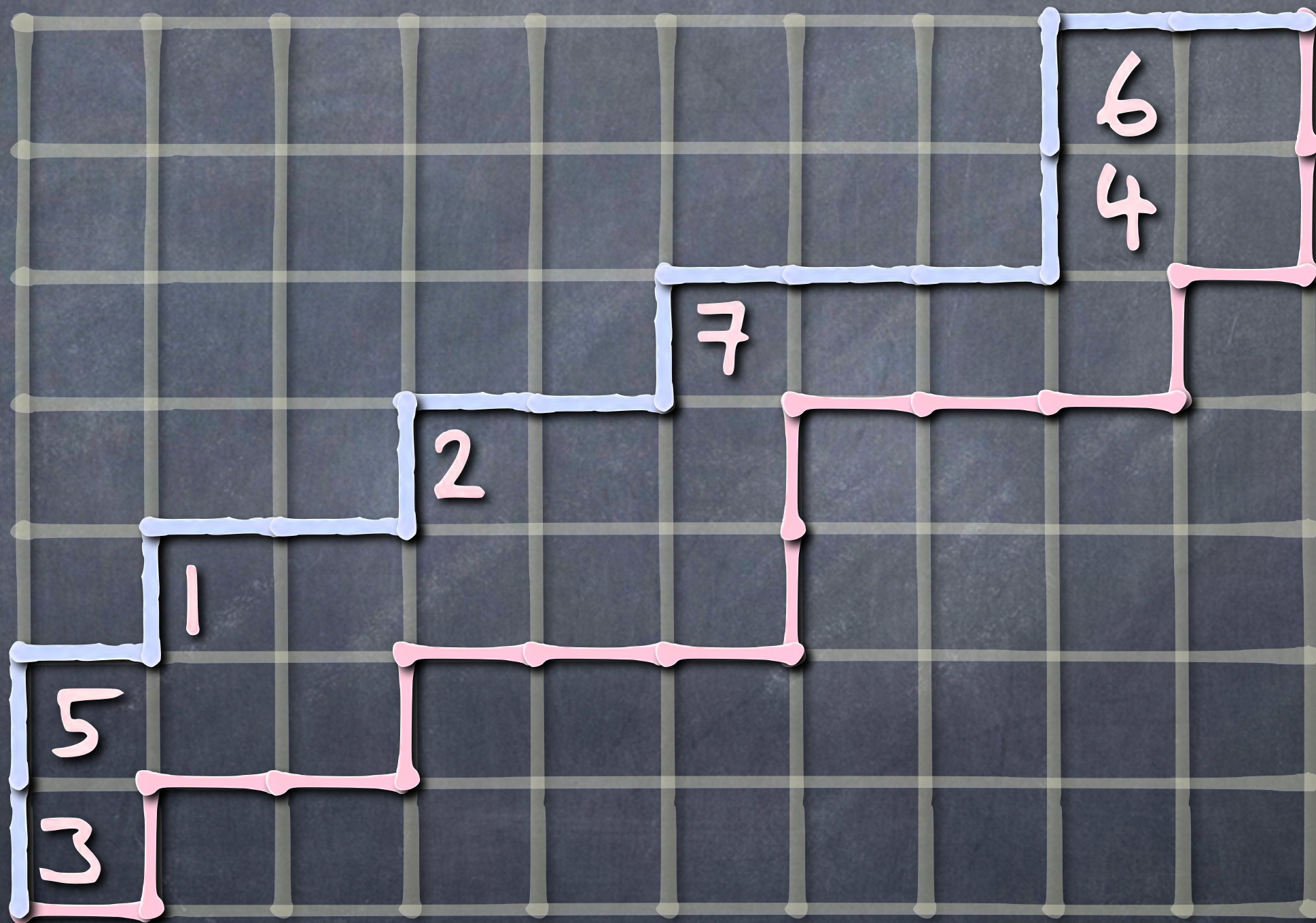
$$= \frac{1}{k} h_m[kz]$$

$$= \sum_{\mu \vdash m} k^{l(\mu)-1} \frac{p_\mu(z)}{z_\mu}$$

$\mathcal{L}_{k,m}$

$$\pi = (\alpha, \beta)$$

$$\lambda(\pi) := \lambda(\beta)$$



FROBENIUS DE $\mathcal{L}_{k,m}$

