

Probabilistic cellular automata and
coalescing random walks

Mairesse & Chassaing, LIAFA & IECN,
S.P.A. 2011

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here
 $d=1$!

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Note:

	Finite state Markov chains	Particle systems $d=1$	PCA $d=1$
② possible?	YES	NO (Mountford-1995)	?

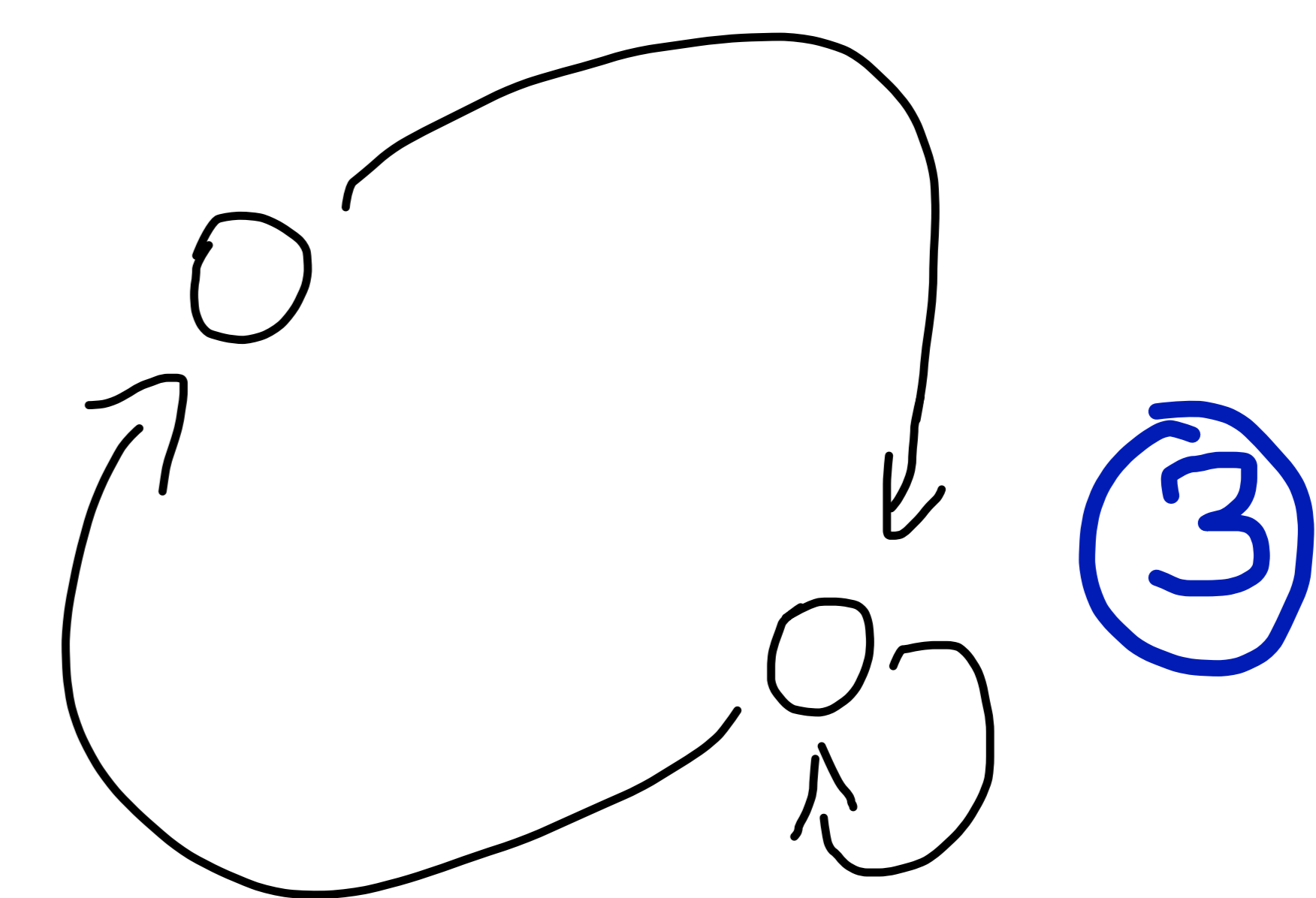
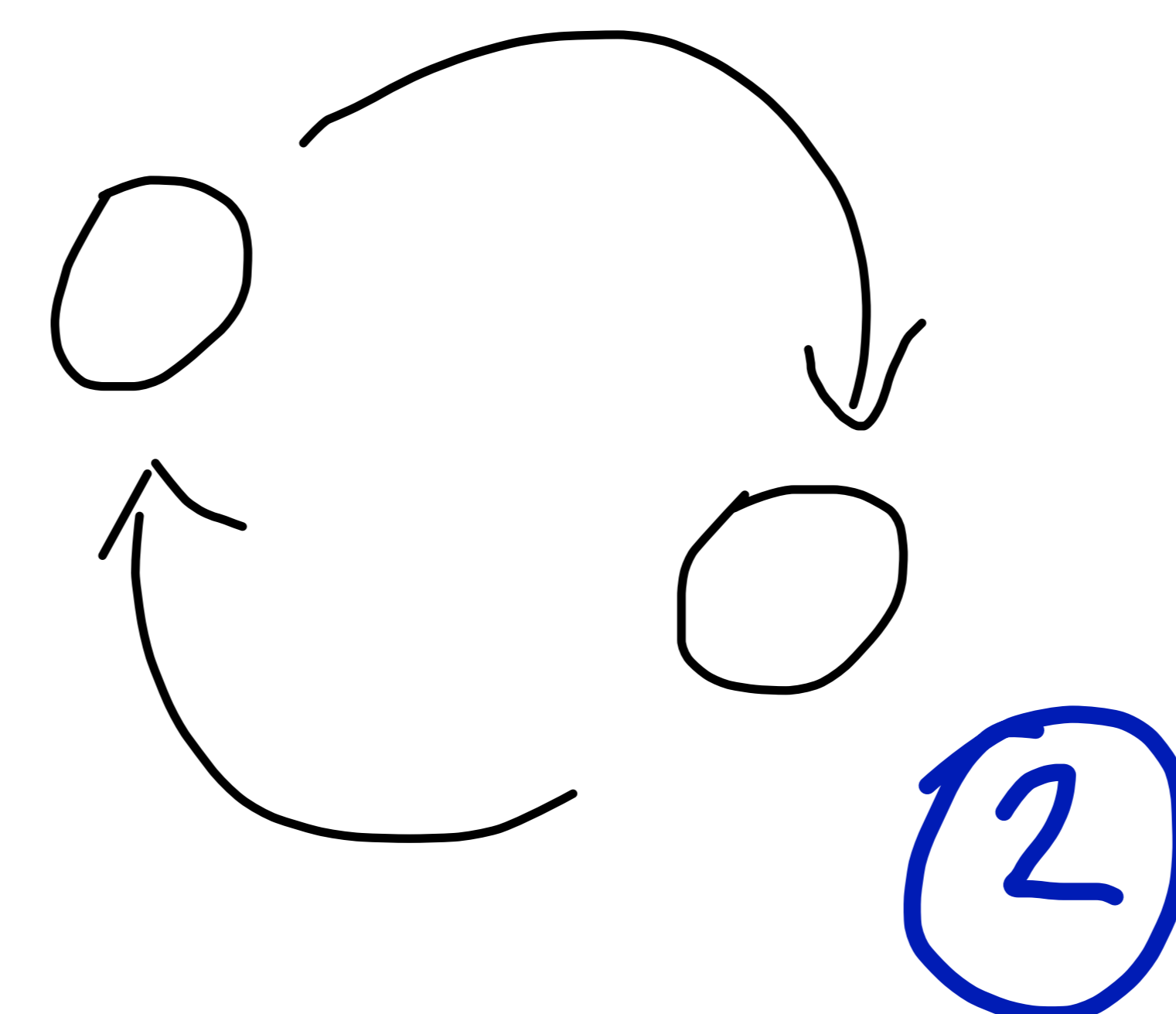
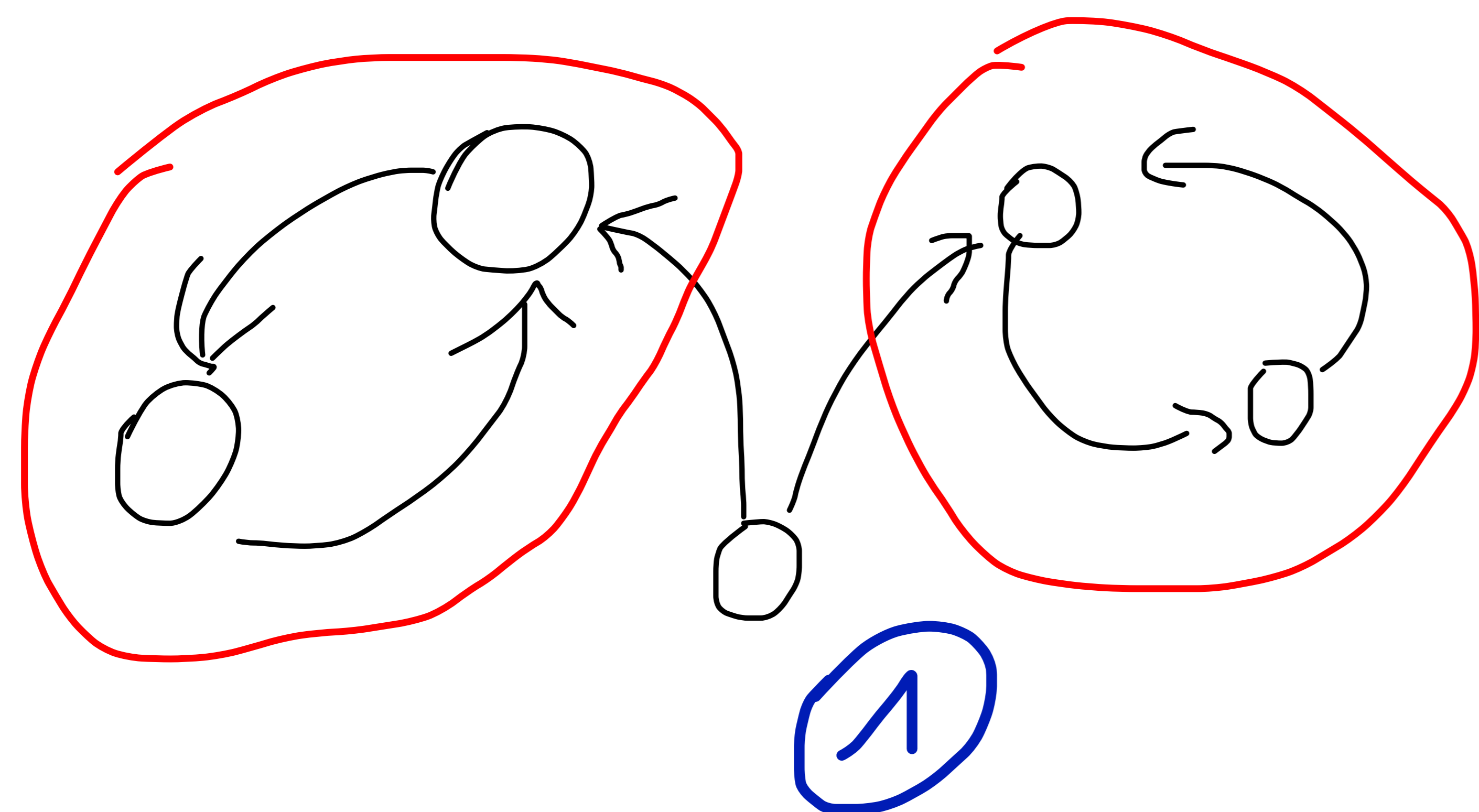
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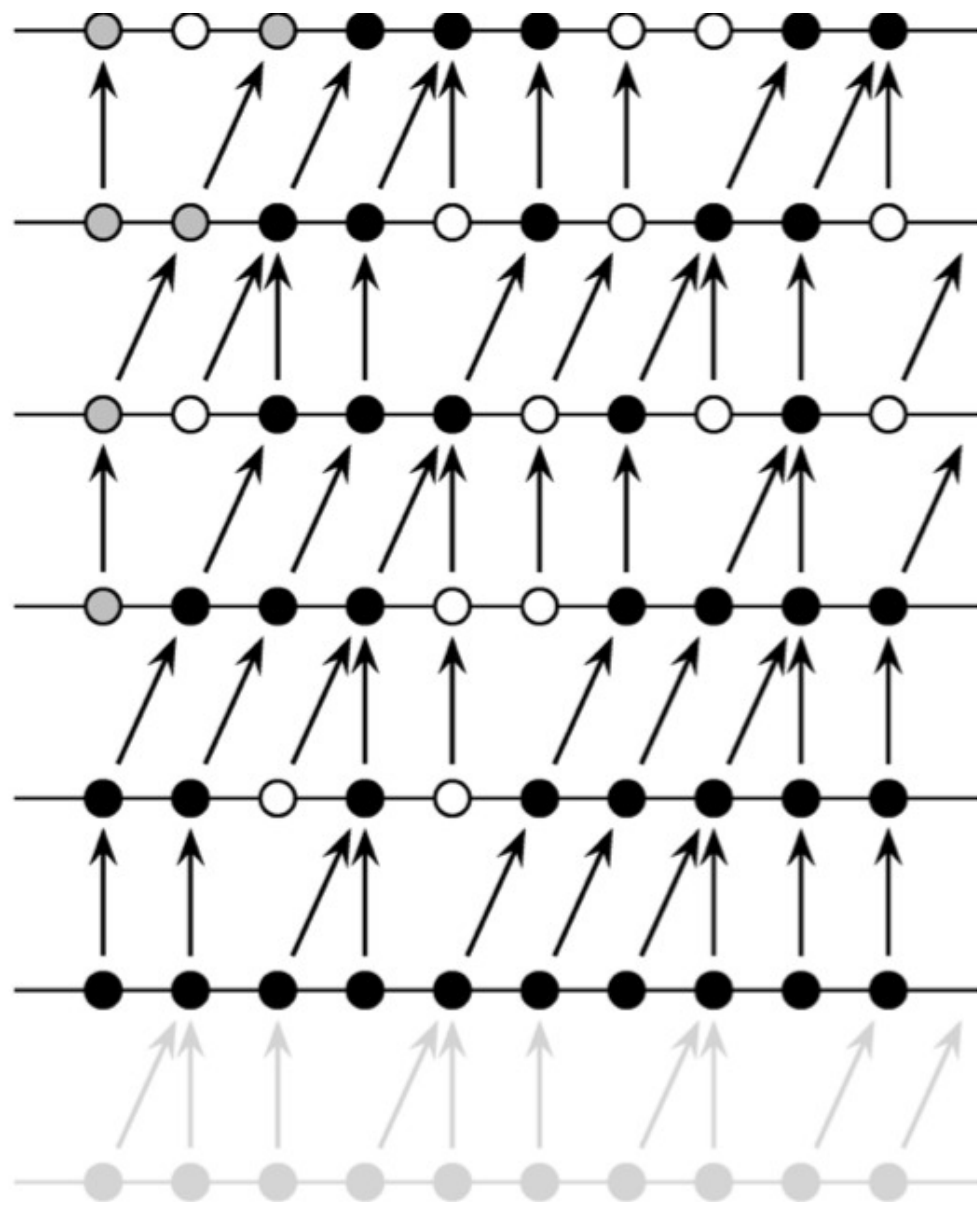
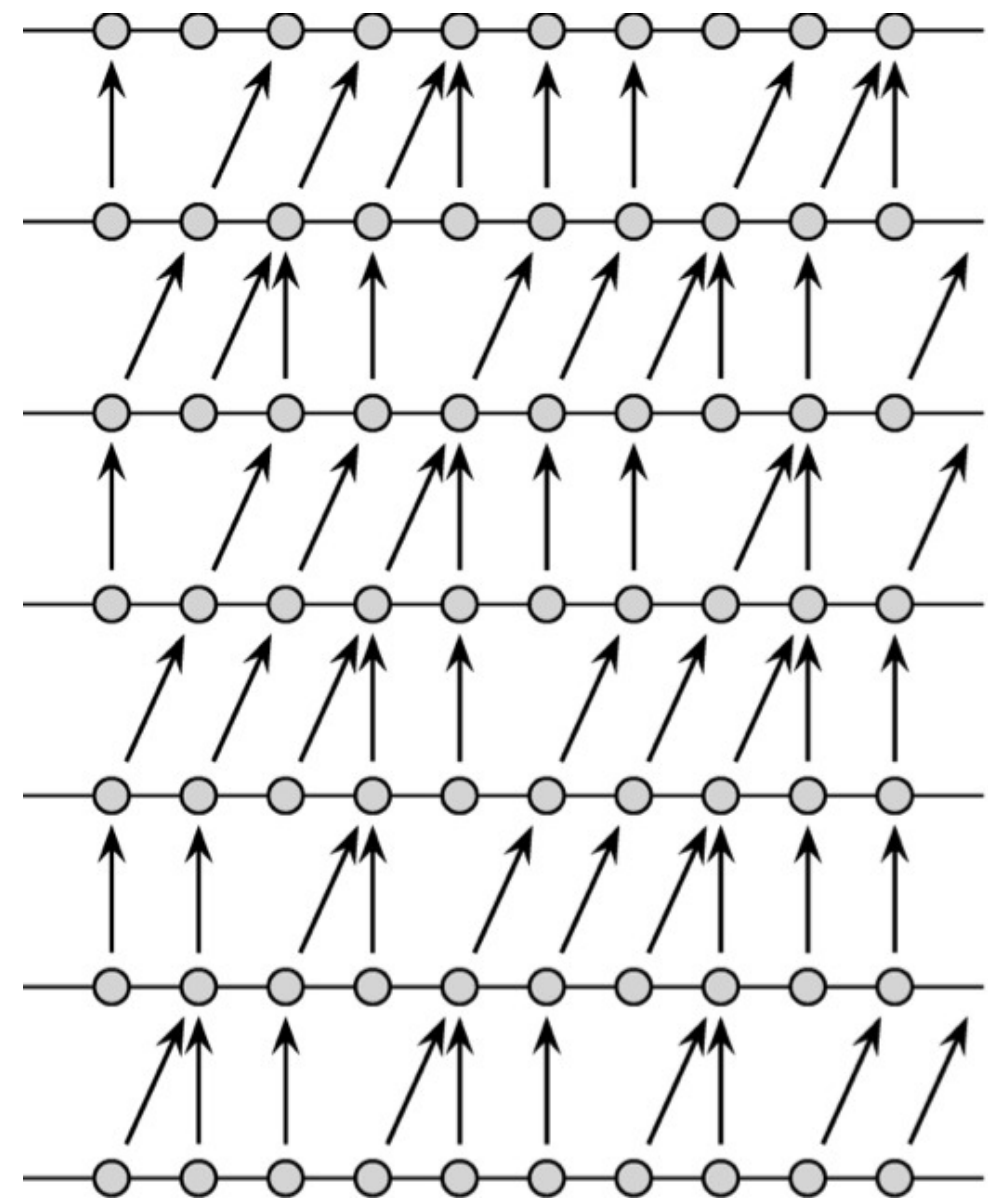
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FINITE STATE MARKOV CHAIN



Ingredients

- * coalescing random walks
- * Catalan numbers
- * enumeration of vicious walkers.



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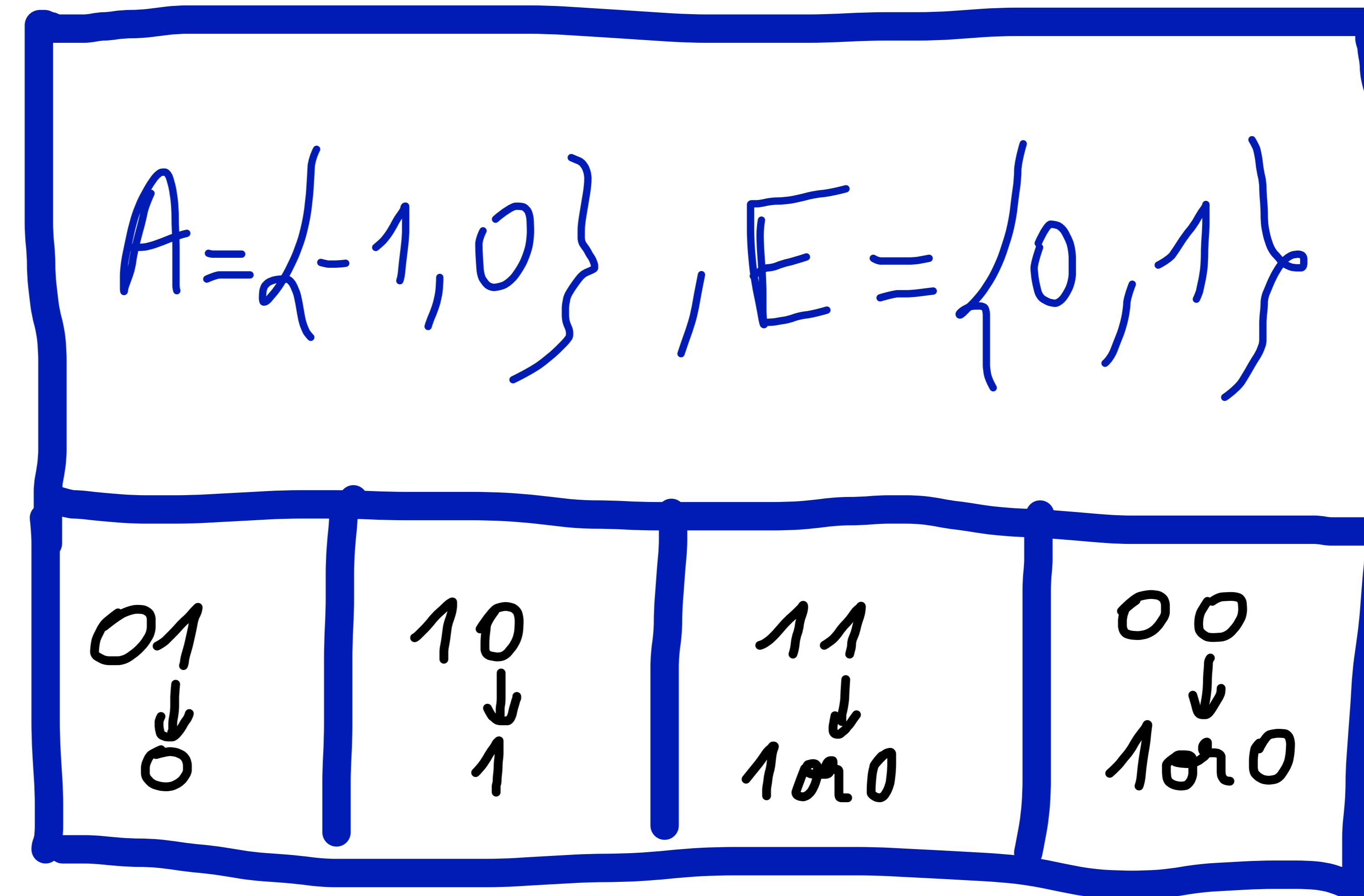
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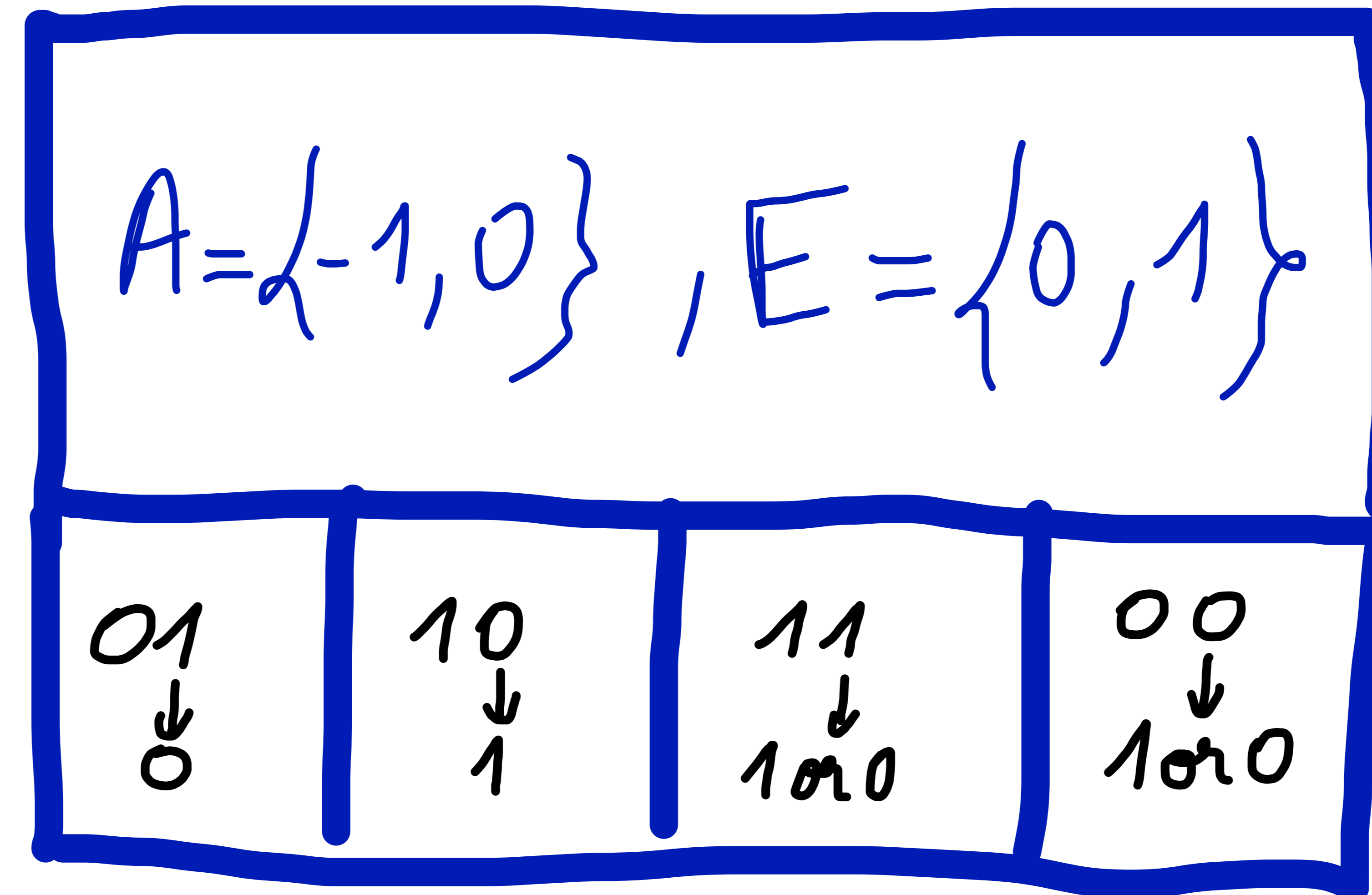
$A = \{-1, 0\}, E = \{0, 1\}$			
01 ↓ 0	10 ↓ 1	11 ↓ 1 or 0	00 ↓ 1 or 0

