Simple maps: distances and convergence

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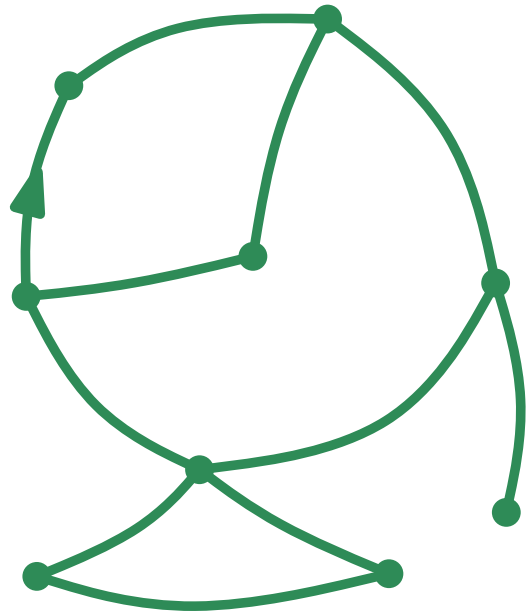
joint work with Marie Albenque (LIX), Olivier Bernardi (Brandeis University) and Éric Fusy (LIX)

JCB 2015

February 3rd 2015

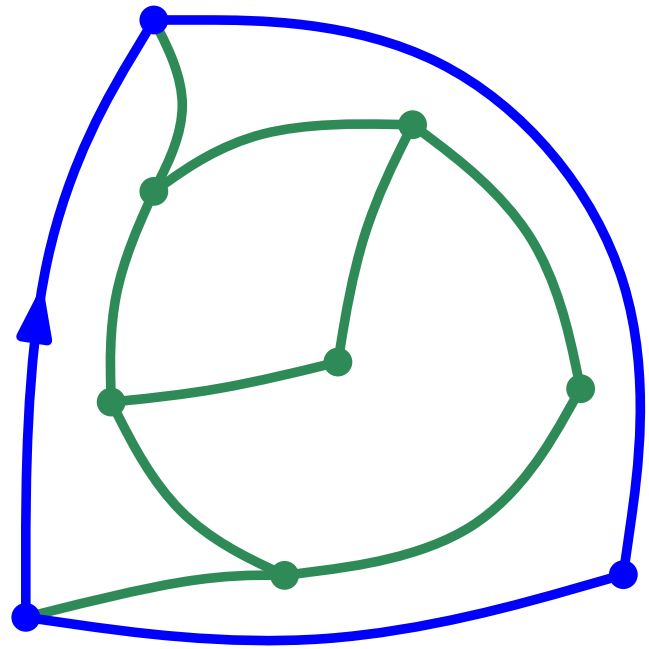
Random simple maps

Simple planar map: rooted planar map with no loops nor multiple edges $=$ planar graph embedded on the sphere, considered up to continuous deformation.



$\mathcal{S}_n = \{$ Simple maps with n edges $\}$
$S_n =$ uniform random element of $\mathcal{S}_n$

Random simple maps



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$(V(S_n), d_{gr})$ metric space for graph distance
Behaviour of S_n when n goes to infinity?
Typical distance? Distance distribution? Scaling limit?

Remark: Study of outertriangular simple maps is enough
$\rightarrow$ Simple map has a.s. a giant outertriangular core

Random simple maps

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Typical distance? Distance distribution? Scaling limit?

Well known:

→ Chassaing–Schaeffer (2004):
typical distance in rooted plane quadrangulations with n faces $\sim n^{1/4}$ when $n \to \infty$
$+$ distance profile converges towards **ISE**

→ Le Gall (2007, 2012), Miermont (2012):
quadrangulations converge towards the **Brownian map** when rescaled by $n^{-1/4}$
extended to $2p$-angulations and triangulations

→ Abraham (2013): Bipartite maps

→ Addario-Berry–Albenque (2013):
simple triangulations and simple quadrangulations

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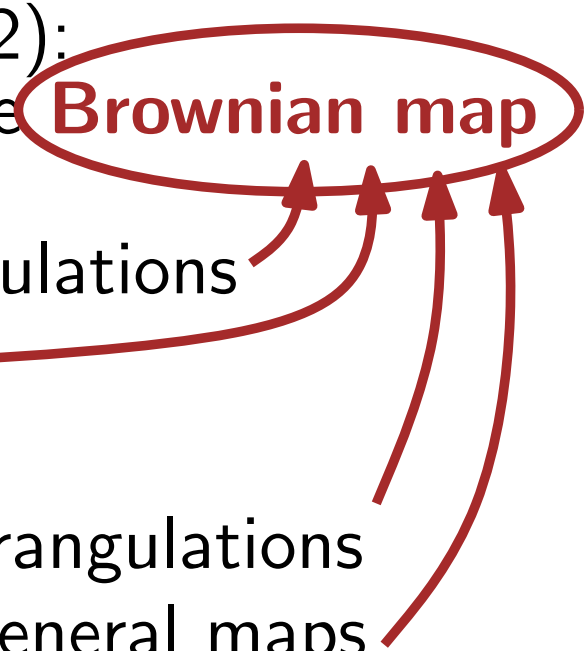

Random simple maps

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Brownian map universal limit for "reasonable" classes of maps

Idea: General method in Le Gall (2013)

- Encode maps with some (labeled, blossoming...) trees
- Study the limit of these trees
- Interpret distance on the map as a function on the tree
- Technical conditions: tightness, invariance upon rerooting

The results

$\forall e \in E(S_n) : d_{S_n}(e) =$ length of the shortest path from e to e_0

Profile : $(f_k)_{k\geq 1}$, where $f_k := \frac{1}{n}\sum_{e\in E_M} \delta_{d_{S_n}(e)}$

Radius : $r(M) := max(d(e), e \in E_M)$

Theorem [Bernardi, C., Fusy 2014]:

$f_{k/(2n)^{1/4}} \xrightarrow{(d)}$ ISE positively shifted

$r/(2n)^{1/4} \xrightarrow{(d)}$ width of ISE (also holds for moments)

Theorem [Albenque, Bernardi, C., Fusy 2015]:

$\left(V(S_n), \left(\frac{1}{2n}\right)^{1/4} d_{S_n}\right) \xrightarrow{(d)} (M, D^\star)$ Brownian map

for the distance of Gromov–Hausdorff on metric spaces.

Remark: Same rescaling factor $n^{1/4}$, typical of planar maps

Orientation for outertriangular simple maps

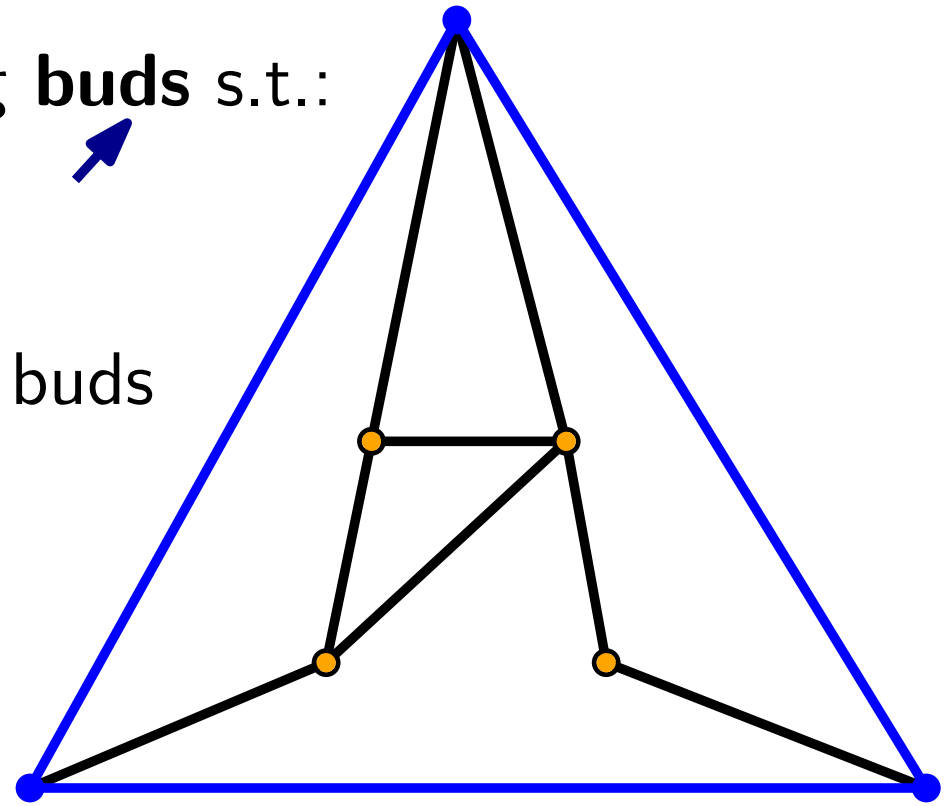
3-orientation with buds=

orientation of edges and outgoing **buds** s.t.:

out(●) $= 1$

out(●) $= 3$

$+$ each face of degree $d+3$ has d buds



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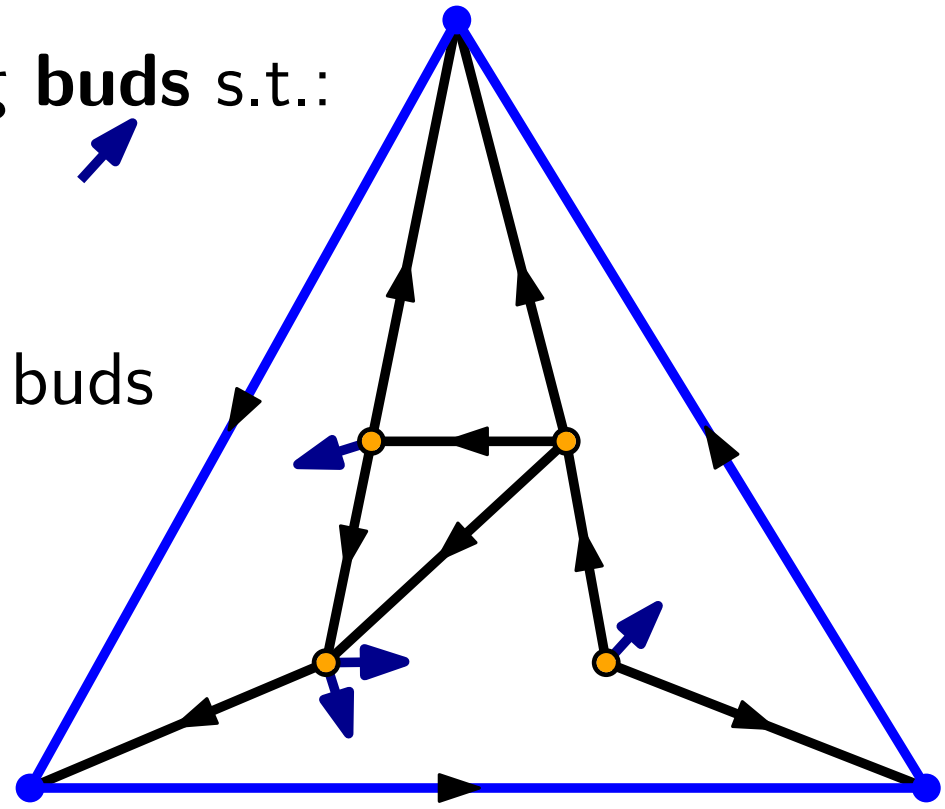


$\text{out}(\bullet) = 1$

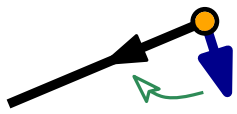
$\text{out}(\circ) = 3$

$+$ each face of degree $d+3$ has d buds

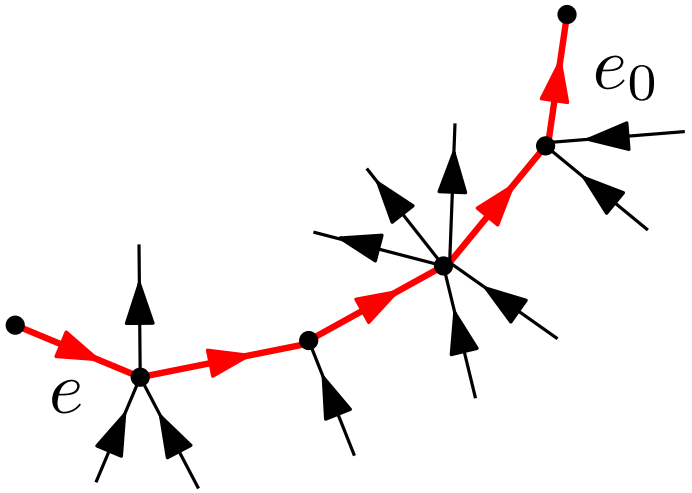
[Bernardi, Fusy '10]

These orientations characterize outertriangular simple maps

Moreover, there is a **unique** one without clockwise circuit and with local rule:



Rightmost paths (RMP) in simple maps



S_n endowed with its 3-orientation
e_0 root, $e \in E(S_n)$:
$d_{RM}(e) =$ length of the rightmost path from e to e_0

[Addario-Berry, Albenque '13] (adapted to simple maps)

For $\epsilon > 0$, let $A_{n,\epsilon}$ be the event that there exists $e \in S_n$ such that $d_{RM}(e) \geq d_{S_n}(e) + \epsilon n^{1/4}$

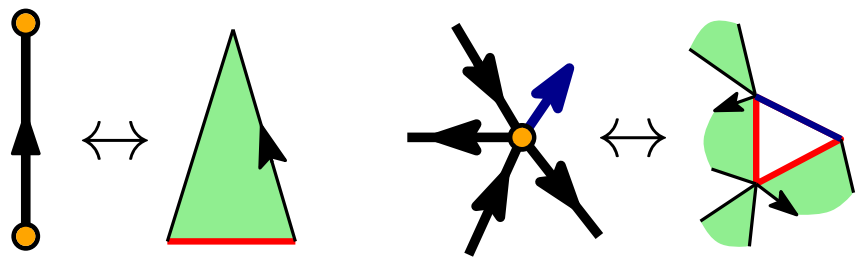
Then under the uniform law on $\mathcal{S}_n$, for all $\epsilon > 0$: $\mathcal{P}(A_{n,\epsilon}) \to 0$

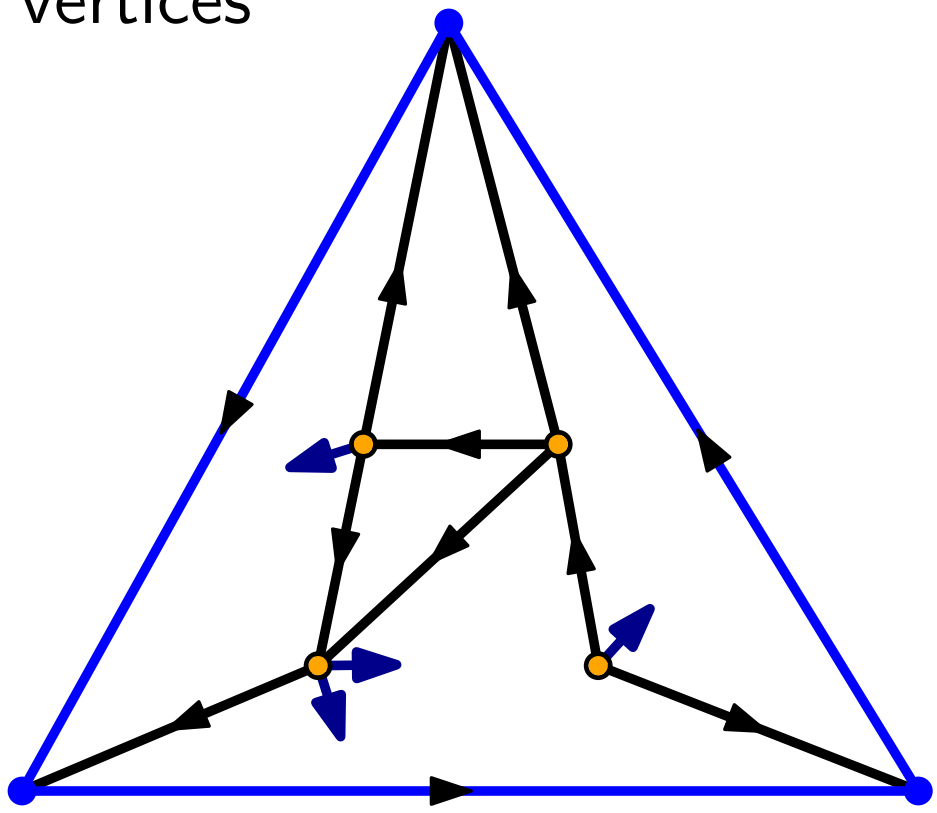
$\Rightarrow$ Rightmost paths are almost surely quasi-geodesic