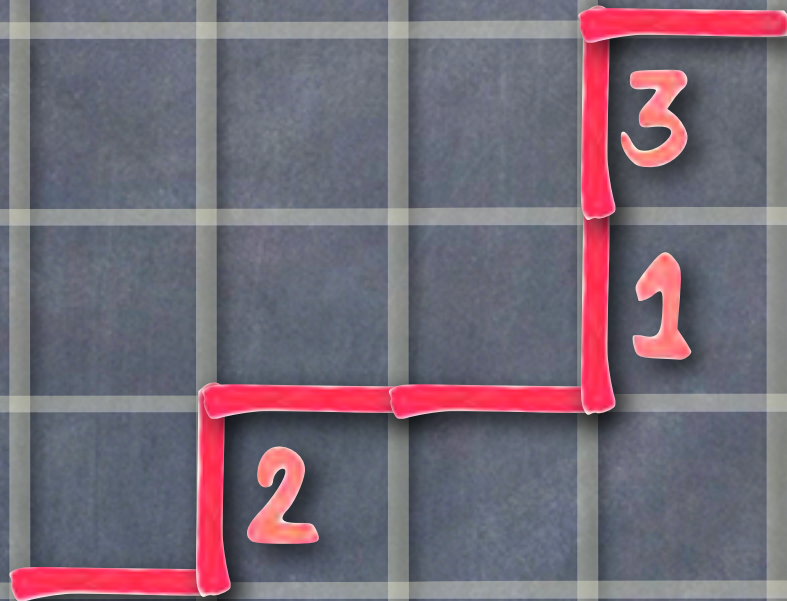
$$\mathcal{L}_{k,m}(z) = \sum_{\pi \in \mathcal{P}_{k,m}} h_{\lambda(\pi)}(z)$$

$$= \frac{1}{k} h_m[kz] \cdot \binom{m+k-2}{k-1}$$

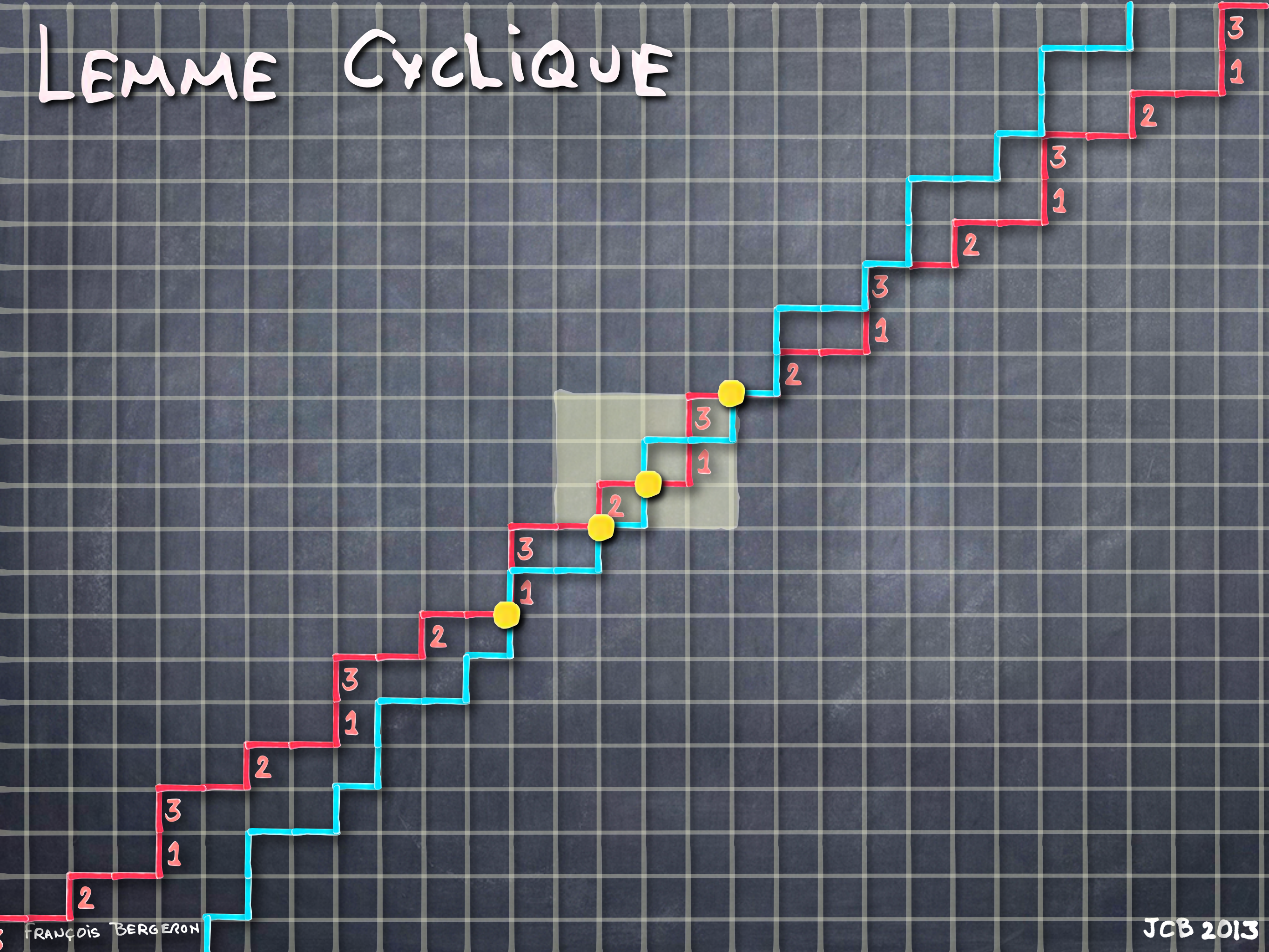
$$\# \mathcal{Z}_{k,m} = k^{m-1} \binom{m+k-2}{k-1}$$

