

The half-plactic monoid

Nathann Cohen and Florent Hivert

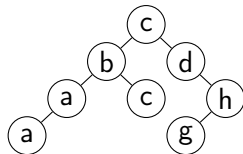
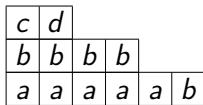
LRI / Université Paris Sud 11 / CNRS

Fevrier 2015

Outline

- 1** Background: plactic like monoïds
 - Schensted's algorithm and the plactic monoïd
 - Binary search trees and the sylvester monoïd
 - The lattice of combinatorial Hopf algebras
- 2** The half plactic monoïd
 - Orienting the relations ?
 - A rewriting rule on Posets !
 - Binary search posets
 - The insertion algorithm
- 3** Work in progress: further questions

Binary Search Trees and Young tableaux ...

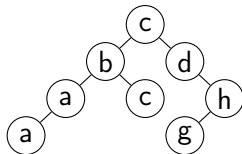
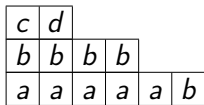


... are **very similar** combinatorial objects:

- Robinson-Schensted correspondence;
- Plactic/Sylvester monoid;
- Hook formula;
- Hopf algebra;
- Tamari lattice/tableau lattice.

I don't think we fully understand why.

Binary Search Trees and Young tableaux ...

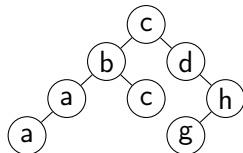
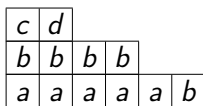


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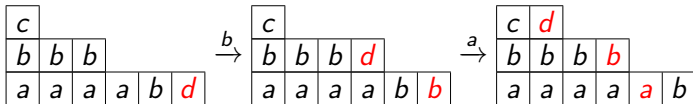
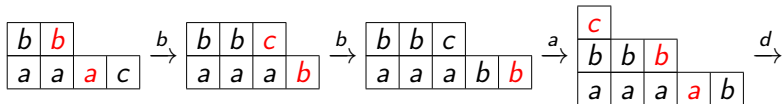
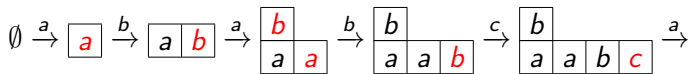
Plactic like monoïds

Motivation: construct combinatorial Hopf algebras by taking quotient in **FQSym** (Malvenuto-Reutenauer algebra of permutations).

- Plactic monoïd \mapsto **FSym** (Poirier-Reutenauer algebra of tableau). Application: proof of the Littlewood-Richardson rule.
- sylvester monoïd \mapsto **BT** (Loday-Ronco algebra of binary tree). Application: dendriform algebra (shuffle/quasi-shuffle ...).
- hypoplactic monoid \mapsto (QSym, **NCSF**). Application: descent of permutations, degenerate Hecke algebra $H_n(0)$.

The Robinson-Schensted's map

Bumping the letters: when a letter is replaced in Schensted algorithm, insert it in a next row (placed on top in the drawing).



The Robinson-Schensted's correspondence

Going backward: If we record the order in which the boxes are added then we can undo the whole process.

$$ababcabbadba \longleftrightarrow \left(\begin{array}{|c|c|} \hline c & d \\ \hline b & b & b & b \\ \hline a & a & a & a & a & b \\ \hline \end{array}, \begin{array}{|c|c|} \hline 9 & 13 \\ \hline 3 & 6 & 7 & 11 \\ \hline 1 & 2 & 4 & 5 & 8 & 10 & 12 \\ \hline \end{array} \right).$$

Theorem

This define a **bijection** between words w and pairs

(semi standard Young tableau, standard Young tableau)

of the same shape.

The plactic monoïd

Quotient of the free monoïd by Knuth's relations:

$$acb \equiv cab \quad \text{if } a \leq b < c$$

$$bac \equiv bca \quad \text{if } a < b \leq c$$

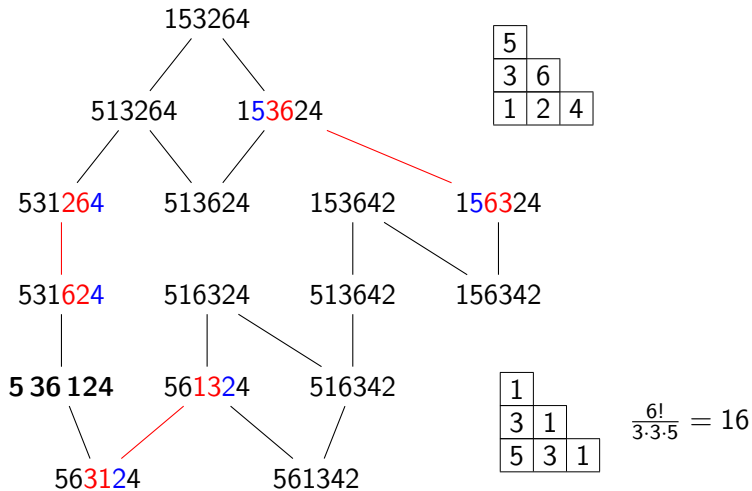
The letter b allowing the exchange is called **catalytic**.

Theorem

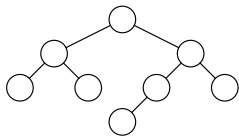
Two words give the same tableau under the RS map if and only if, is it possible to go from one to the other using Knuth's relations.

Proof: rewriting to the row reading of the tableau + green invariants.

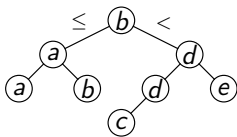
A plactic class



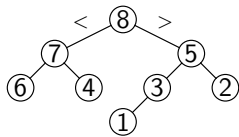
Words and trees !



binary tree (BT)

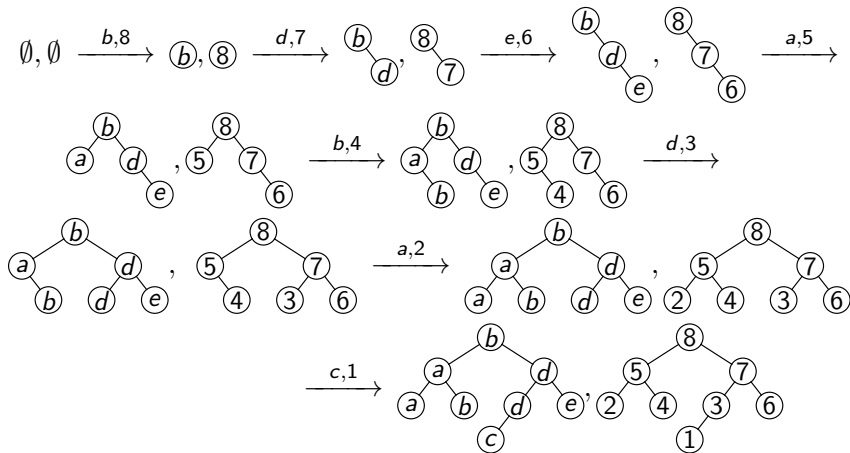


search tree (BST)



decreasing tree (DT)

The underlying tree of a BST or a DT is called its **shape**.

Sylvester RS correspondence for the word *cadbaedb*

The sylvester monoid

Quotient of the free monoid by the relations:

$$ac - b \equiv ca - b \quad \text{if } a \leq b < c$$

($-$ stands for any number of letters).

Theorem

Two words give the same binary search tree if and only if, is it possible to go from one to the other using the sylvester relations.

Proof: orienting the relations + permutations avoiding 312.

Standardization of a word

word of length n \longmapsto permutation of \mathfrak{S}_n

$w = l_1 l_2 \dots l_n$ \longmapsto $\sigma = \text{Std}(w)$

for $i < j$ then $\sigma(i) > \sigma(j)$ iff $l_i > l_j$.

Example: $\text{Std}(abcadbcaa) = 157296834$

a	b	c	a	d	b	c	a	a
a_1	b_5	c_7	a_2	d_9	b_6	c_8	a_3	a_4
1	5	7	2	9	6	8	3	4

The lattice of Combinatorial Hopf Algebra

Definition

A monoid is *compatible with the standardization* if

$$v \equiv w \quad \text{iff} \quad \text{eval}(v) = \text{eval}(w) \text{ and } \text{Std}(v) \equiv \text{std}(w)$$

$I := [a, b]$ interval of the alphabet, w/I : erase the letter of w which are not in I .

Definition

A monoid is *compatible with the restriction of interval* if

$$v \equiv w \quad \text{implies} \quad v/I \equiv w/I$$

The lattice

Theorem (H Nzeutchap 2007)

*If a monoid is standardization / restriction compatible then one can take the quotient in **FQSym** and get a Hopf algebra on the standard (permutations) classes.*

Proposition (Priez 2013)

The intersection and the union of two standardization / restriction compatible monoids are also compatible.