$\Rightarrow$ We can consider profile of d_{RM}, up to $n^{1/4}$ scaling

Bijection with eulerian triangulations

First step: Inflate inner edges and vertices





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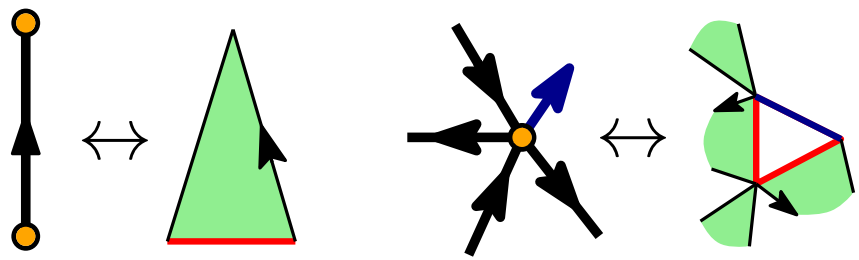

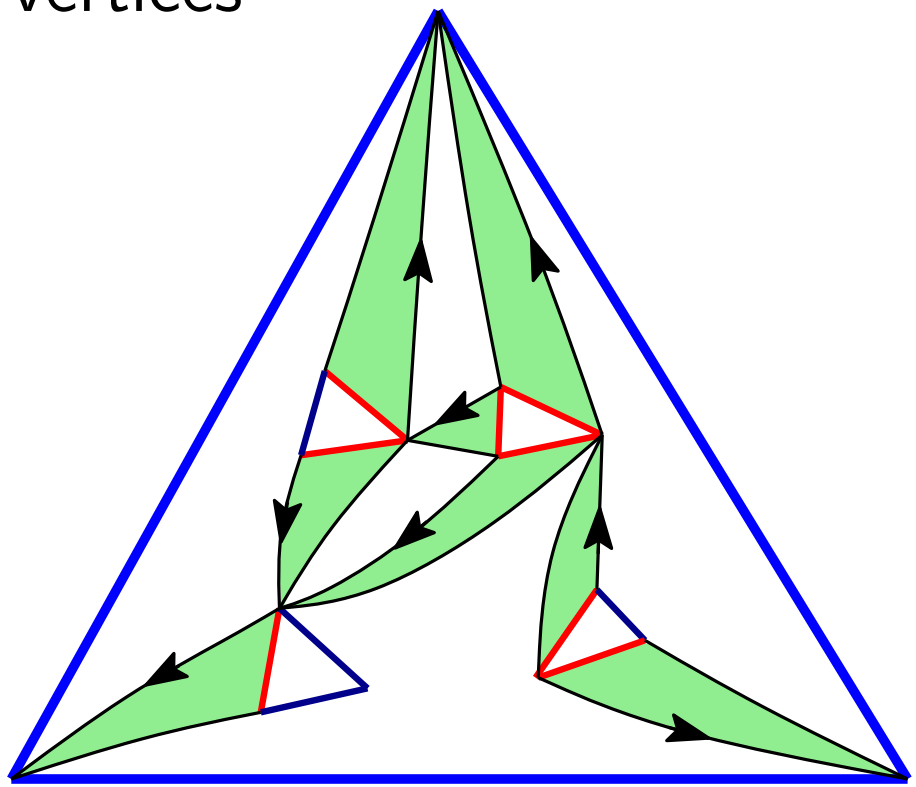

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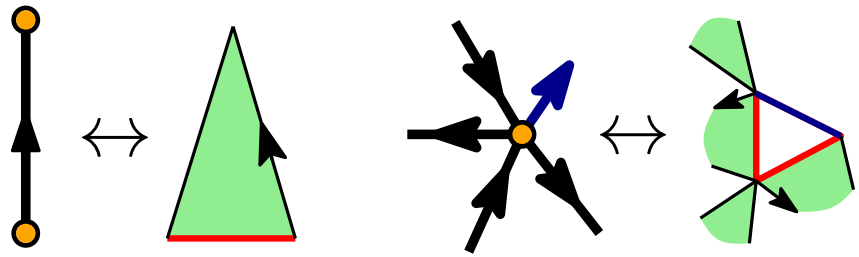
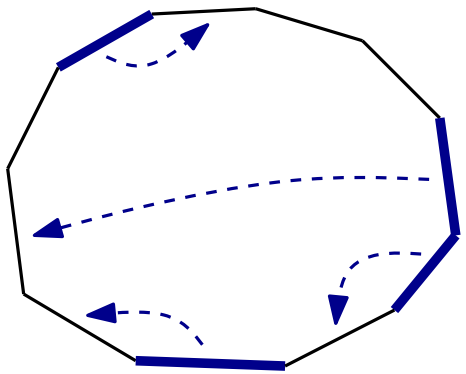
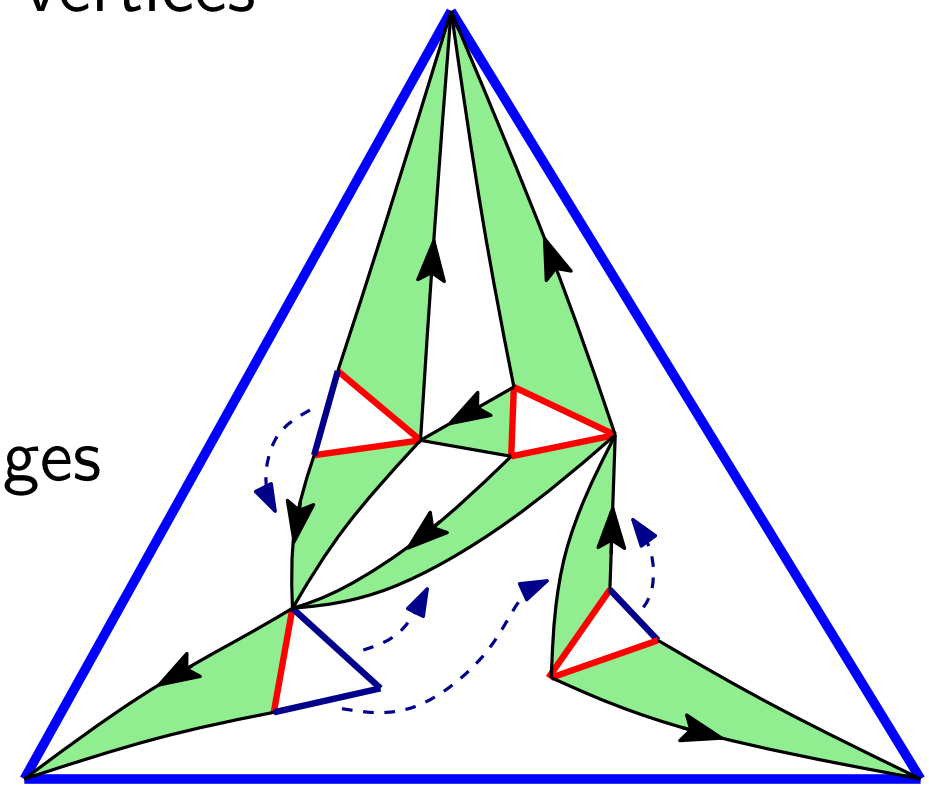


Second step: Merge remaining edges

degree $2d+3$





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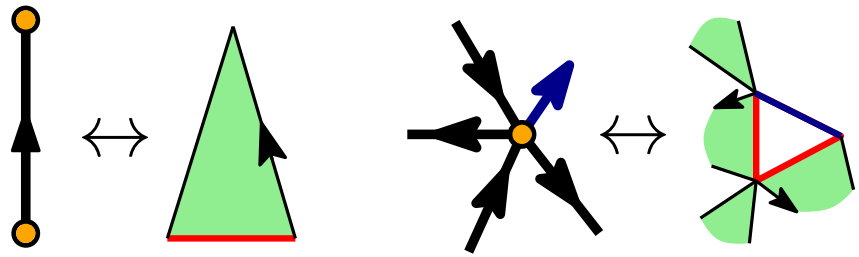

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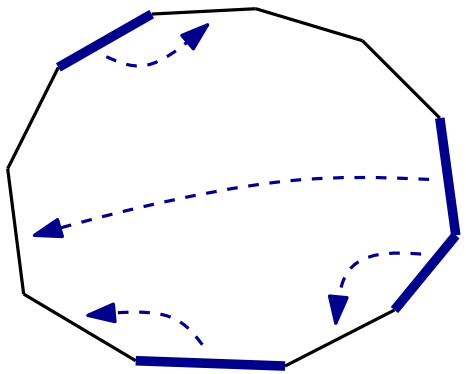

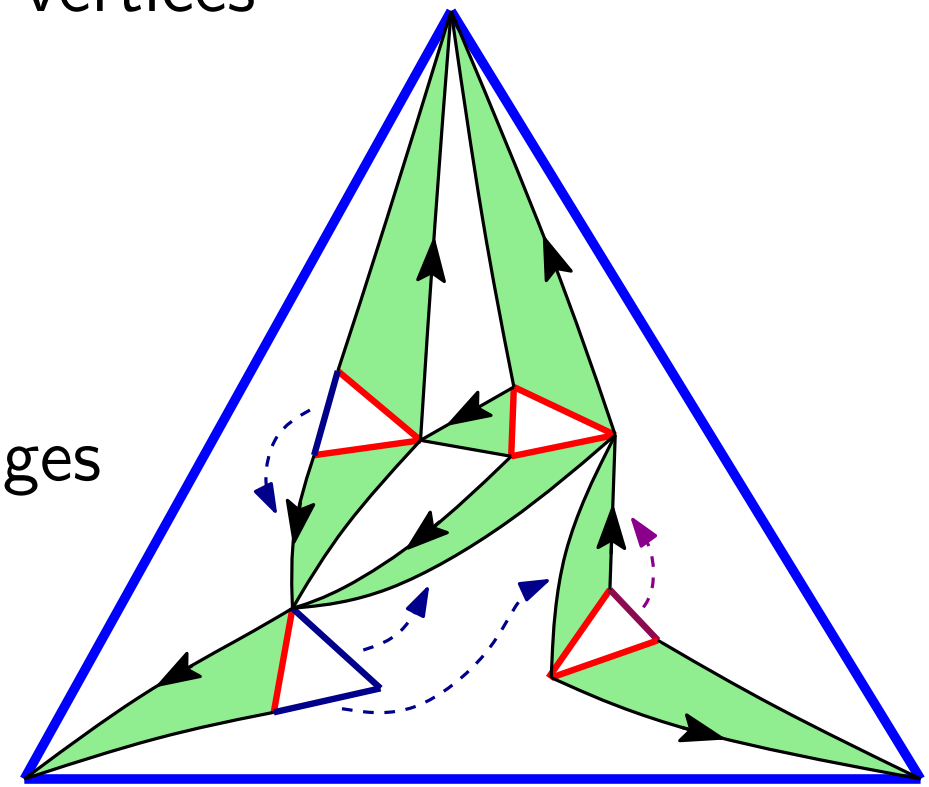

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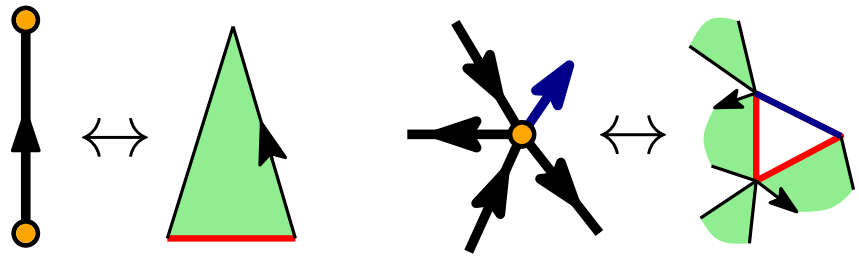

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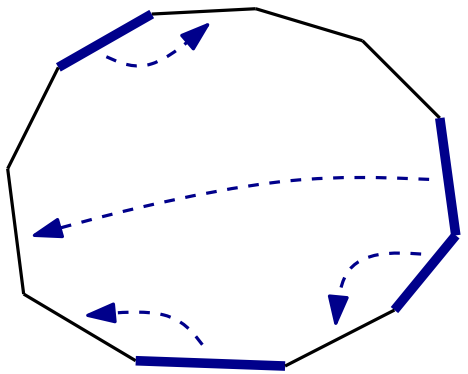

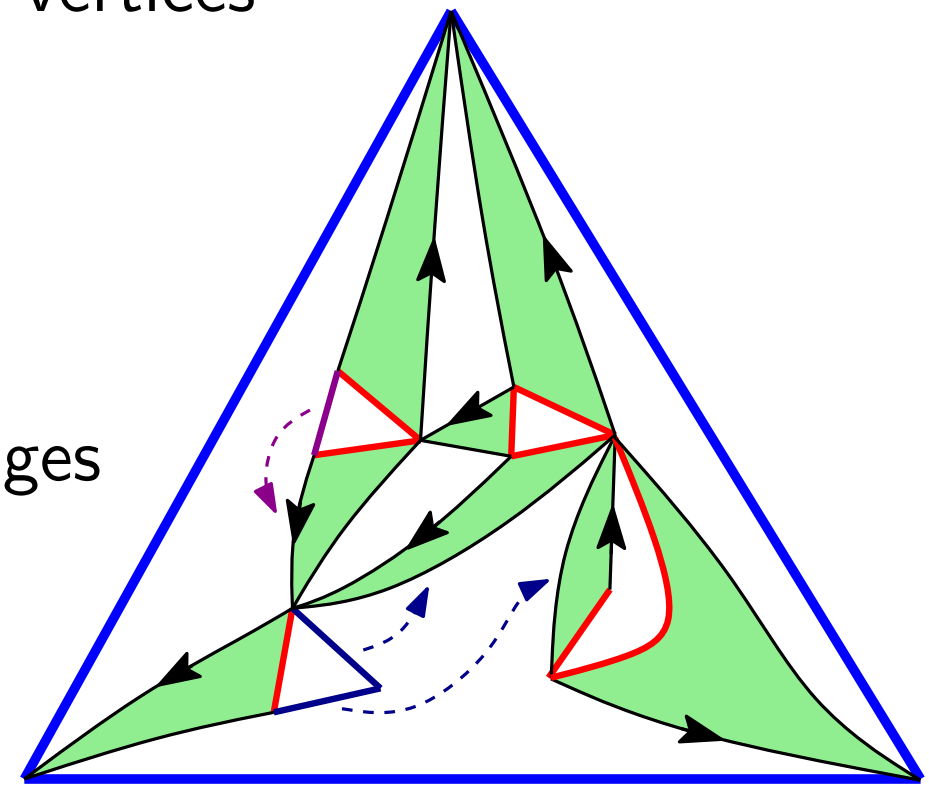

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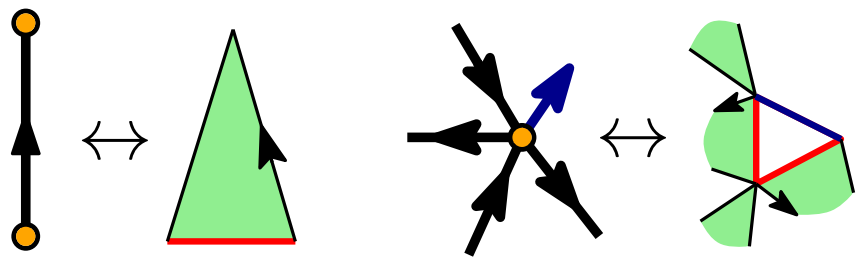