with equiprobability

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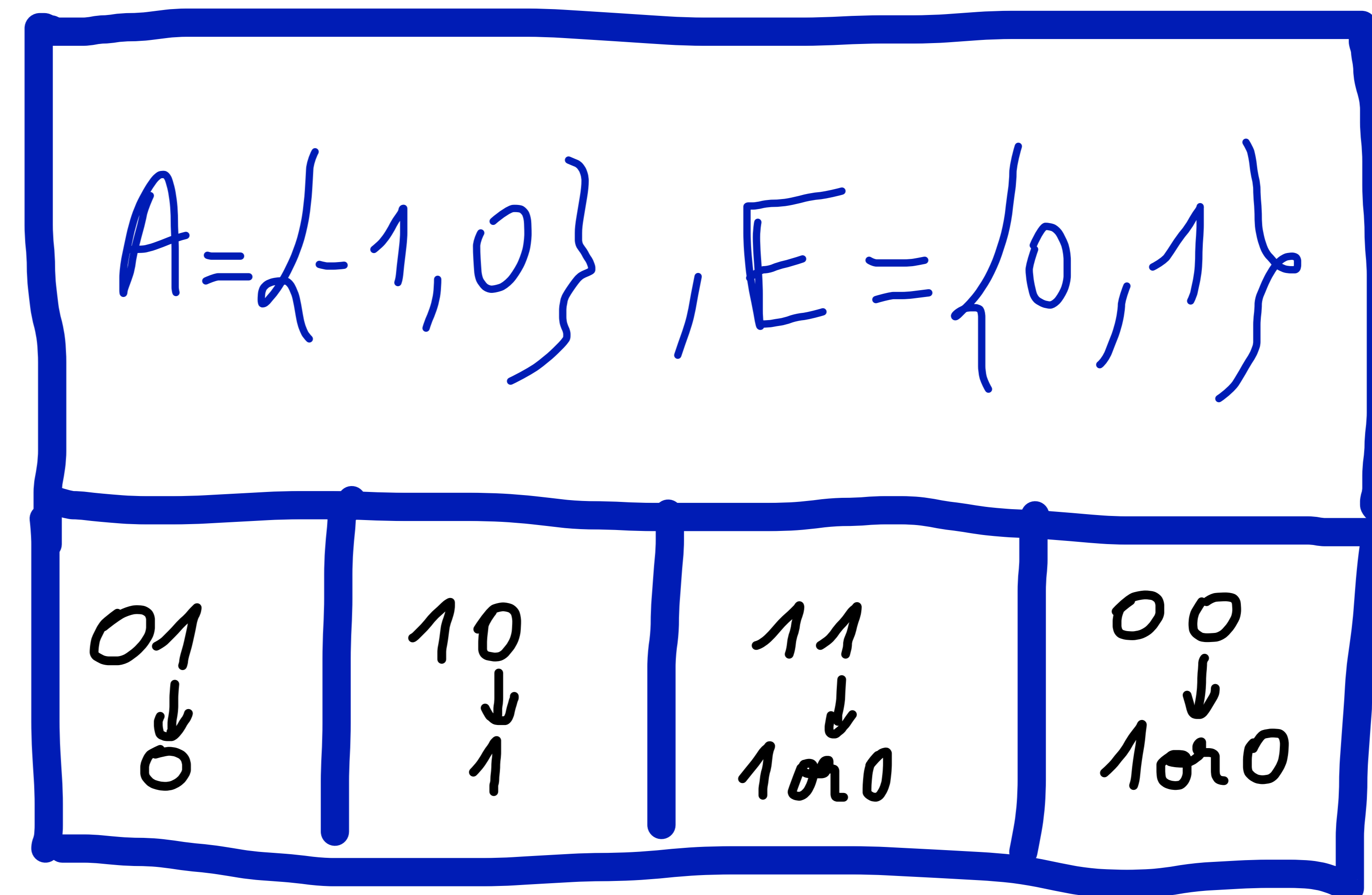
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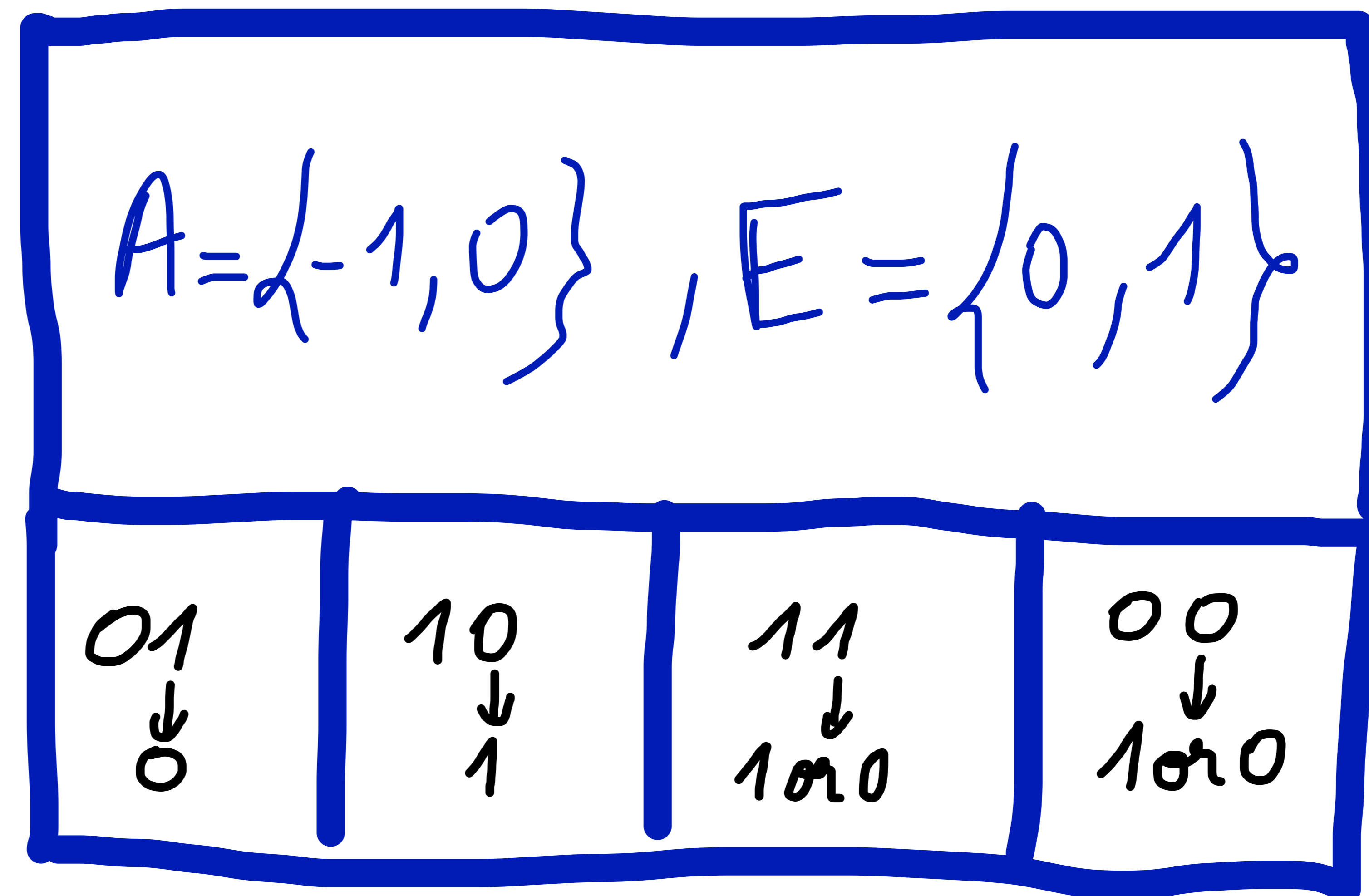
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Idea: take $a = (01)^{\mathbb{Z}}$ and $b = (10)^{\mathbb{Z}}$ two configurations

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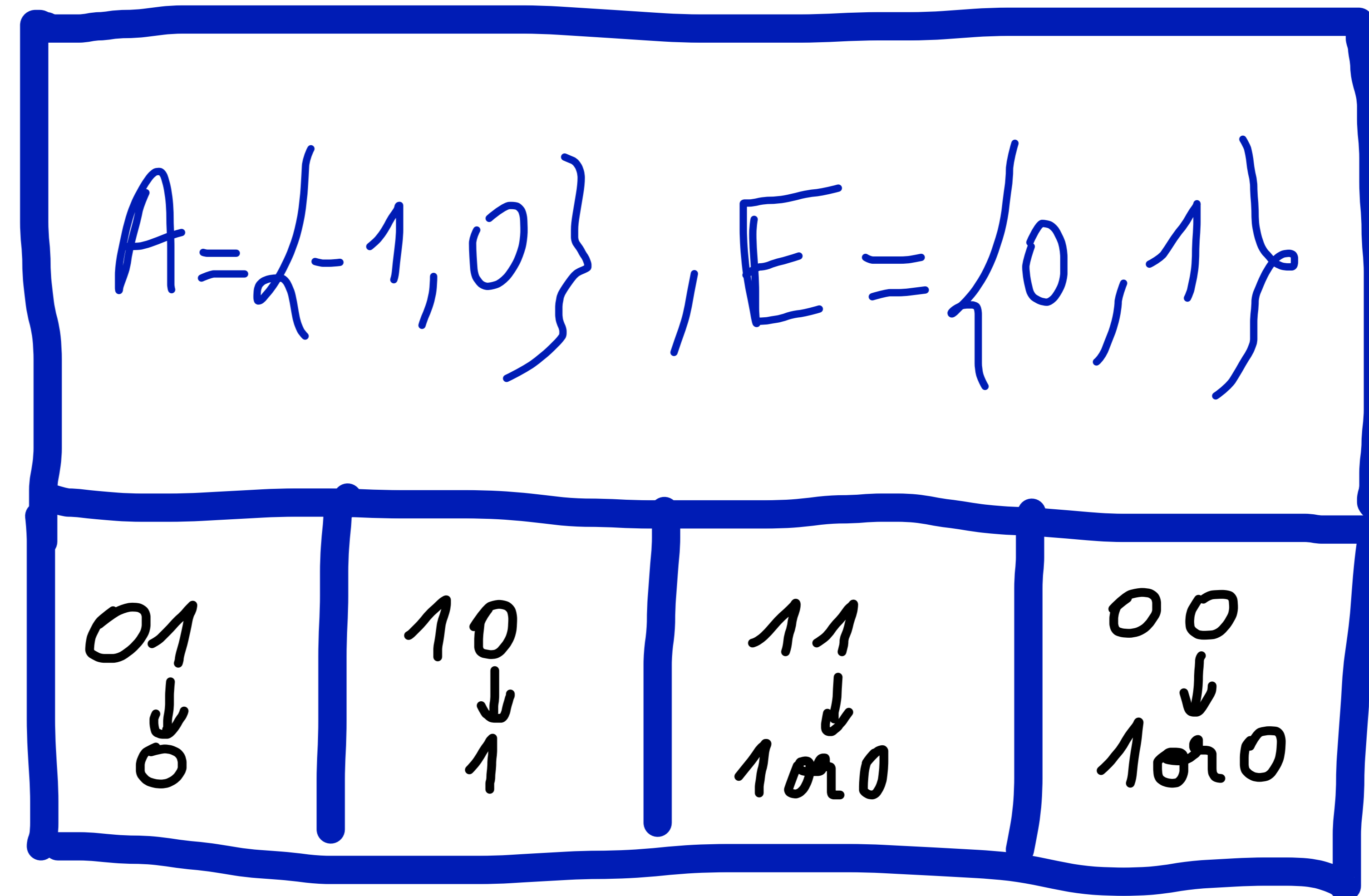
Then $a \rightleftharpoons b$ because, clearly

...	0	1	0	1	0	...
			↓			
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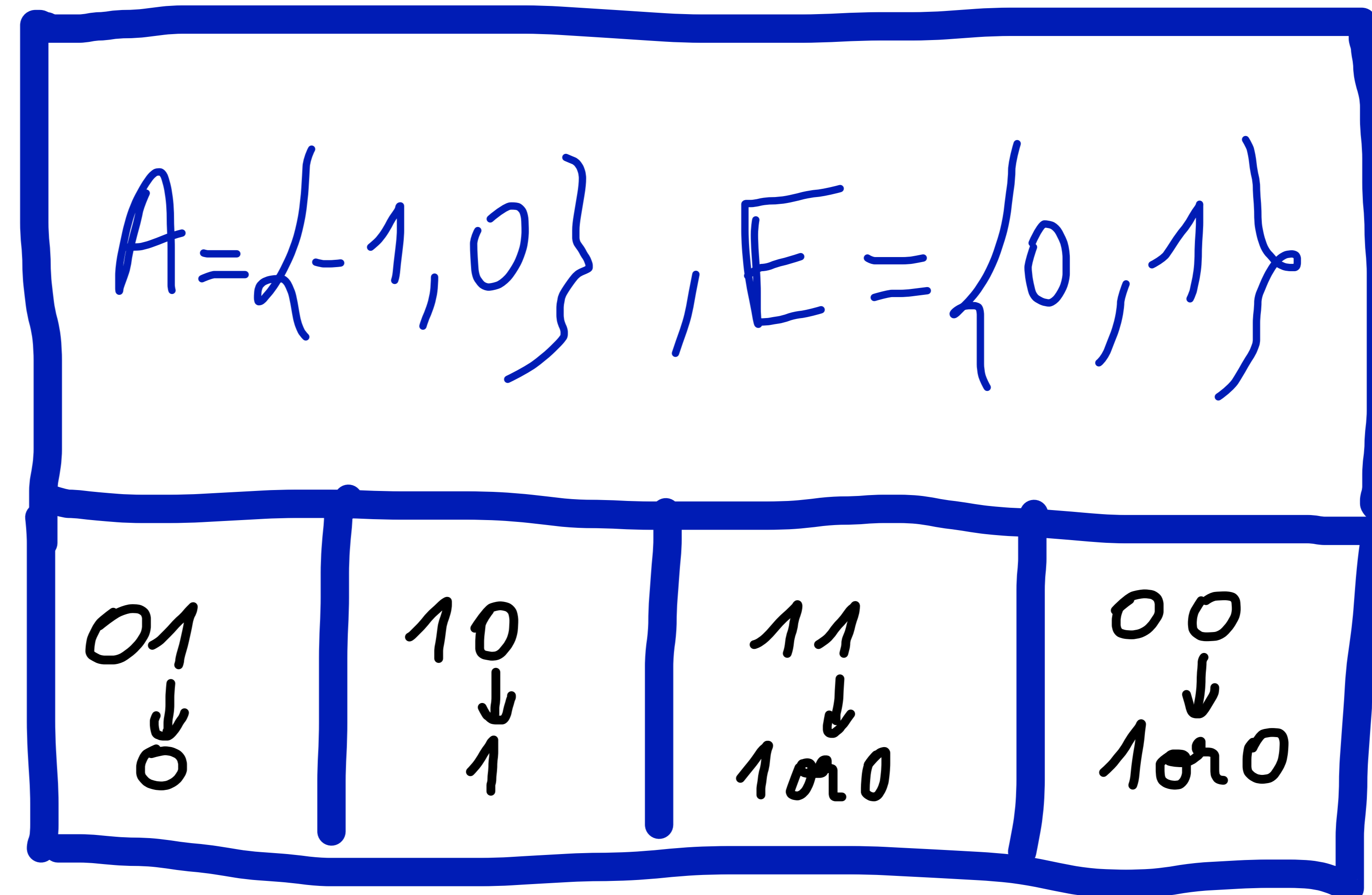
$$\begin{array}{ccccccc} \dots & 0 & 1 & 0 & 1 & 0 & \dots \\ & & & \downarrow & & & \\ \dots & 1 & 0 & 1 & 0 & 1 & \dots \end{array}$$

Thus $\frac{1}{2} \delta_a + \frac{1}{2} \delta_b = \mu$ is stationary, and is **not** attractive

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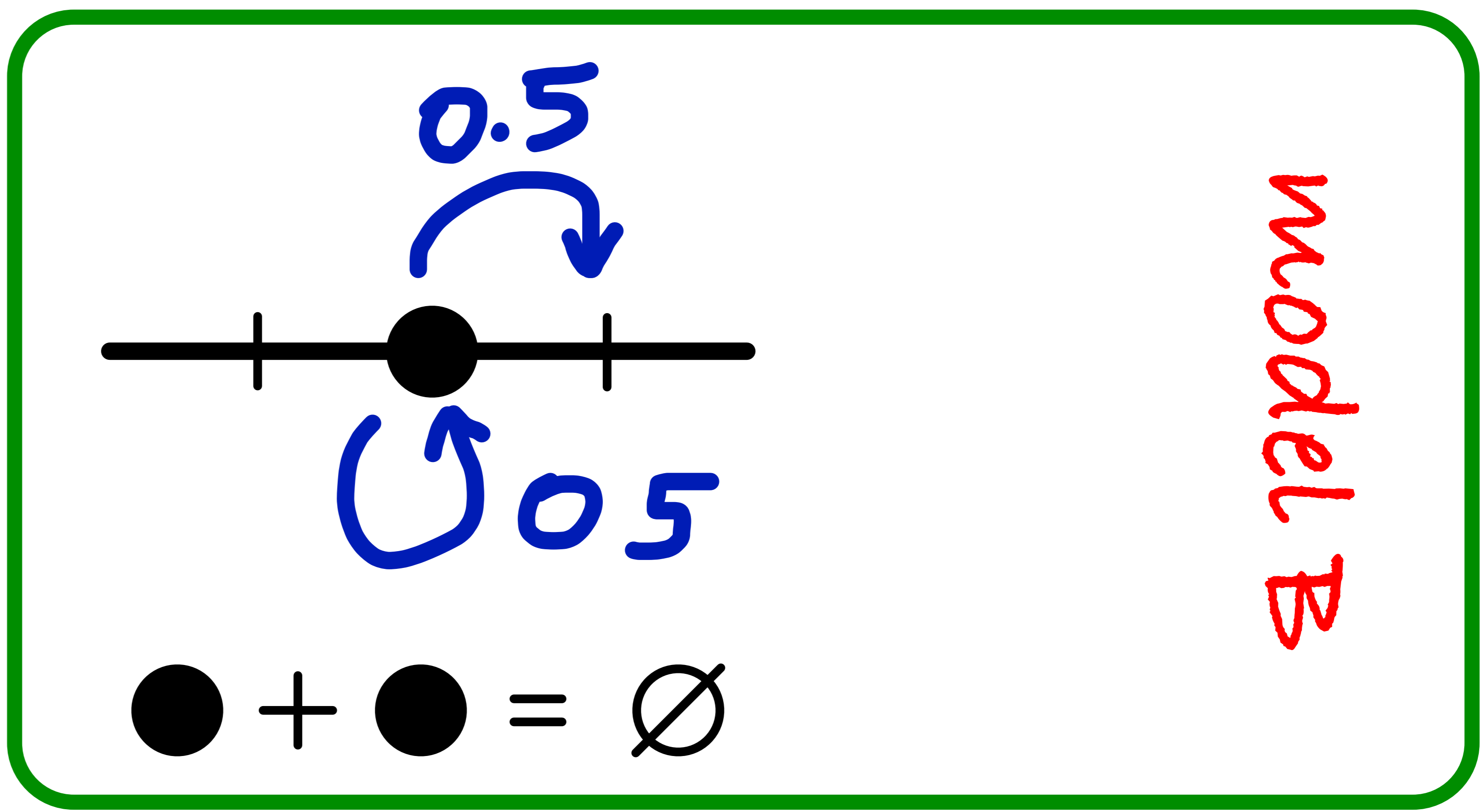
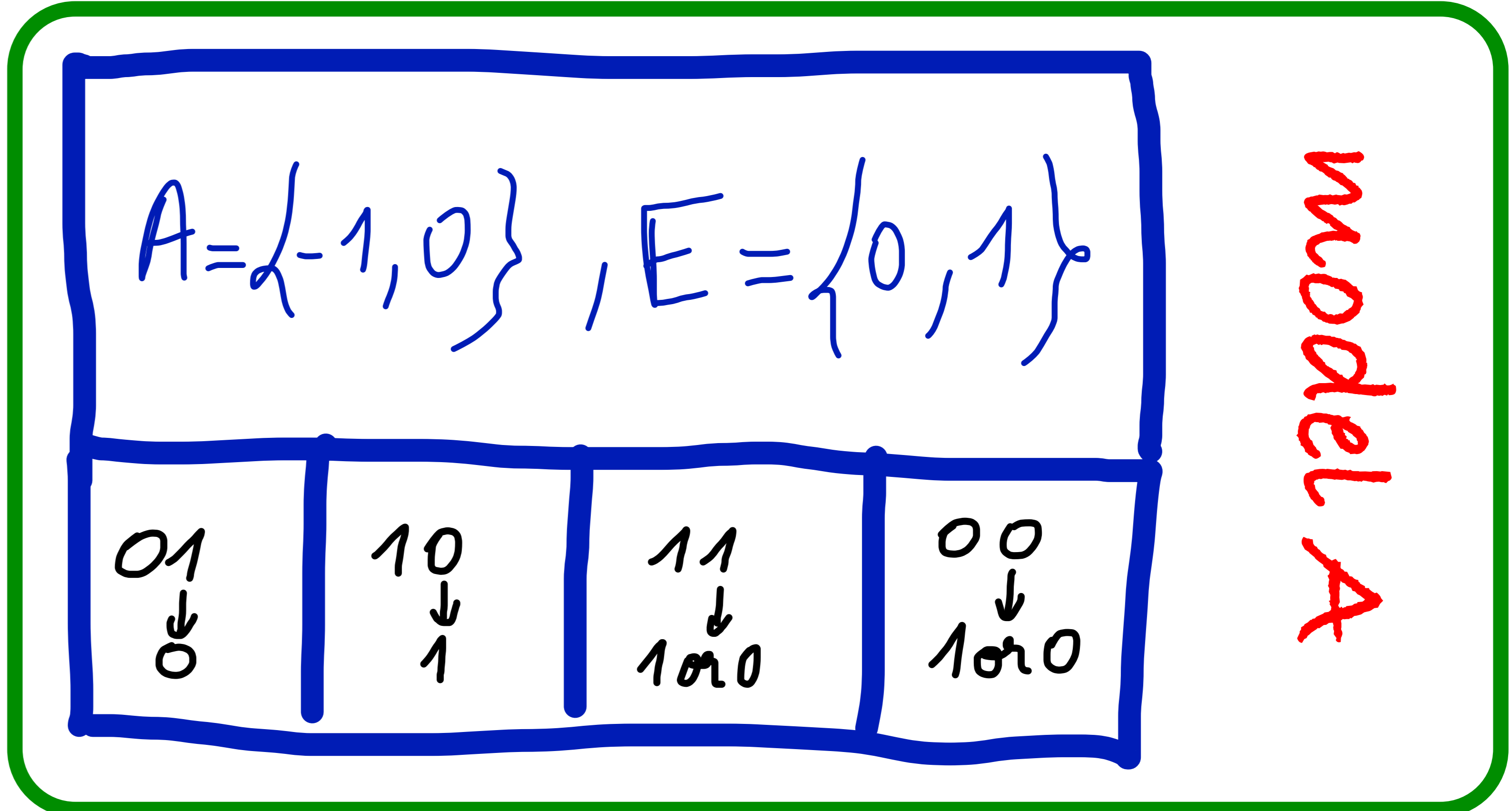
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Question: is it **unique**?