$$= \sum_{\pi \in \mathcal{P}_{k,m}} \binom{m}{\lambda(\pi)}$$

LEMME CYCLIQUE

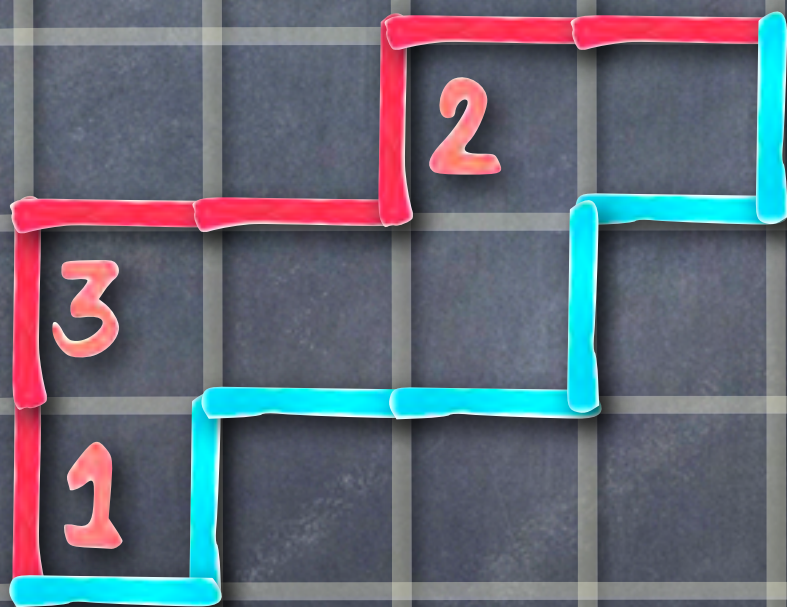


LEMME CYCLIQUE

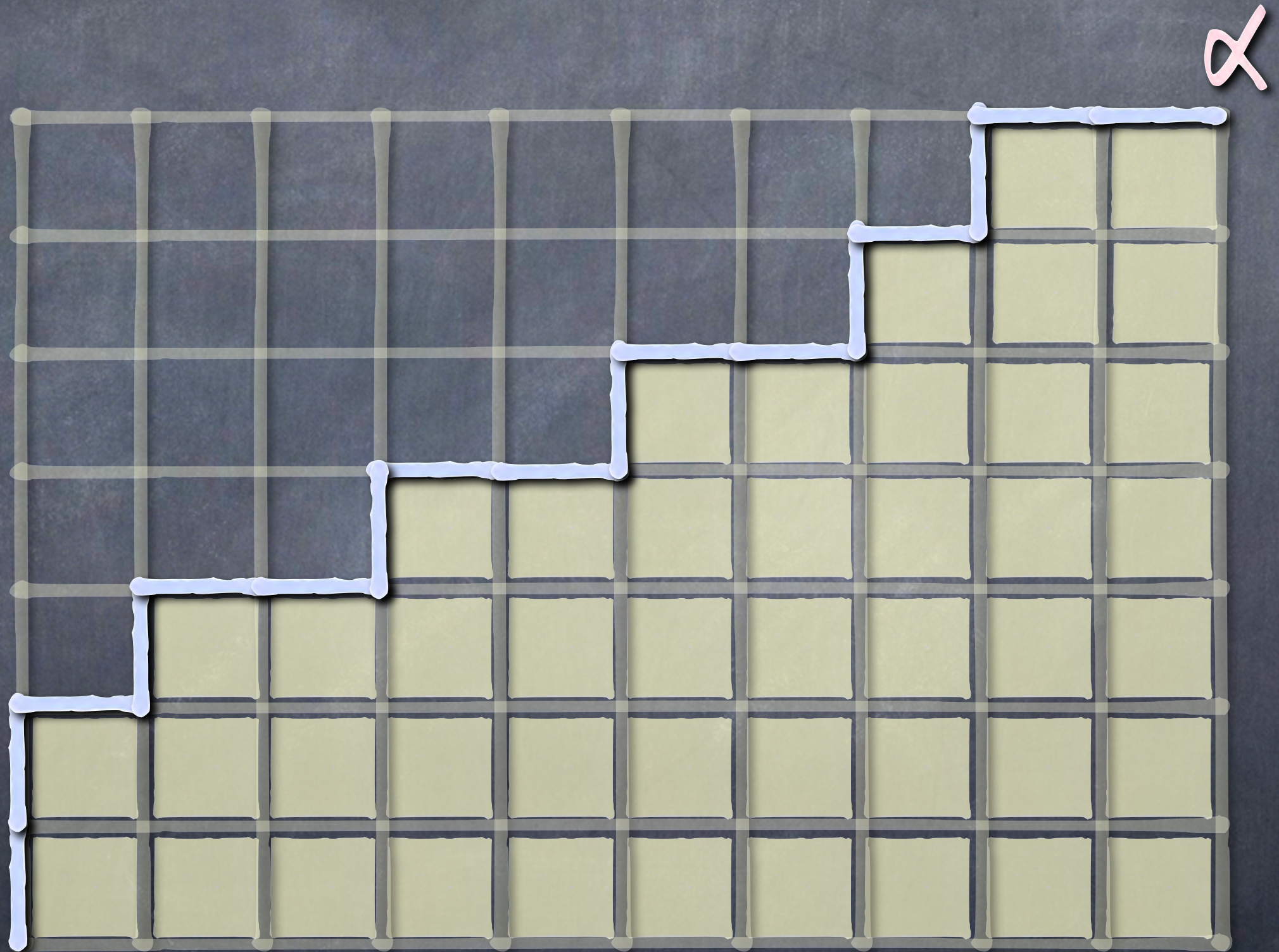


LEMME CYCLIQUE

$$\frac{1}{k} h_m[kz] \cdot \binom{n+k-2}{k-1}$$



