

Un survol de la domination de puissance dans les graphes

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Electrical system monitoring (with PMU)

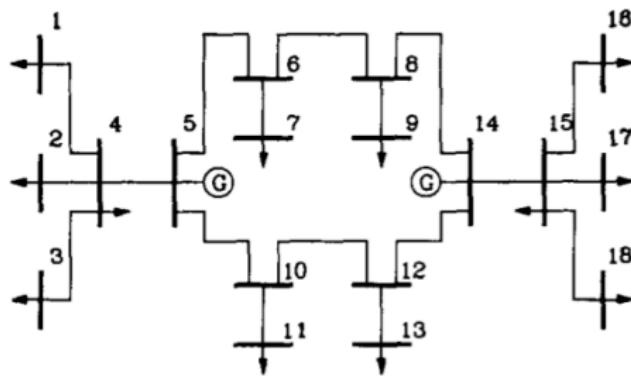


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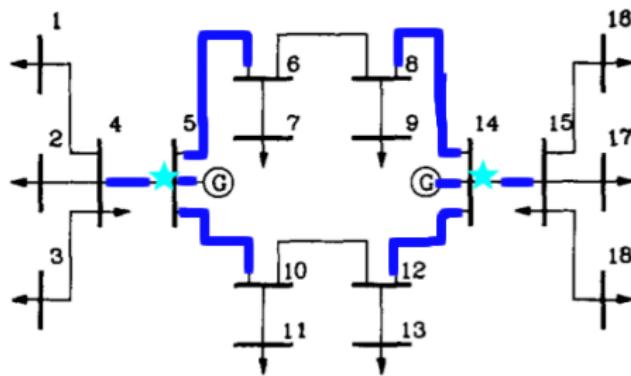


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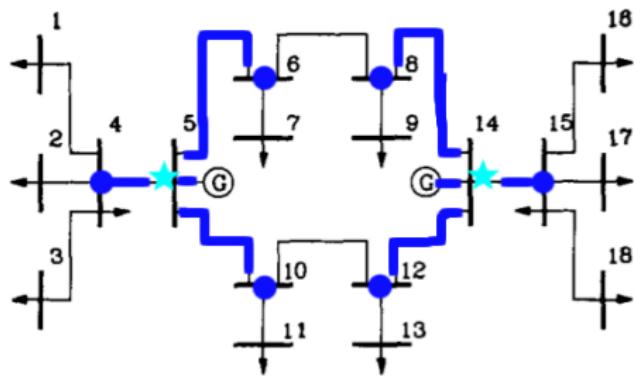


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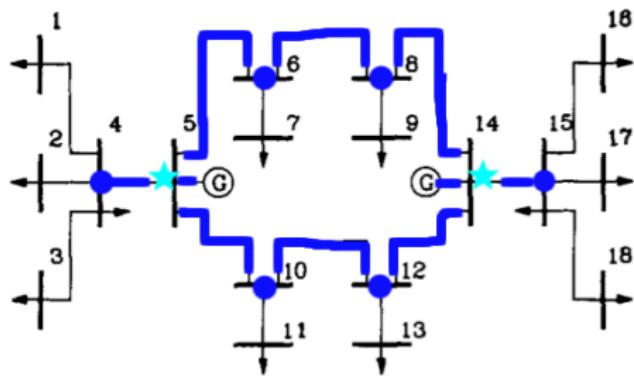


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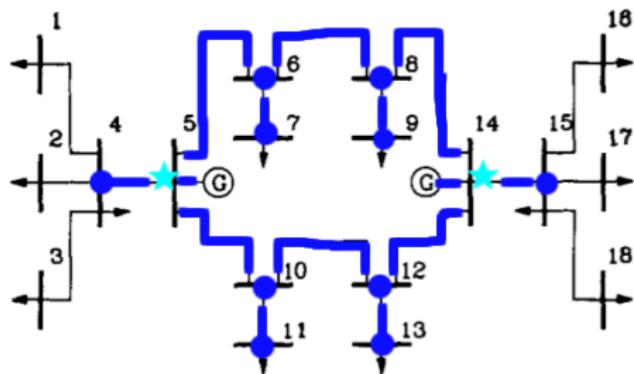


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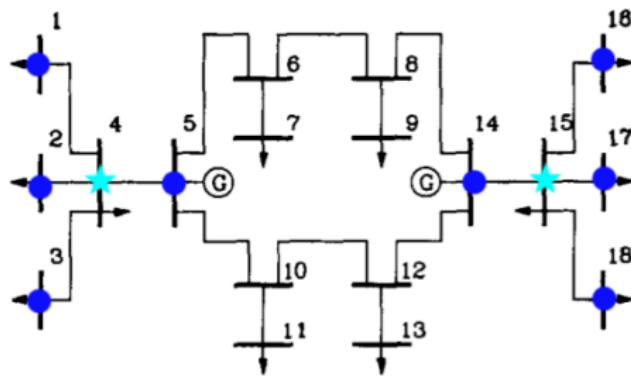