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PCA

annihilating
random walk

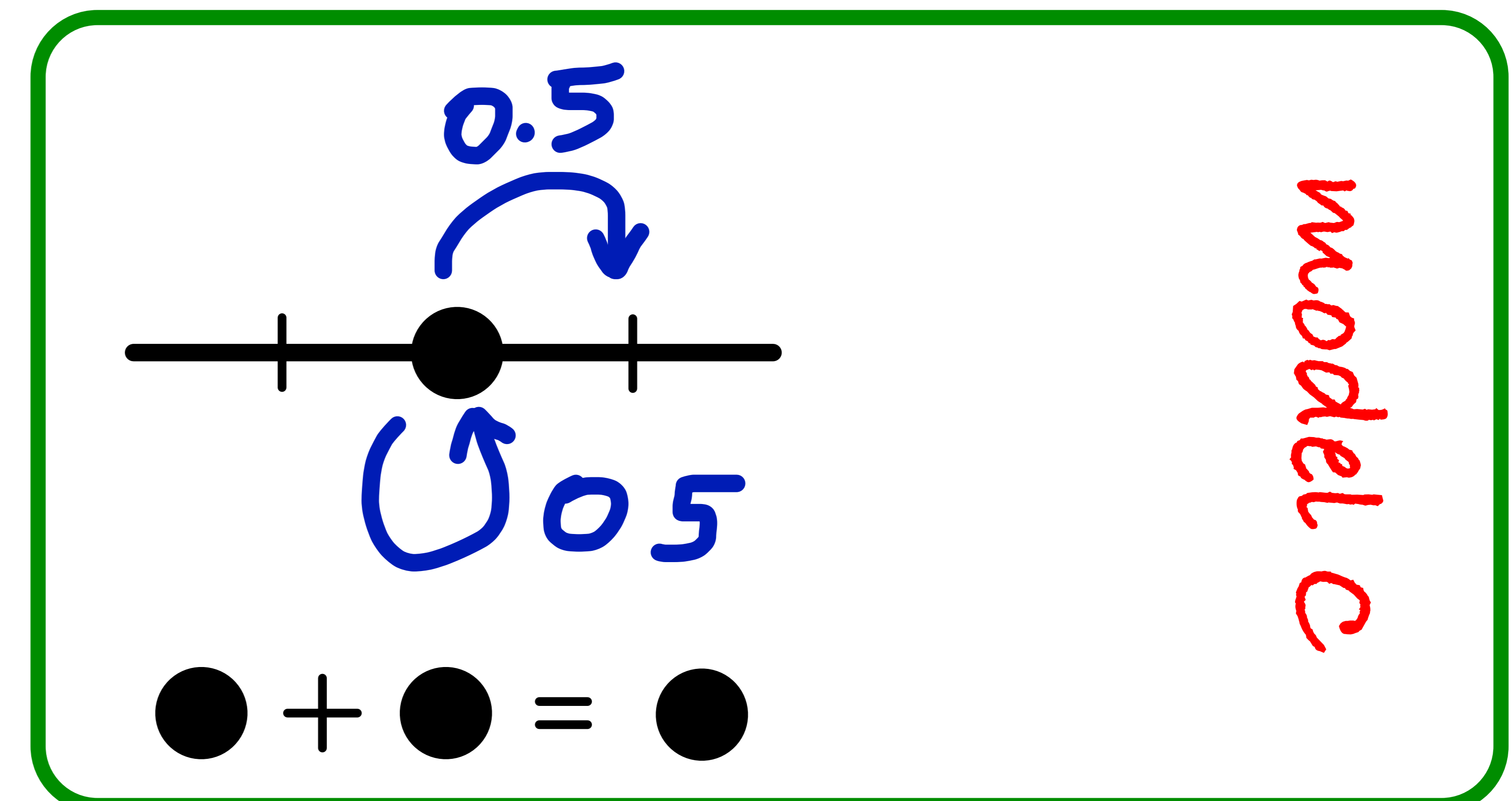
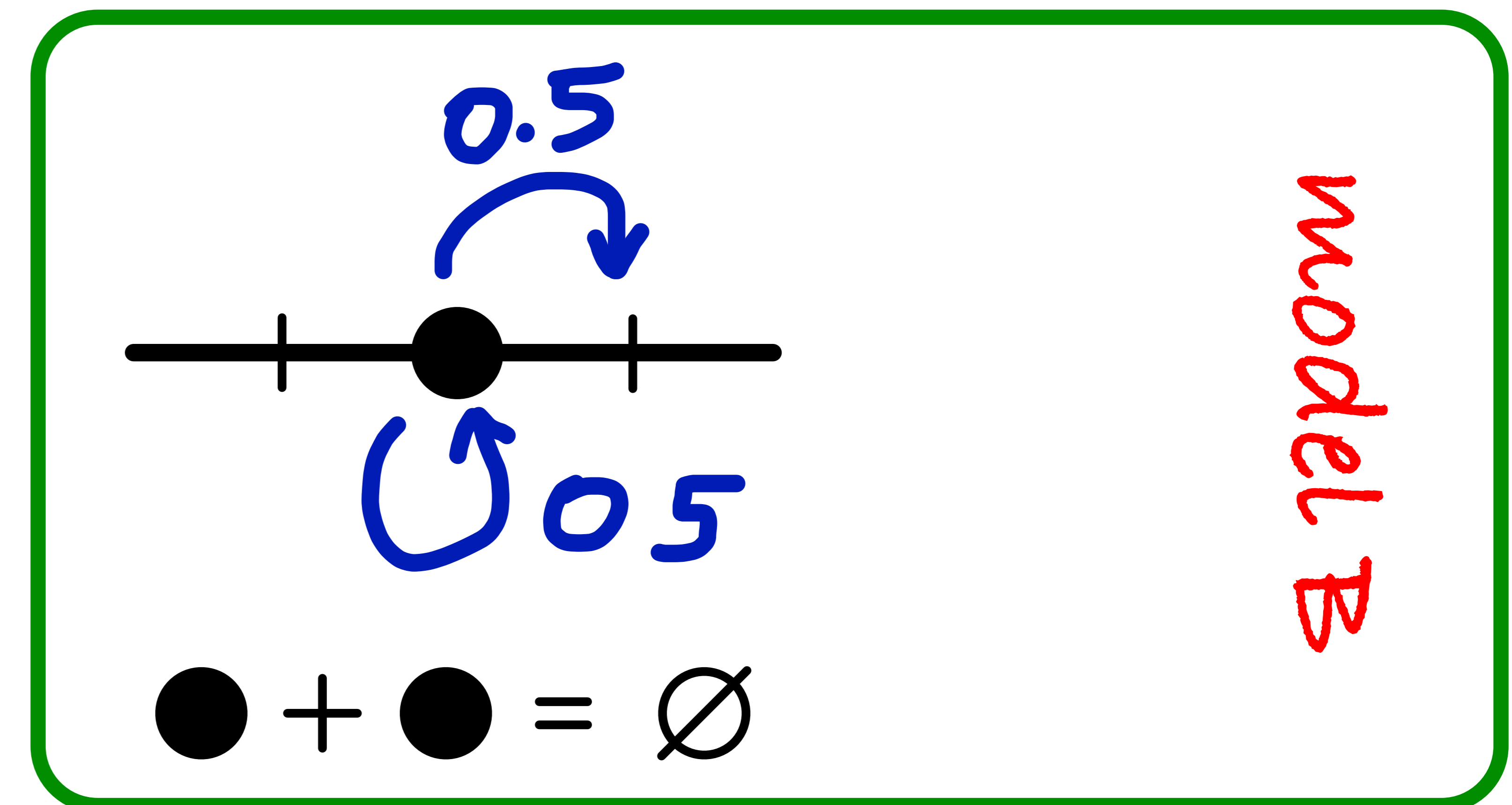
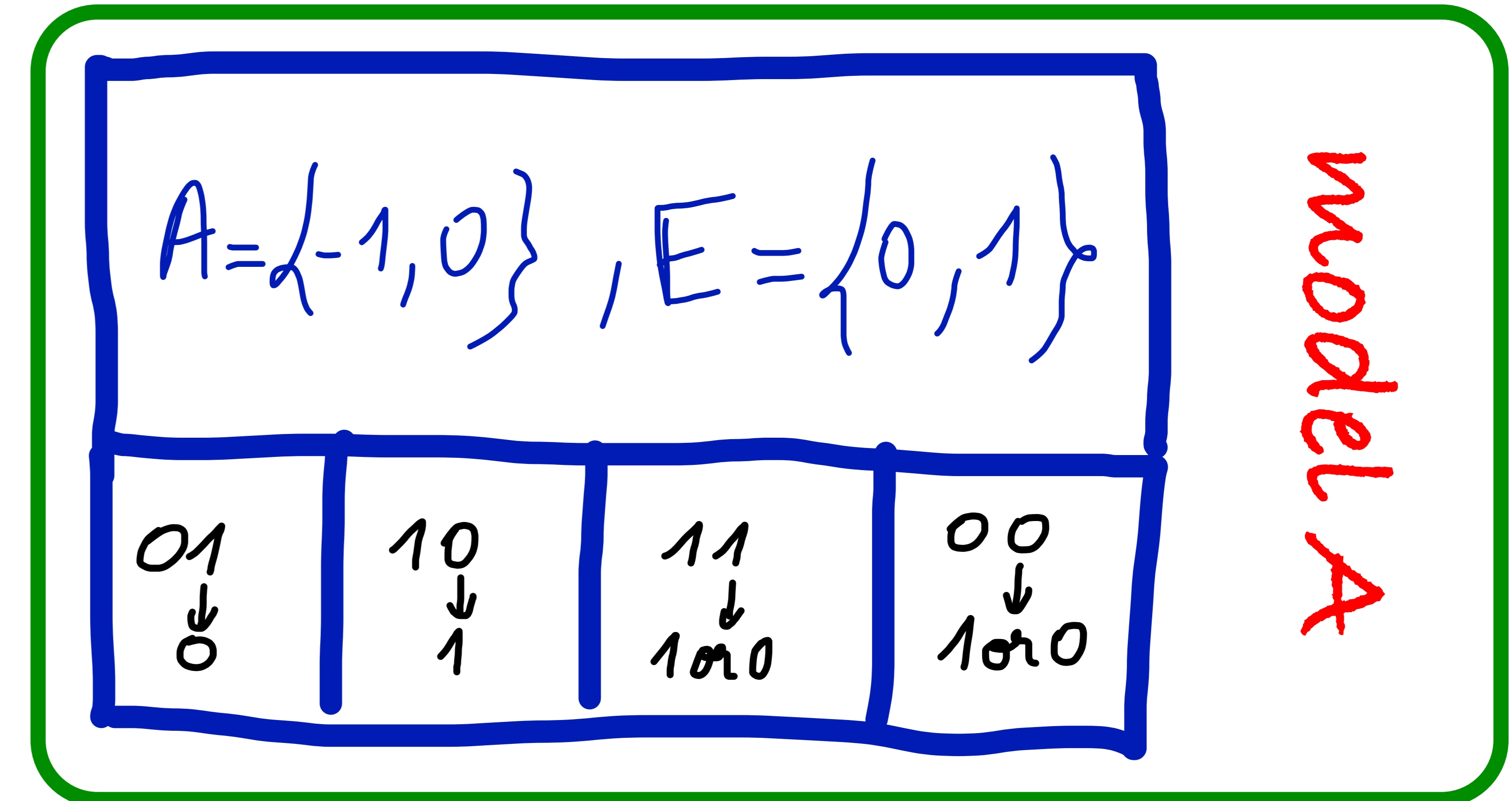


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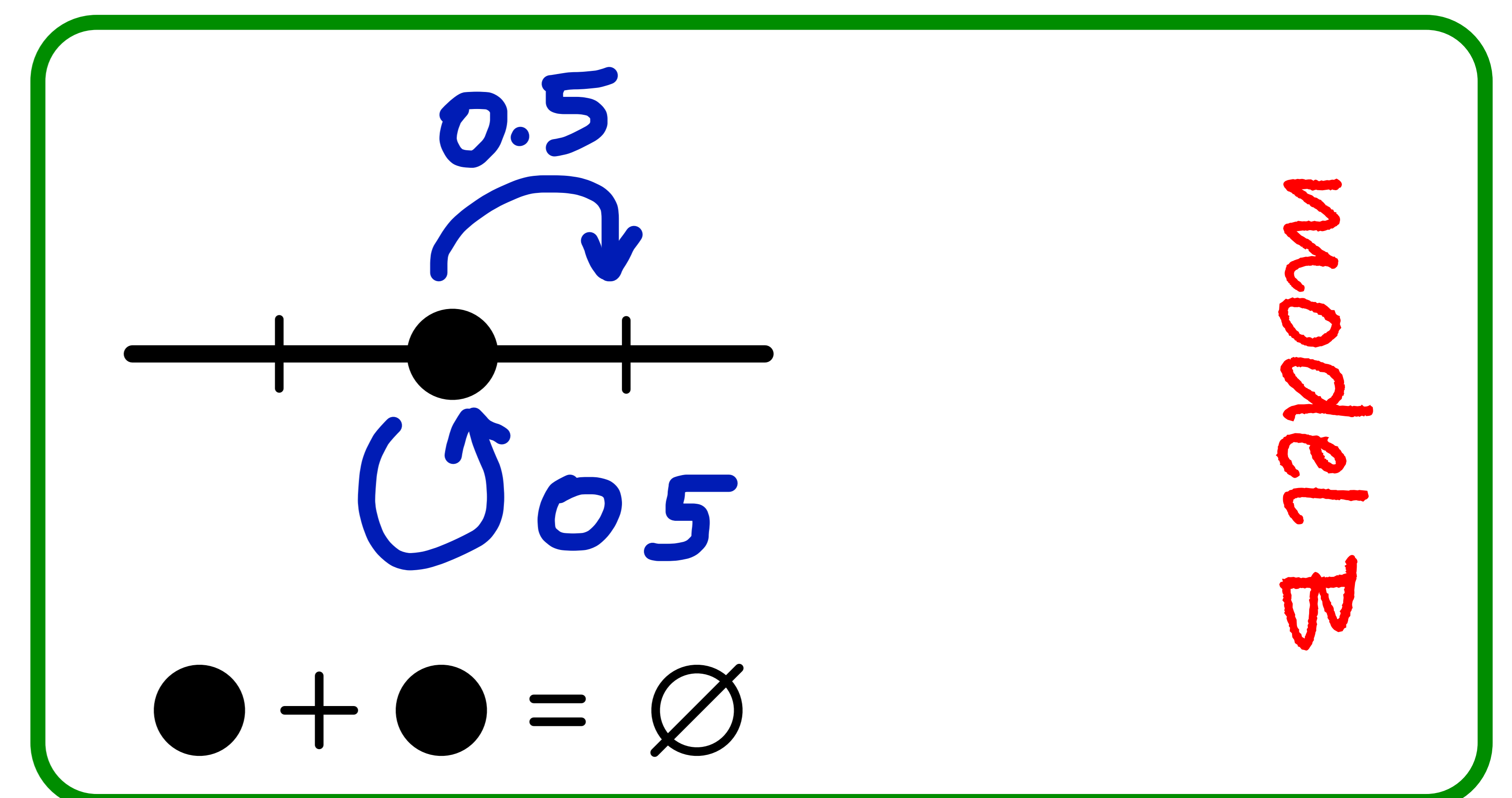
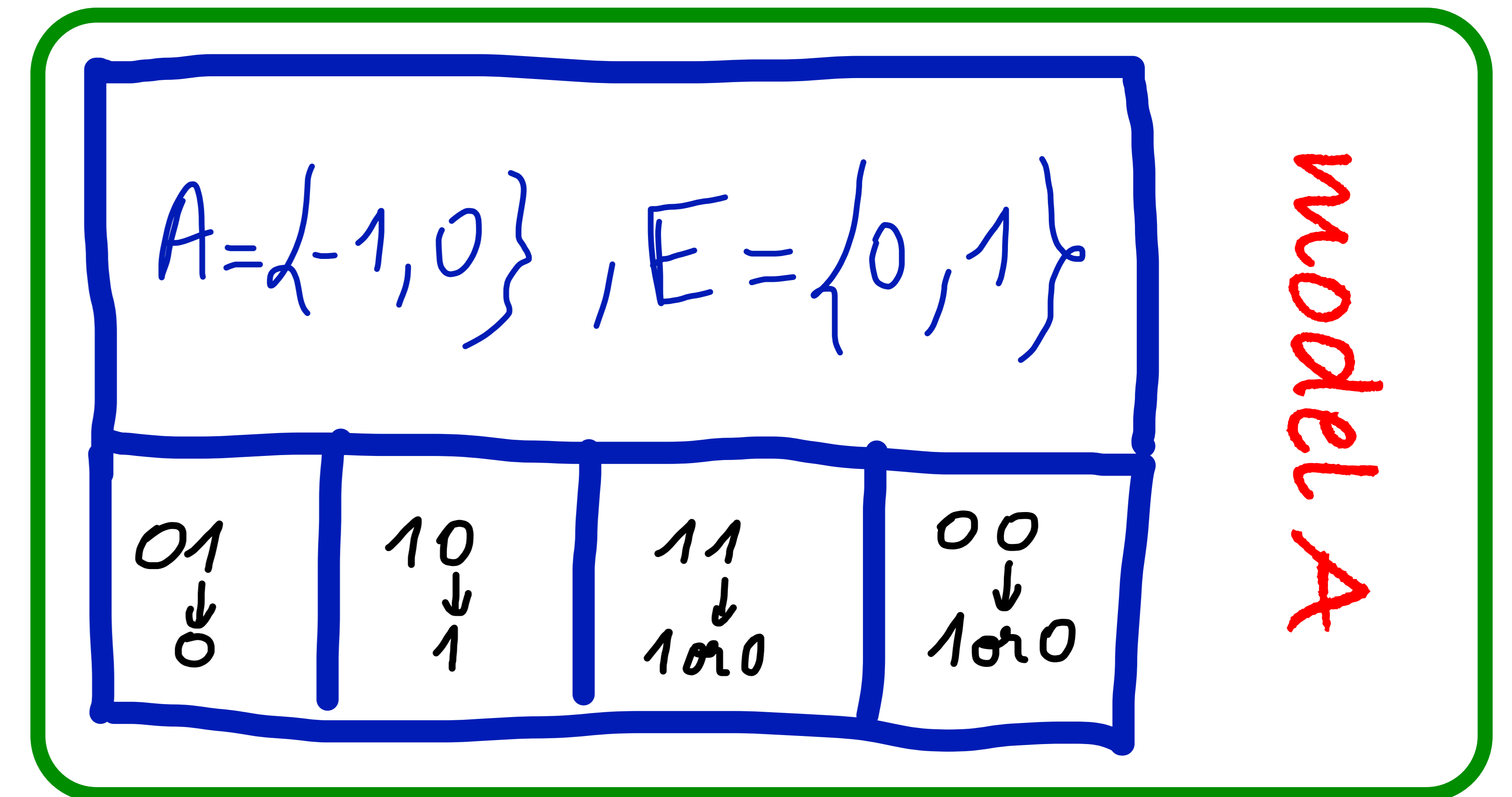


$$\text{Model A} \xrightarrow{\phi} \text{Model B}$$

For $x \in \{0,1\}^{\mathbb{Z}}$, $\eta \in \{0, \bullet\}^{\mathbb{Z}}$, $\eta = \phi(x)$ if:

$$\{x_i = x_{i+1}\} \iff \{\eta_i = \bullet\}$$

0 stands for \emptyset or "empty"



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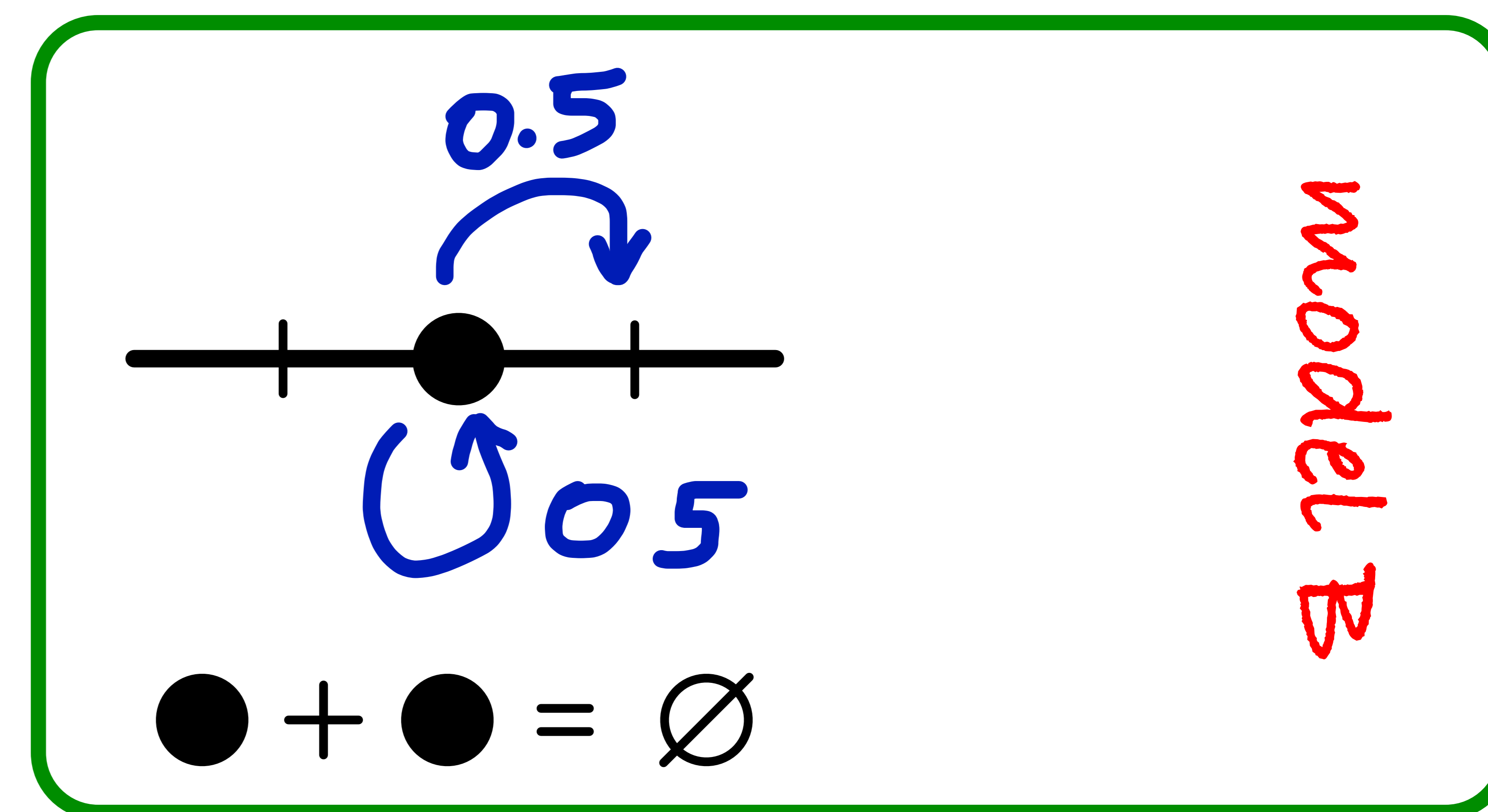
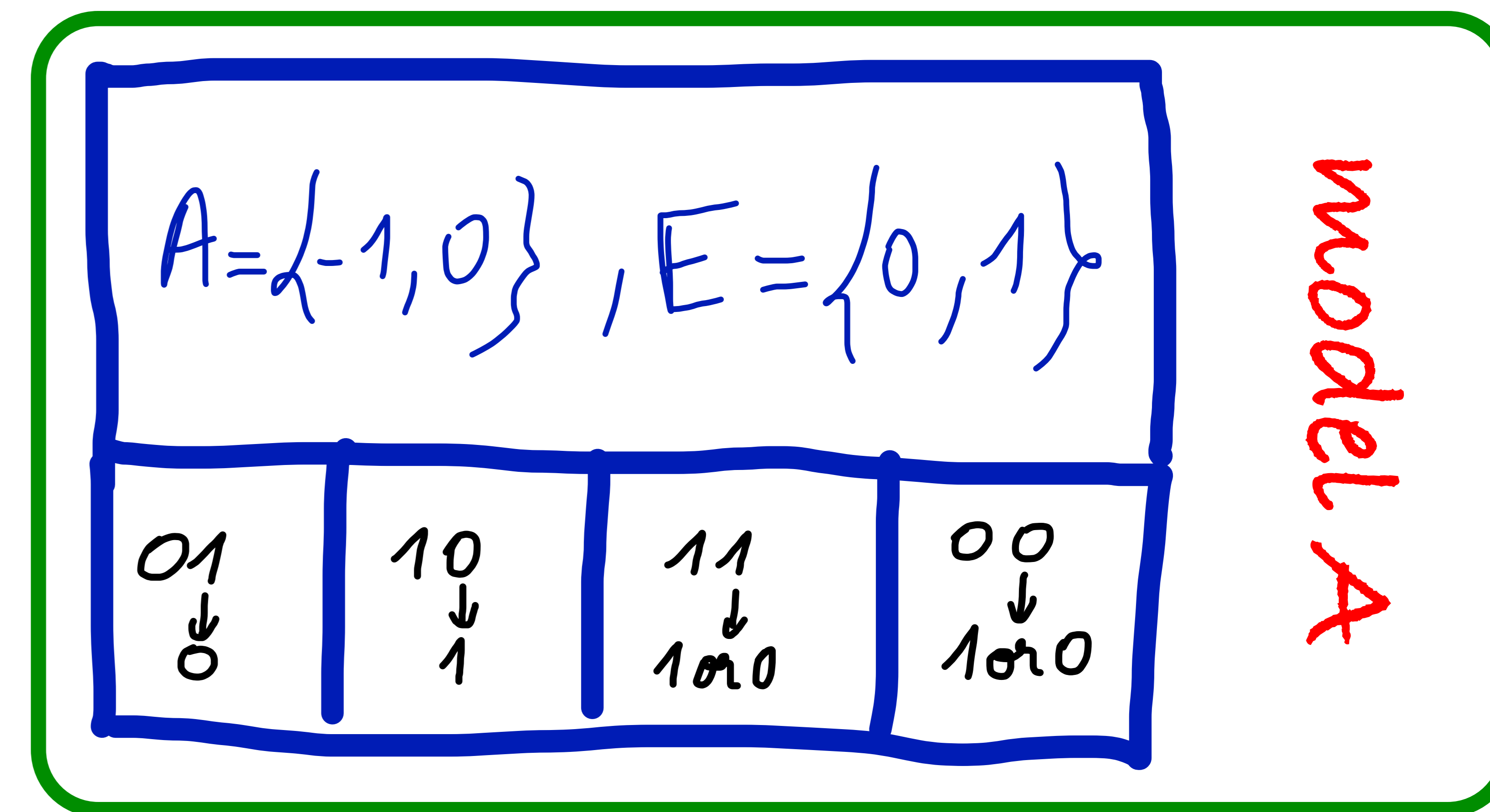
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t 001110101000
 t+1 ? 1 0 1 1 1 0 1 0 1 0 1



t ● ○ ● ● ○ ○ ○ ○ ○ ● ● ?
 t+1 ? ○ ○ ● ● ○ ○ ○ ○ ○ ○ ○ ?



Recall that $a = (01)^{\mathbb{Z}}$ $b = (10)^{\mathbb{Z}}$ and
 $\frac{1}{2} \delta_a + \frac{1}{2} \delta_b = \mu$.