- $\text{Plactic} \cup \text{Sylvester} = \text{Hypoplactic} \mapsto \text{QSym}$.
- $\text{Plactic} \cap \text{Sylvester} = \frac{1}{2}\text{-plactic} \mapsto ???$.

The half plactic monoïd

Definition

Quotient of the free monoid by the relations:

$$acb \equiv cab \quad \text{if } a \leq b < c$$

The catalytic letter must be right after the exchanged letters.

How to describe the classes ?

The half plactic monoïd

Definition

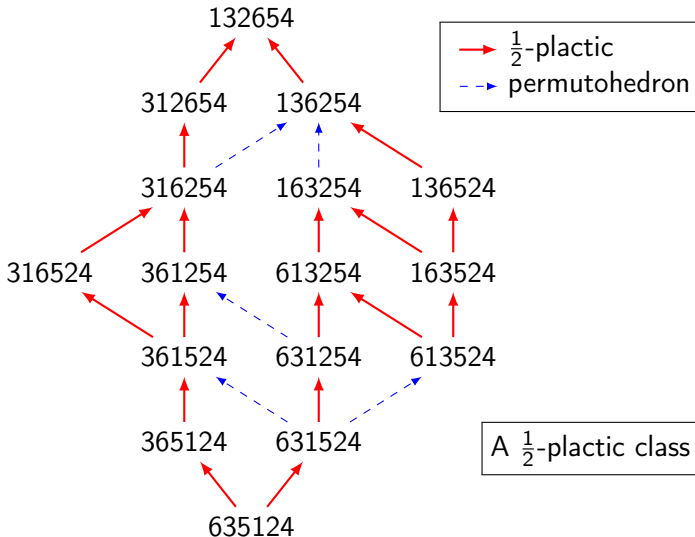
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Numerology

Number of classes:

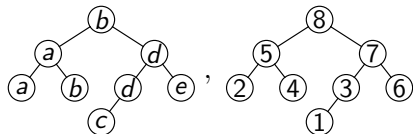
n	0	1	2	3	4	5	6	7	8	9
#classes	1	1	2	5	16	61	274	1413	8266	54099

Distribution of size of the classes for $n = 7$:

size	1	2	3	4	5	6	7	9	10
#classes	454	320	194	90	132	44	40	54	16
size	12	14	15	16	19	20	21	26	35
#classes	18	16	2	14	2	1	10	2	4

Poset and linear extensions

cadbaedb

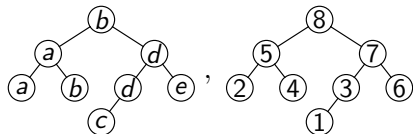


a
↓ : a is after b
 b

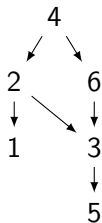


153624
513624
153264
513264
531264
531624
536124

Poset and linear extensions

 $cadbaedb$ 

a
 \downarrow : a is after b
 b

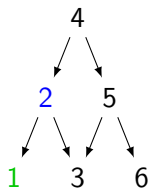
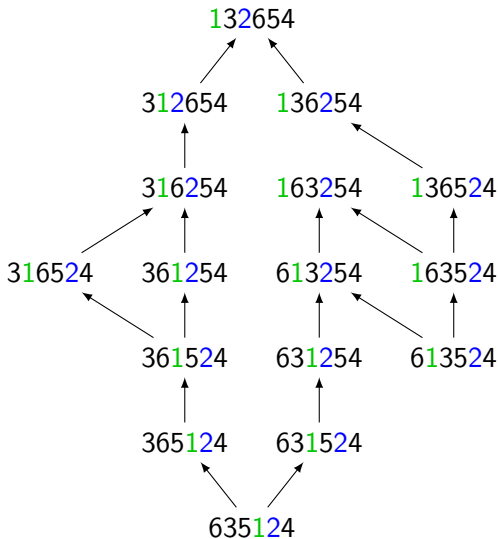


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Main Theorem

Theorem

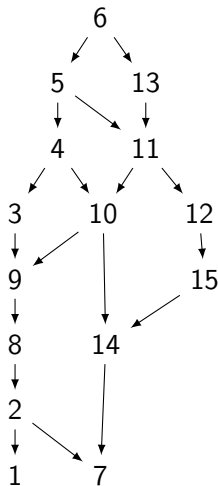
For all $\frac{1}{2}$ -plactic class C of size n permutations there exists a partial order on $\{1, \dots, n\}$ whose linear extensions are exactly the elements of C .