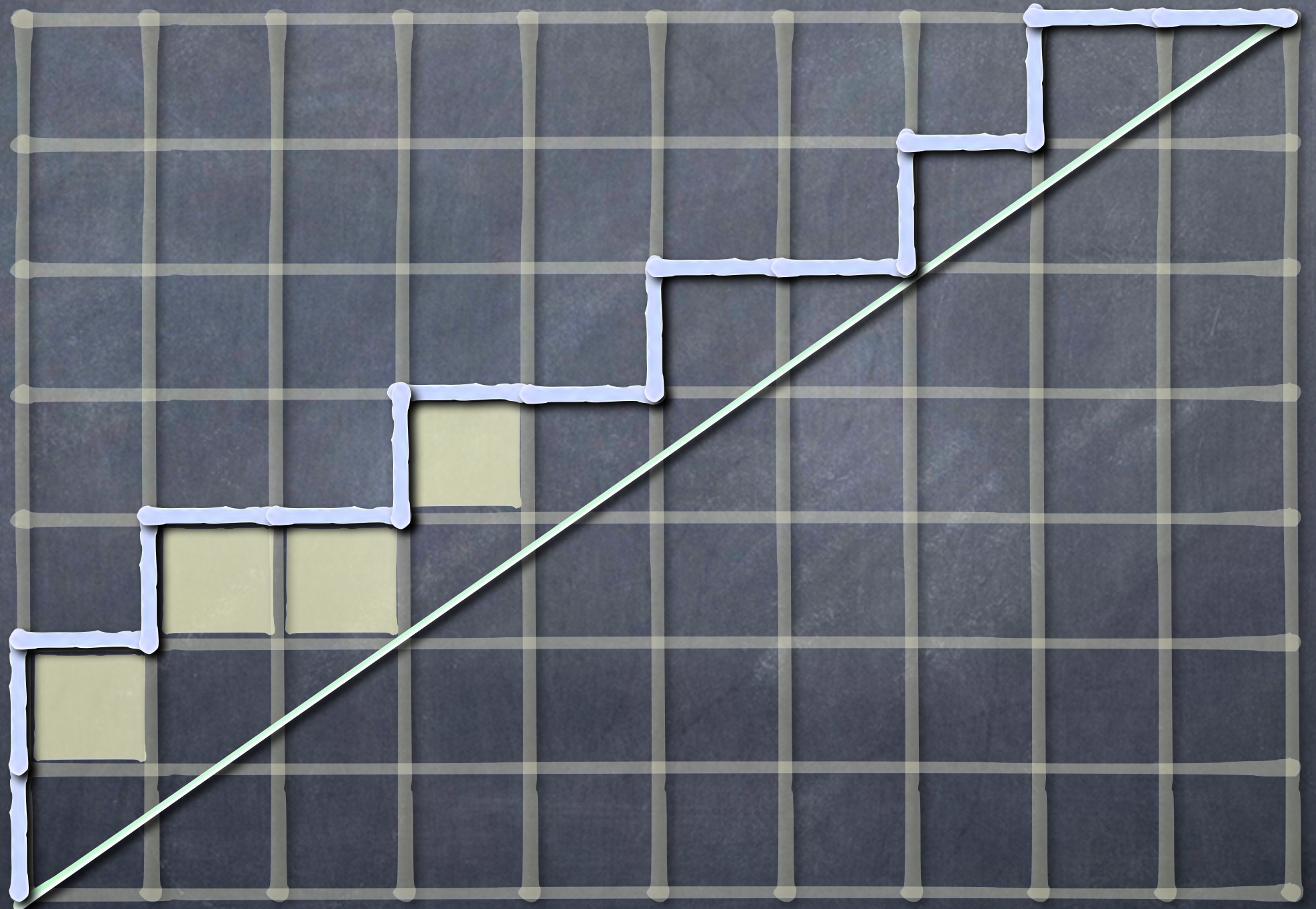
AIRE D'UN CHEMIN $\text{aire}(x)$



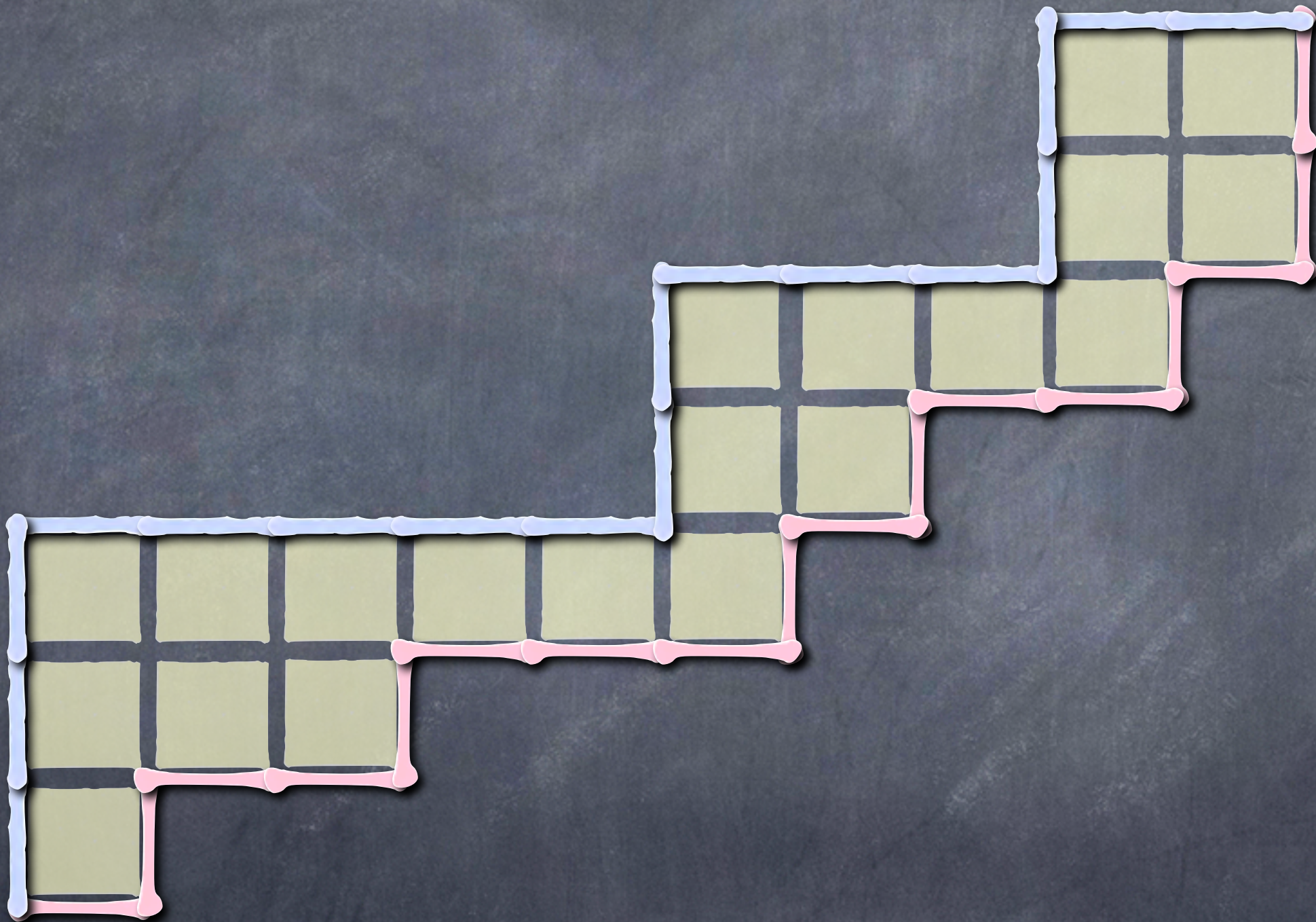
AIRE D'UN CHEMIN DE DYCK

$aire(\alpha)$

α



aire (π)



$$\left[\begin{smallmatrix} n+k \\ k \end{smallmatrix} \right]_q = \sum_{\alpha \in \mathcal{B}_{k,m}} q^{\text{aire}(\alpha)}$$

$$\left[k+1 \right]_q^n = \sum_{\alpha \in \mathcal{L}_{k,m}} q^{\text{aire}(\alpha)}$$

$$\mathcal{L}_{k,m}(\mathbb{Z}; q) = \sum_{\mu \vdash m} S_{\mu}(1, q, \dots, q^k) S_{\mu}(\mathbb{Z})$$