Since $\varphi(a) = \varphi(b) = \emptyset^{\mathbb{Z}} = 0^{\mathbb{Z}}$,

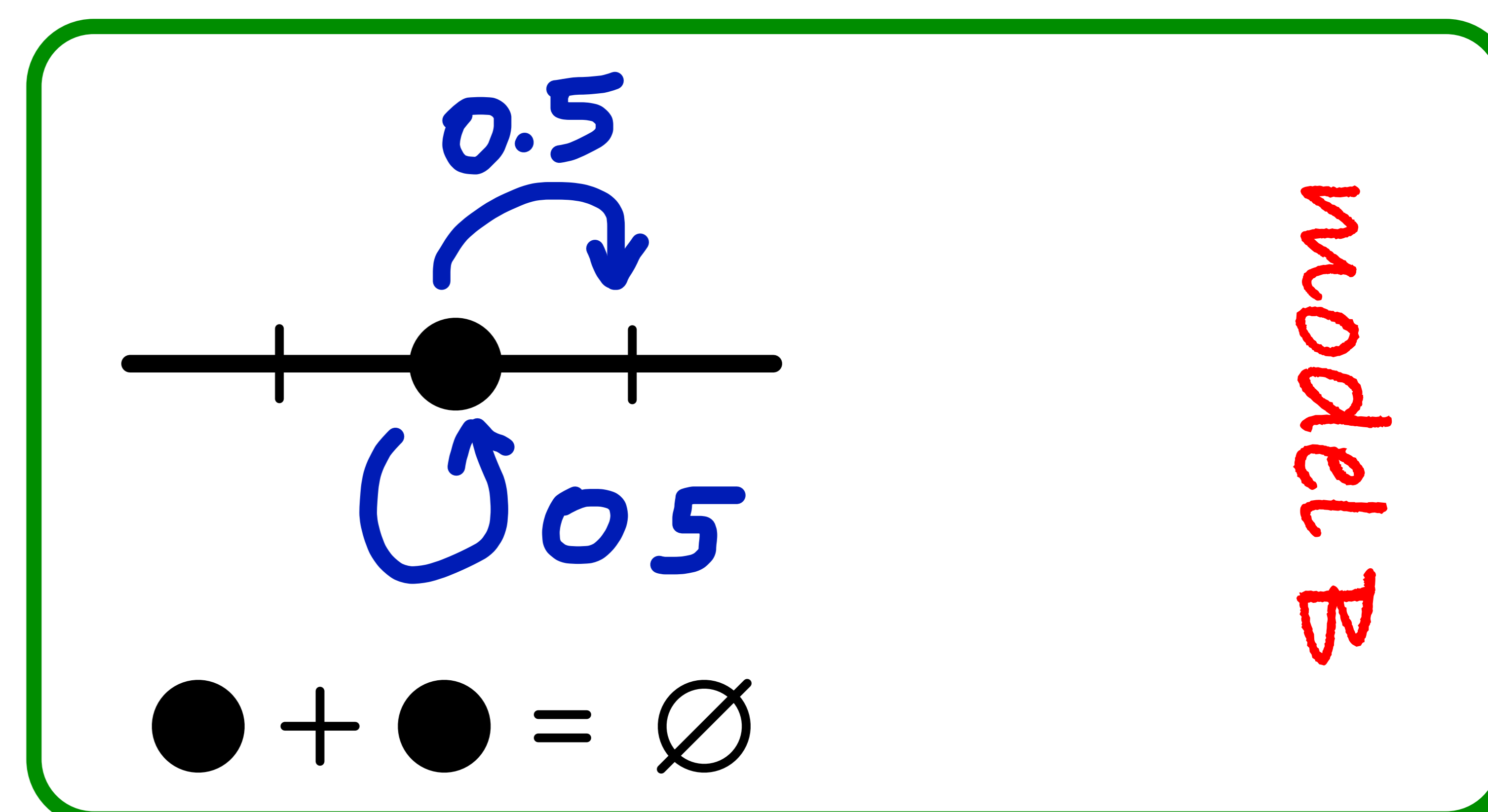
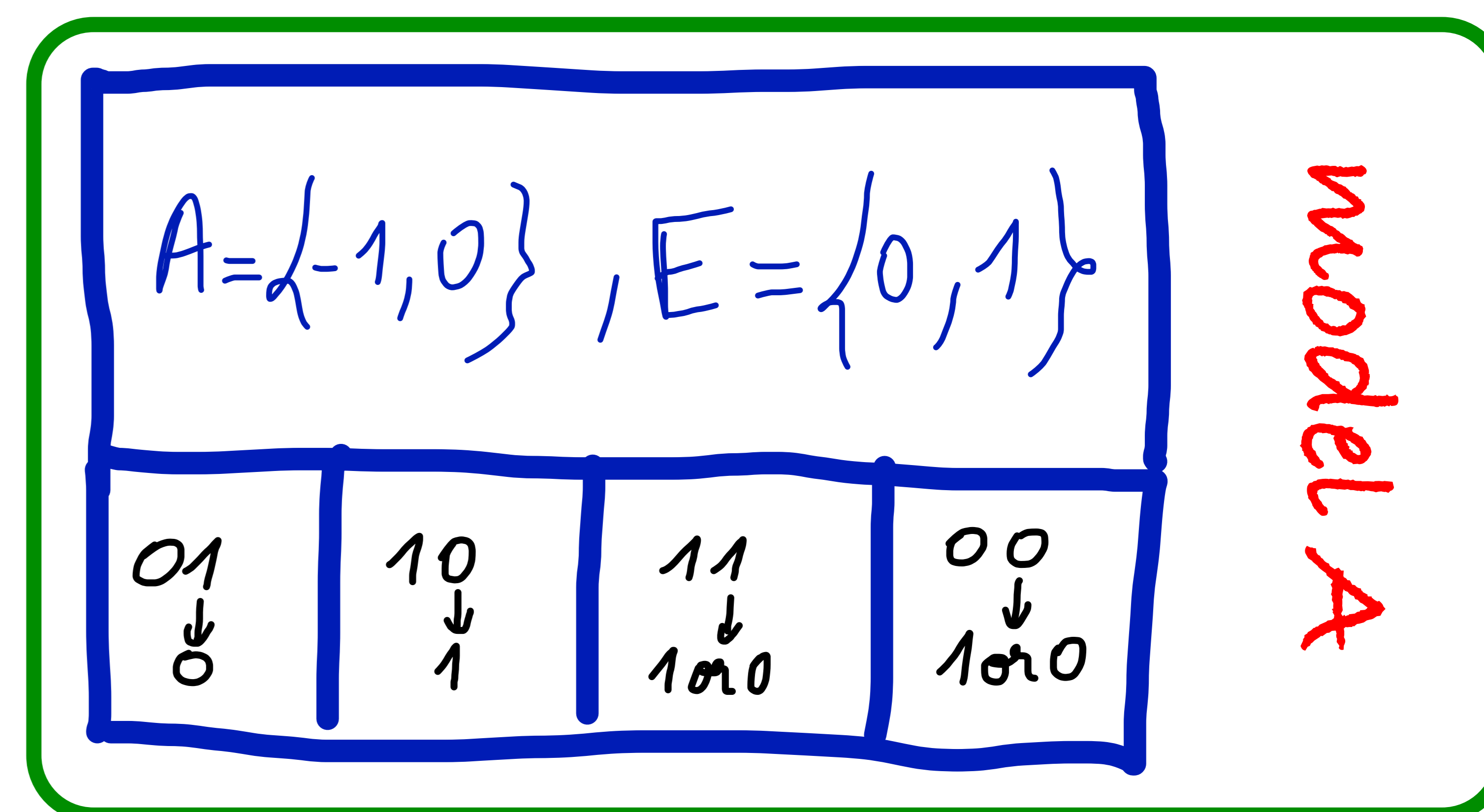
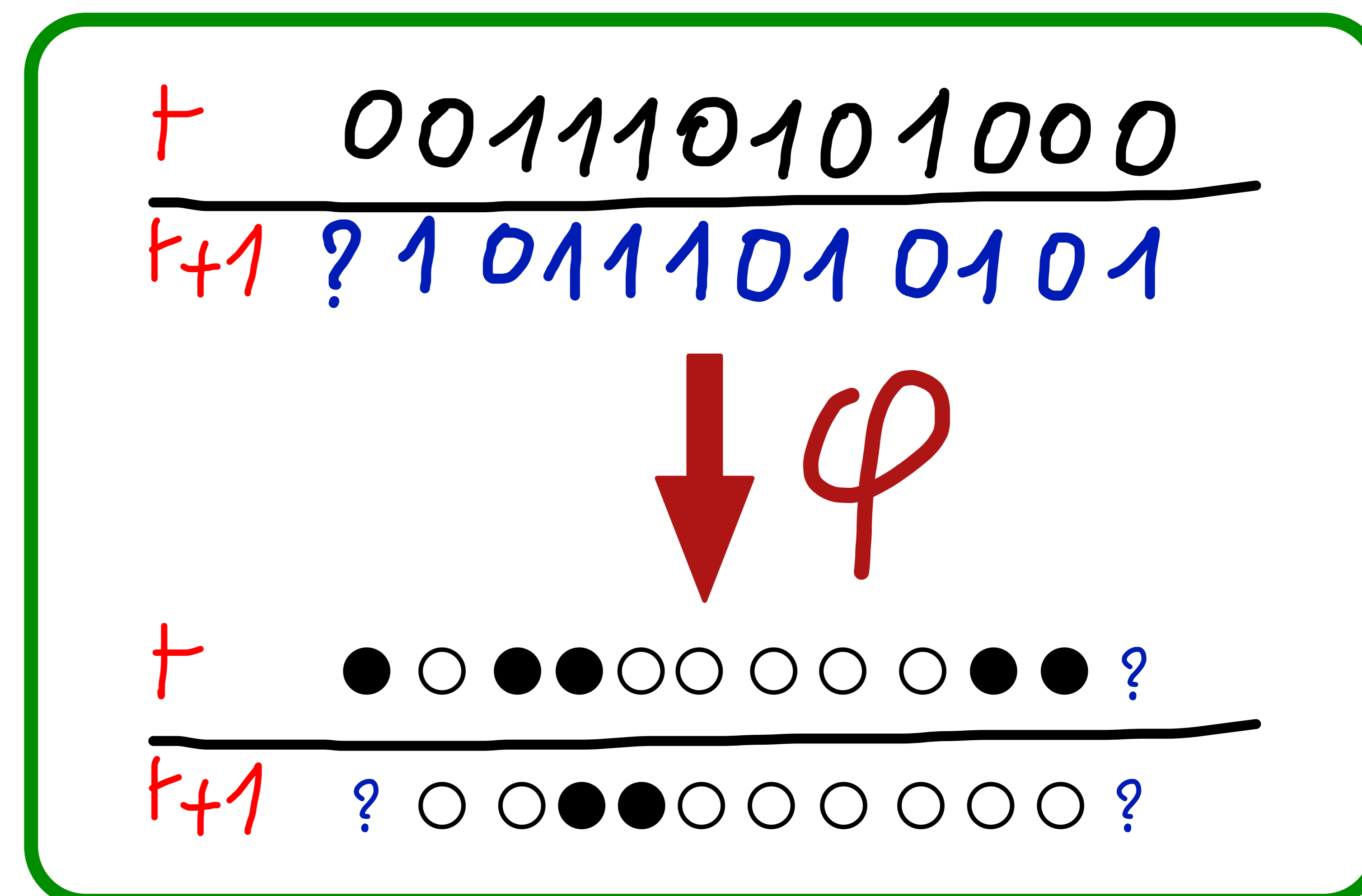
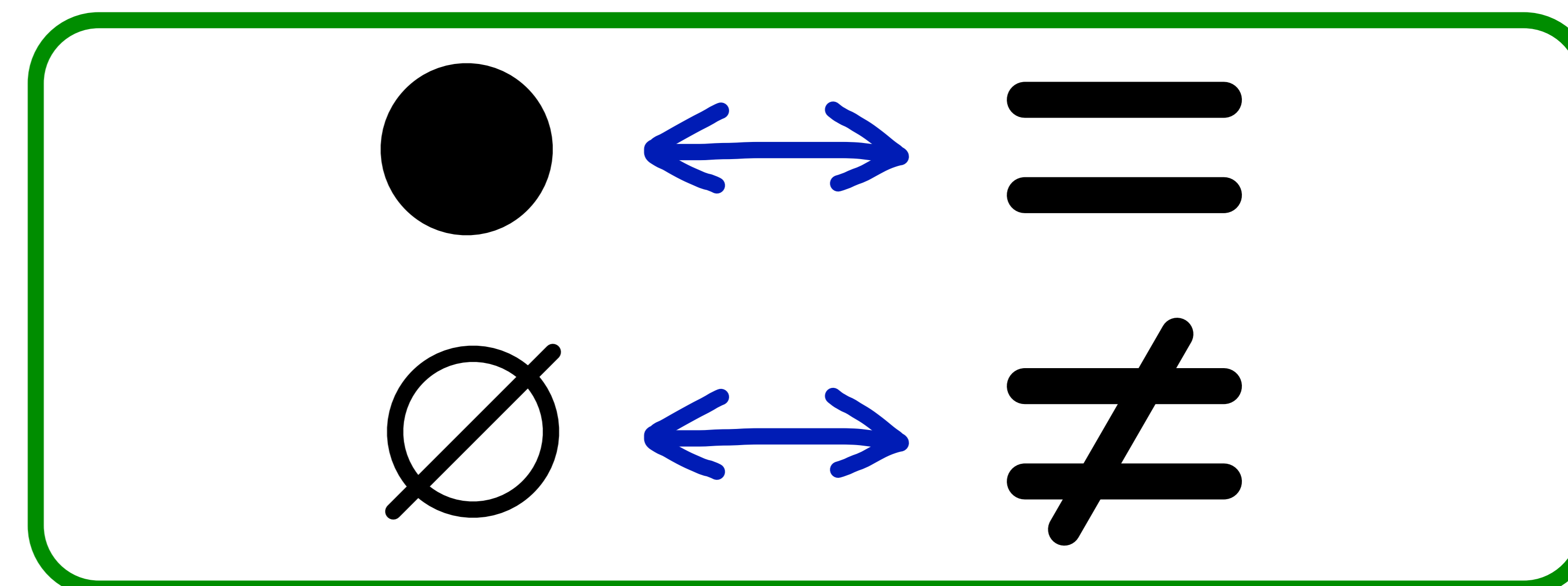
we have:

$$\varphi(\mu) = \delta_{\emptyset^{\mathbb{Z}}}$$

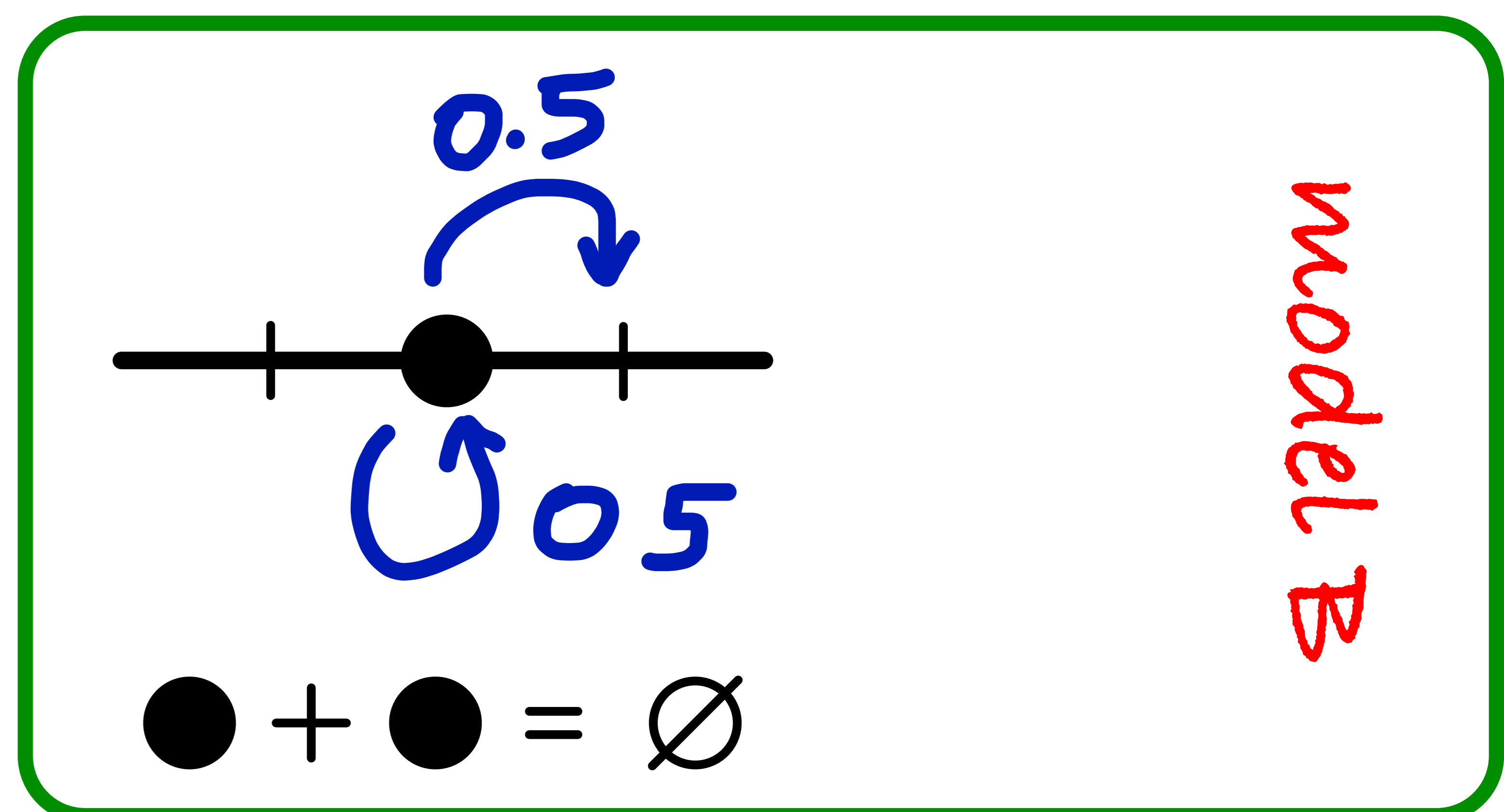
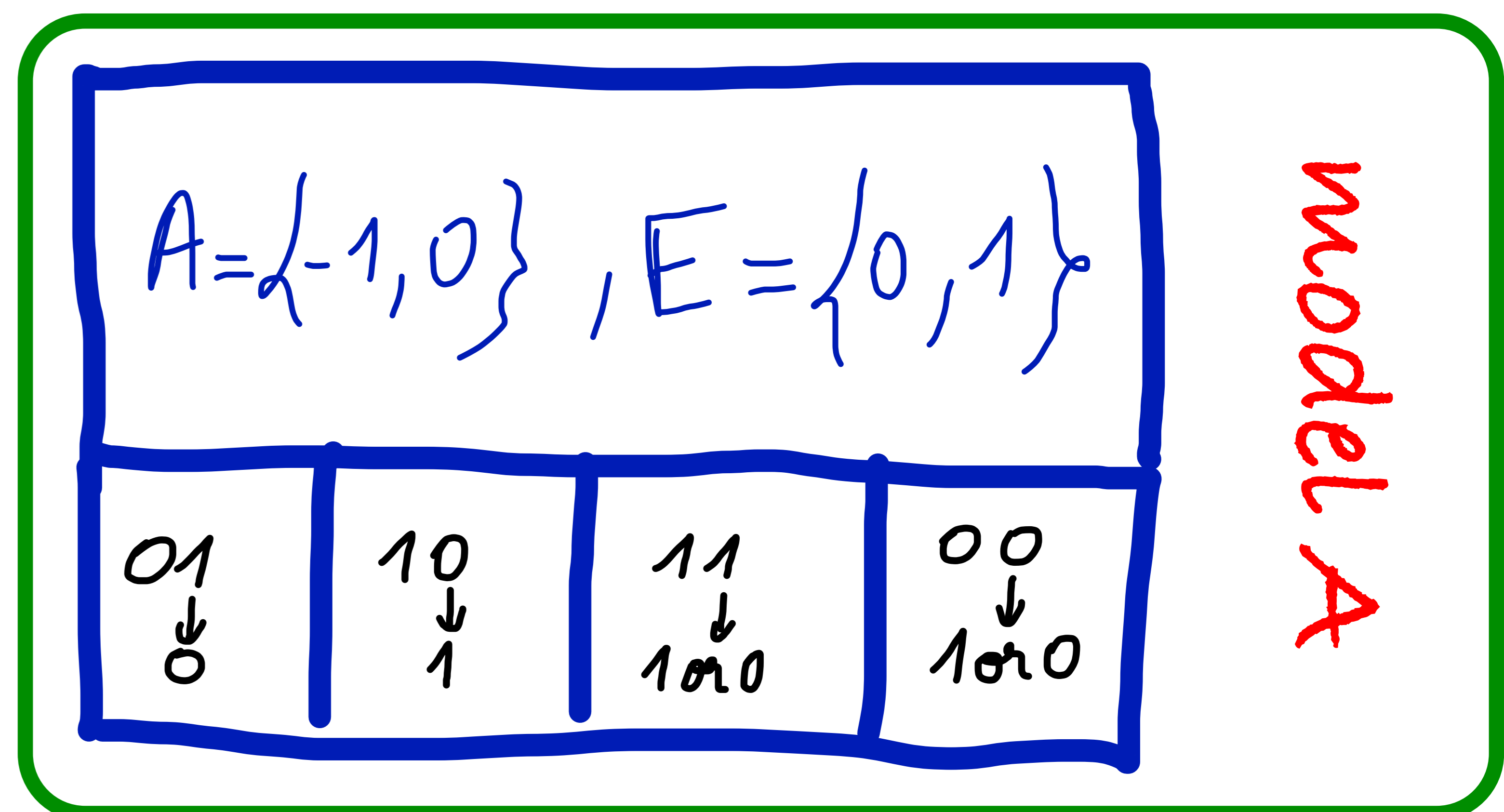
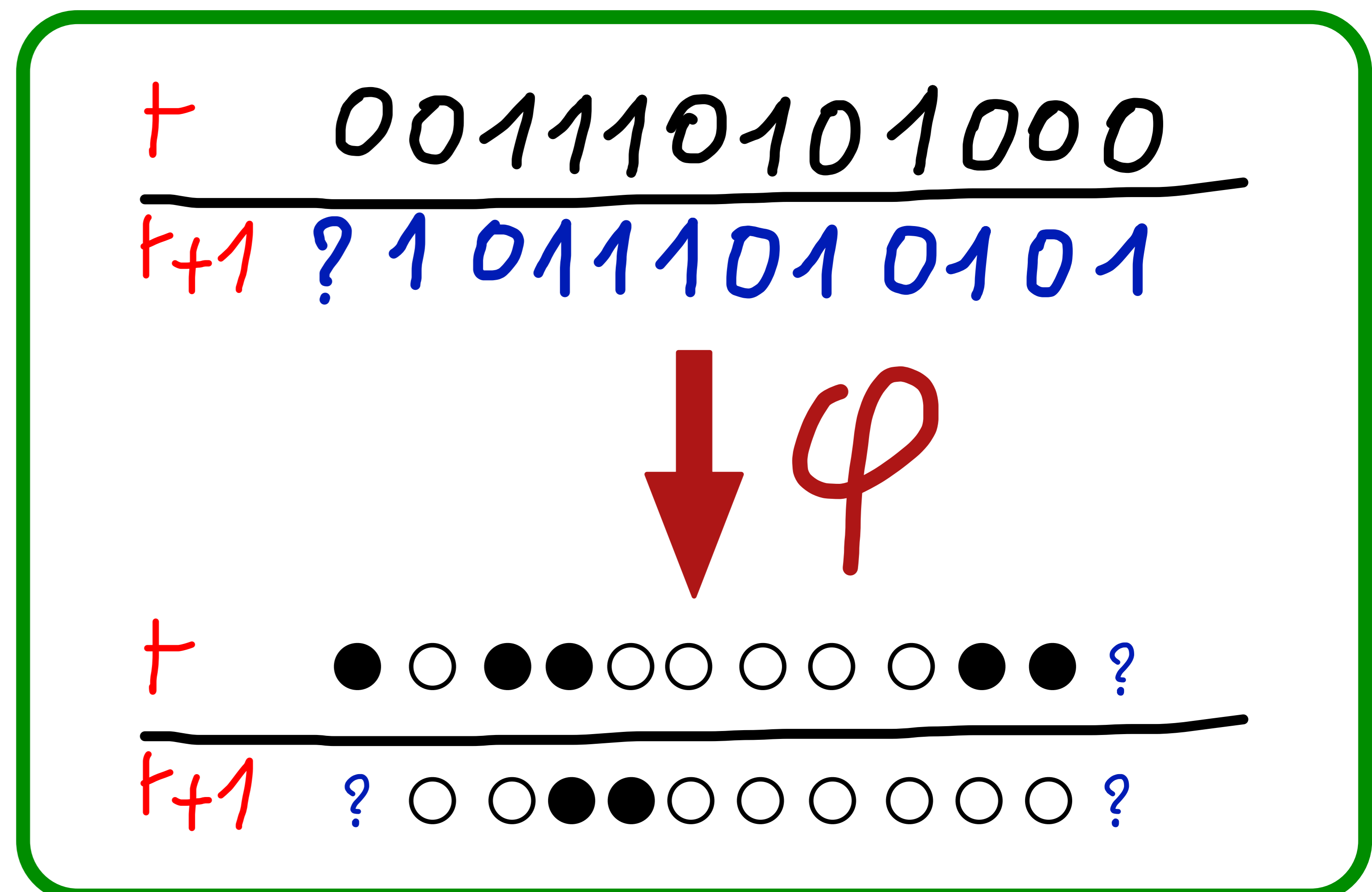
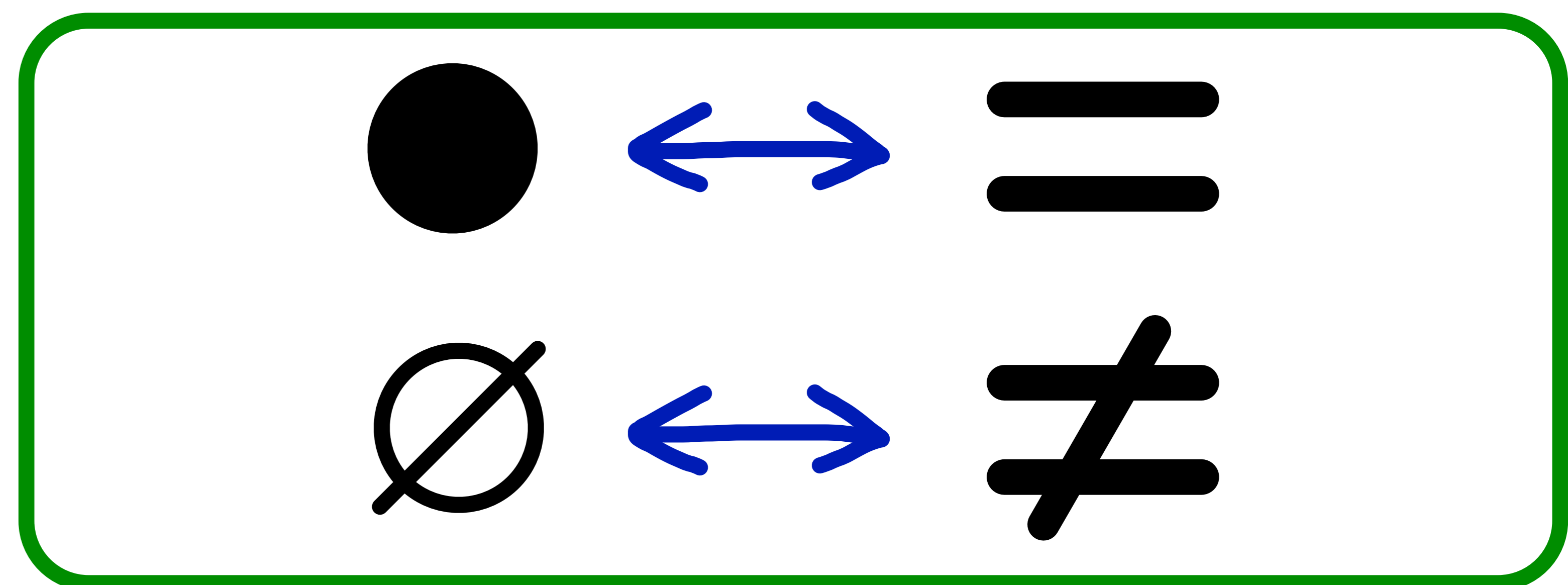
but also:

$$\varphi(\alpha \delta_a + (1-\alpha) \delta_b) = \delta_{\emptyset^{\mathbb{Z}}}$$

and $\alpha \delta_a + (1-\alpha) \delta_b$ are the only solutions of $\varphi(v) = \delta_{\emptyset^{\mathbb{Z}}}$.