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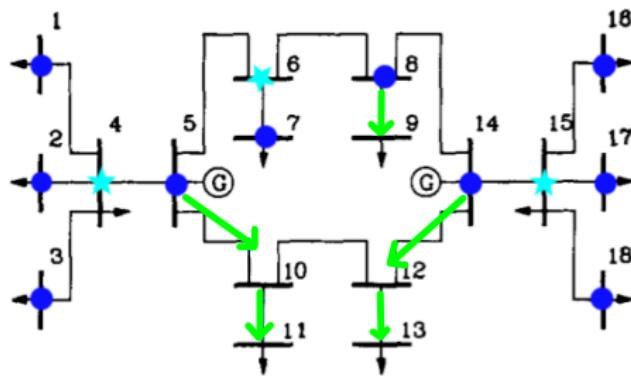
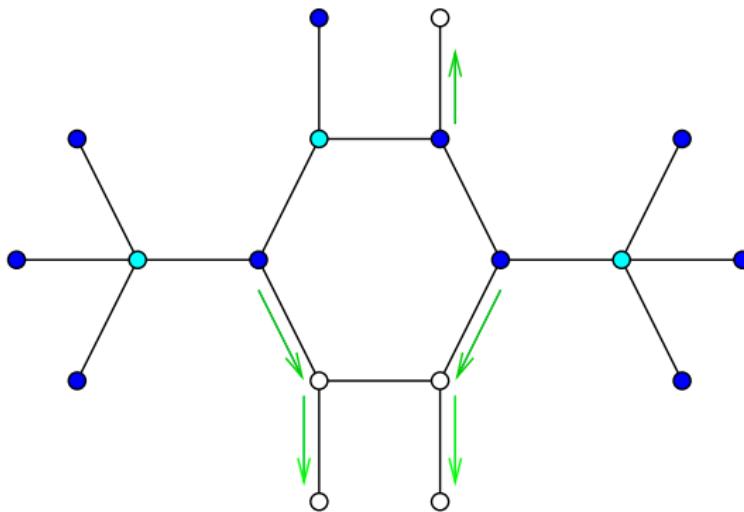


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Electrical system monitoring (with PMU)



A graph problem with simpler rules

- ▶ Only take care of vertices (edges follow...)
- ▶ 1st step = **domination** : a vertex with a PMU monitors its closed neighbourhood
- ▶ 2nd step = **propagation** : if a monitored vertex has a single neighbour not monitored, it gets monitored.

The minimum number of PMU that (eventually) monitors the whole graph = *Power domination number* : $\gamma_P(G)$

Monitored vertices

Formal definition :

G a graph, S a subset of vertices

The set $\mathcal{P}^i(S)$ of vertices monitored by S at step i is defined by

- ▶ (domination)

$$\mathcal{P}^0(S) = N[S]$$

- ▶ (propagation)

$$\mathcal{P}^{i+1}(S) = \bigcup N[v] \text{ for } \begin{cases} v \in \mathcal{P}^i(S), \\ |N[v] \setminus \mathcal{P}^i(S)| \leq 1 \end{cases}$$