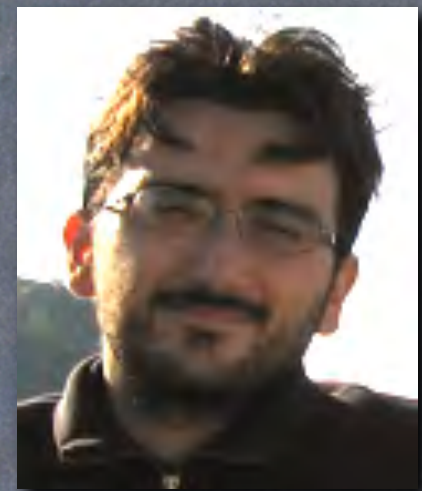
FROBENIUS

$$\mathcal{L}_{k,m}(z; q) = \sum_{\pi \in \mathcal{X}_{k,m}} q^{a(\pi)} h_{\lambda(\pi)}$$

CONJECTURE DE D'ADDERIO

$$\mathcal{L}_{k+1,m}(z; q) = \omega \tilde{\Delta}_{h_k}(e_{\tilde{n}}(z))$$

$$\tilde{\Delta}_f := \Delta_f|_{t=1}$$



$$L_n(z; q) := \sum_{k \geq 1} \mathcal{L}_{k, n}(z; q) x^k$$

$$= w \sum_{n \vdash k} f_n[1-q] \prod_{i=1}^{\ell(n)} \frac{x^{h_{n_i}} [z/(1-q)]}{(1-x) \cdots (1-x q^{n_i-1})}$$

$$L_n(z; q) := \sum_{k \geq 1} \mathcal{L}_{k, n}(z; q) x^k$$

$$L_1(z; q) = \frac{qx}{1-qx} h_1(z)$$

$$L_2(z; q) = \frac{q^2 x}{(1-qx)(1-q^2 x)} \left(h_2(z) + \frac{qx}{1-qx} h_{11}(z) \right)$$

$$L_3(z; q) = \frac{q^3 x}{(1-qx)(1-q^2 x)(1-q^3 x)} \left(h_3(z) + \frac{q(1+q)x}{1-qx} h_{21}(z) + \frac{q^2(1+q)x^2}{(1-qx)^2} h_{111}(z) \right)$$

FROBENIUS

k, m -FUNCTIONS PARKING

$$\text{PARK}_{k, m}(\mathbb{Z}; q) := \sum_{\alpha \in \text{DYCK}_{k, m}} q^{\text{aire}(\alpha)} h_{\lambda(\alpha)}$$

THM

$$\text{PARK}_{k+m, m}(\mathbb{Z}; q) = \tilde{\nabla} w \text{PARK}_{k, m}(\mathbb{Z}; q)$$