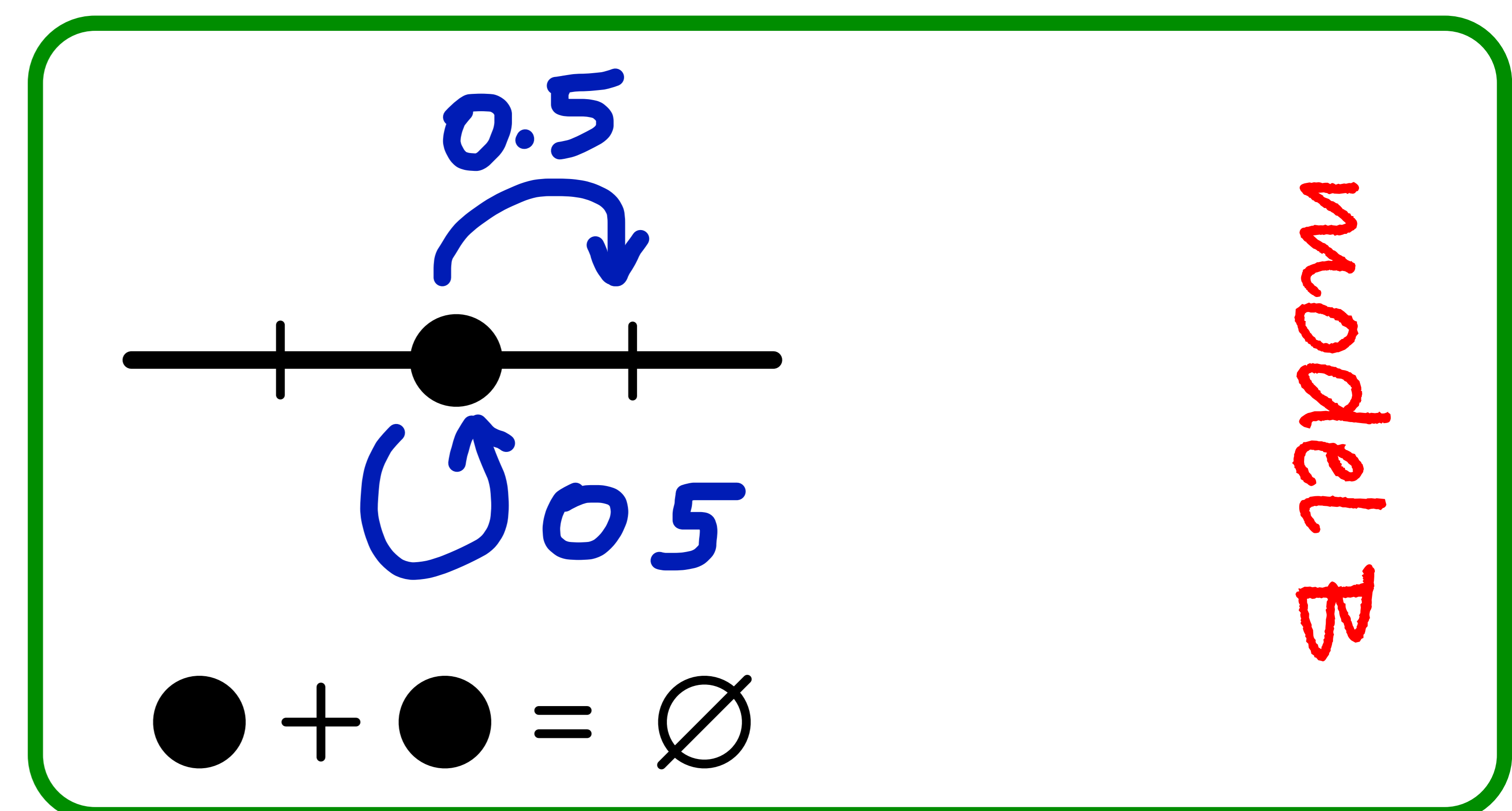
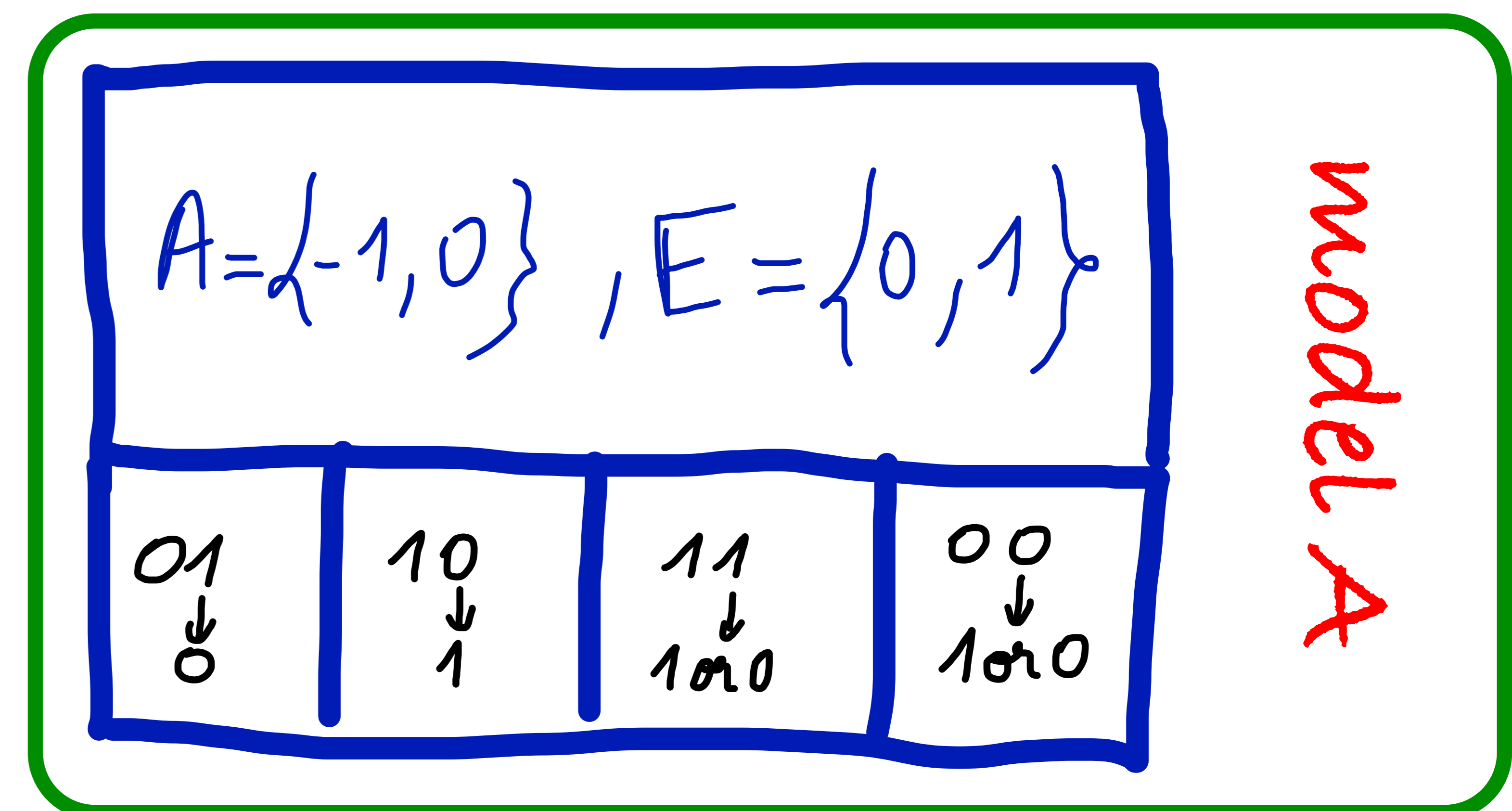
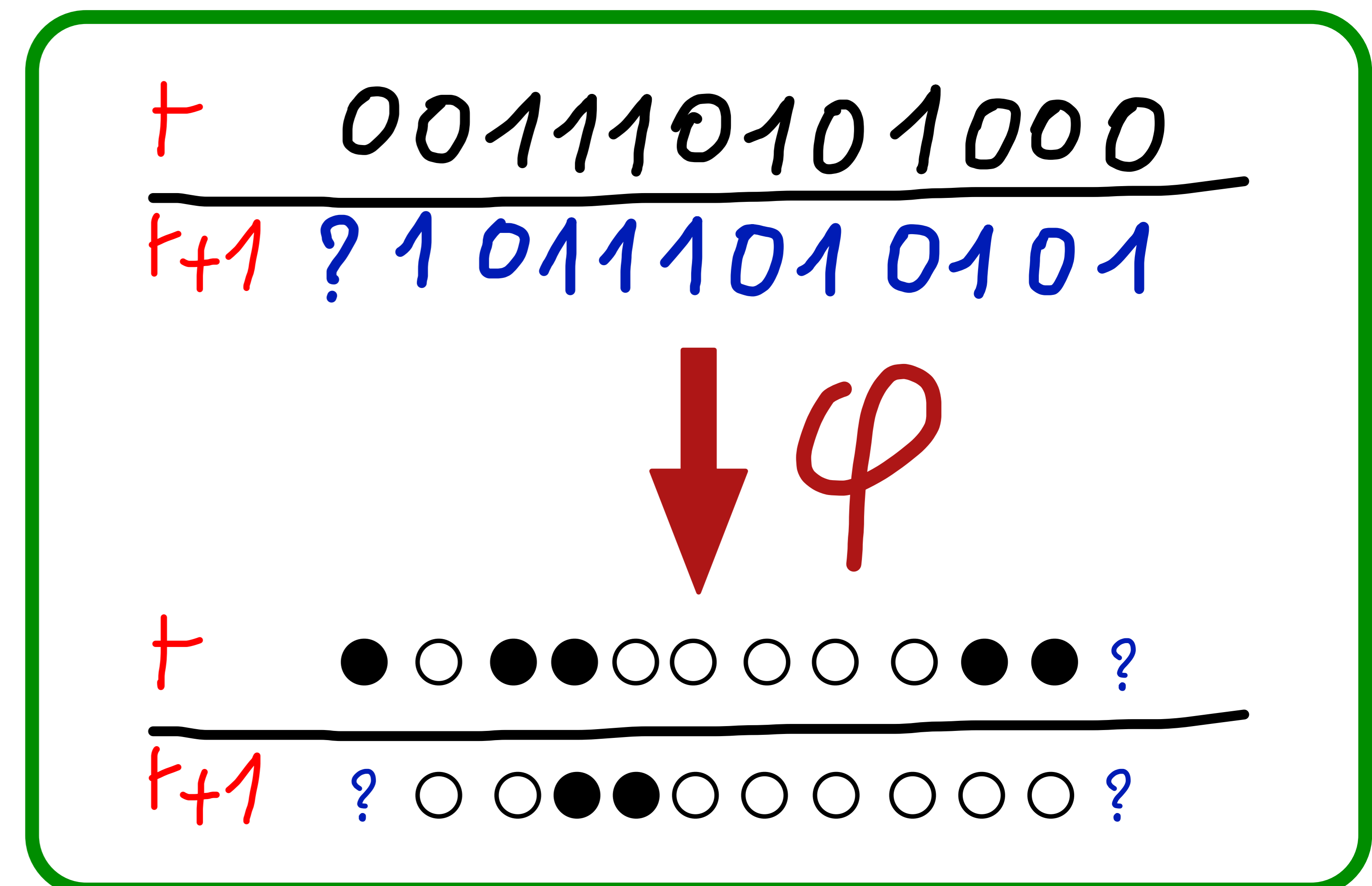
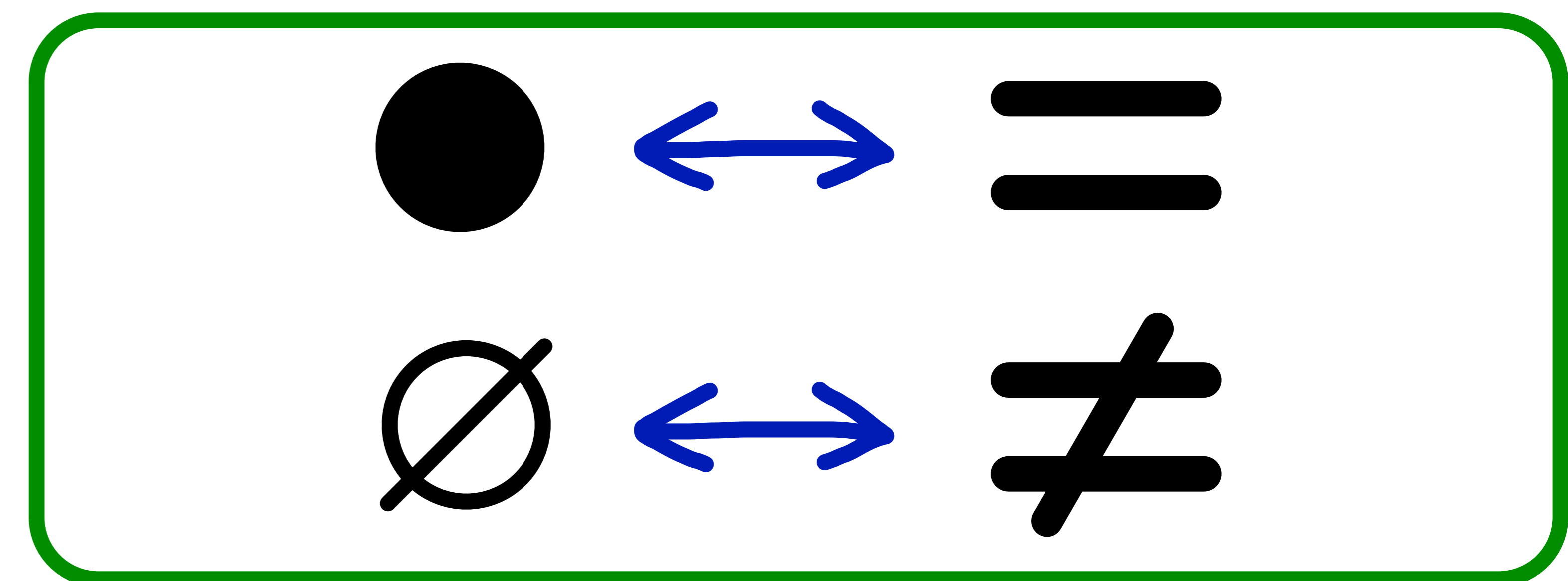


★ The $\alpha \delta_a + (1-\alpha) \delta_b$ are the only solutions of $\varphi(v) = \delta_{\emptyset^2}$.



★ The $\alpha \delta_a + (1-\alpha) \delta_b$ are the only solutions of $\varphi(v) = \delta_{\emptyset^z}$.

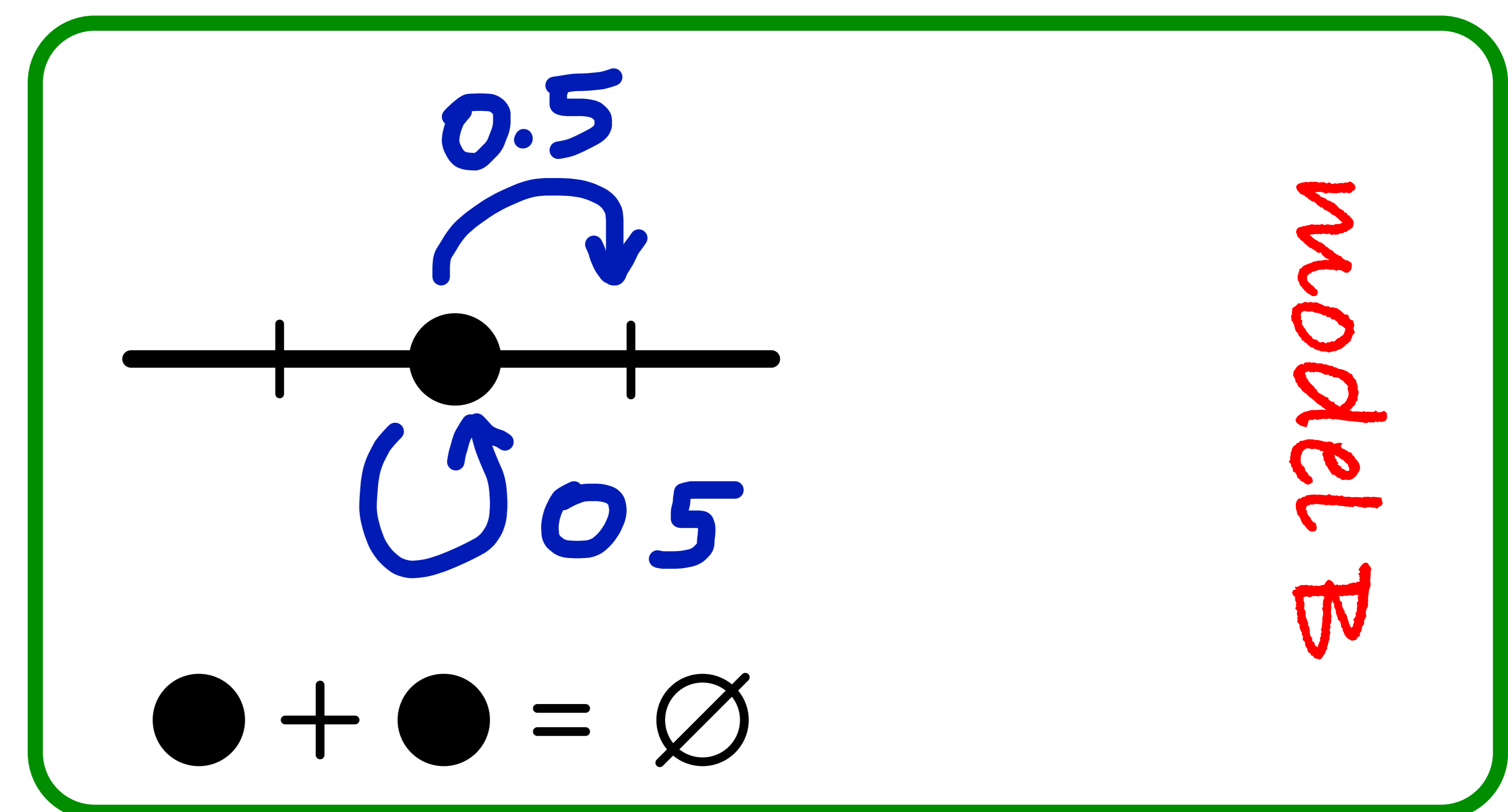
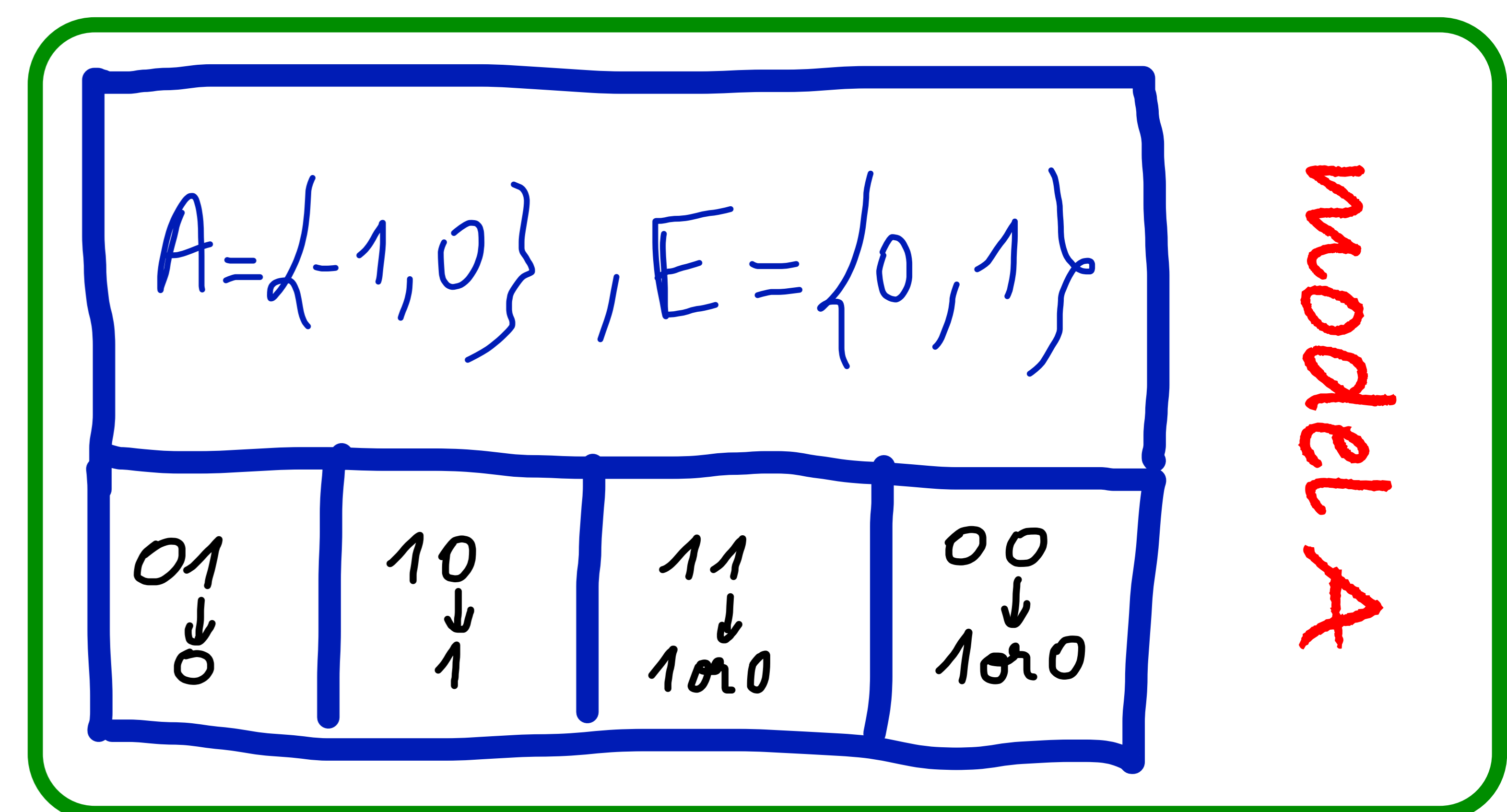
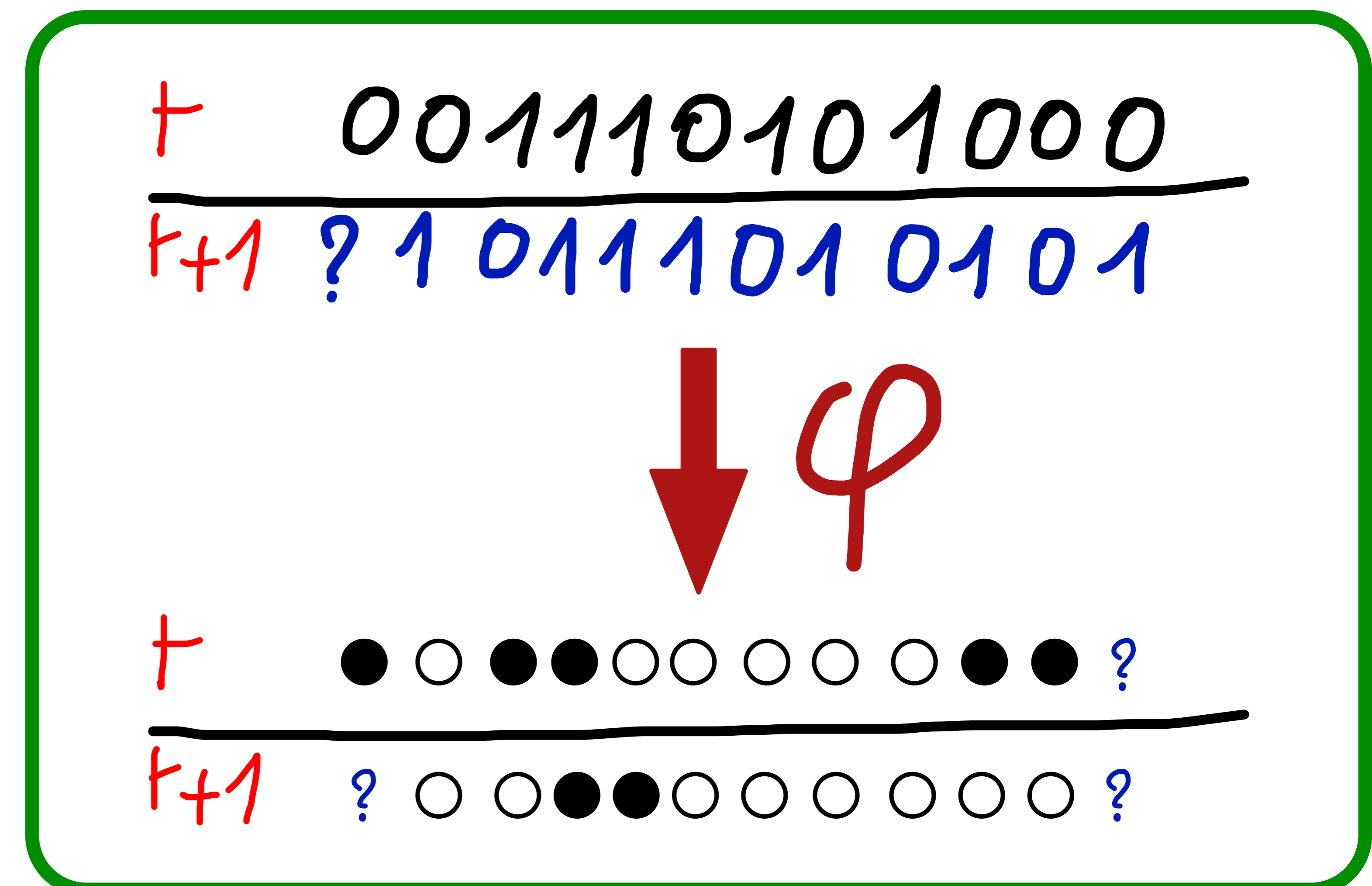
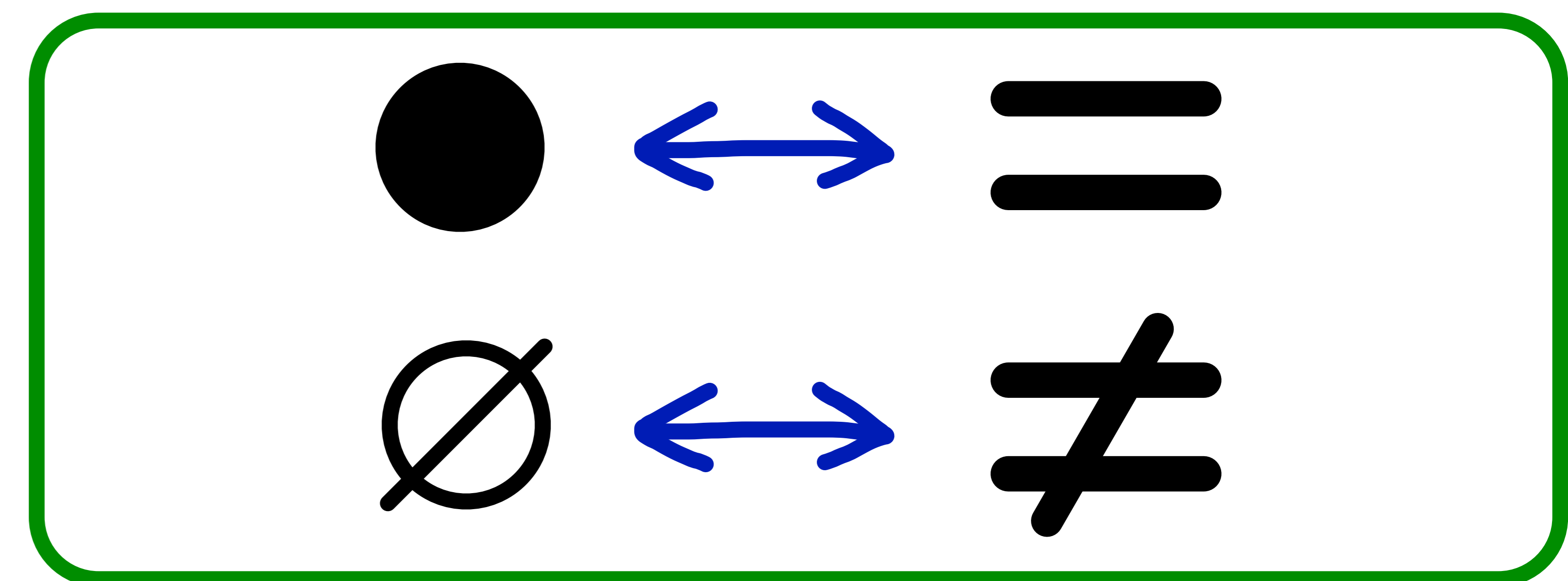
★ If, starting from any configuration, model B converges to δ_{\emptyset^z} , then δ_{\emptyset^z} is the only stationary probability of model B.