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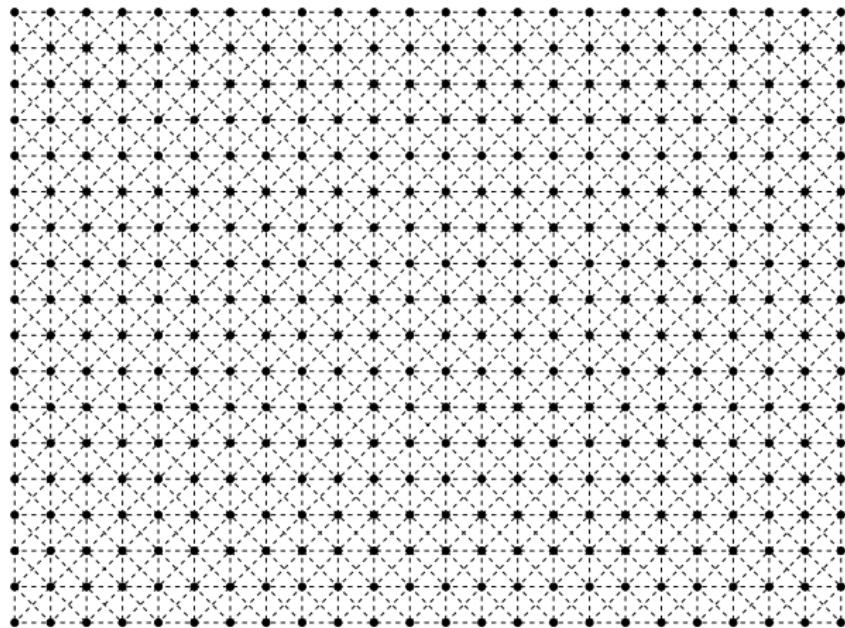
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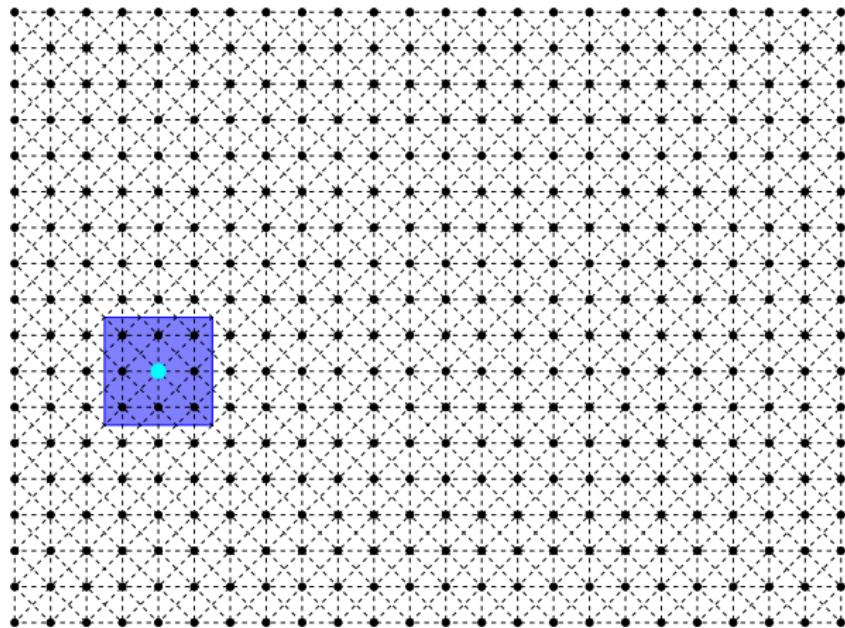
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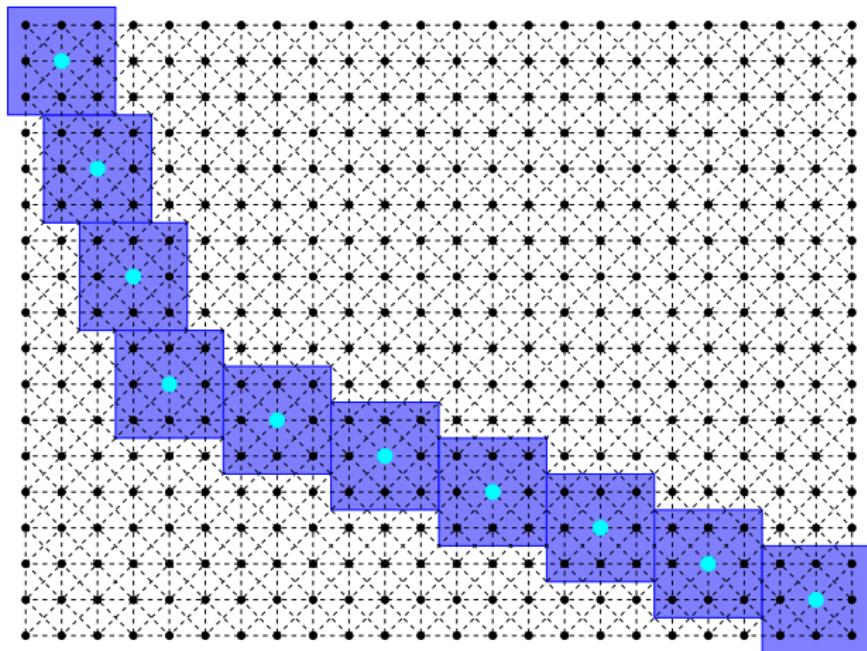
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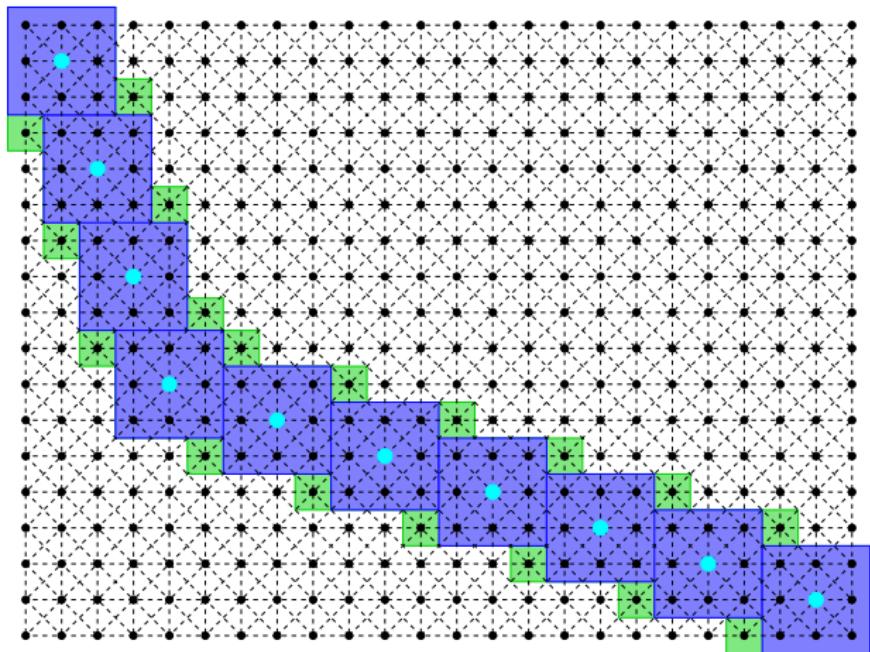