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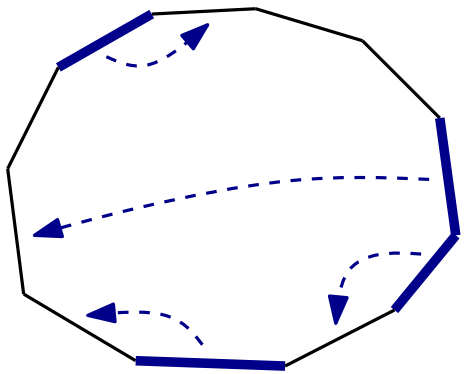

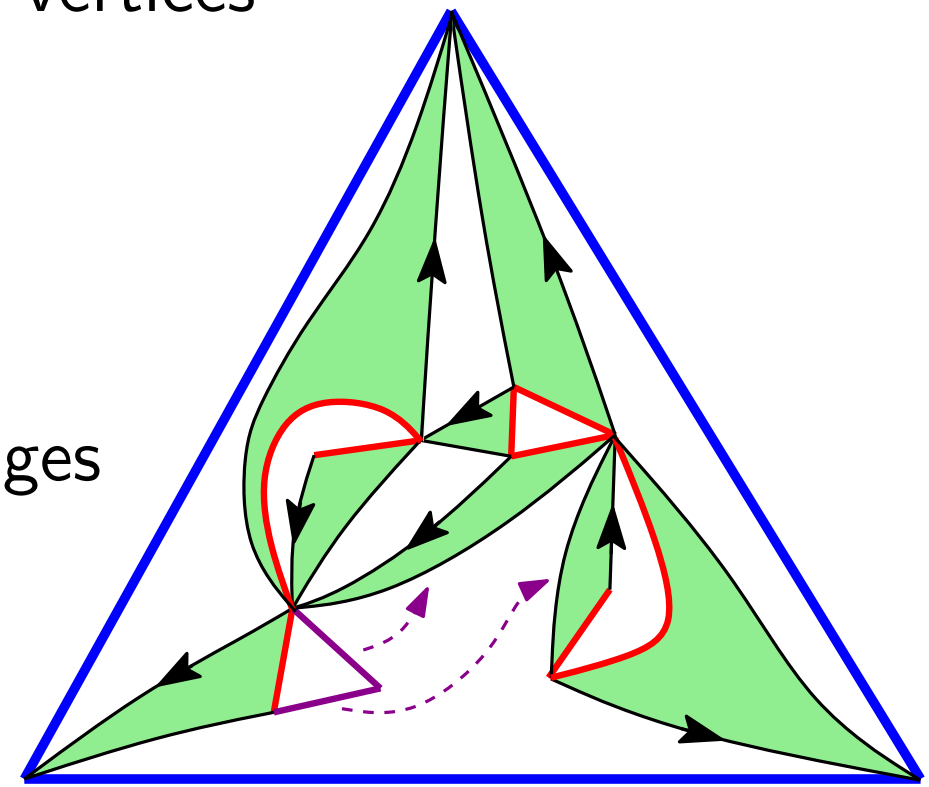

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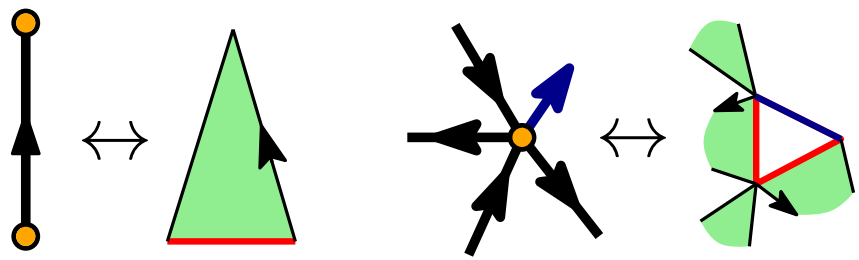

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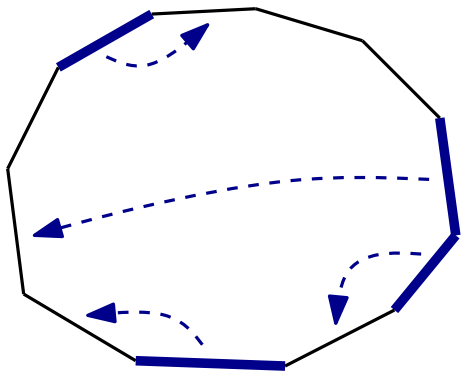
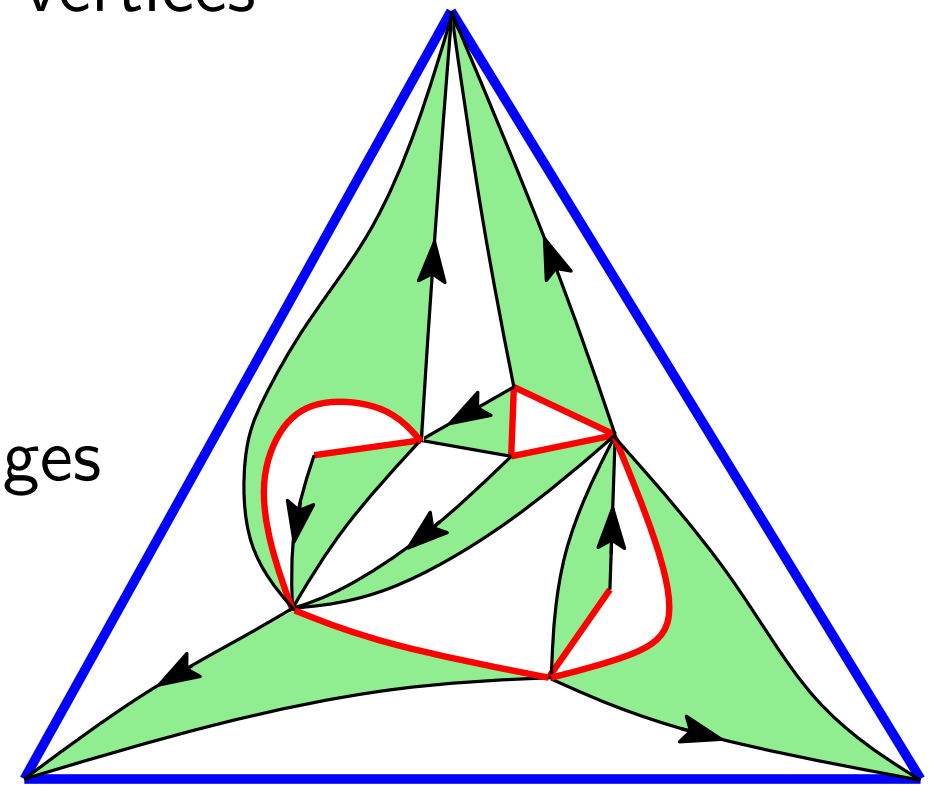
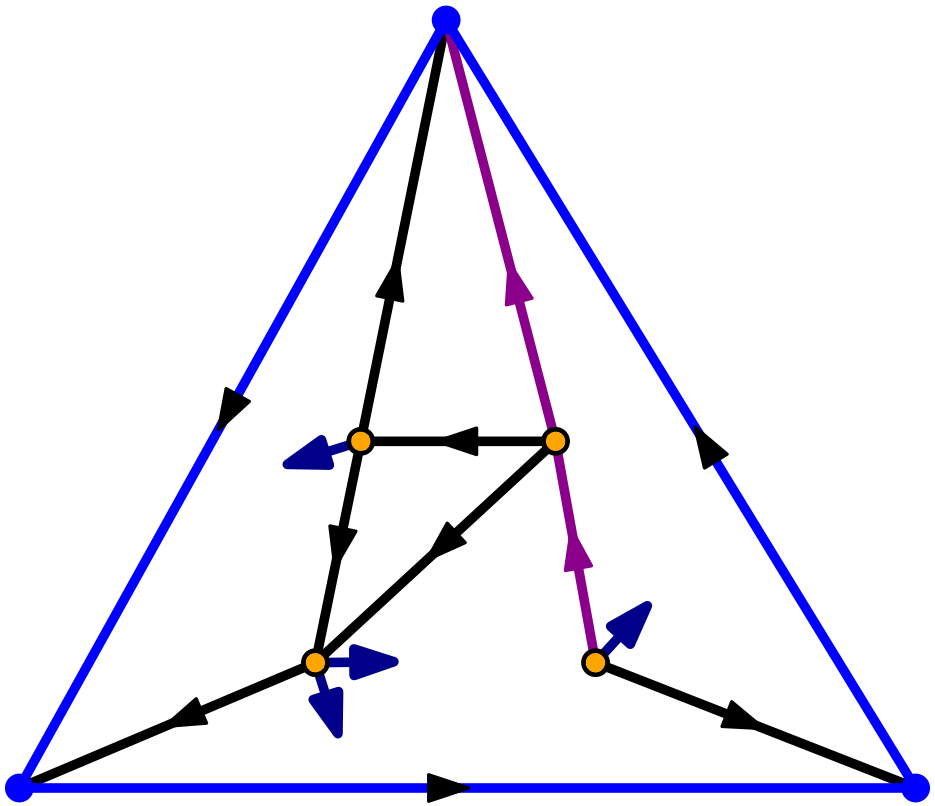


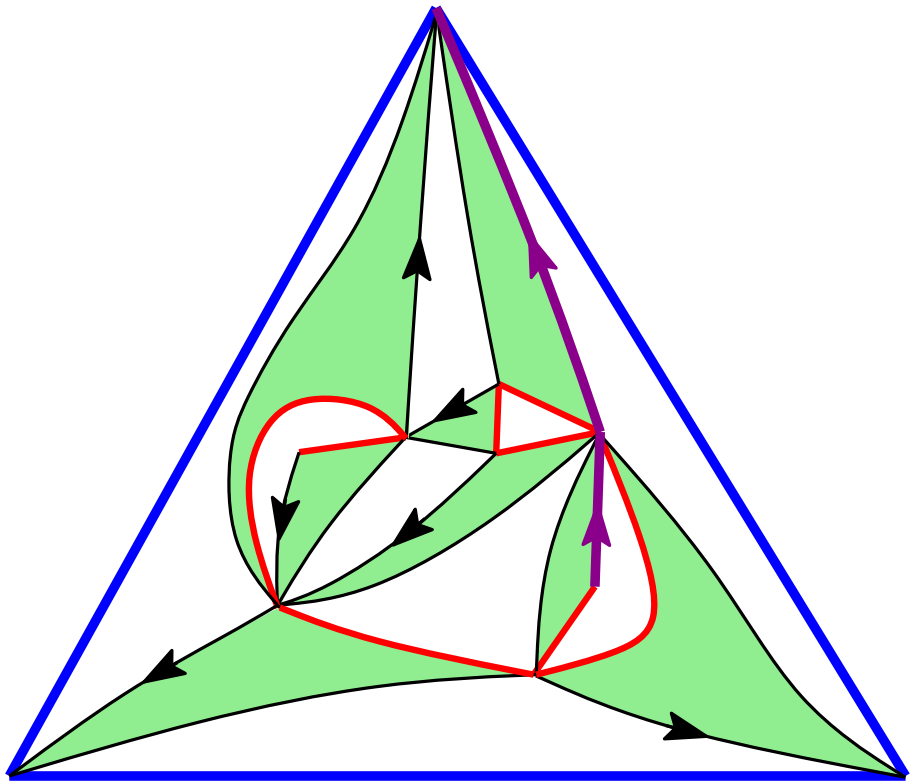


Eulerian triangulation

Distance profile



Outertriangular simple map



Eulerian triangulation

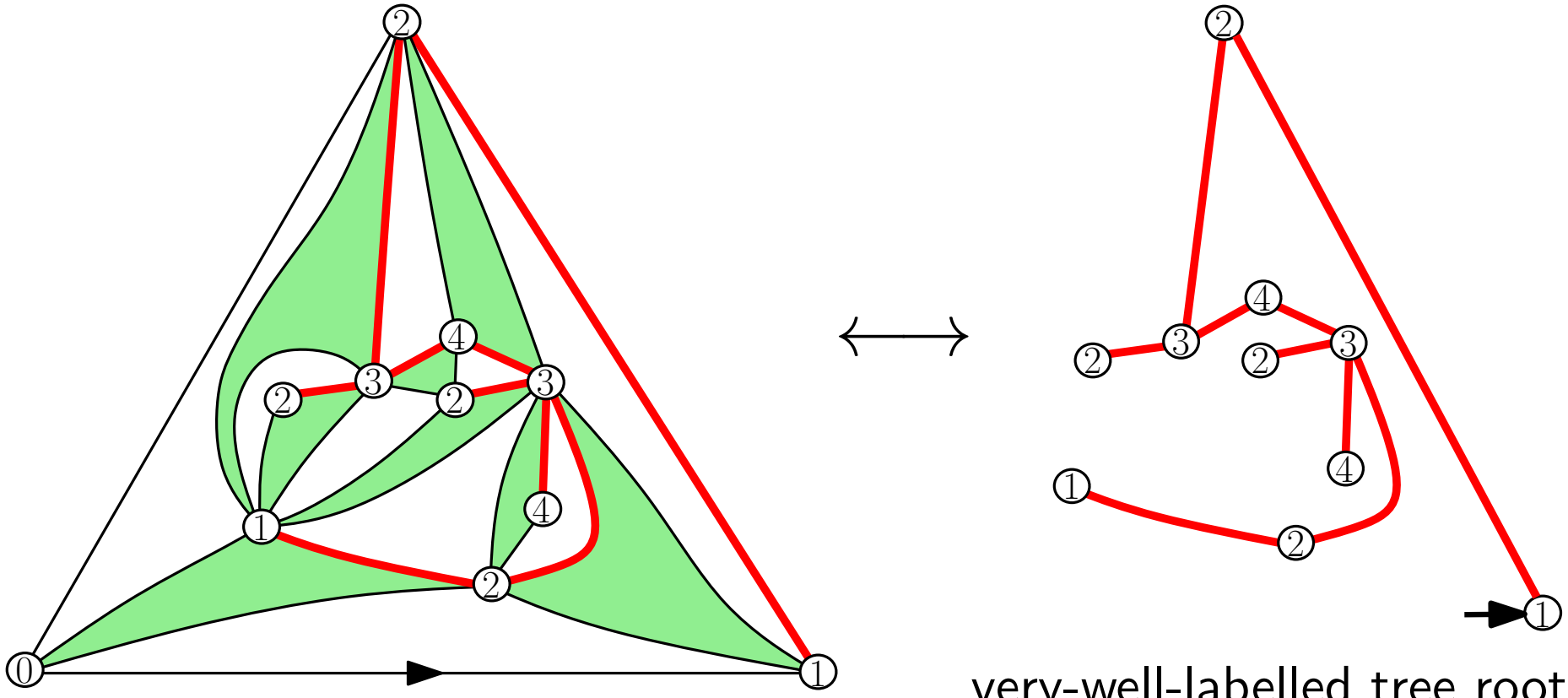
Remark: Preserves rightmost paths

$\Rightarrow d_{S_n} \approx d_{RM} \approx$ distance in oriented forest

Distance profile

[Chassaing–Schaeffer 2004], [Le Gall 2006]:

The distance profile in the oriented forest of a rooted eulerian triangulation, rescaled by $(2n)^{-1/4}$, converges to positively shifted **ISE**.