Another $\frac{1}{2}$ -plactic class

7 14 1 2 8 9 15 12 3 10 4 11 5 13 6

+ 1926 other linear extensions



Main Theorem (2)

Theorem

For all $\frac{1}{2}$ -plactic class C of size n permutations there exists a partial order on $\{1, \dots, n\}$ whose linear extensions are exactly the elements of C .

Sketch of the proof:

- a rewriting system on the posets: removable edges;
- rewriting commutes \Rightarrow the system is confluent;
- For each permutation there is a unique minimal non rewritable poset whose linear extensions contains its class;
- induction on minimal elements \Rightarrow actual equality.

Problem

Starting from a permutation σ , how to compute the poset $P(\sigma)$ associated to the class of sigma, without enumerating the class ?

Assuming the theorem is true, we work at minima: which are the pair (i, j) which are not comparable ?

If $\bar{u}cab\bar{v} = \bar{u}acb\bar{v}$ is a legible rewriting, then a and b are not comparable in b .

Idea of the construction

Start with the linear poset associated to σ , and when we find two element that are not comparable, remove the comparison.

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Removable edges

Notations:

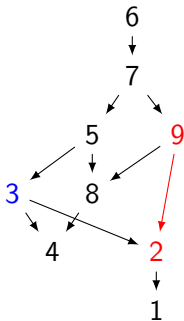
- $<$ for comparison of integers, \prec for the poset
- Poset interval: $]a, b[:= \{x \mid a \prec x \prec b\}$
- Cover: $a \leftarrow b$ means $a \preceq b$ and $]a, b[= \emptyset$.

Definition

A cover $a \leftarrow c$ is **removable** if there exists a b such that

- either $a < b < c$ or $c < b < a$
- $b \not\prec c$
- if $a \prec b$ then $]a, b[\subset \{c\}$

$a \prec b$ and $]a, b[\subset \{c\}$



418293576

Lemma

A cover $a \leftarrow b$ is removable if and only if there exists a linear extension $\bar{u}acb\bar{v}$ of P such that $\bar{u}acb\bar{v} \equiv \bar{u}cab\bar{v}$.

Note: since $a \prec b$ then $\bar{u}cab\bar{v}$ is not a linear extension of P .

Proposition

If a poset P has *no removable edge*, then its set of linear extension is closed by $\frac{1}{2}$ -plactic rewriting; otherwise said, it is a *union* of $\frac{1}{2}$ -plactic classes.

Problem

Given a permutation σ compute the *minimal* poset without removable edge which contains σ as linear extension.

Proposition

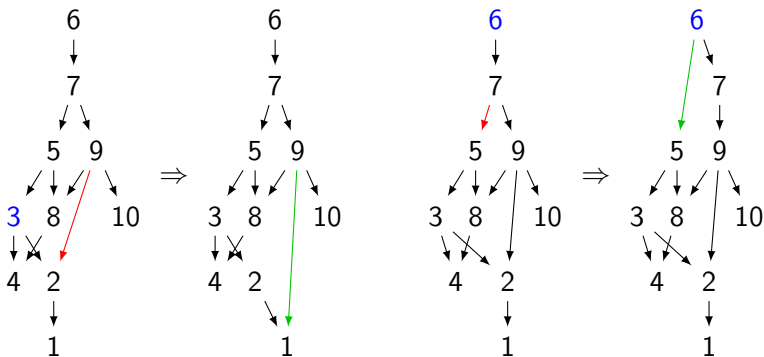
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Removing the edges

Warning !!! Removing is on the Poset (\neq Hasse diagram)



Removing commutes

Inclusion on poset $Q \subset P$ means $a \prec_Q b$ implies $a \prec_P b$.

Proposition

*Suppose that $Q \subset P$ and $a \leftarrow_P b$ is a removable edge.
If $a \prec_Q b$ then $a \leftarrow_Q b$ is a removable edge.*

Corollary

Suppose $(a, b) \neq (c, d)$. If both $a \leftarrow_P b$ and $c \leftarrow_P d$ are removable in P , then $c \leftarrow d$ is removable in $P/a \leftarrow b$.

One can do the removing in any order !

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One can do the removing in any order !

The algorithm

Algorithm

Given a permutation σ compute the poset $P(\sigma)$.

- *set $P := \sigma$ seen as a total order;*
- *while there is a removable edge pick one and remove it from P ;*
- *return P .*

Theorem

The $\frac{1}{2}$ -plactic class of σ is exactly the linear extensions of $P(\sigma)$.