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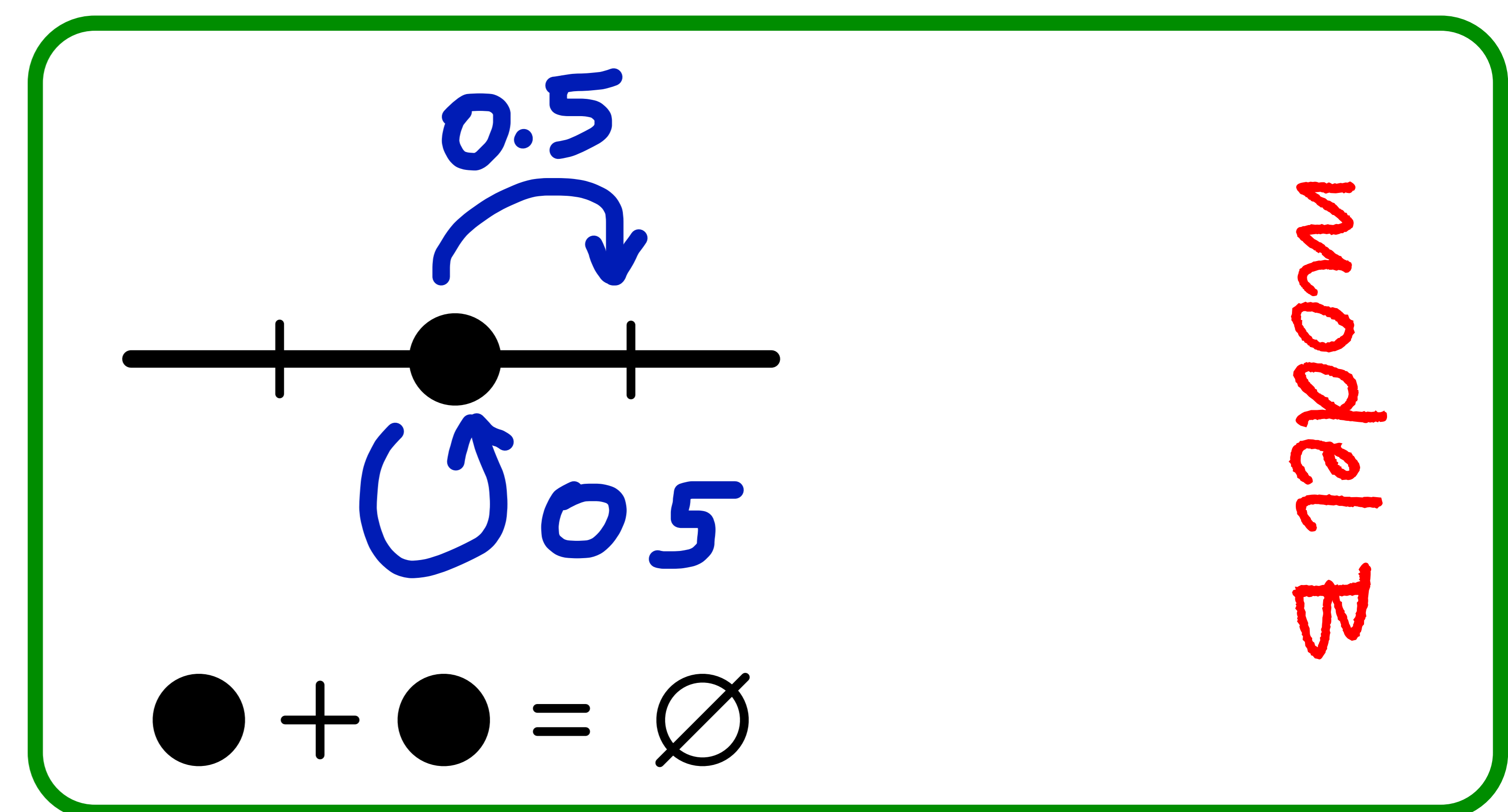
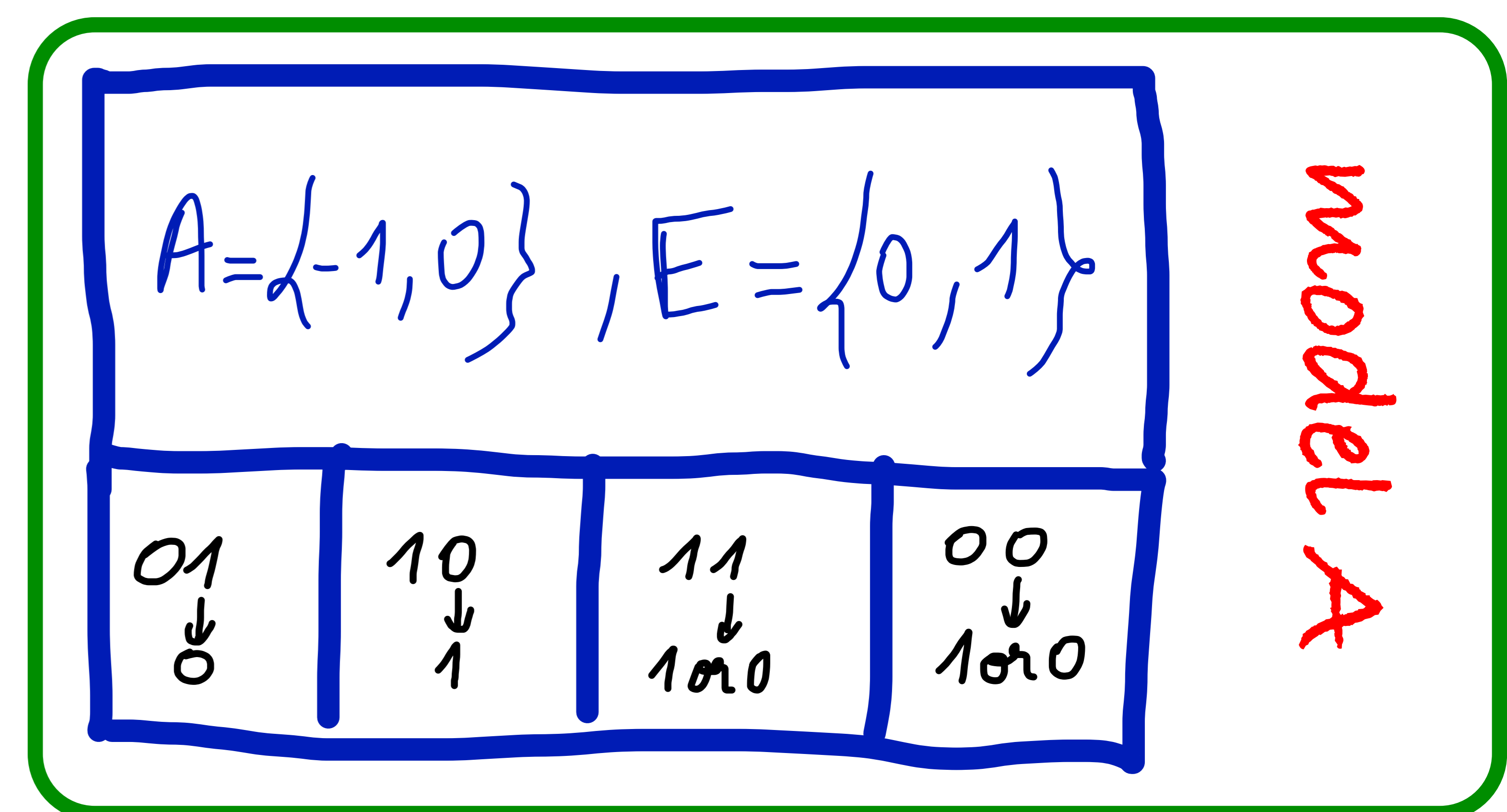
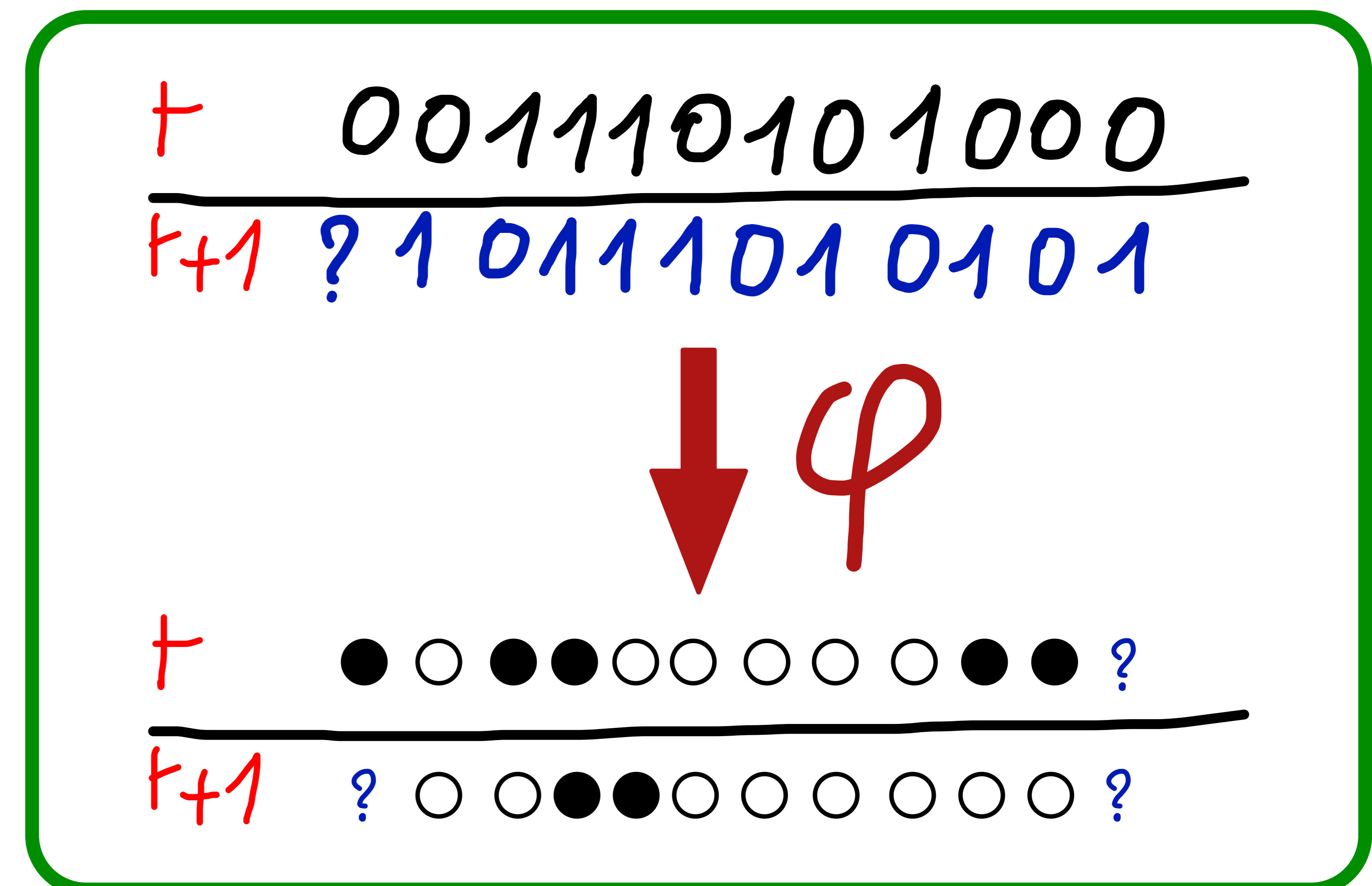
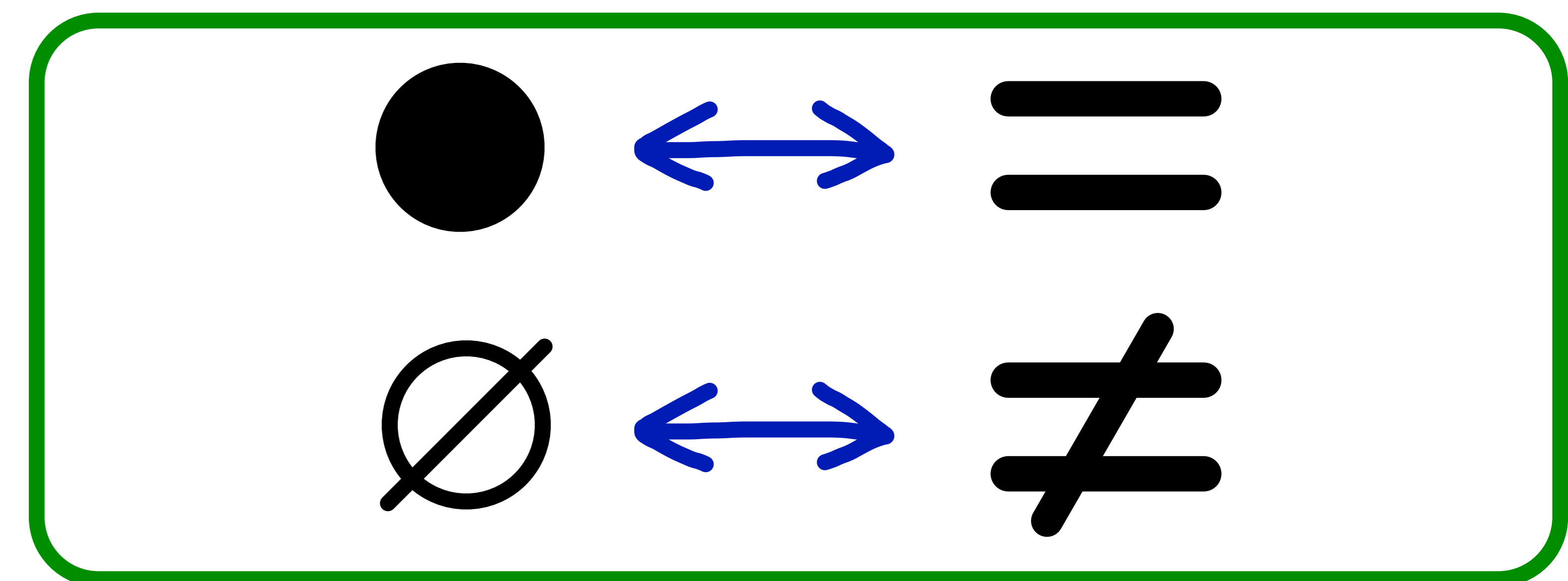


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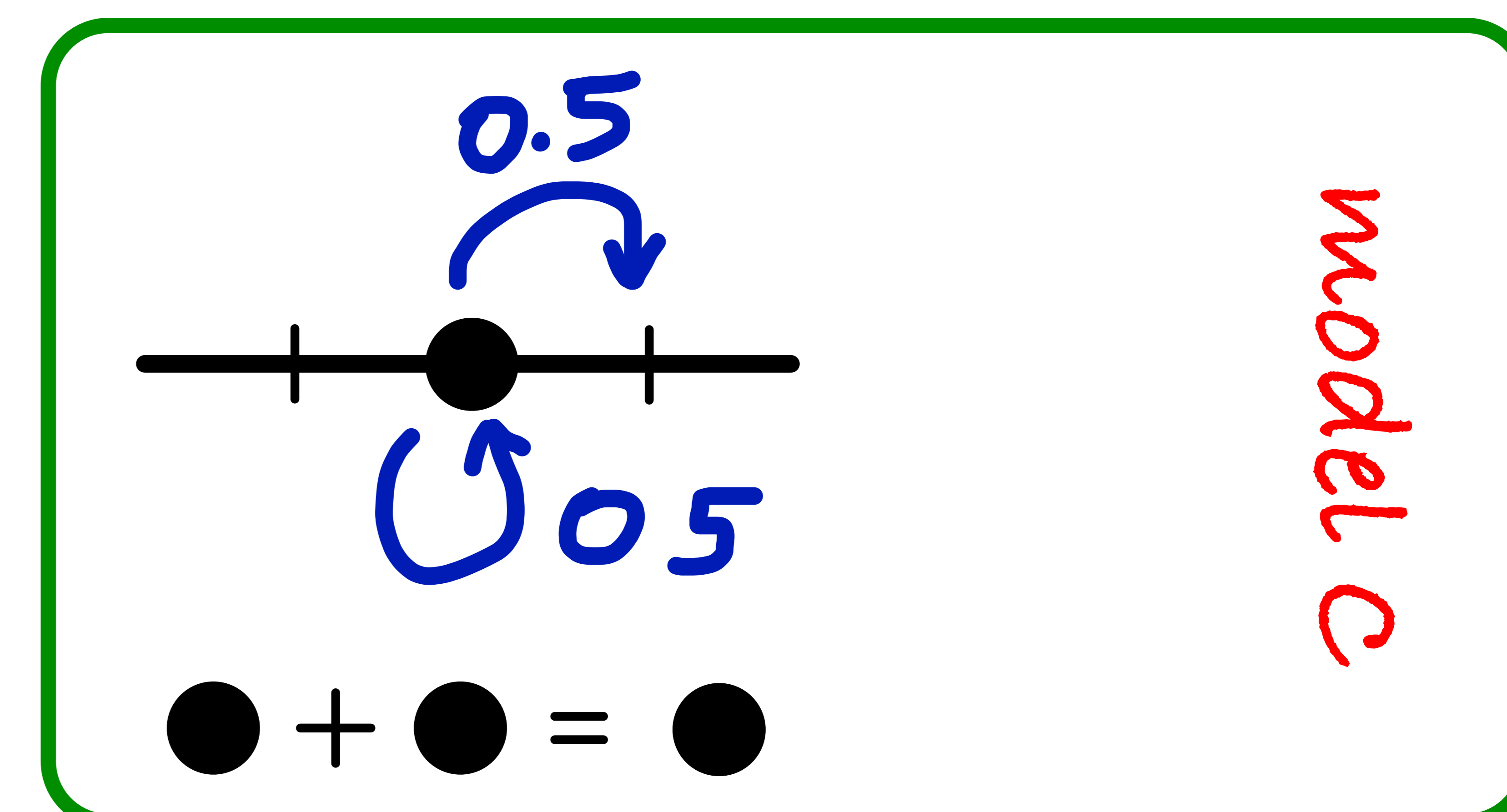
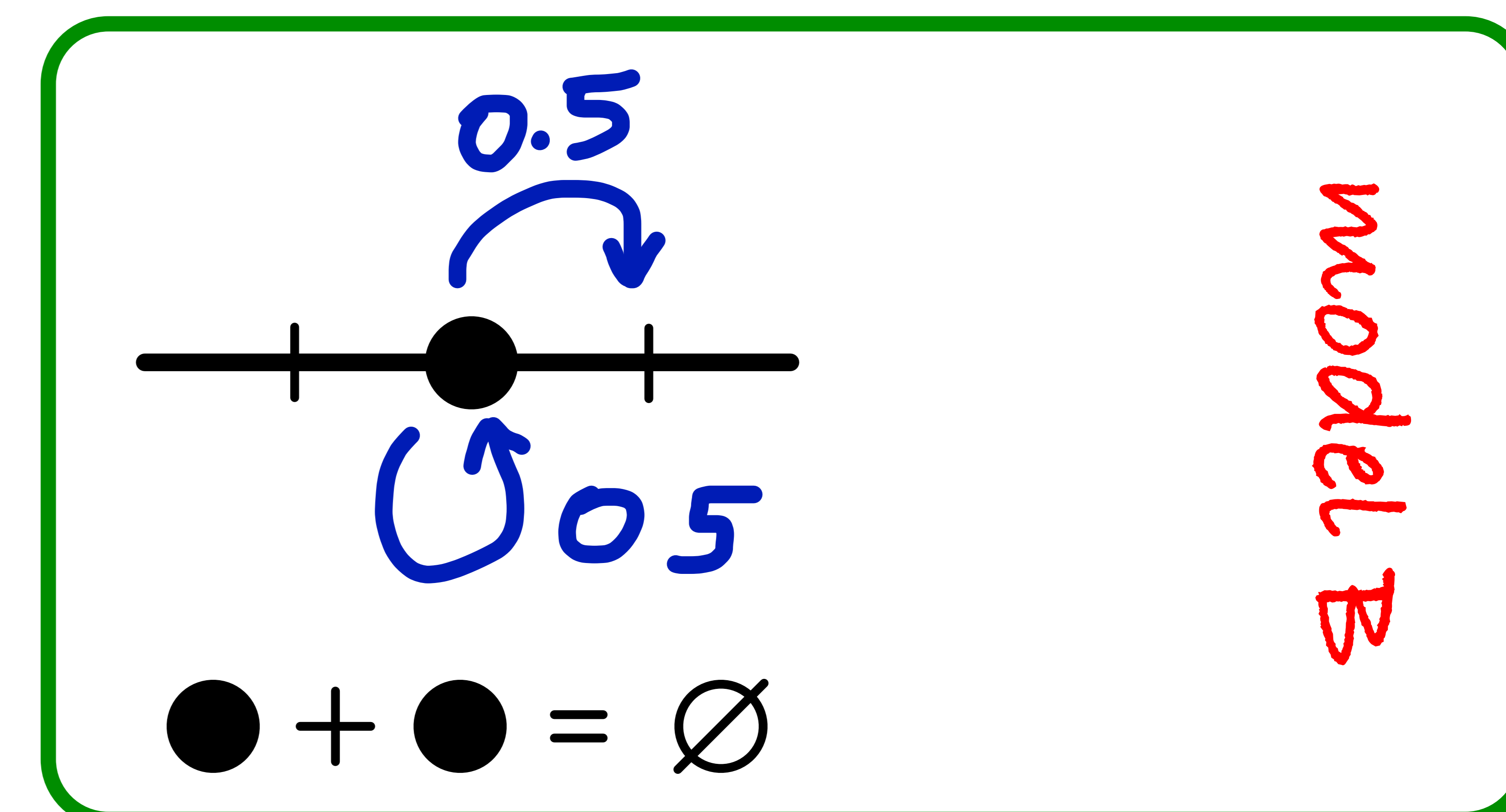
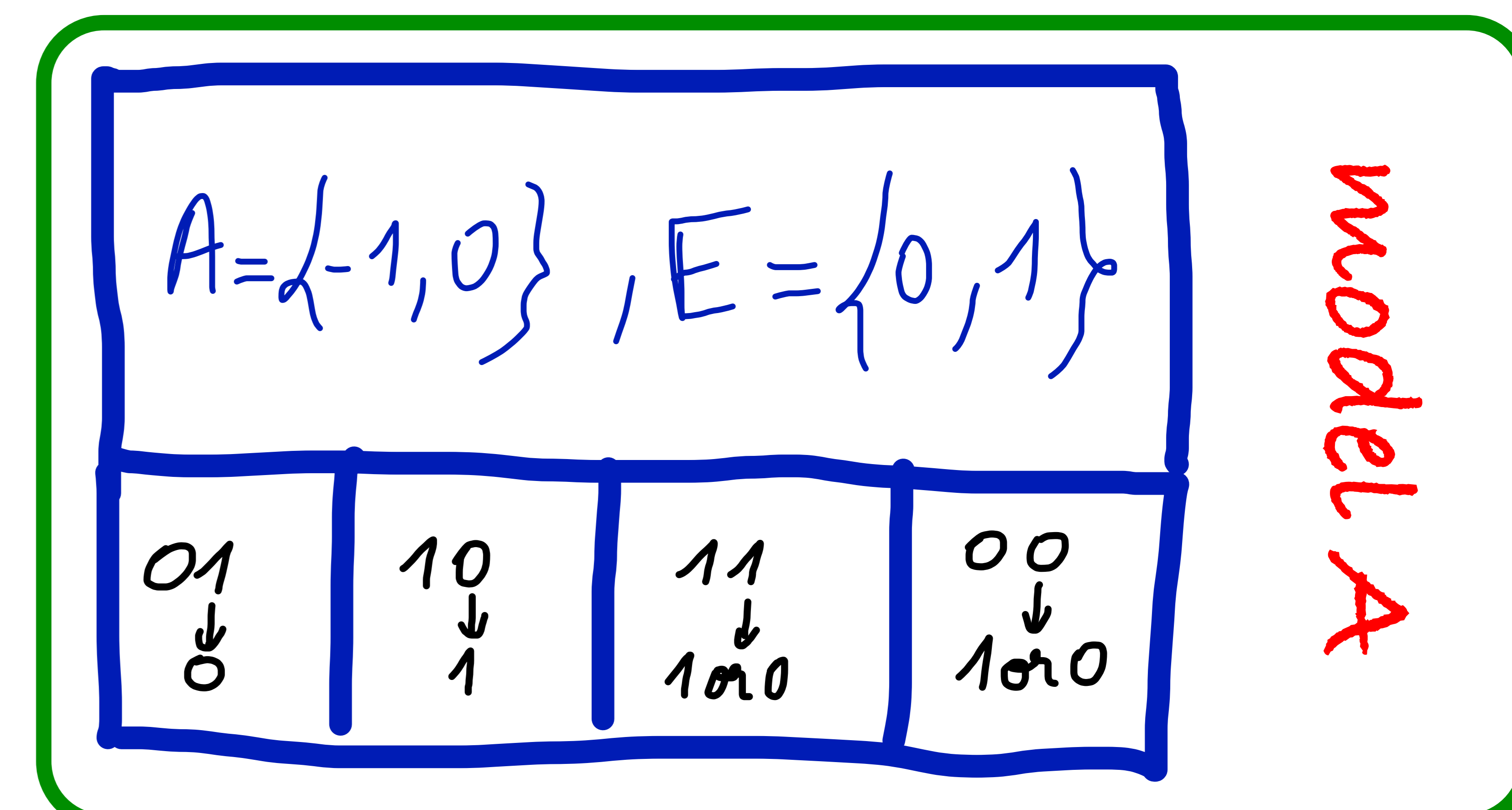
★ Thus the $\alpha \delta_a + (1-\alpha) \delta_b$ are the only candidates for the status of stationary probability of model A.

★ Among them, only μ is stationary.



★ Model B is not monotone while model C is :

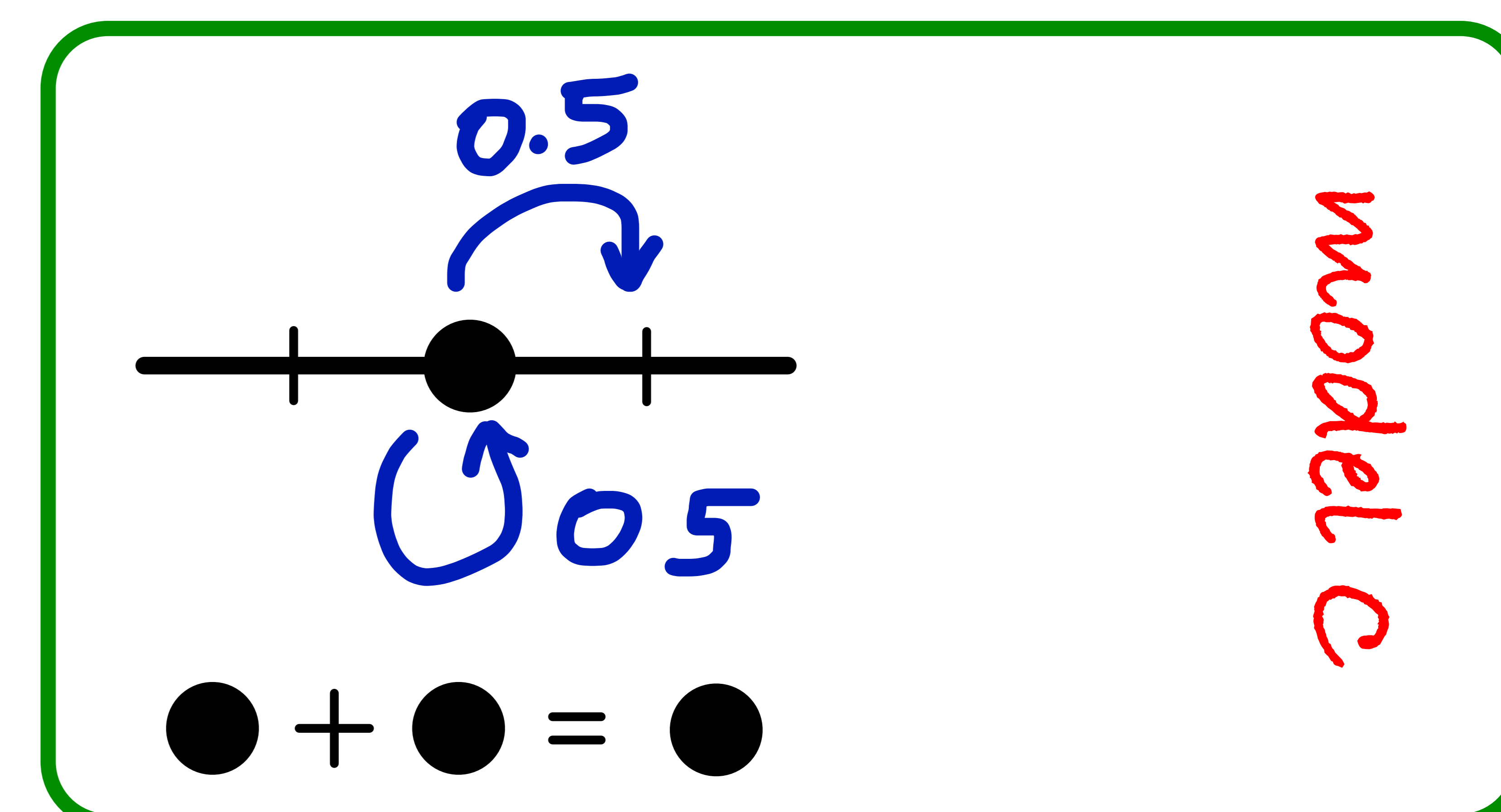
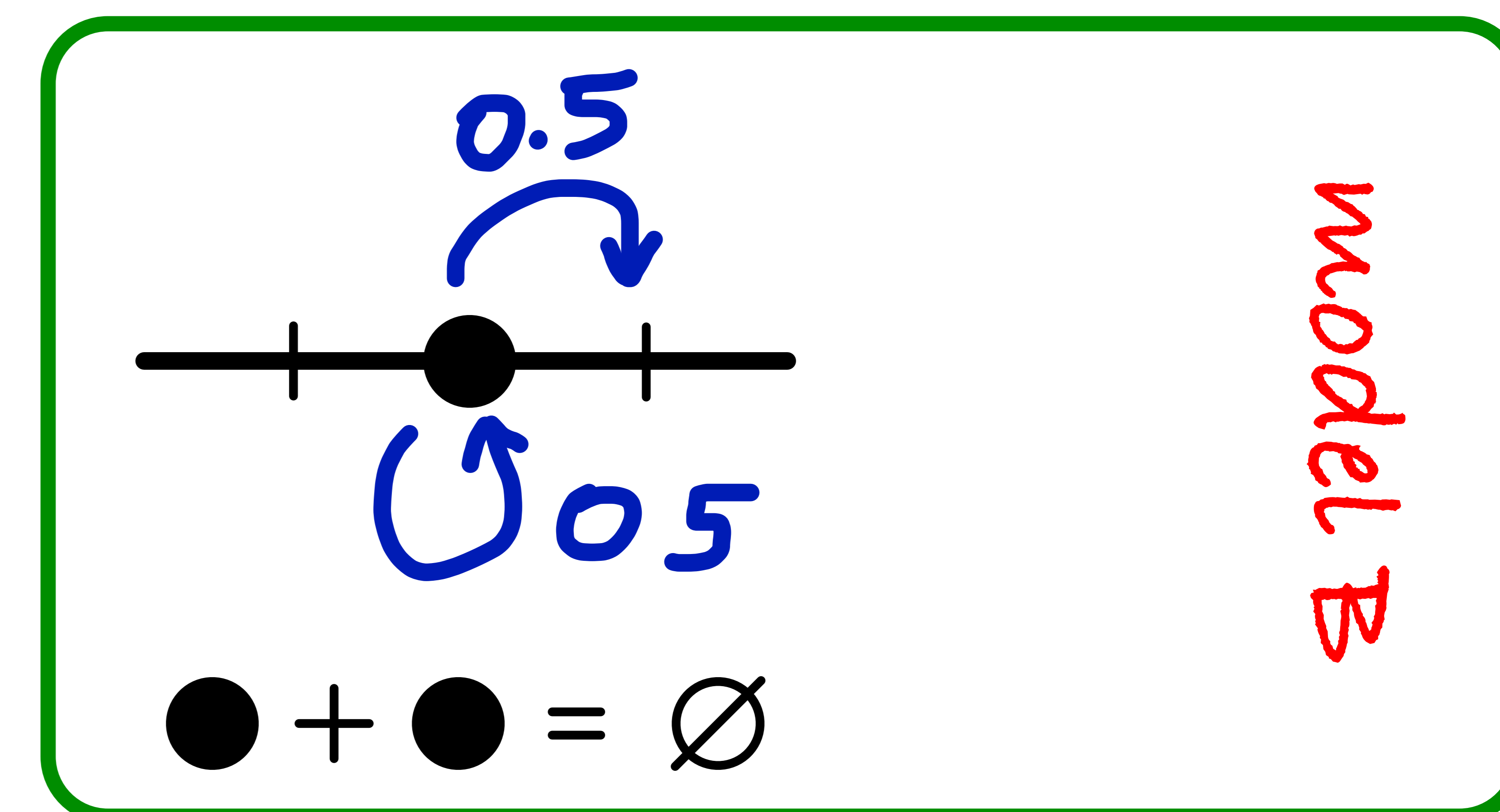
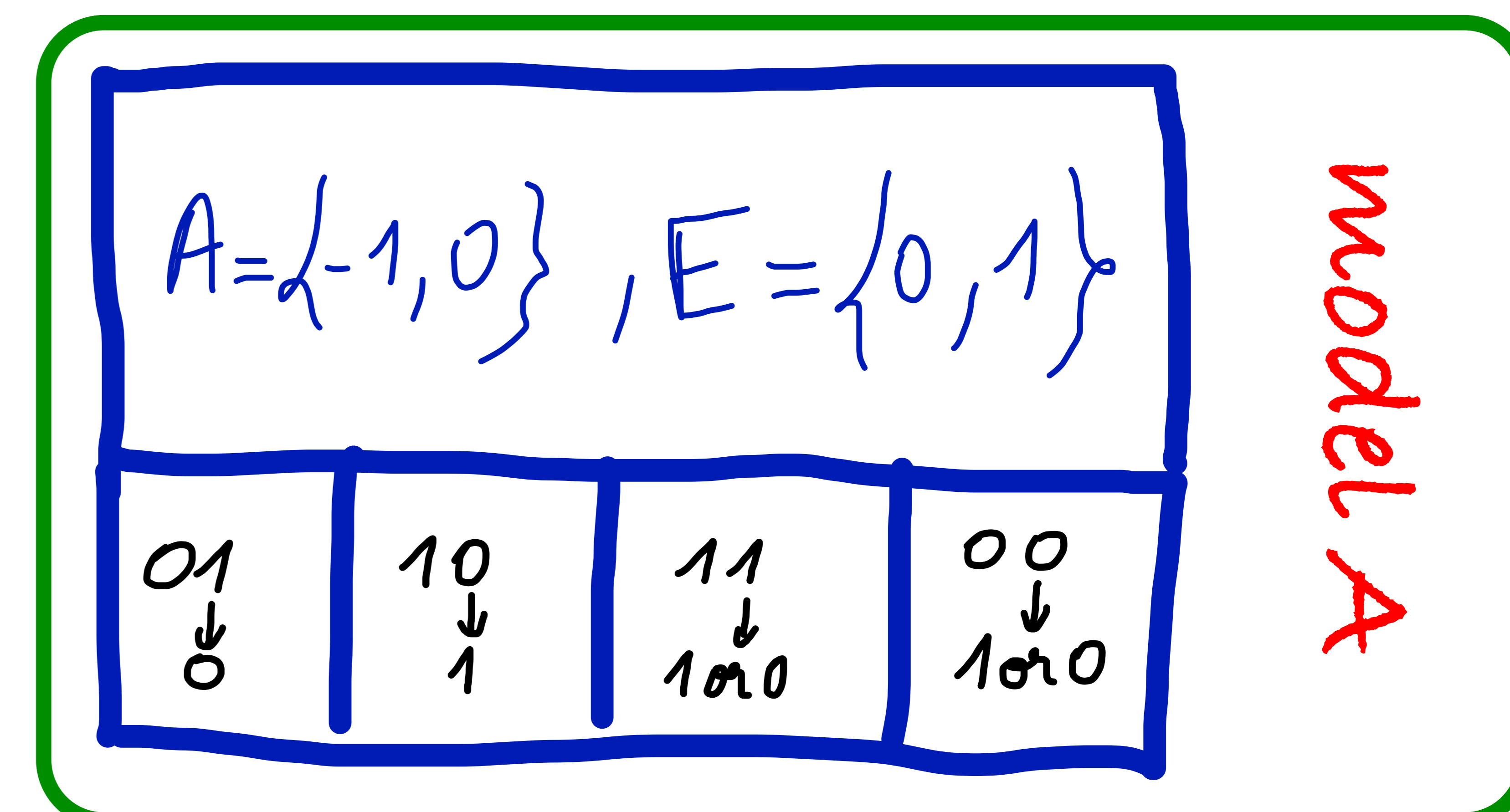
If, starting from $\bullet^{\mathbb{Z}}$, model C goes to $\emptyset^{\mathbb{Z}}$, then it goes to $\emptyset^{\mathbb{Z}}$ starting from *anywhere*