Eulerian triangulation with $2n$ faces with vertices labelled by geodesic distance

very-well-labelled tree rooted on a vertex of minimum label

Distance profile

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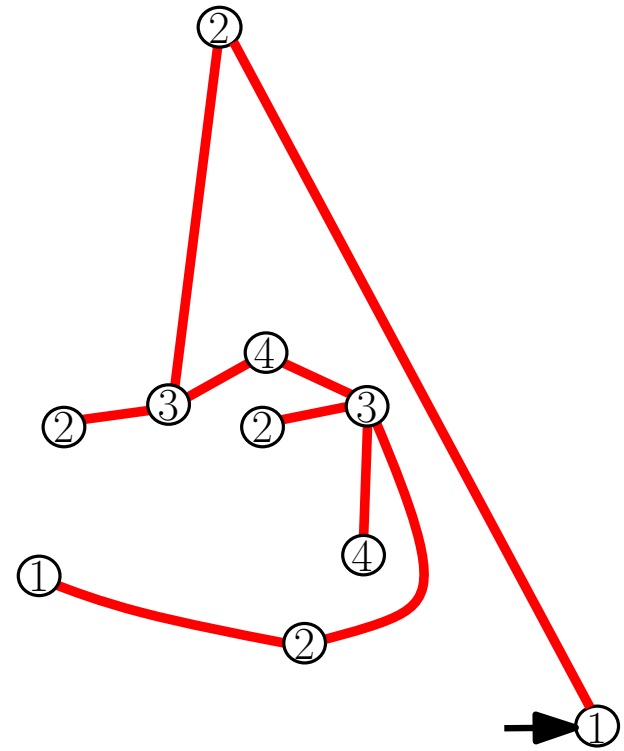
labelled Galton-Walson tree
where the labels behave as a
random walk on the tree

$\longleftrightarrow$



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Encoding with trees: first bijection

[Bousquet-Mélou–Schaeffer 2004] (specialization):

There is a bijection between:

Rooted eulerian triangulations with $2n$ faces $\leftrightarrow$ Balanced oriented binary trees with $n-1$ inner edges

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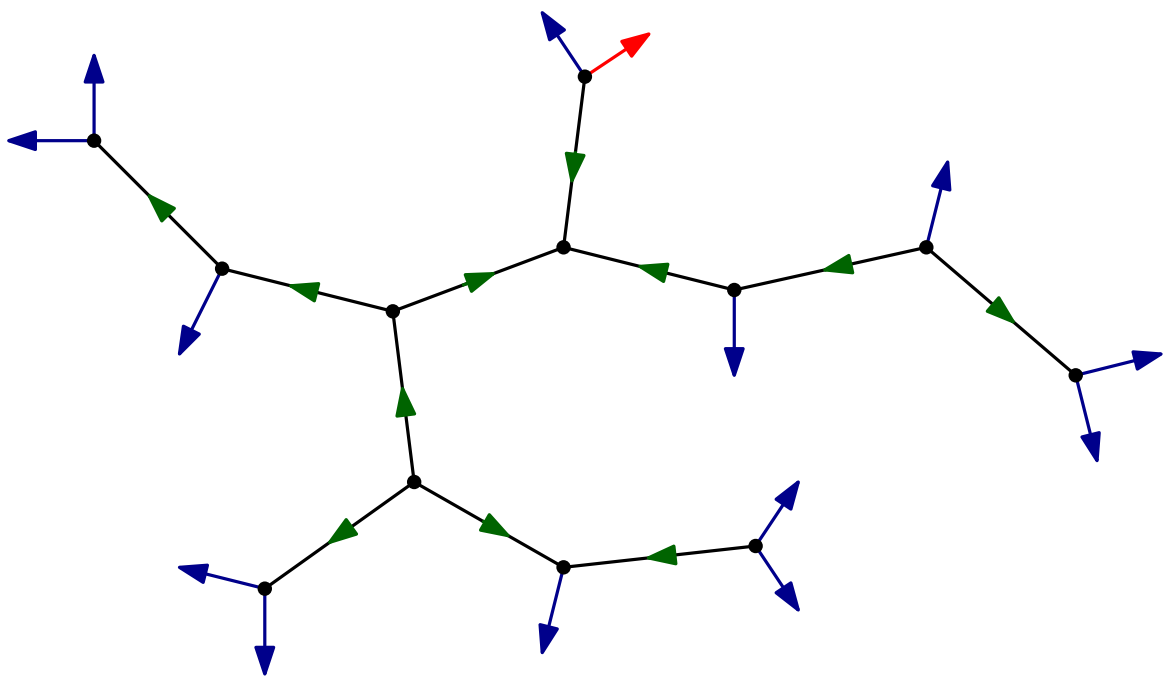

- Rooted binary tree with oriented inner edges

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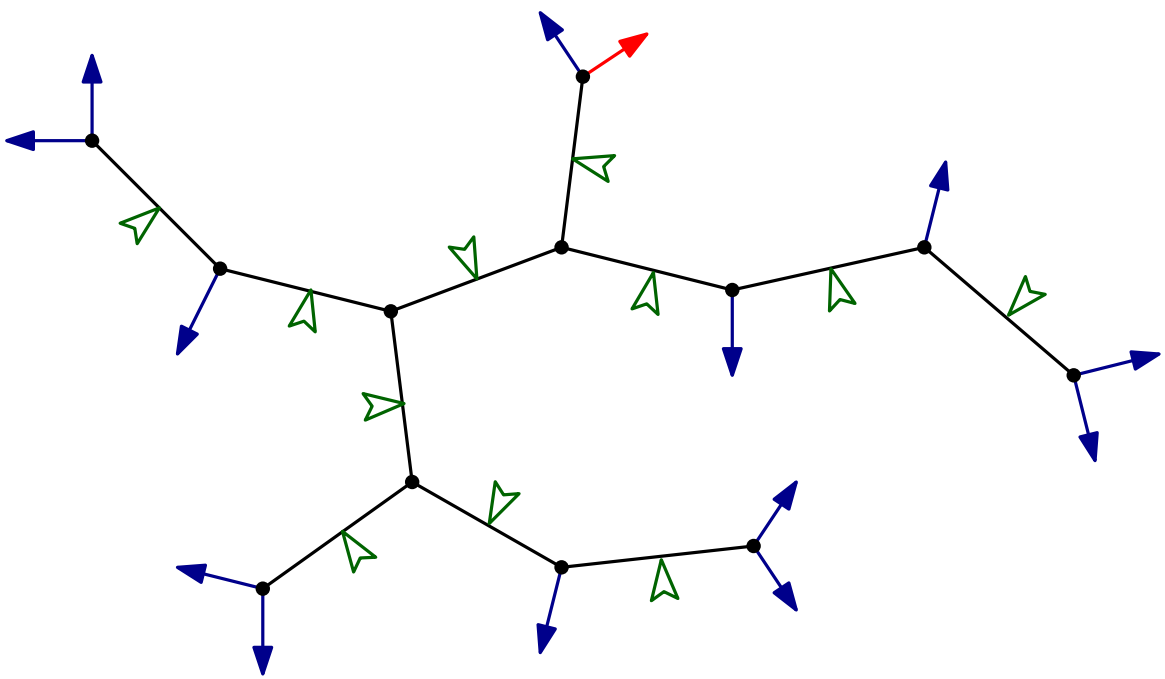

- Rooted binary tree with oriented inner edges
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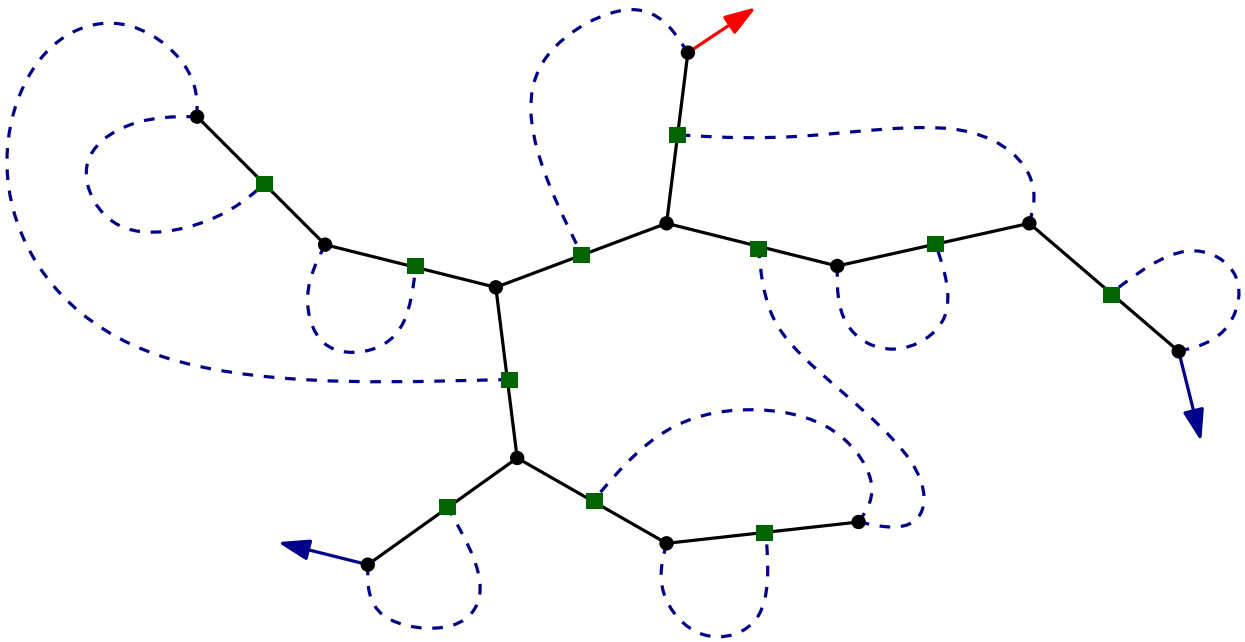

- Rooted binary tree with oriented inner edges
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- Closure *à la* Schaeffer 3 unmatched edges

root unmatched = balanced

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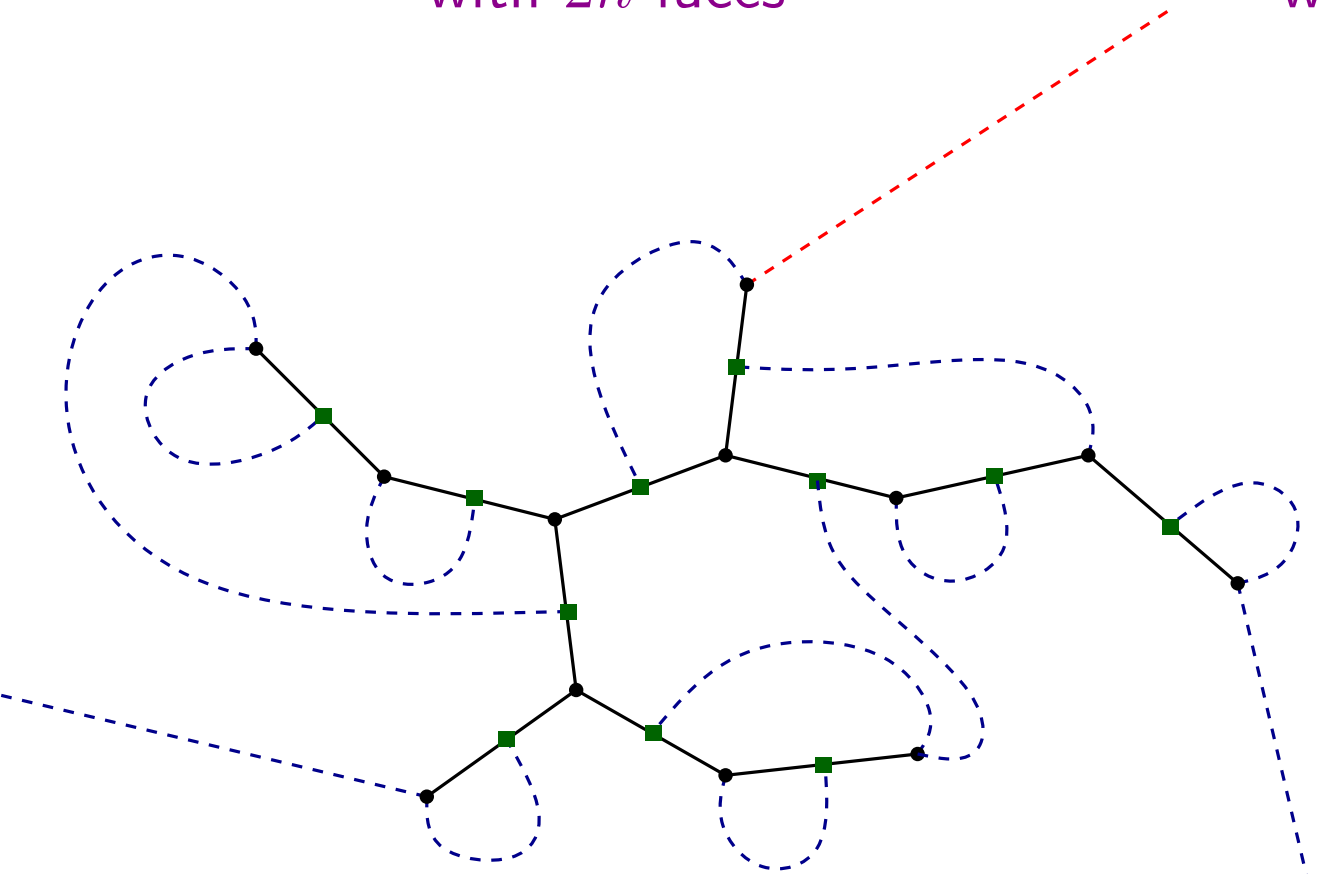

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- Unmatched edges closed at ∞ to obtain a rooted bipartite cubic map

$\rightarrow$ dual of eulerian triangulations

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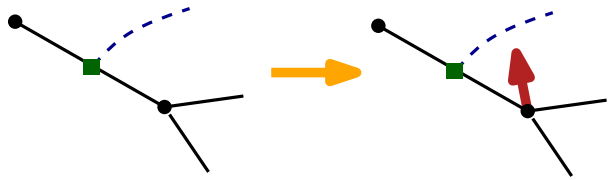