$$\tilde{\nabla}_w h_m := \text{PARK}_{m,m}(z)$$

LINEAIRE ET MULTIPLICATIF

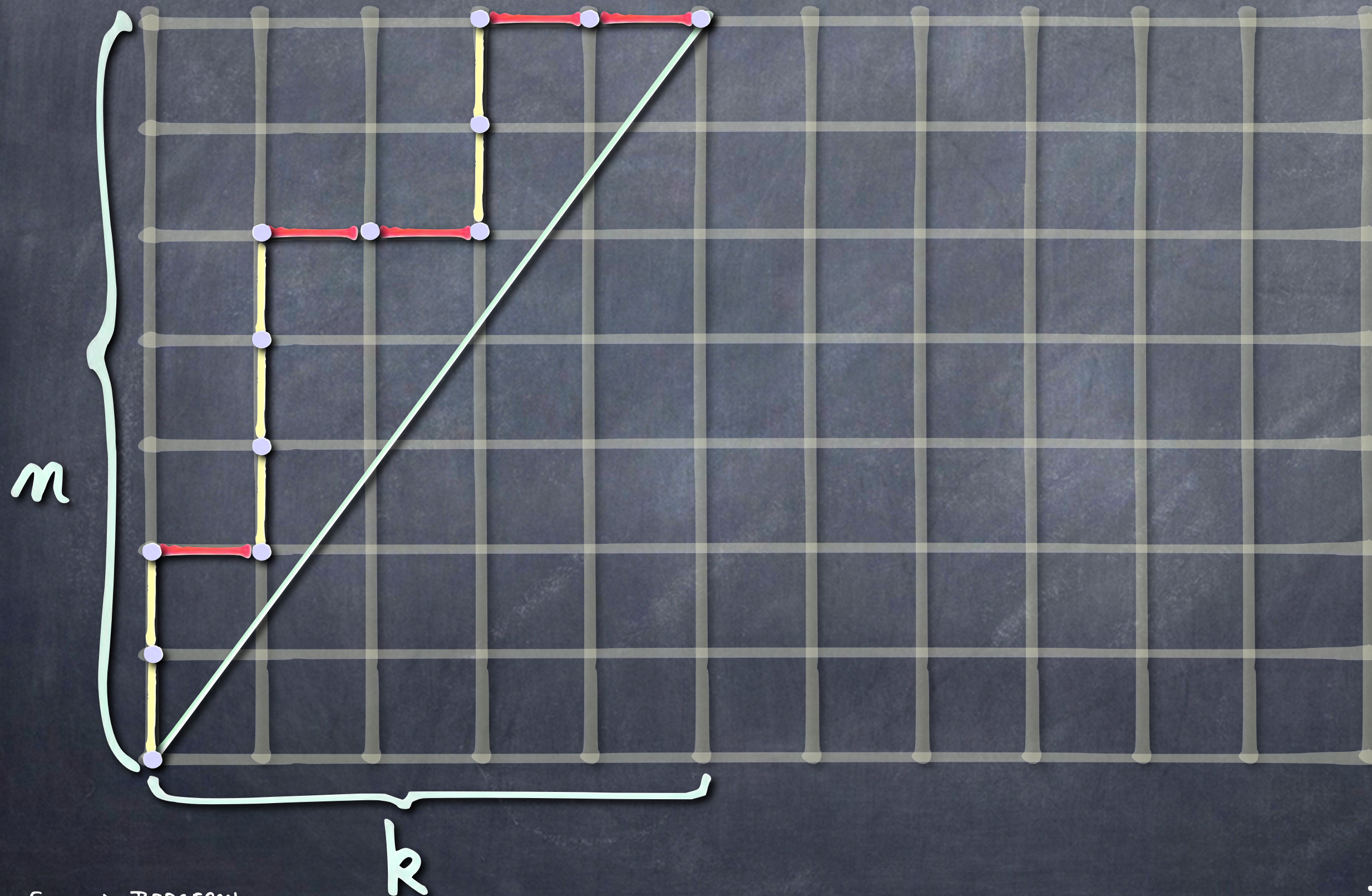
$$h_1 \xrightarrow{\tilde{\nabla}_w} h_1$$