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★ ⚡ Model C dominates model B

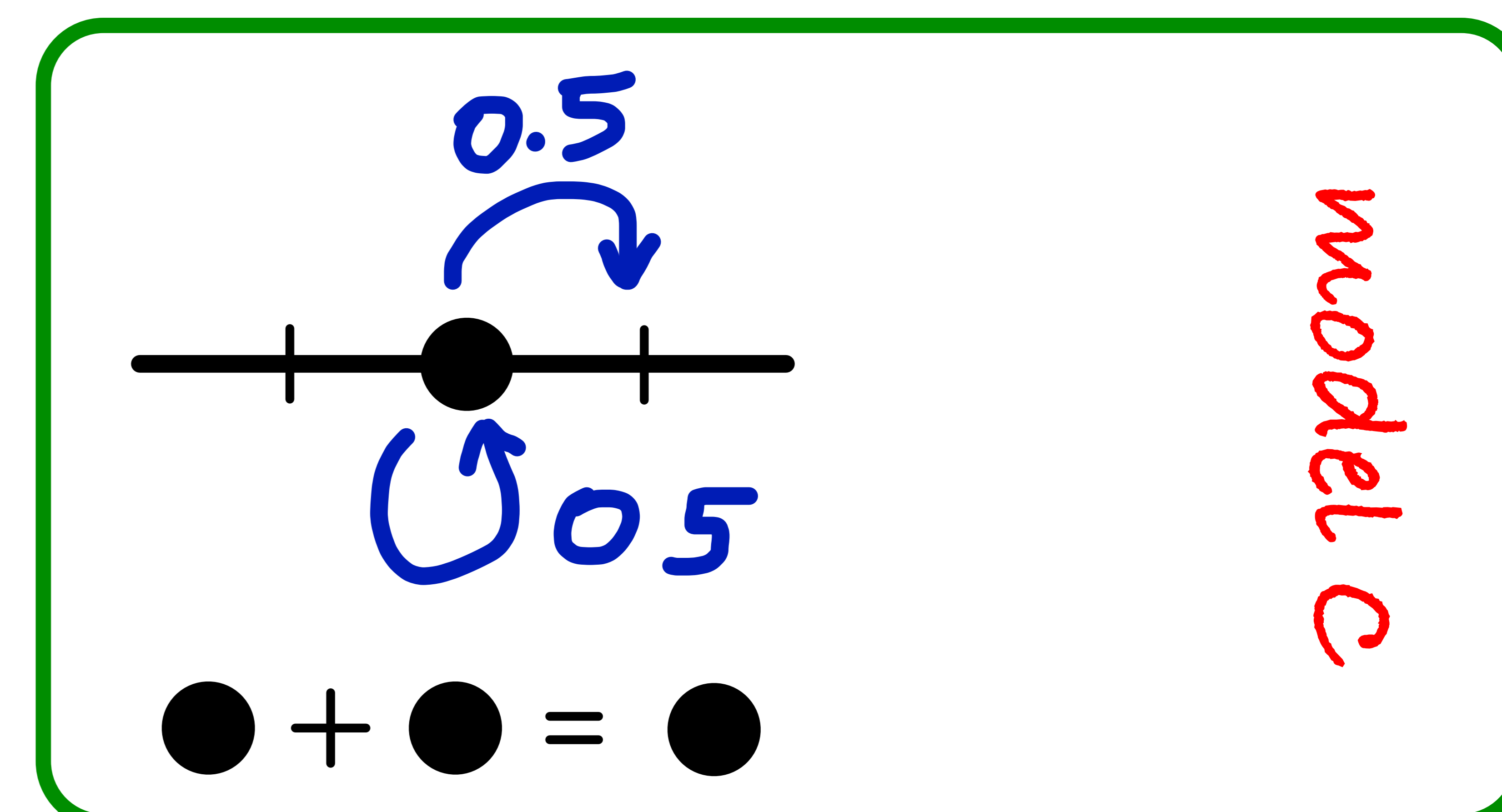
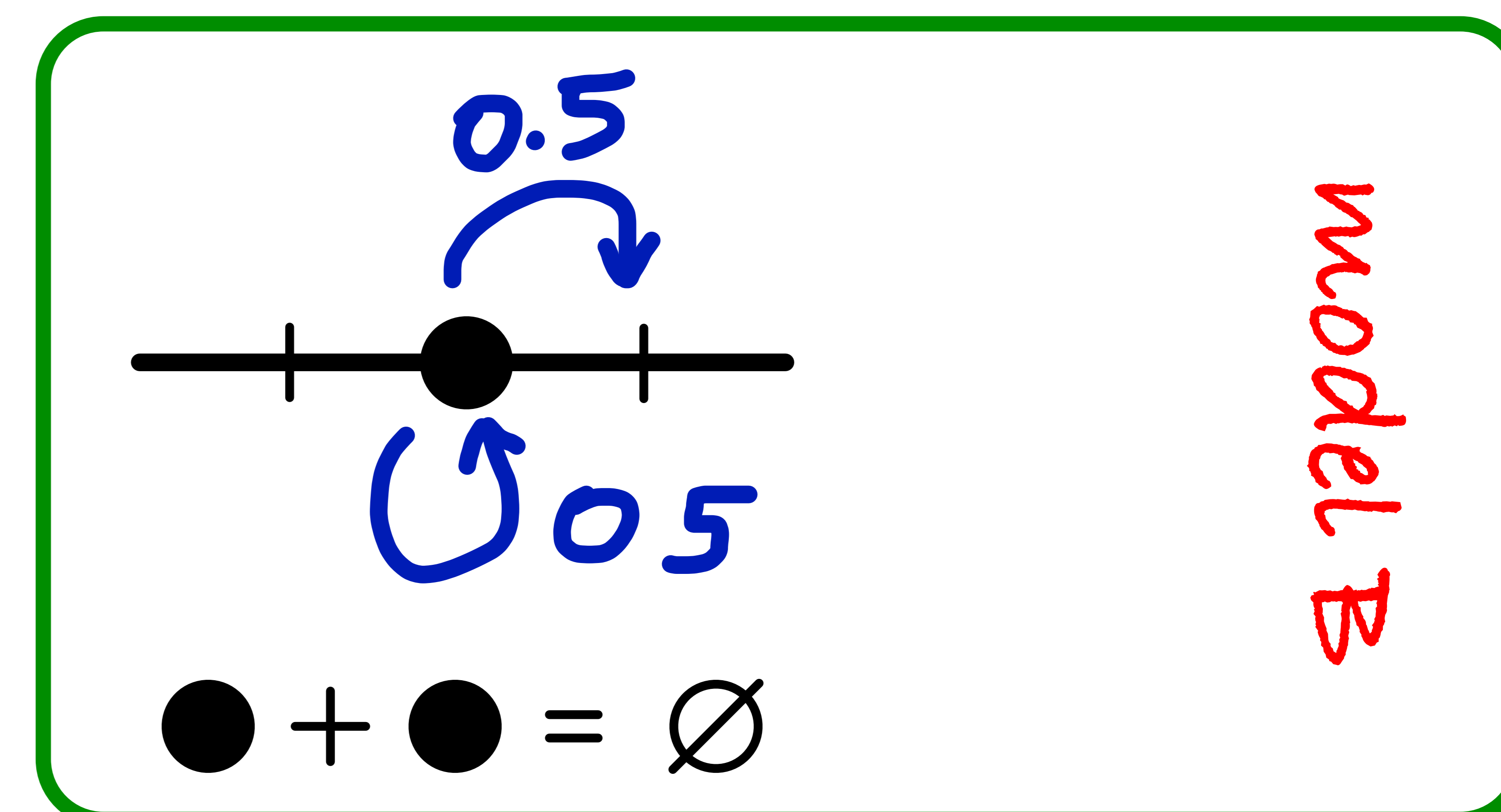
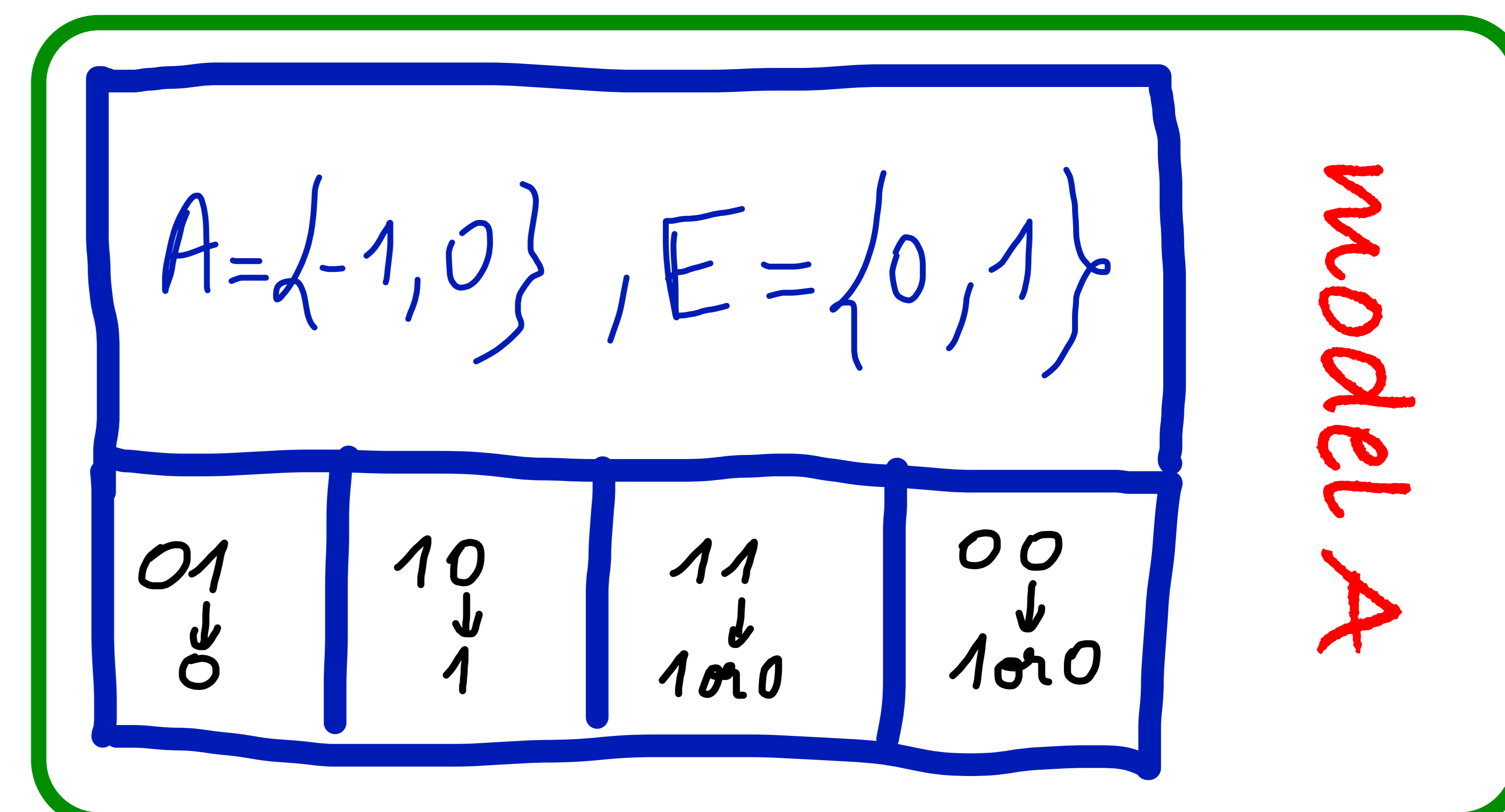


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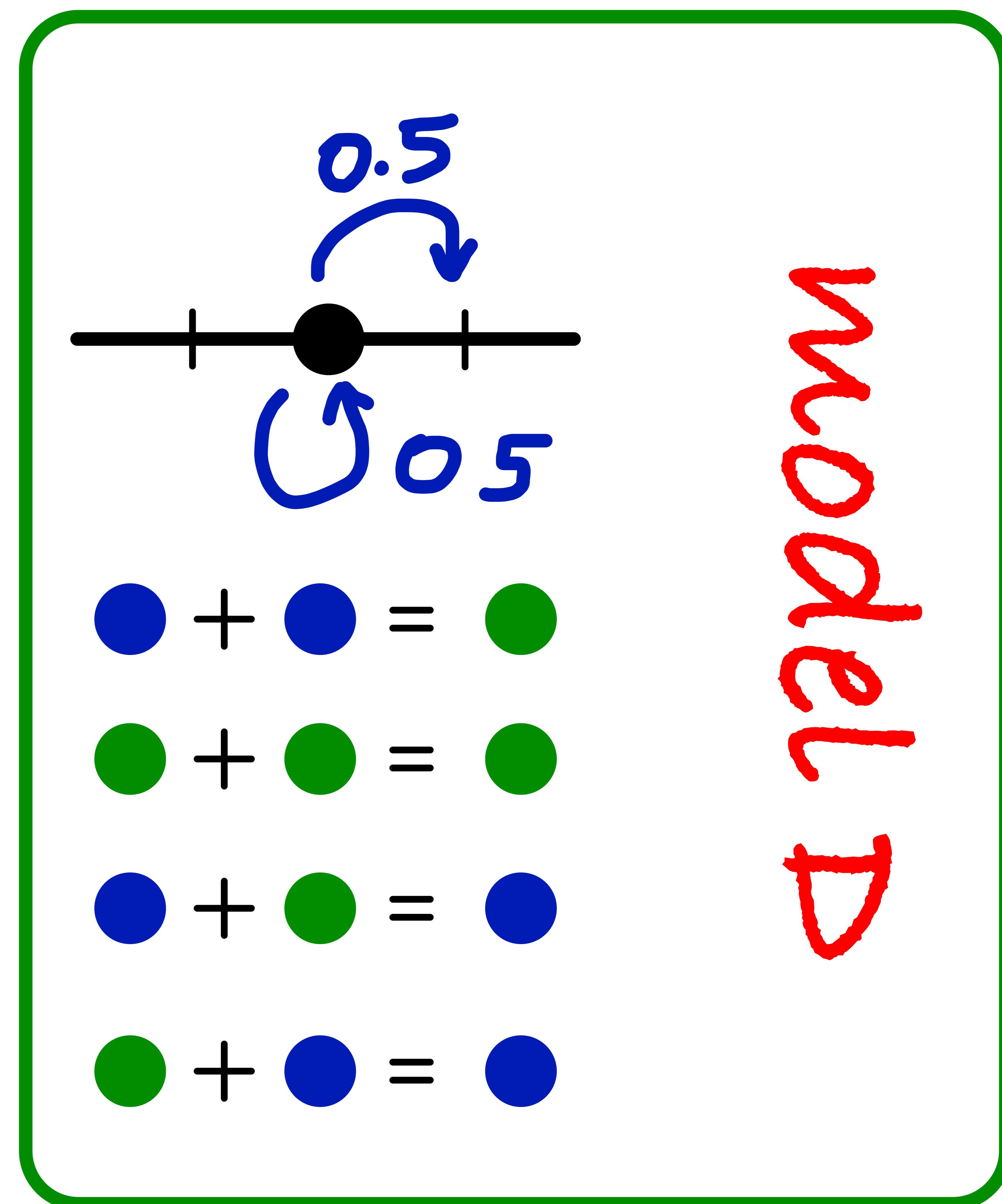
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That is

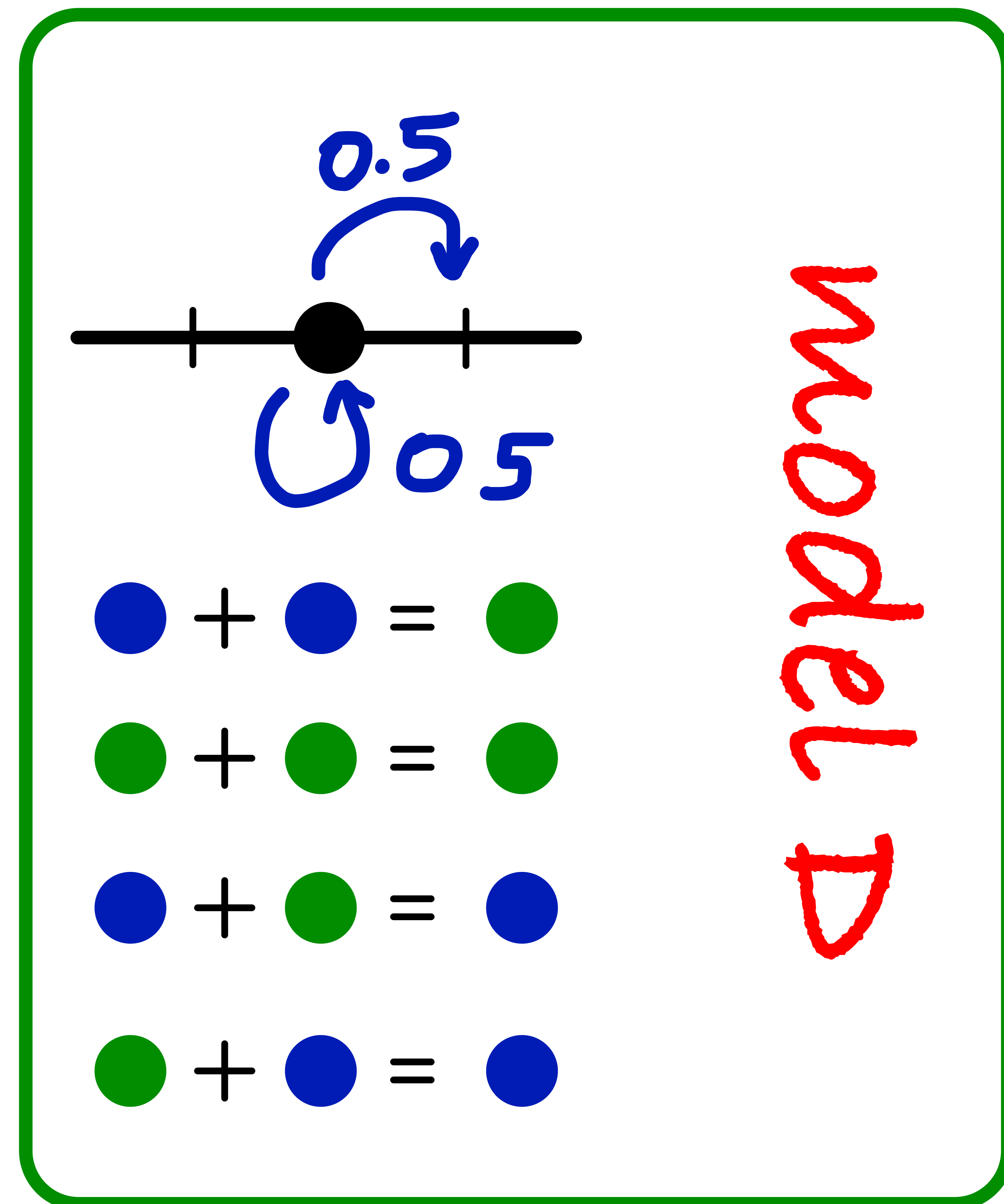


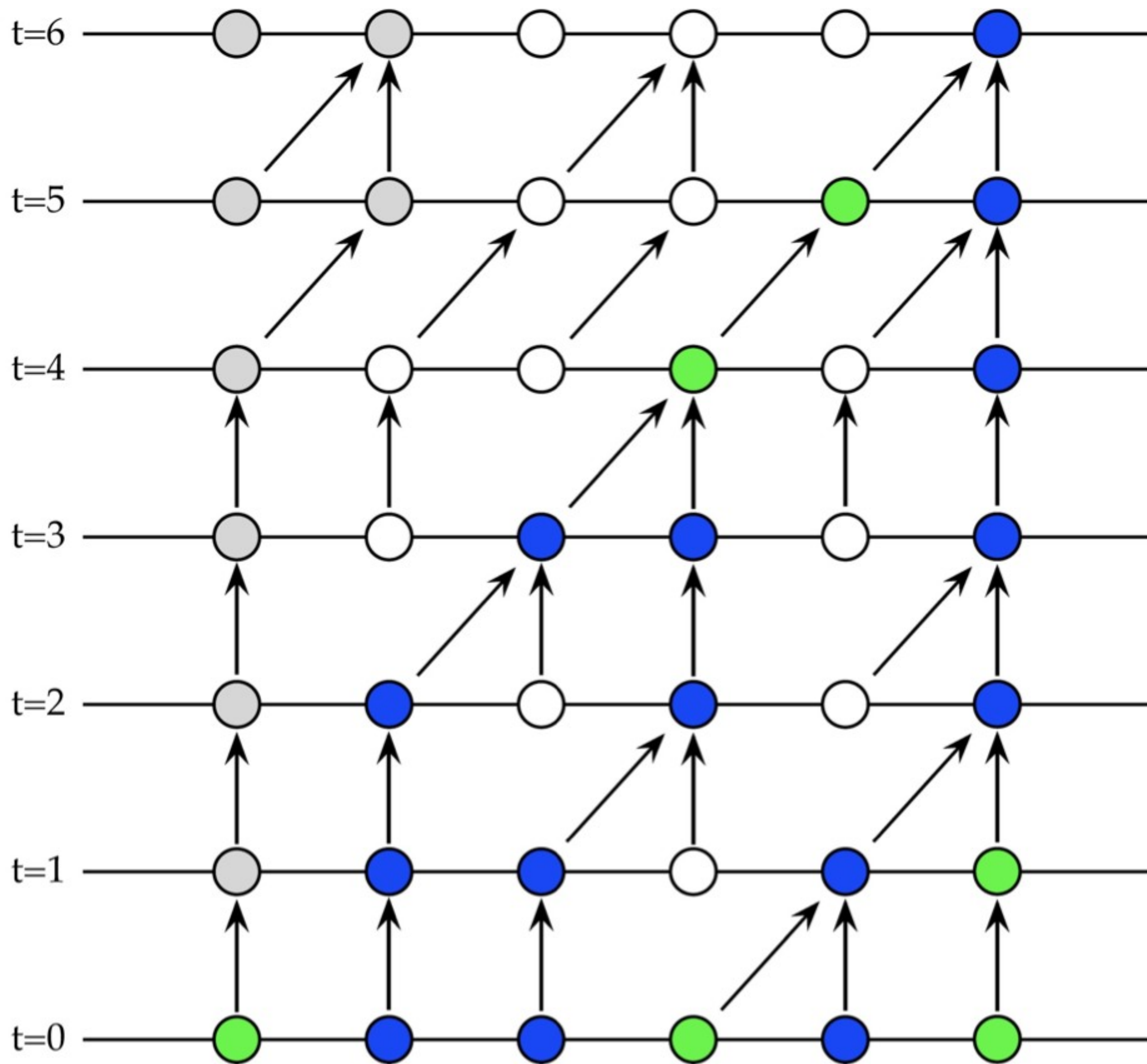
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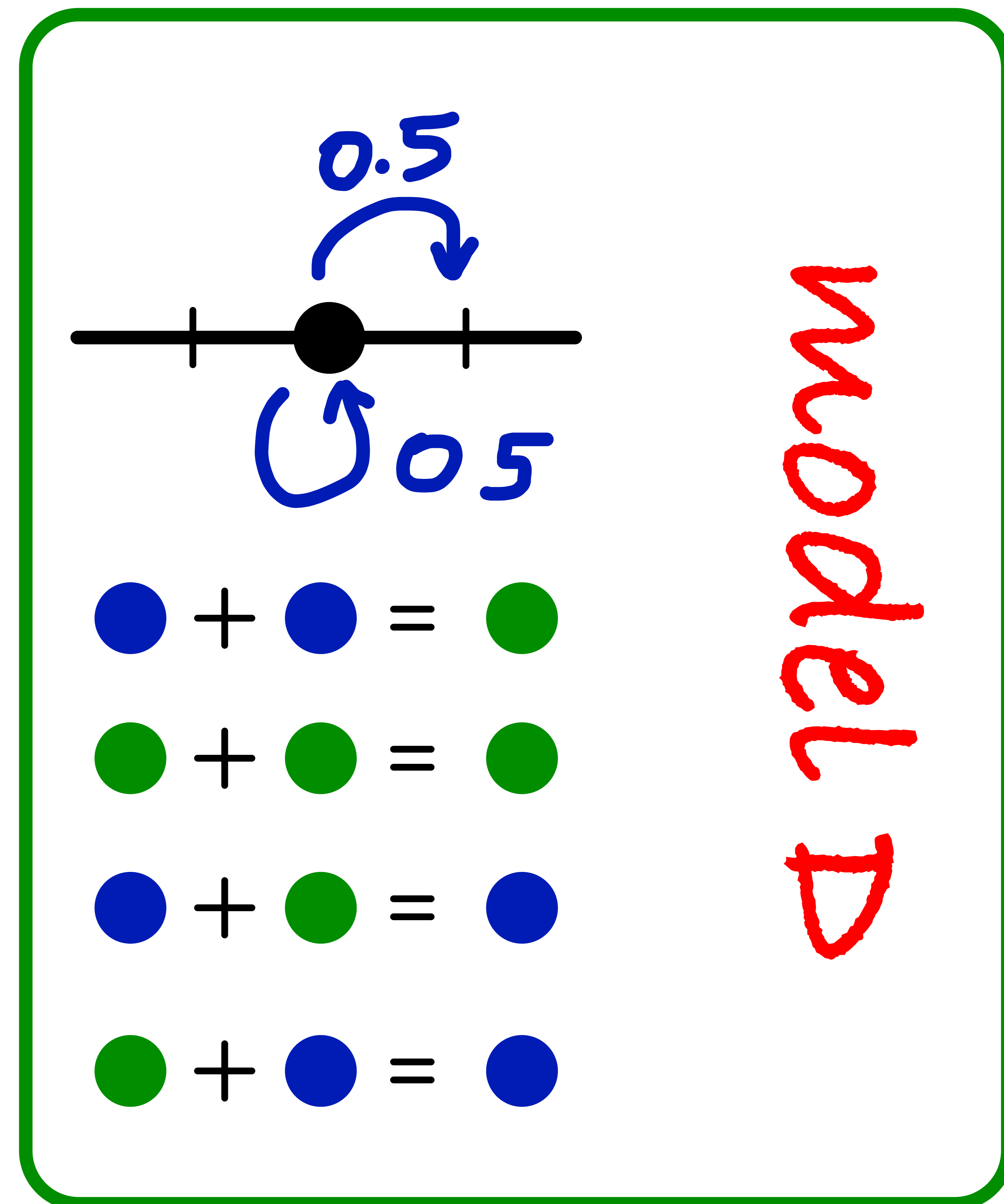
★ ⚡ Thus, start with any blue configuration,
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anywhere, by full model C.