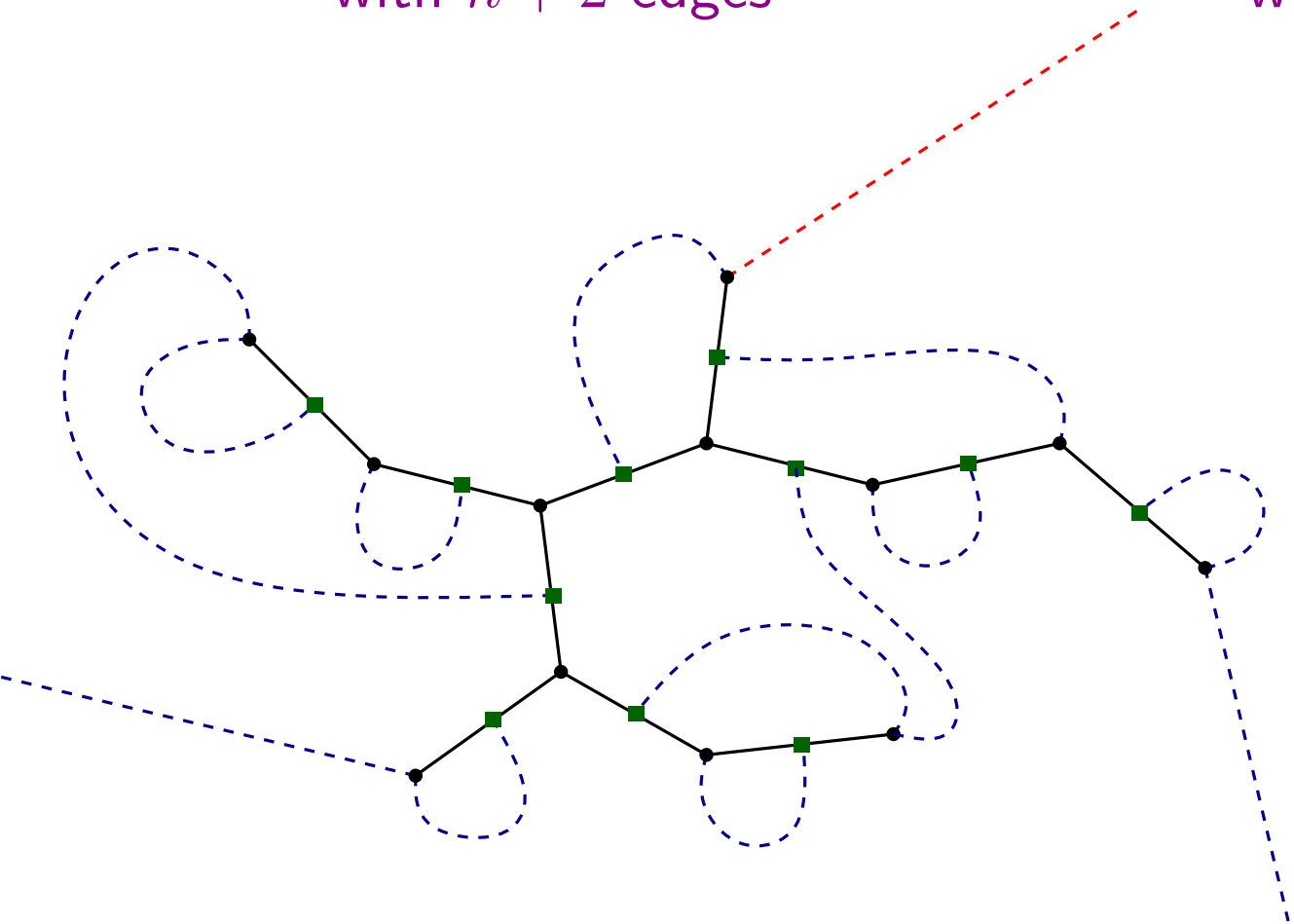

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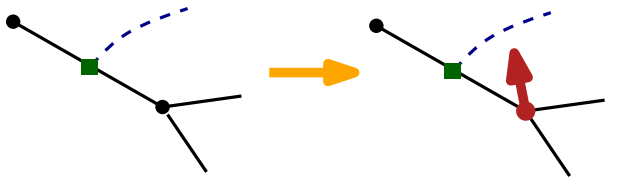

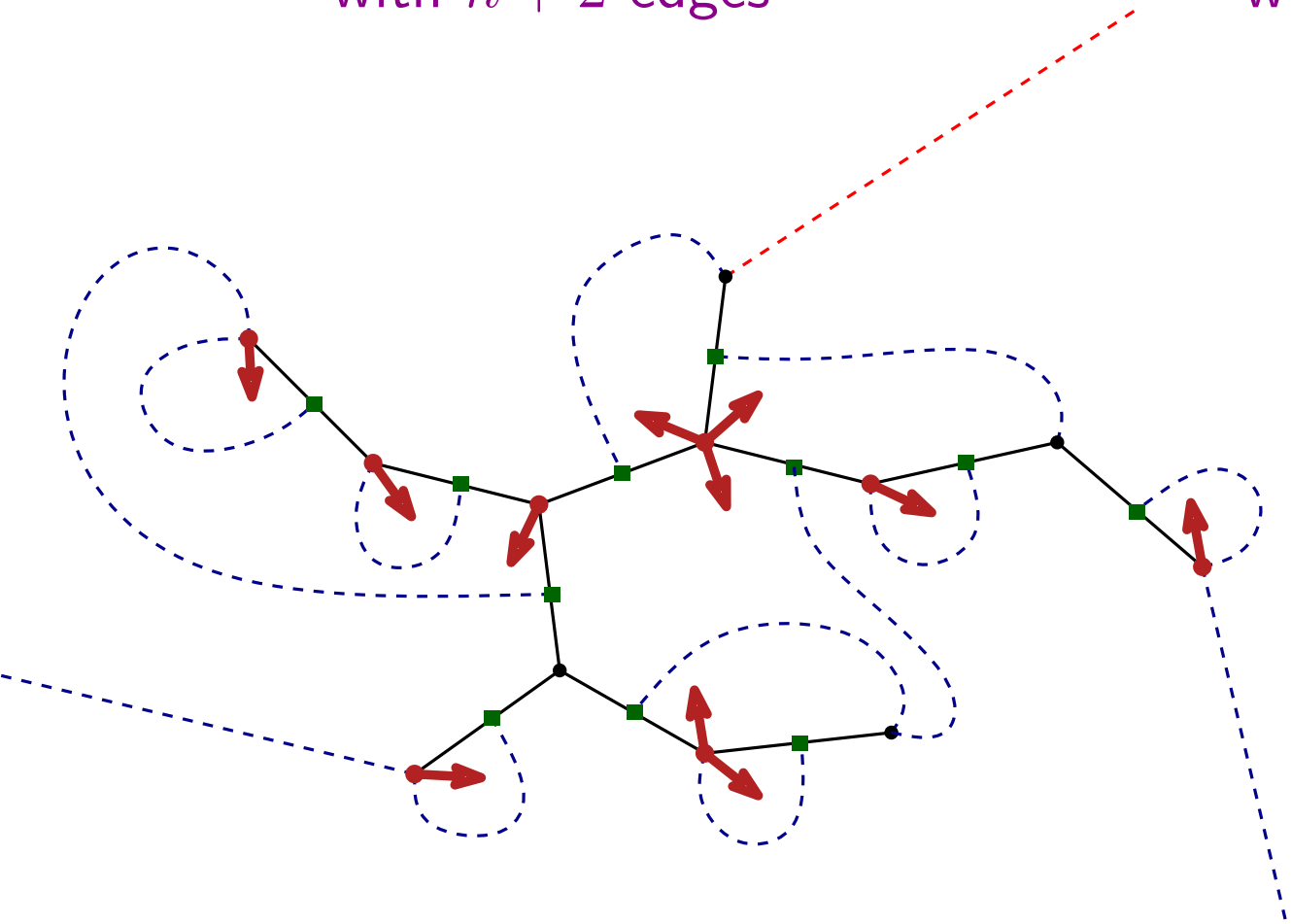

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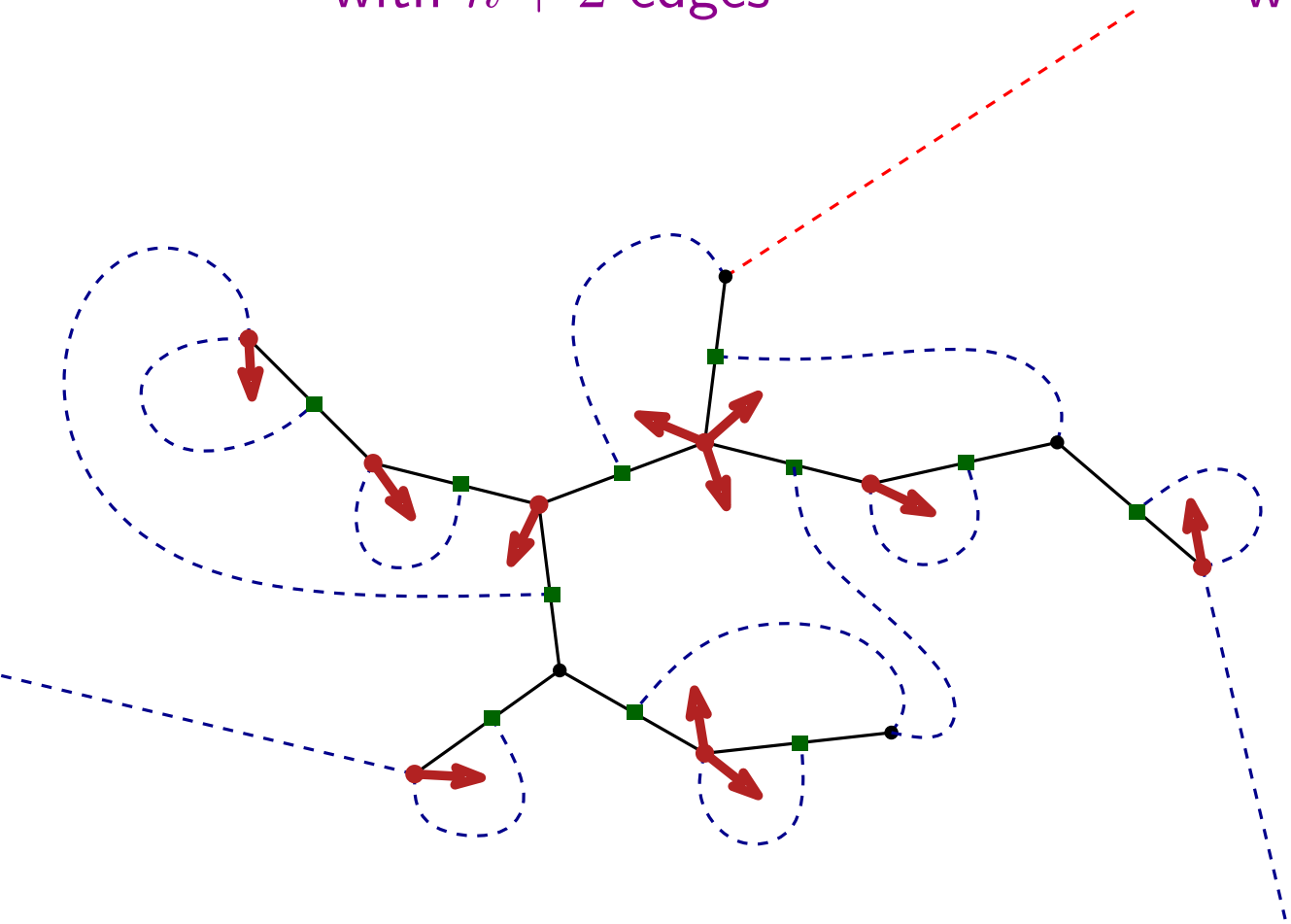

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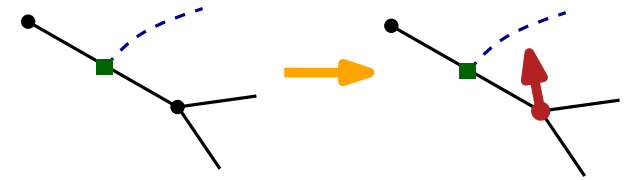
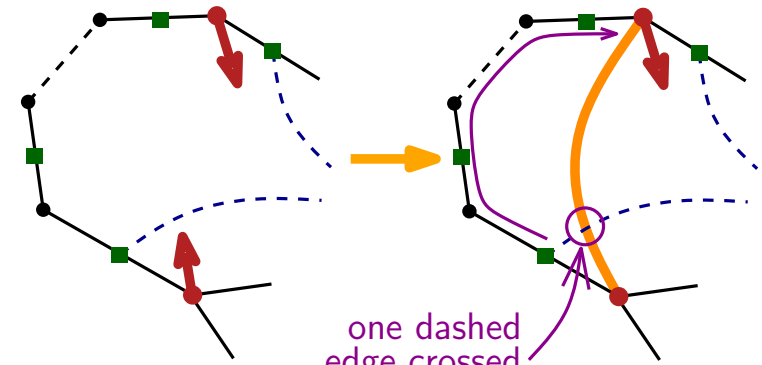


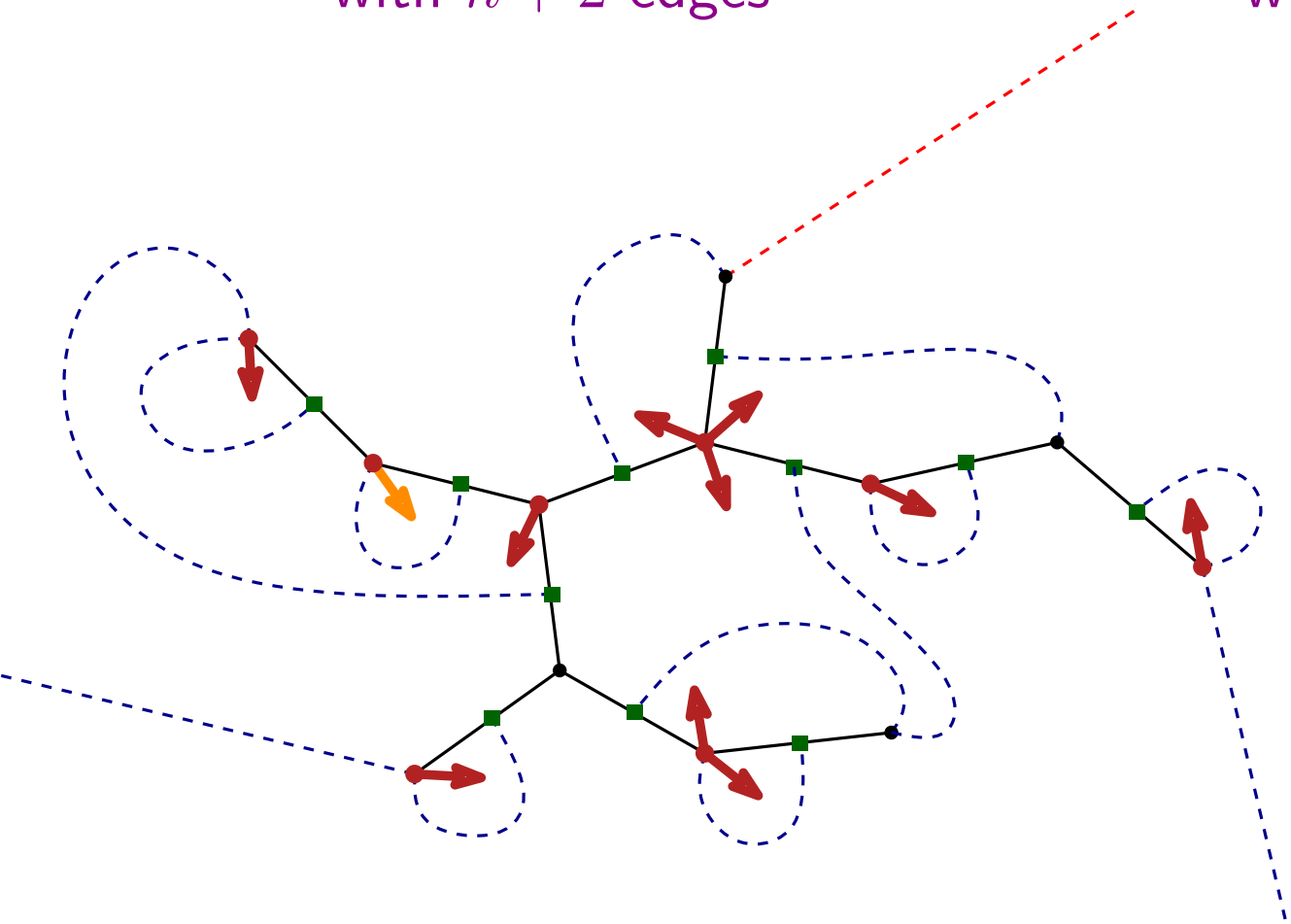
- Counterclockwise closure:



Encoding with trees: first bijection

There is a bijection between:

Rooted outertriangular simple maps with $n+2$ edges $\leftrightarrow$ Balanced oriented binary trees with $n-1$ inner edges



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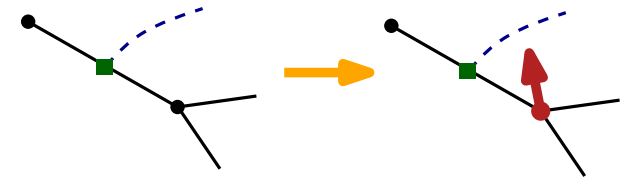