$$h_2 \xrightarrow{\tilde{\nabla}_w} qh_2 + h_1^2$$

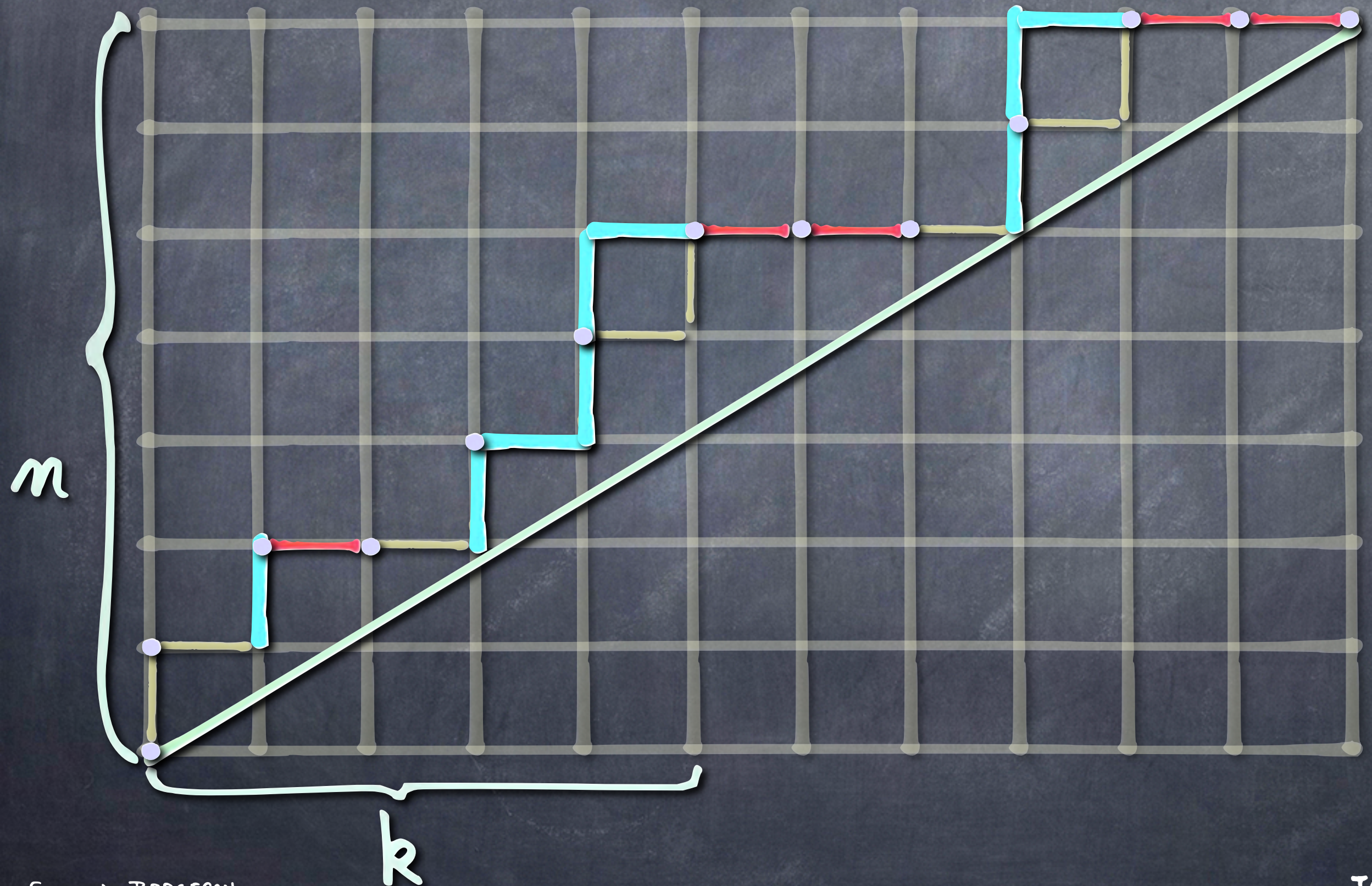
$$h_3 \xrightarrow{\tilde{\nabla}_w} q^3 h_3 + (q^2 + 2q)h_2 h_1 + h_1^3$$

⋮

THM PREUVE COMBINATOIRE



THM PREUVE COMBINATOIRE



TROUVER UNE STATISTIQUE TELLE
QUE

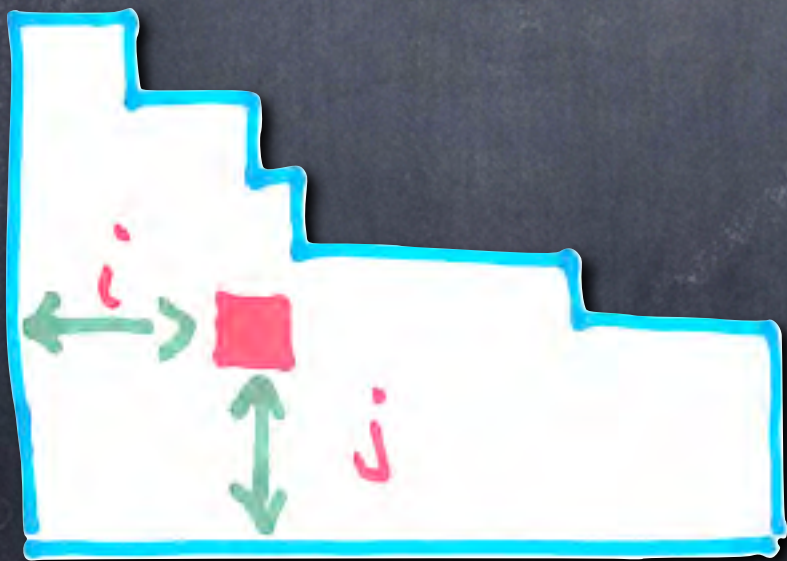
$$\chi_{k+m}(\mathbf{z}; q, t) = w \Delta_{h_k}(e_m(\mathbf{z}))$$

Δ_f OPÉRATEUR SUR LES
FONCTIONS SYMÉTRIQUES

POLYNÔMES DE MACDONALD

$$H_{\mu} = H_{\mu}(z; q, t)$$

$$\Delta_f(H_{\mu}) := f[B_{\mu}] \cdot H_{\mu}$$



$$B_{\mu}(q, t) = \sum_{(i, j) \in \mu} q^i t^j$$

$$H_1 = S_1$$

$$H_2 = S_2 + q S_{11}$$

$$H_{11} = S_2 + t S_{11}$$

$$H_3 = S_3 + (q^2 + q) S_{21} + q^3 S_{111}$$

$$H_{21} = S_3 + (q + t) S_{21} + qt S_{111}$$

$$H_{111} = S_3 + (t^2 + t) S_{21} + t^3 S_{111}$$

Fin