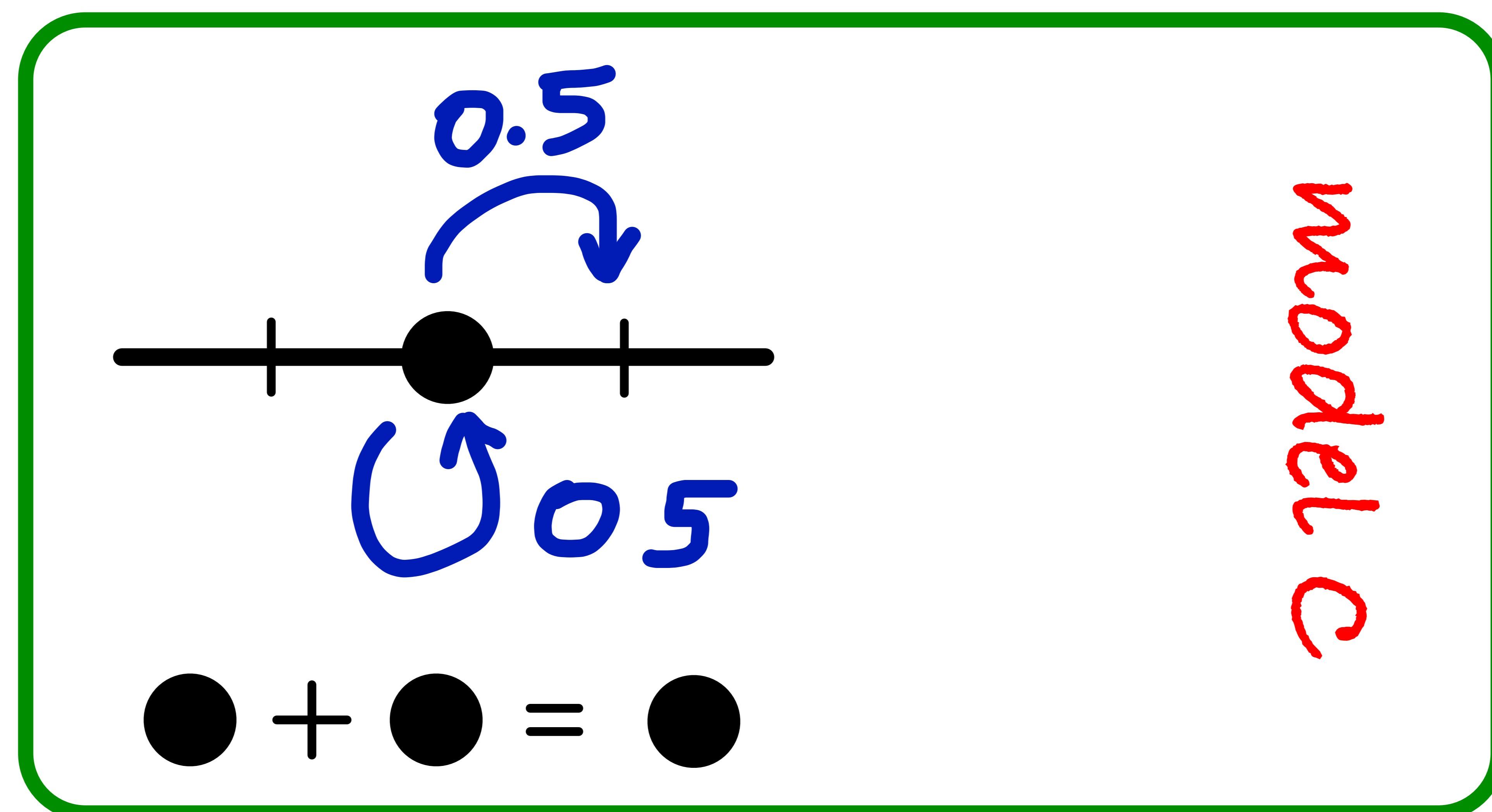


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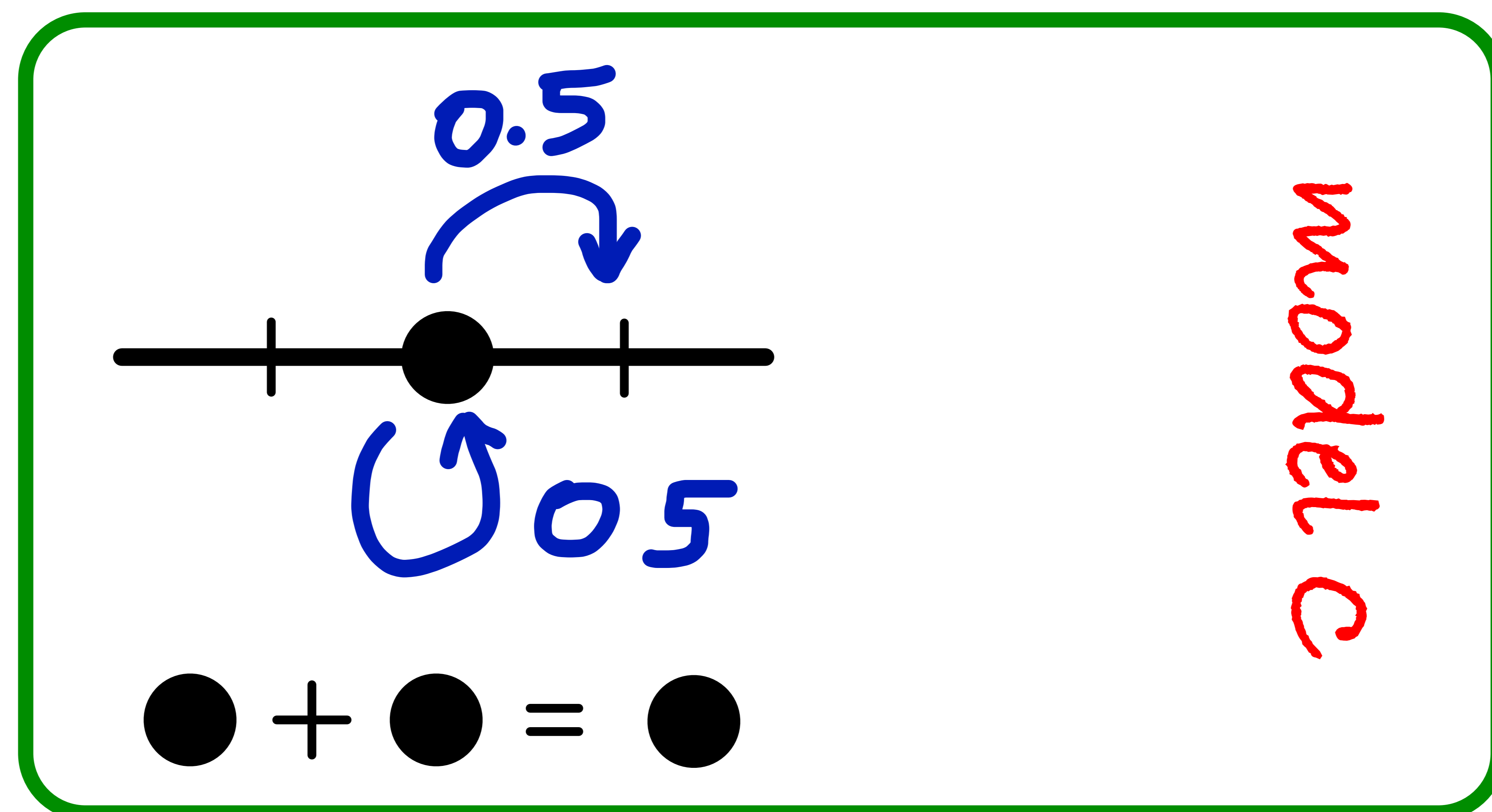


Toom conjectures an exponential rate of convergence for any
 ergodic PCA Actually ...



d_n . density of presence at time n .

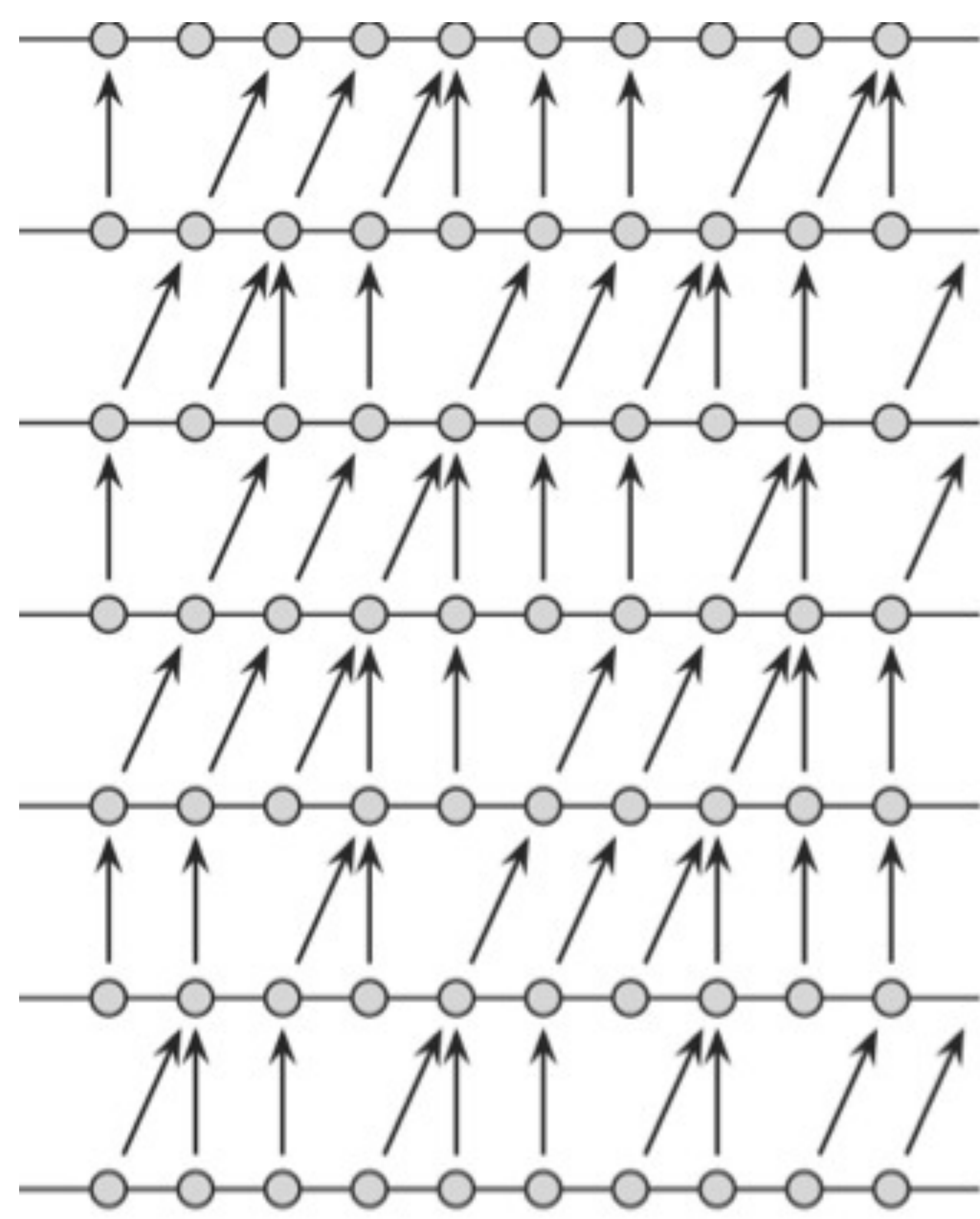
$$d_n = 4^{-n} \binom{2n+1}{n} \sim \frac{2}{\sqrt{\pi n}}$$



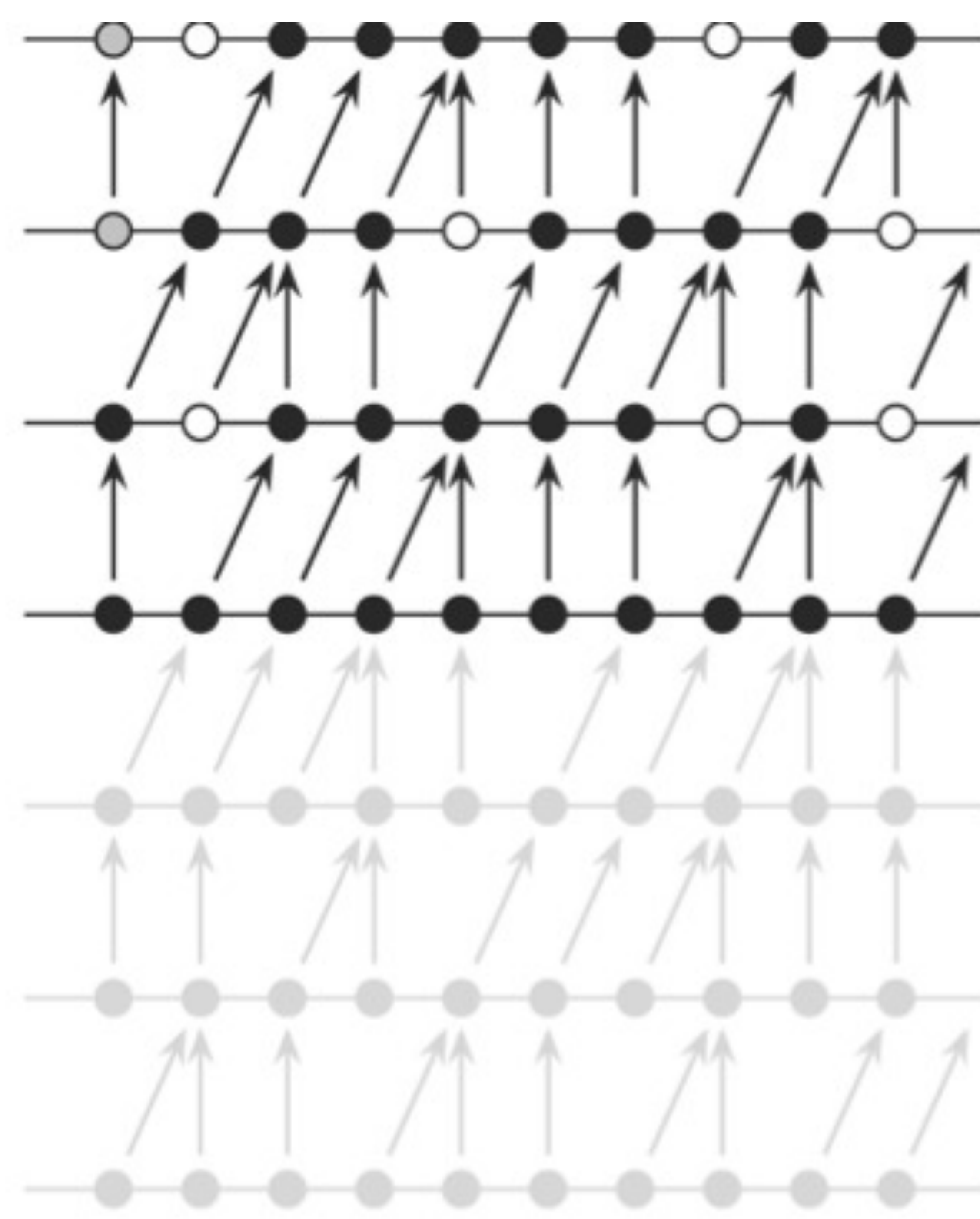
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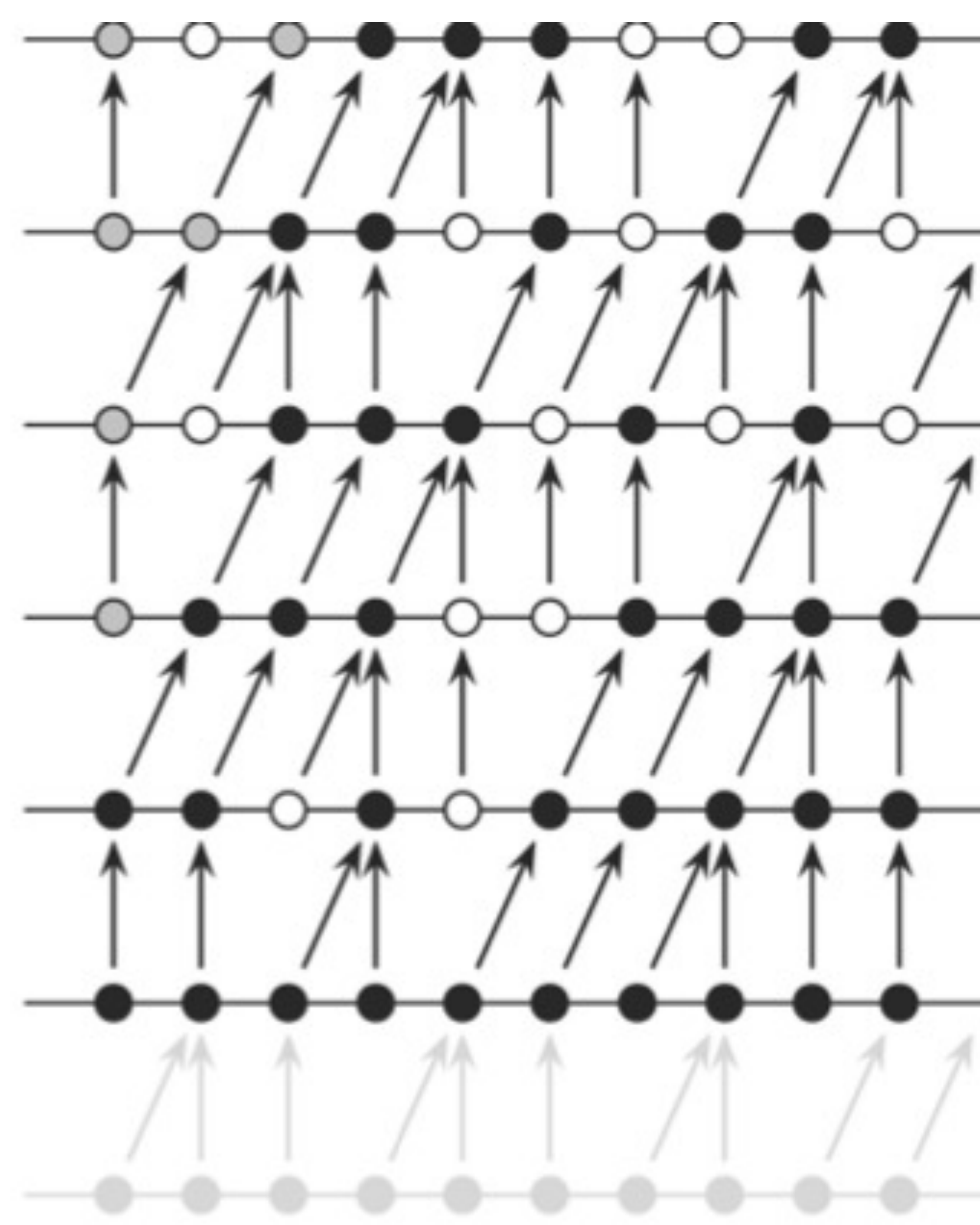
$d_n = \mathbb{P}(T > 2n)$, in which T is the hitting time of 2 for a symmetric random walk starting from 0



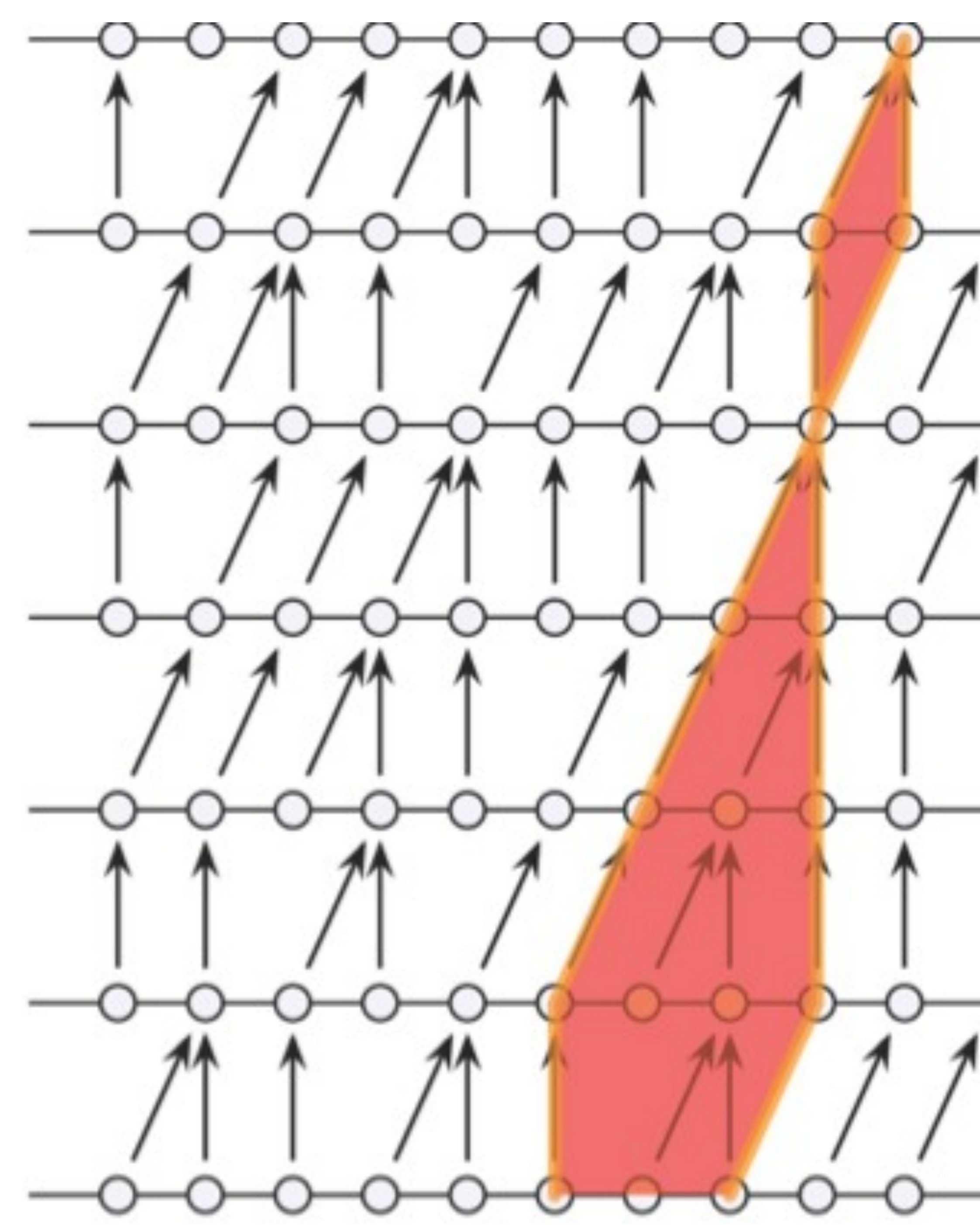
a.



b.



c.



d.

Model C, starting from $\bullet^{\mathbb{Z}}$, what is the law of η_t ,
the state of the whole line at time t ?

* asymptotics for $(\frac{1}{\sqrt{t}} \eta_{tn})_{t \geq 0}$: the **Brownian web**
1990's. Tóth & Werner

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Fact: the process $k \longrightarrow \eta_t(k) \in \{0, \bullet\}$
 $\mathbb{Z} \longrightarrow \{0, \bullet\}$

is stationary, reversible and ergodic, but
is **not** Markov, at any order...

Theorem (Chassaing & Mercier, 2011)

$$P_{t,m} \stackrel{\text{def}}{=} \mathbb{P}(\eta_+(\llbracket m \rrbracket) = \bullet^m) = 2^{-t(m+1)} \prod_{1 \leq i < j \leq t} \frac{m+i+j}{i+j-1}$$

with the help of:

C. Krattenthaler, A.J. Guttmann, and X. G. Viennot. Vicious walkers, friendly walkers and Young tableaux: II. with walls.
J. Phys. A: Math. Gen., 33:8123-8135, 2000.