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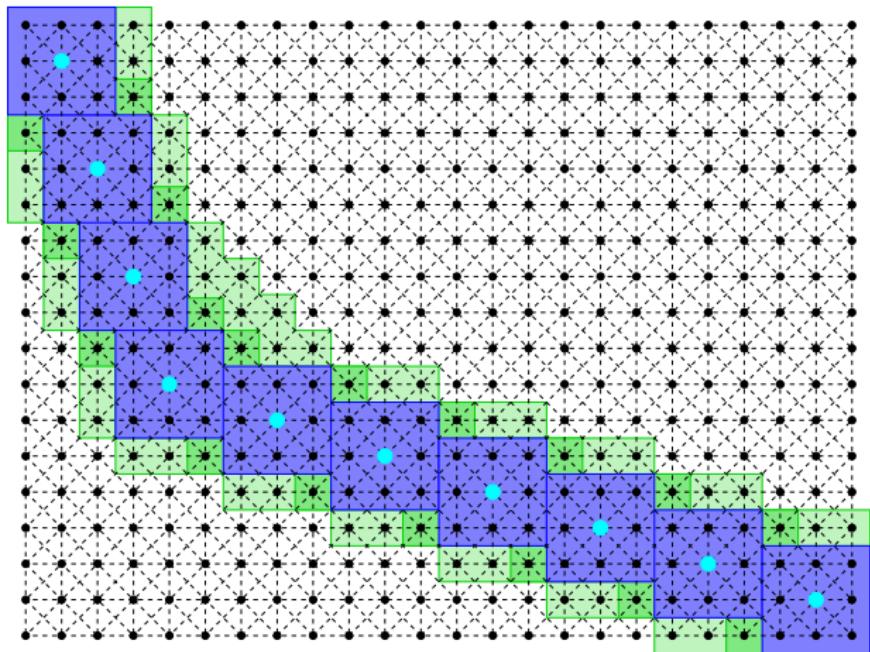
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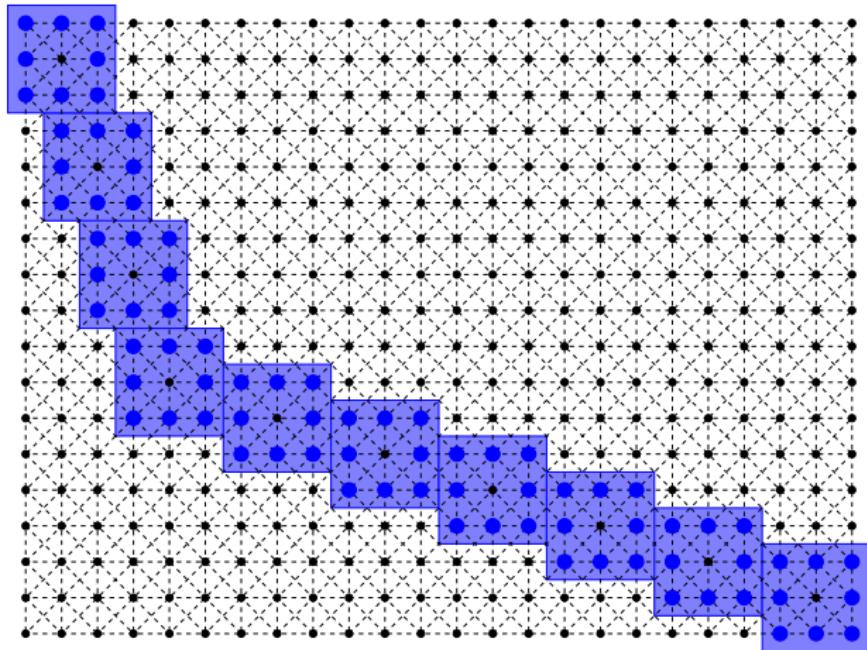
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