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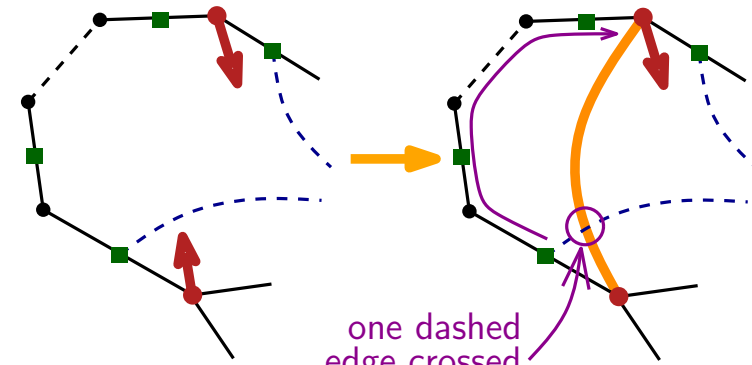

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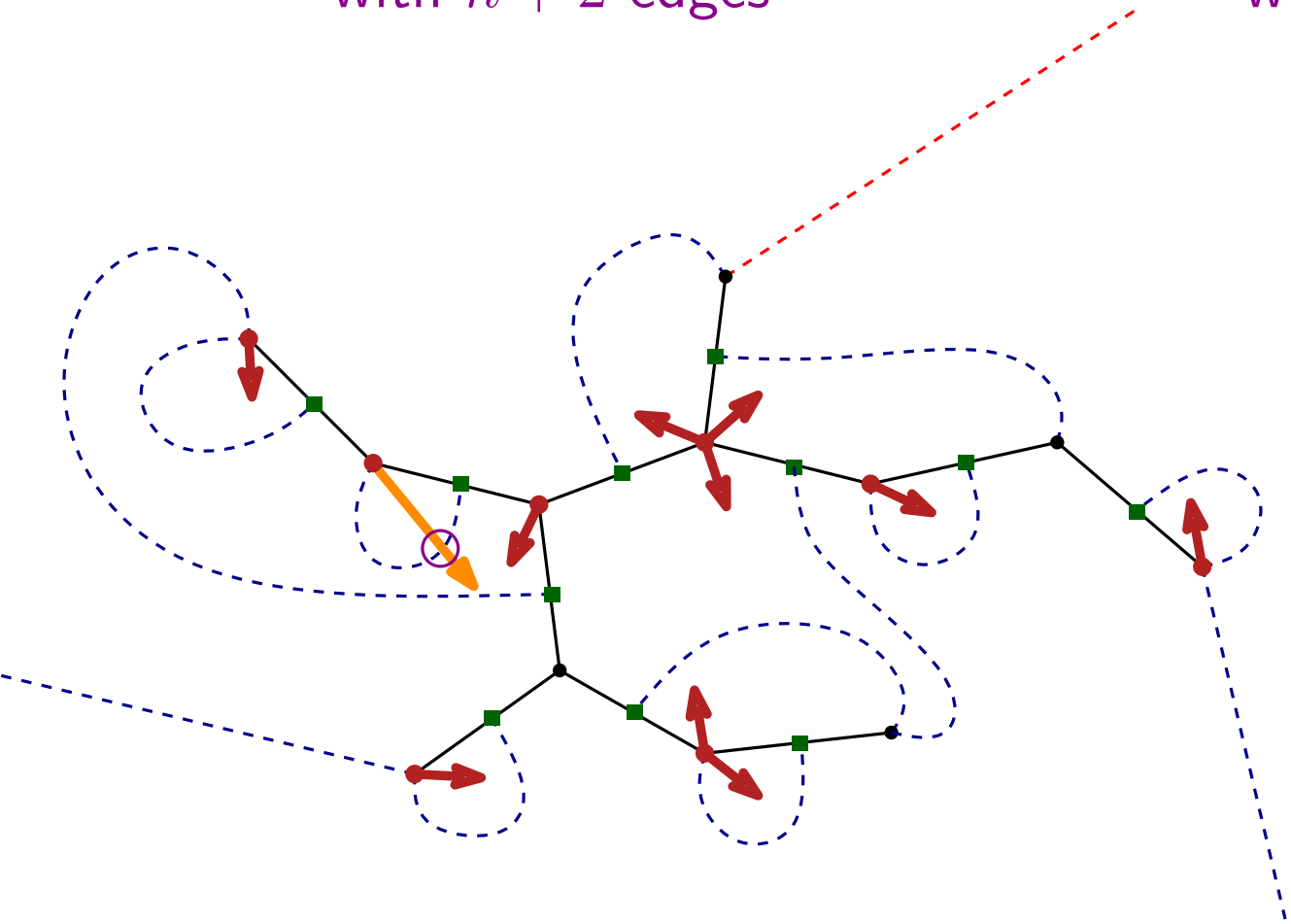

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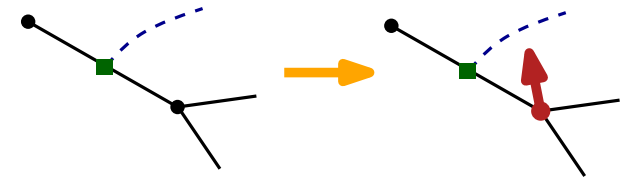

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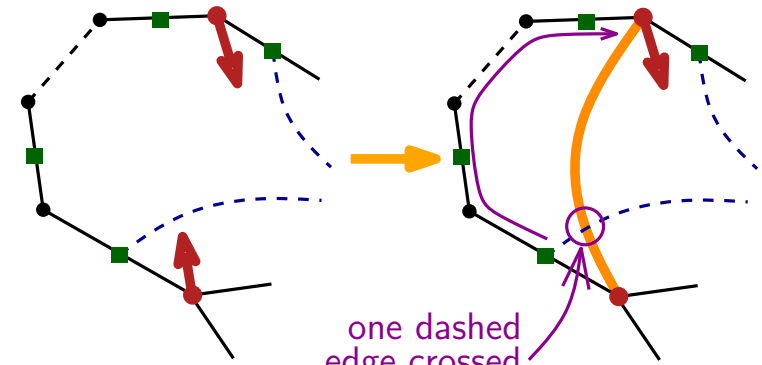

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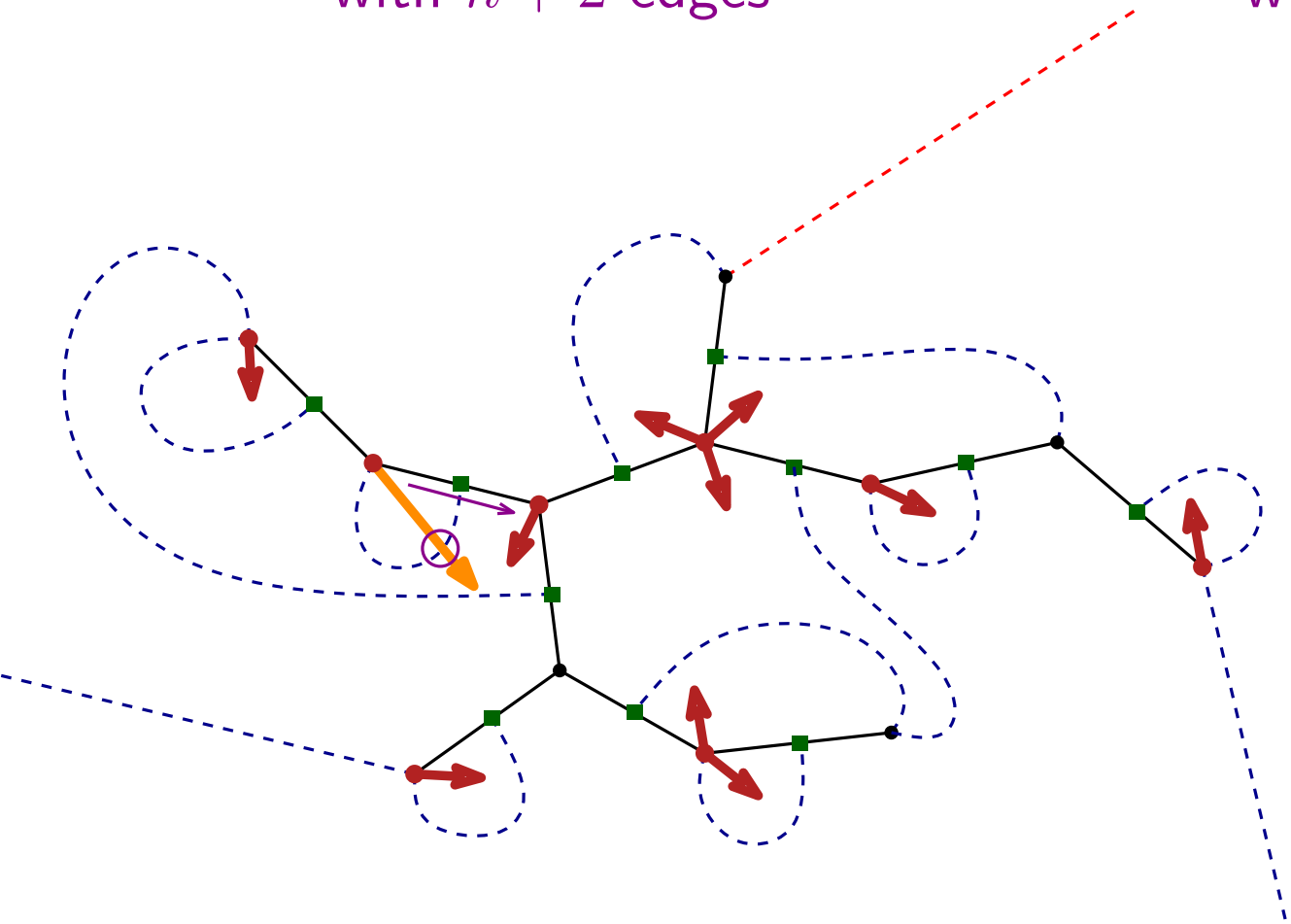

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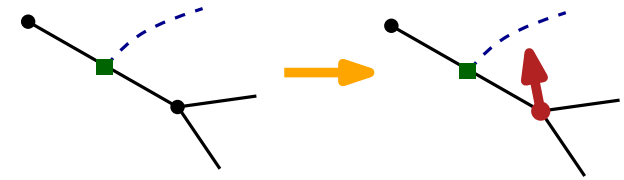

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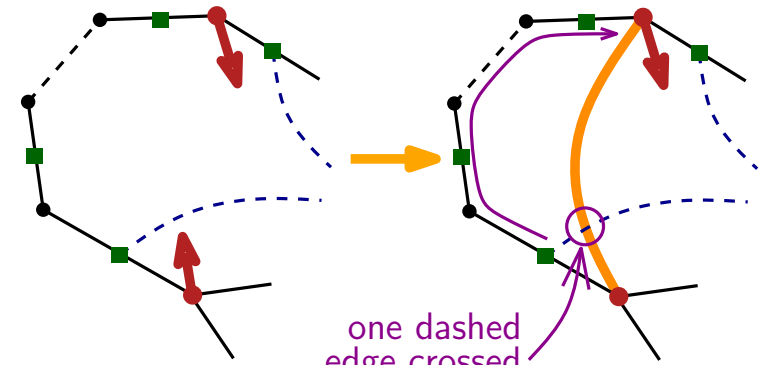

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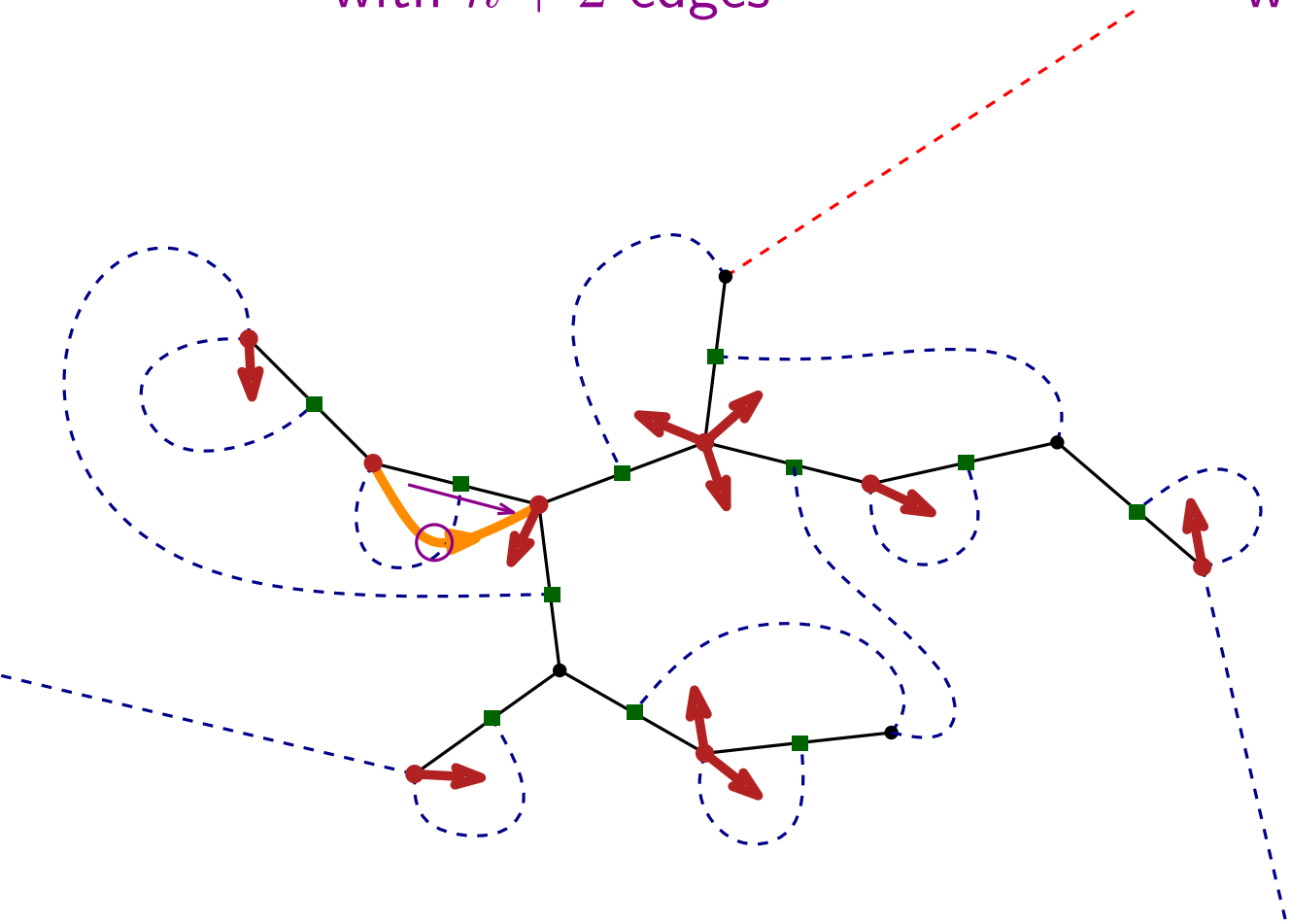

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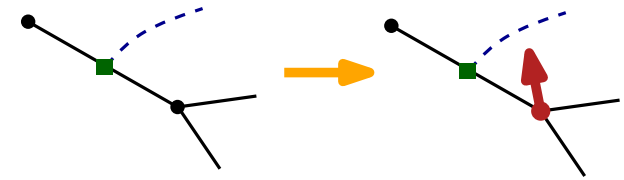