Proof by induction on the minimums.

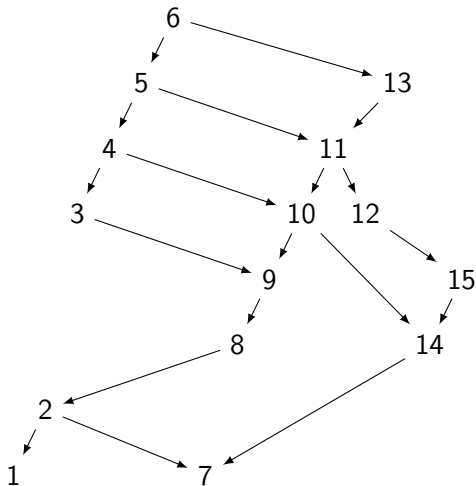
Binary search posets

Proposition

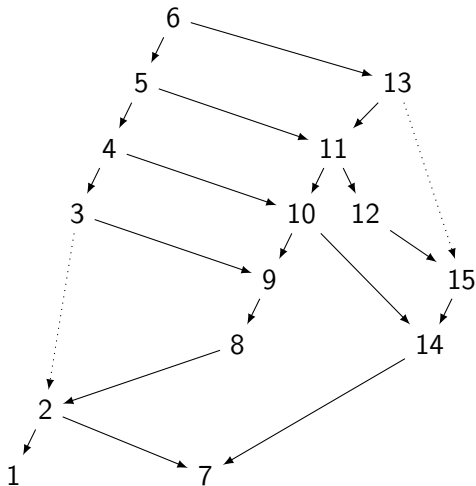
Let P be a poset without a removable edge, then for any x ,

- *there is at most one $a < x$ s.t. $a \leftarrow x$*
- *there is at most one $a < x$ s.t. $x \leftarrow a$*
- *there is at most one $a > x$ s.t. $a \leftarrow x$*
- *there is at most one $a > x$ s.t. $x \leftarrow a$*

A binary search posets



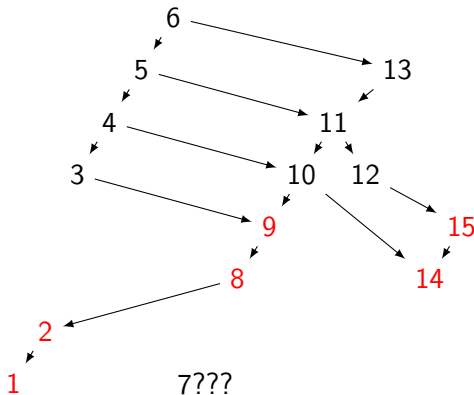
A binary search posets



The insertion algorithm

Lemma

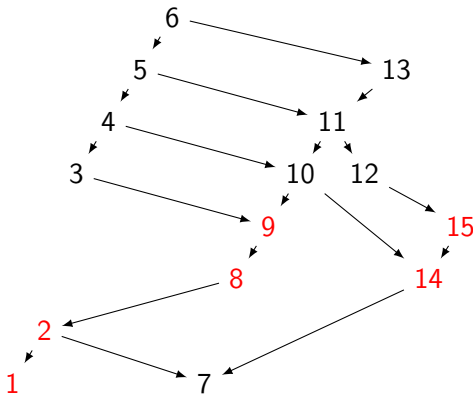
For any letter x and word w , $P(xw)/w = P(w)$.



The insertion algorithm

Lemma

For any letter x and word w , $P(xw)/w = P(w)$.



Work in progress: further questions

- Simple characterization of the posets
- Classes seems to be intervals of the permutohedron
- k -sylvester monoïds:

$$ac\bar{w}b \equiv ca\bar{w}b \quad \text{if } a \leq b < c \text{ and } |w| < k$$

The catalytic letter must not be too far

1	1	2	5	16	61	274	1413
1	1	2	5	14	45	162	636
1	1	2	5	14	42	136	472
1	1	2	5	14	42	132	434
1	1	2	5	14	42	132	429

- analogues of the associahedron

Thank You

A characterisation of the posets

Inclusion on poset $Q \subset P$ means $a \prec_Q b$ implies $a \prec_P b$.

Proposition

*Suppose that $Q \subset P$ and $a \leftarrow_P b$ is a removable edge.
If $a \prec_Q b$ then $a \leftarrow_Q b$ is a removable edge.*

Corollary

The posets associated to the $\frac{1}{2}$ -plactic classes are exactly the maximal (for the inclusion) posets among the posets without removable edges.