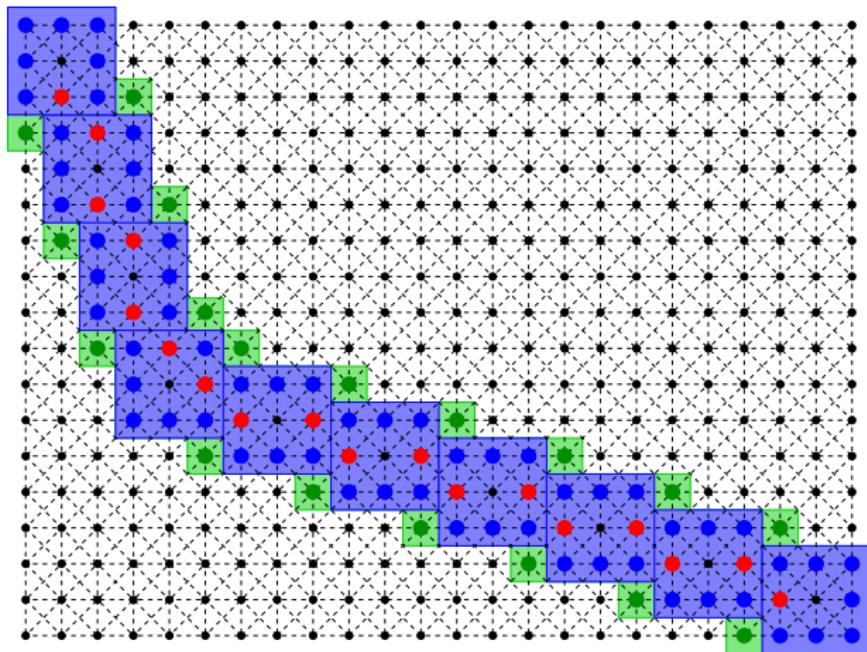
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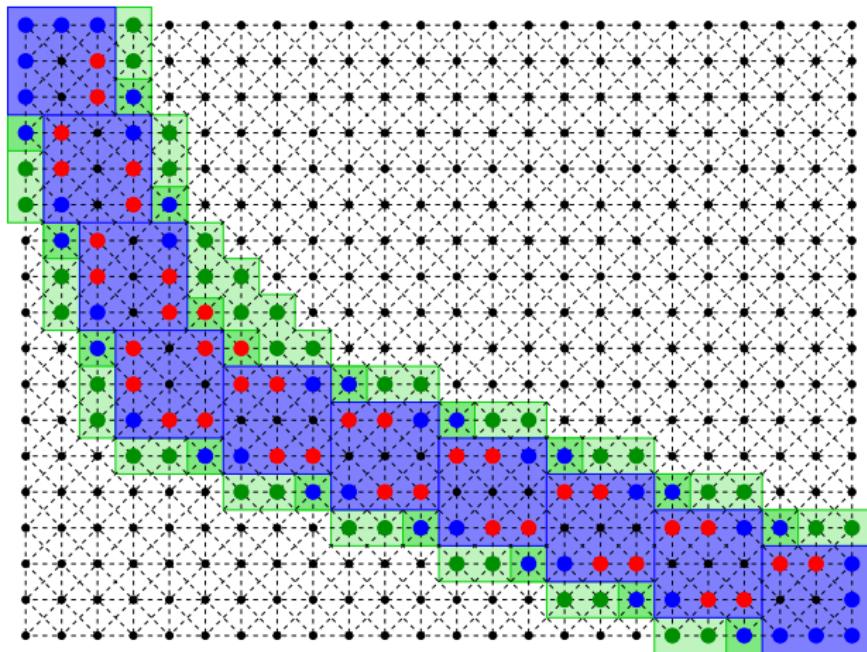
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$$\gamma_P \geq \frac{2n+2m-4}{8}$$