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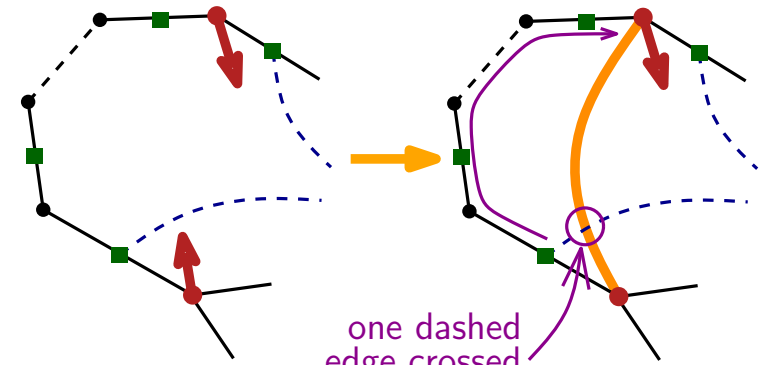

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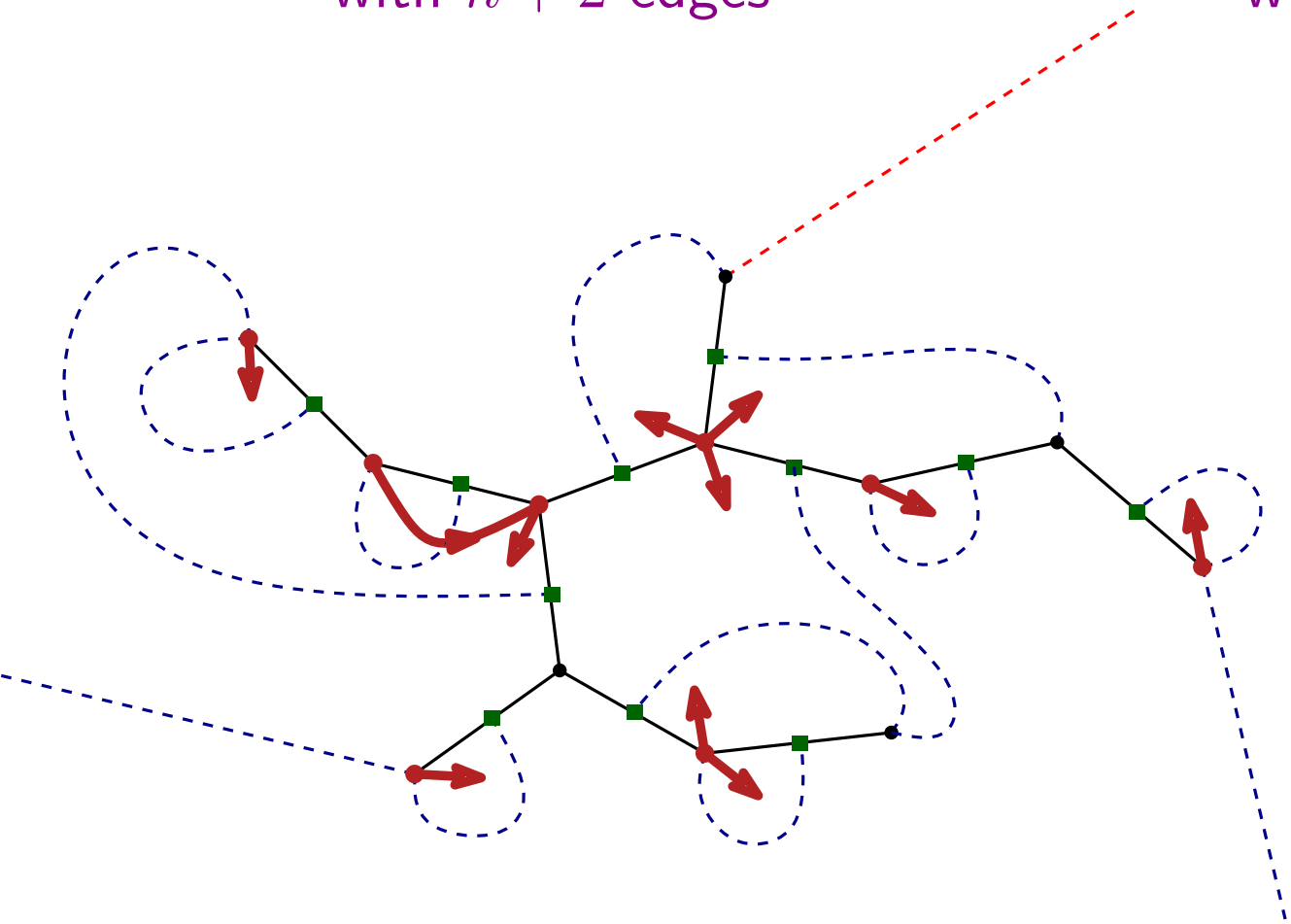

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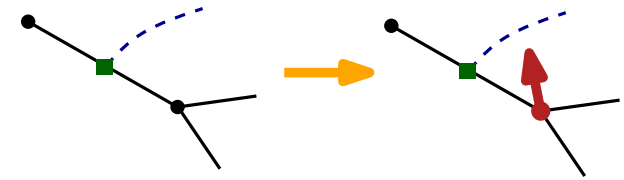

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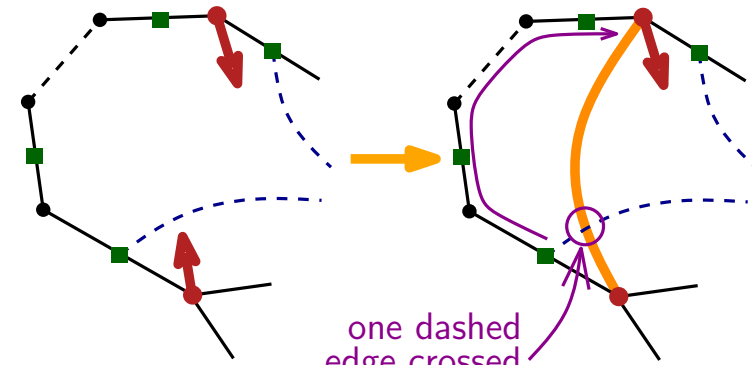

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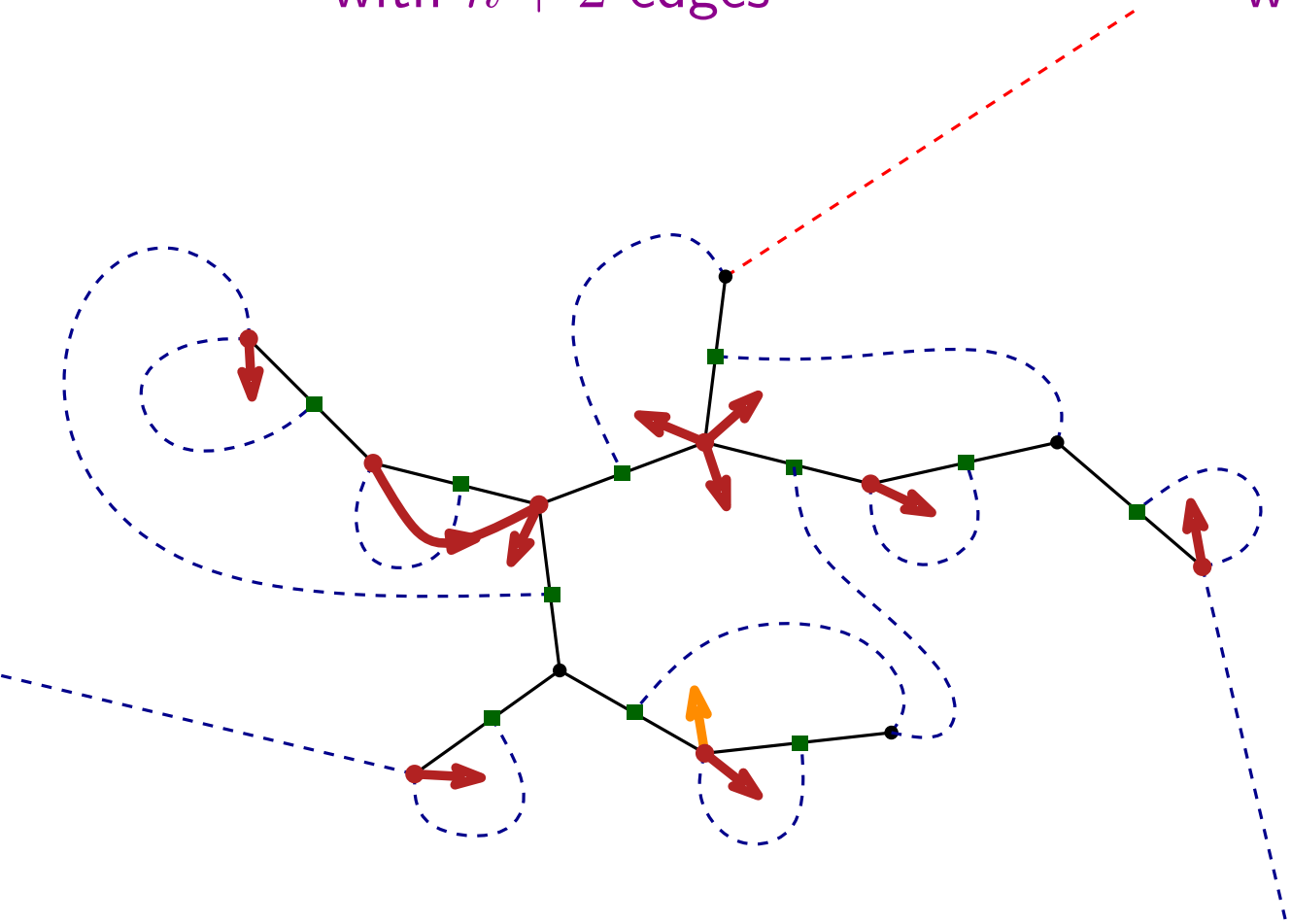

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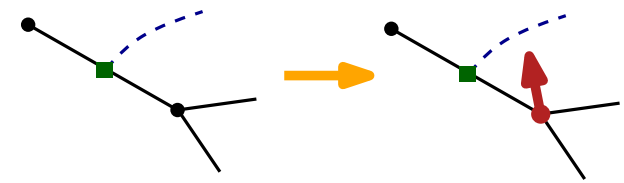

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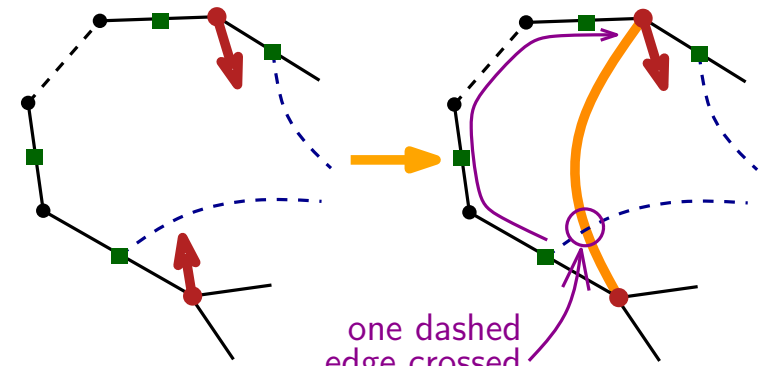

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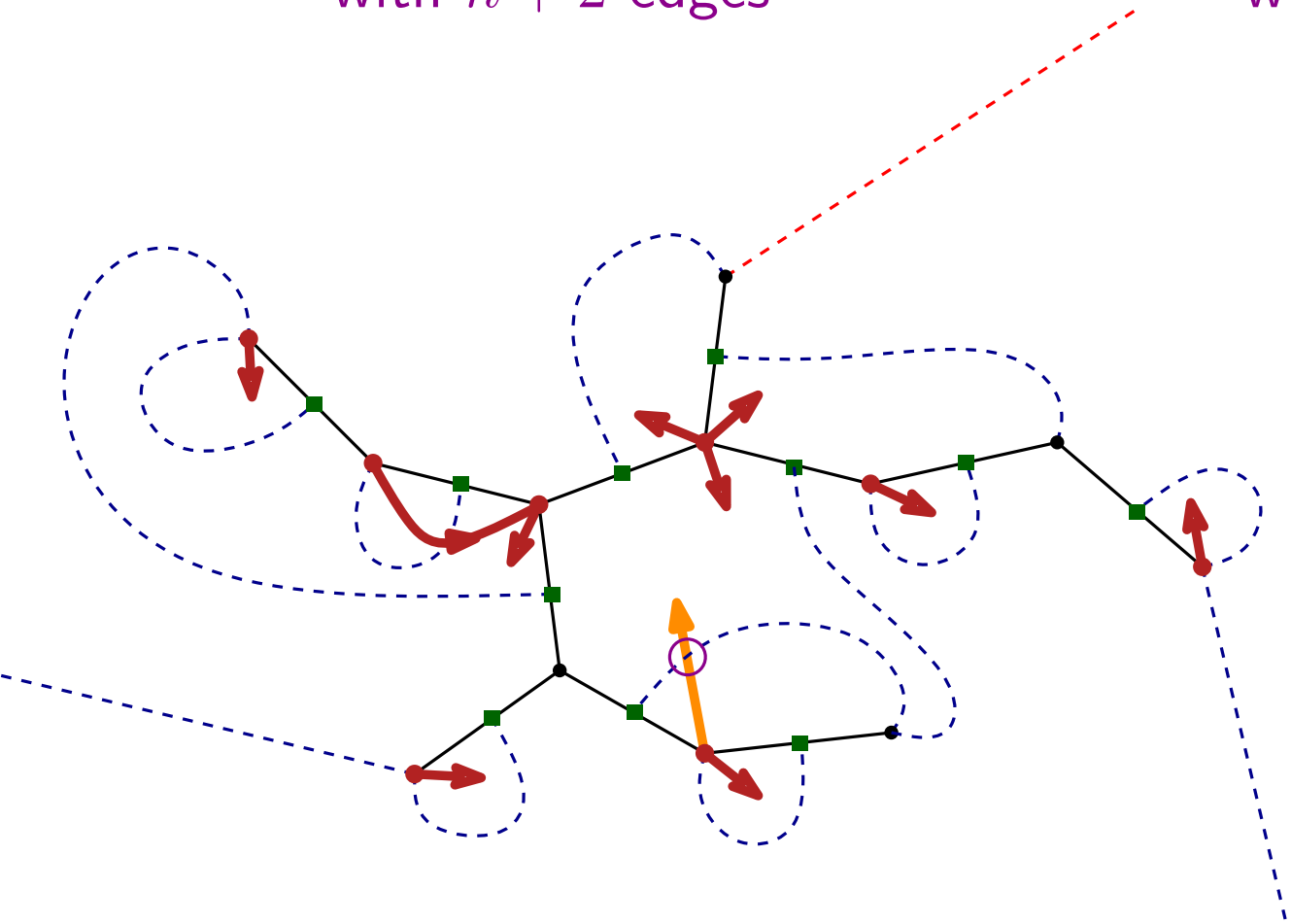

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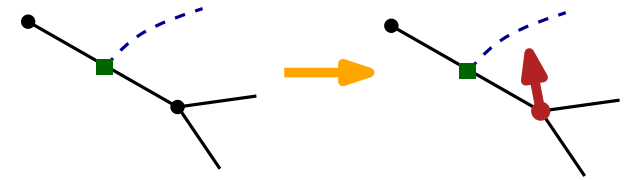

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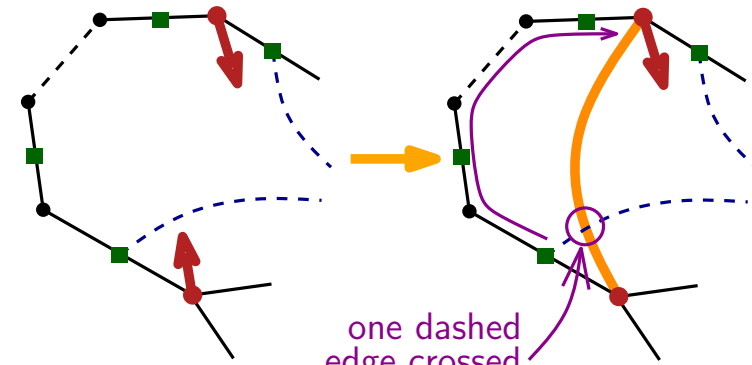