Bounds on general graph families

First, let us generalize.

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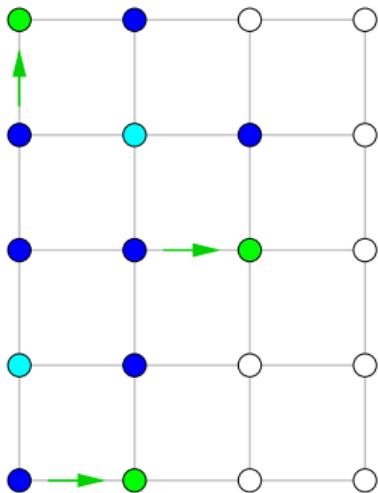
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$$k = 1, \mathcal{P}^1(S)$$



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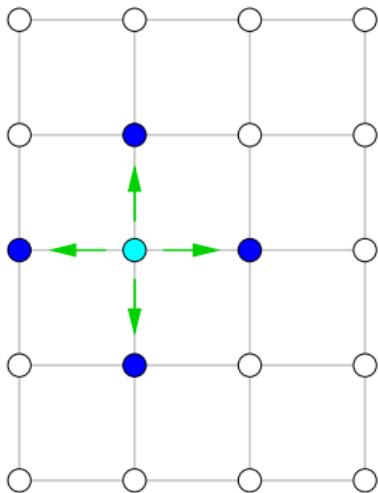
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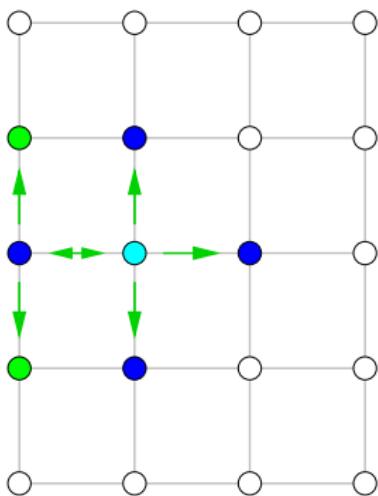
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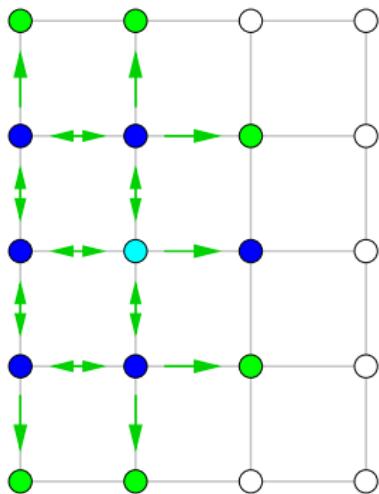
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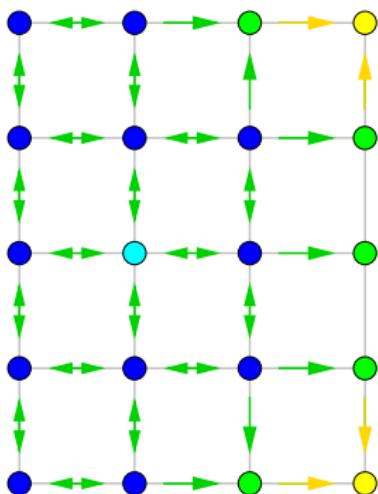
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Generalized power domination

Problem

Given a graph G , find its k -power domination number $\gamma_{P,k}(G)$
= smallest size of S such that $\mathcal{P}^\infty(S) = V(G)$.

- ▶ generalizes power domination ($\gamma_{P,1} = \gamma_P$)
- ▶ generalizes domination ($\gamma_{P,0} = \gamma$)
- ▶ helps to understand how power-domination is related to domination :
 - ▶ critical graphs : $(k + 1)$ -crowns
 - ▶ general bounds
 - ▶ common linear algorithm on trees (and bounded treewidth)
 - ▶ other bounds for families of graphs...

Relation between $\gamma_{P,k}$ for different k

Question

Clearly, $\gamma_{P,k}(G) \geq \gamma_{P,k+1}(G)$. Can we say more ?

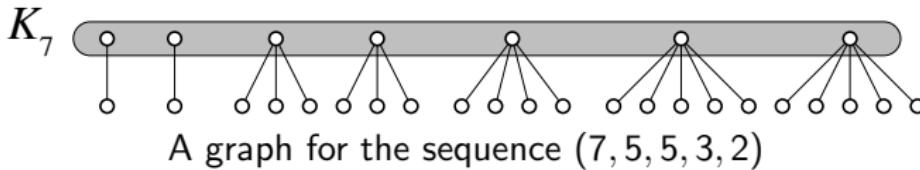
Relation between $\gamma_{P,k}(G)$ for different k

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Obs : No

For any sequence $(x_k)_k > 0$ finite and non-increasing, there exist G such that $\gamma_{P,k}(G) = x_k$.



On regular graphs

Theorem [Zhao,Kang,Chang,2006]

G connected claw-free cubic

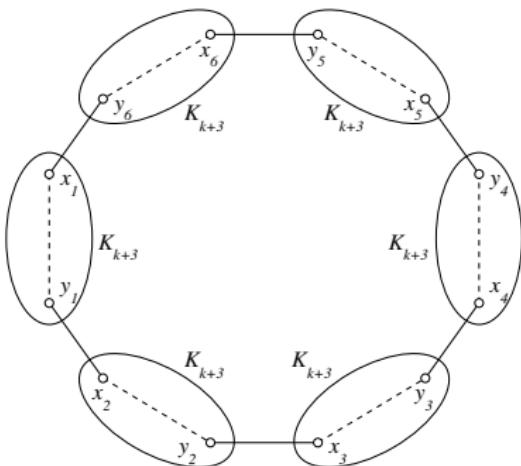
$$\Rightarrow \gamma_P(G) \leq \frac{n}{4}.$$

Theorem [CDMR2012]

G connected claw-free $(k+2)$ -regular

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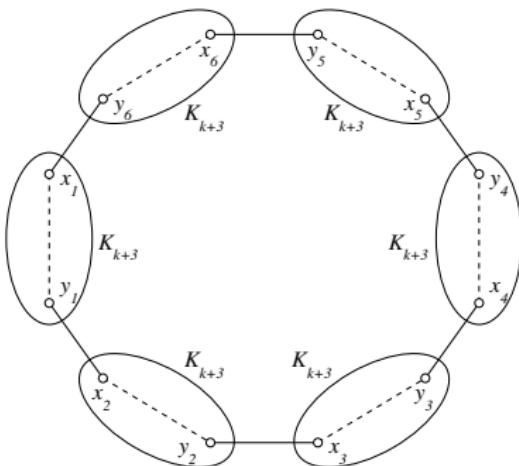
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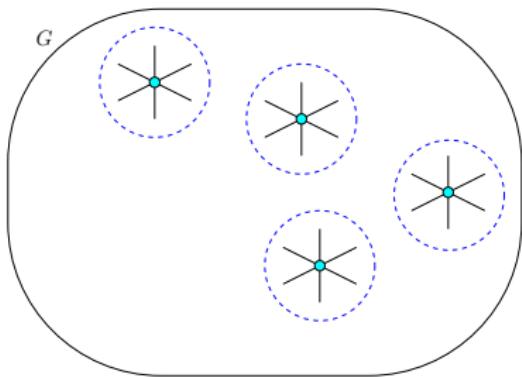
Theorem [DHLMR2013+]

G connected $(k+2)$ -regular $\Rightarrow \gamma_{P,k}(G) \leq \frac{n}{k+3}$ unless $G = K_{k+2,k+2}$.

Idea of the proof $(\gamma_P(G) \leq n/(k+3))$

Let G be a connected $(k+2)$ -regular graph.

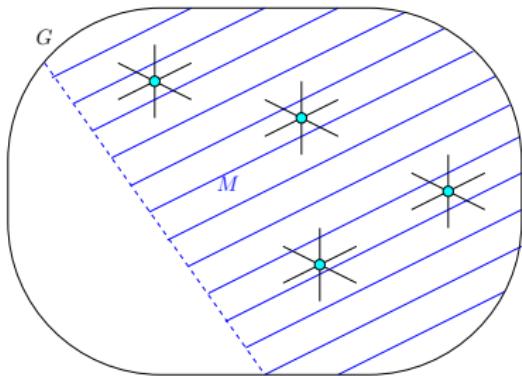
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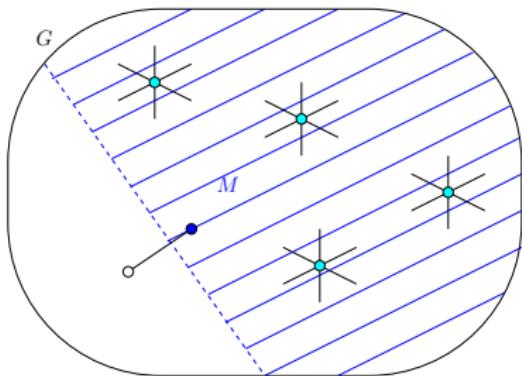
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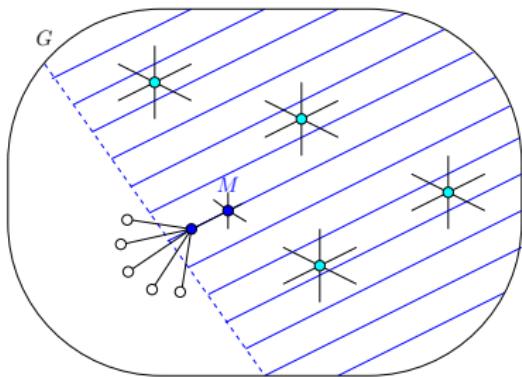
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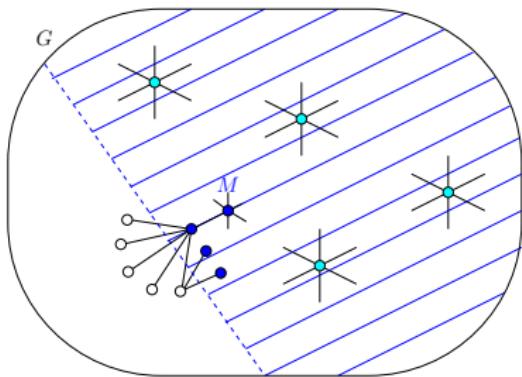
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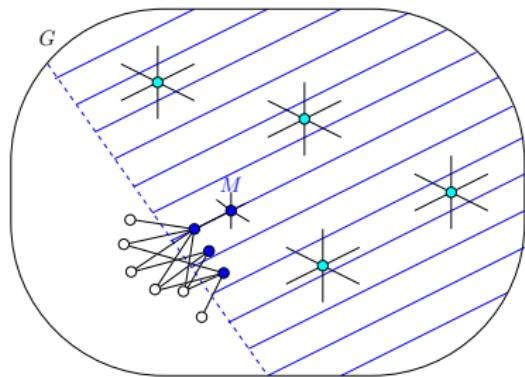
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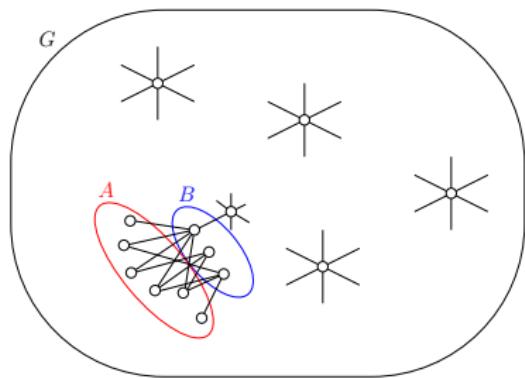
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 - ▶ Thus, $N[u] \cap M$ is a stable set.
 - ▶ likely to find a vertex that will monitor $k+3$ new vertices
- ▶ difficulty is we don't know M



(A, B) -configurations

Definition : (A, B) -configurations

- (P1).** $|A| \in \{k + 1, k + 2\}$.
- (P2).** $B = N(A) \setminus A$.
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► We can prove more :

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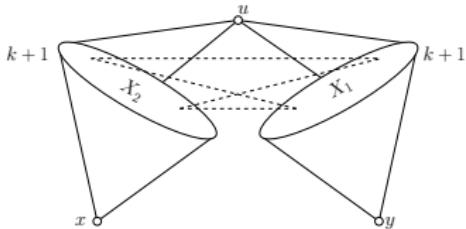
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- ▶ Remains some family \mathcal{F}_k :



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- ▶ take a vertex in each \mathcal{F}_k
- ▶ take a vertex in every other (A, B) -configurations.
- ▶ complete into a maximal packing of G .
- ▶ propagate, then increase the set iteratively : possible since no (A, B) -configurations left...

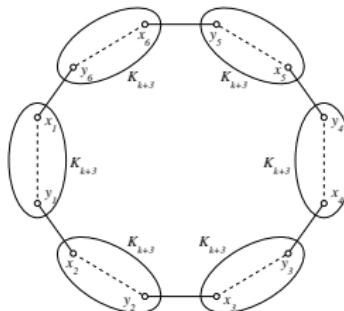
Summary

Recall that if $\Delta(G) \leq k + 1$, $\gamma_{P,k}(G) = 1$

We proved :

Theorem [DHLMR2013+]

G connected $(k + 2)$ -regular, $G \neq K_{k+2,k+2}$, $\Rightarrow \gamma_{P,k}(G) \leq \frac{n}{k+3}$.



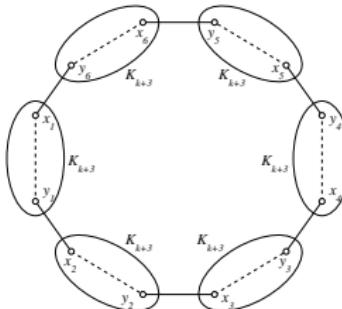
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Conjecture

A similar bound holds for all r -regular graphs (with $r > k + 1$) :

$$\gamma_{P,k}(G) \leq \frac{n}{r+1}$$

Thanks for your attention.

DMKŠ2008 : Dorbec, Mollard, Klavžar, Špacapan, SIAM J. Disc. Math.

CDMR2012 : Chang, Dorbec, Montassier, Raspaud, Discrete Appl. Math.

DHLMR2012+ : Dorbec, Henning, Lowenstein, Montassier, Raspaud,
manuscript.