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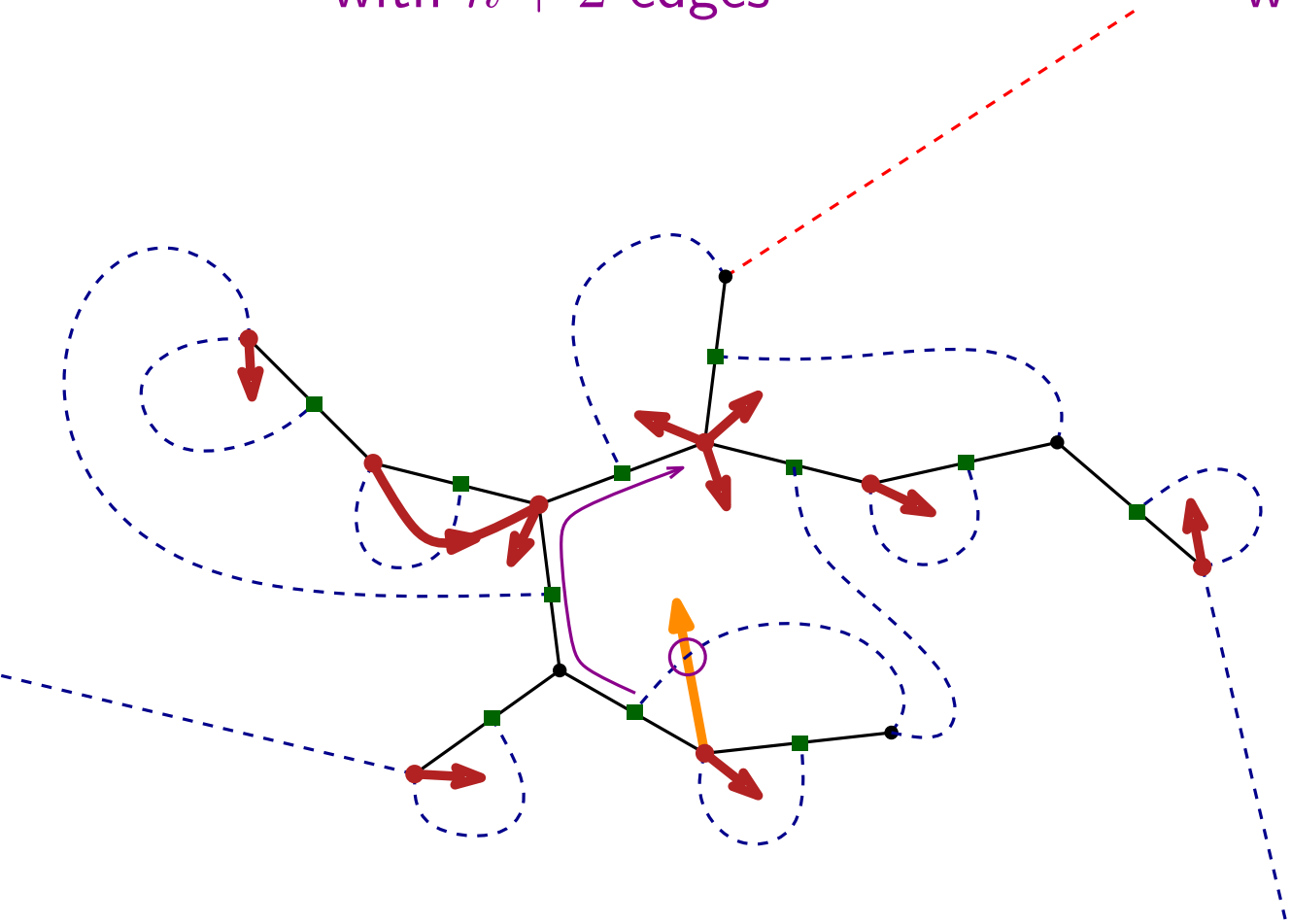

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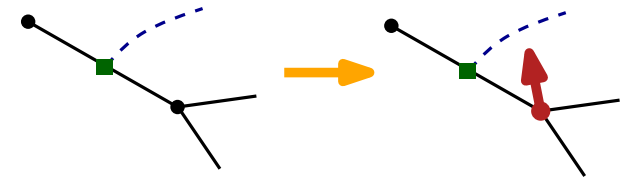

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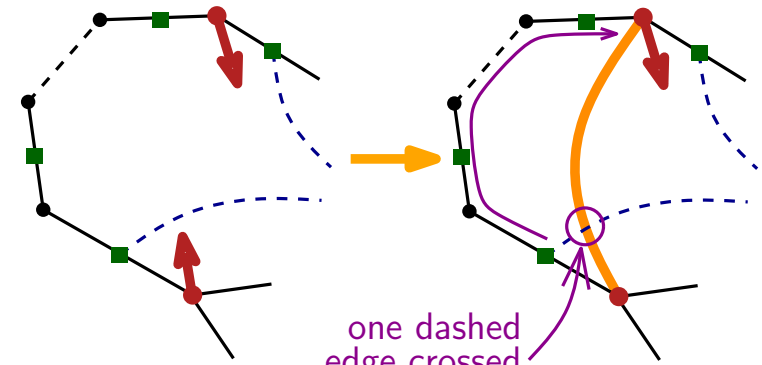

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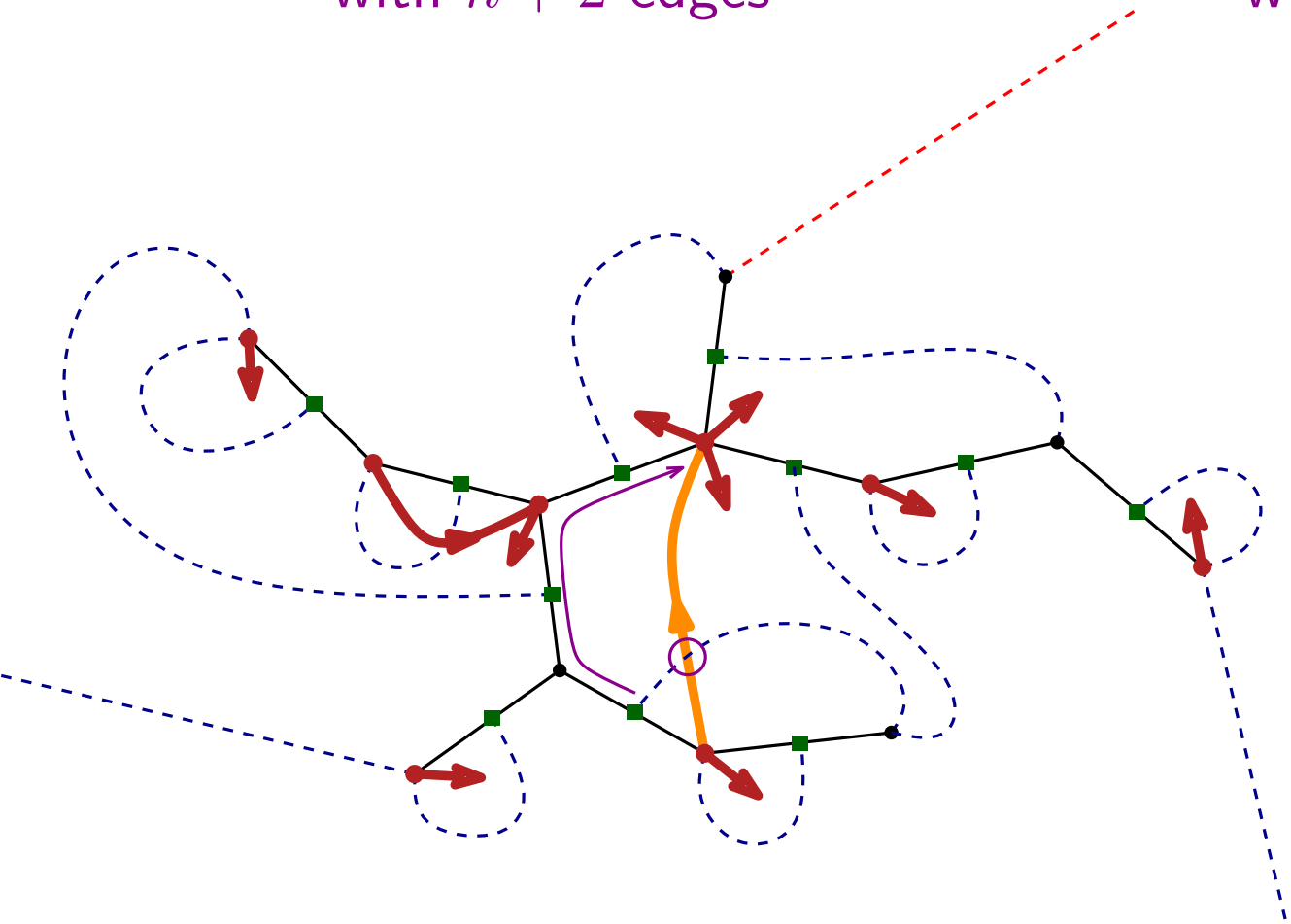

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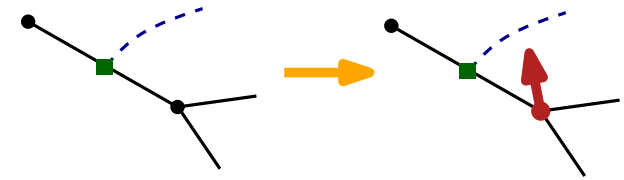

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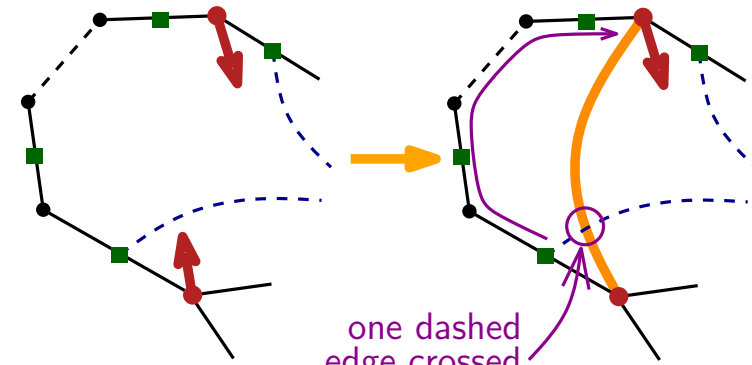

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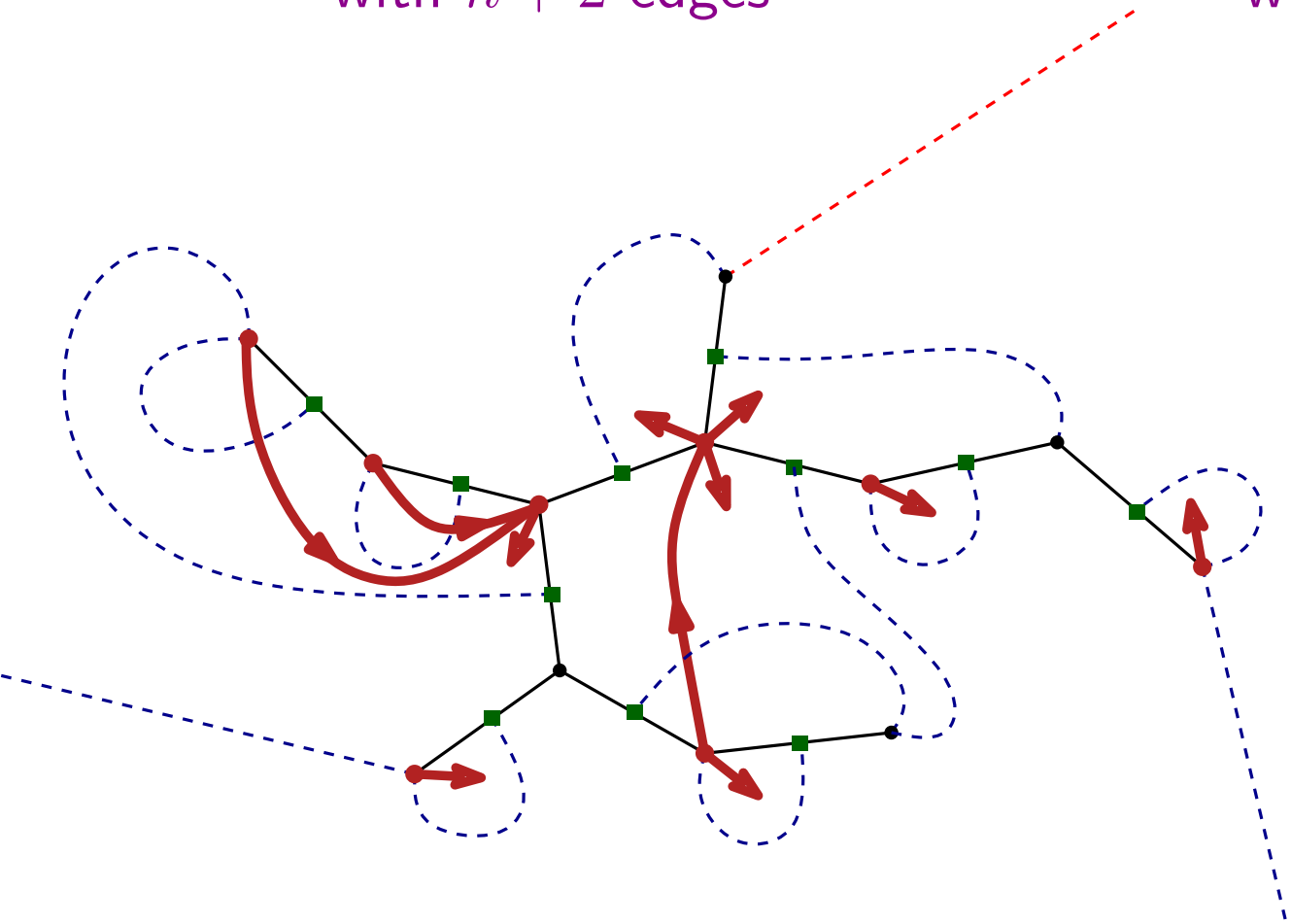

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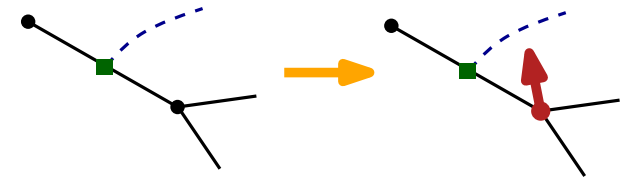

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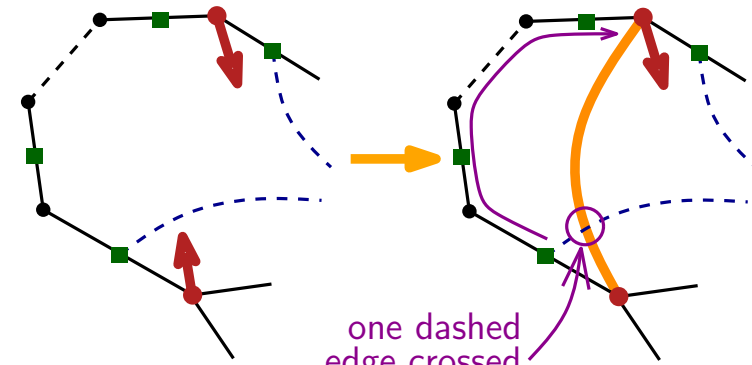

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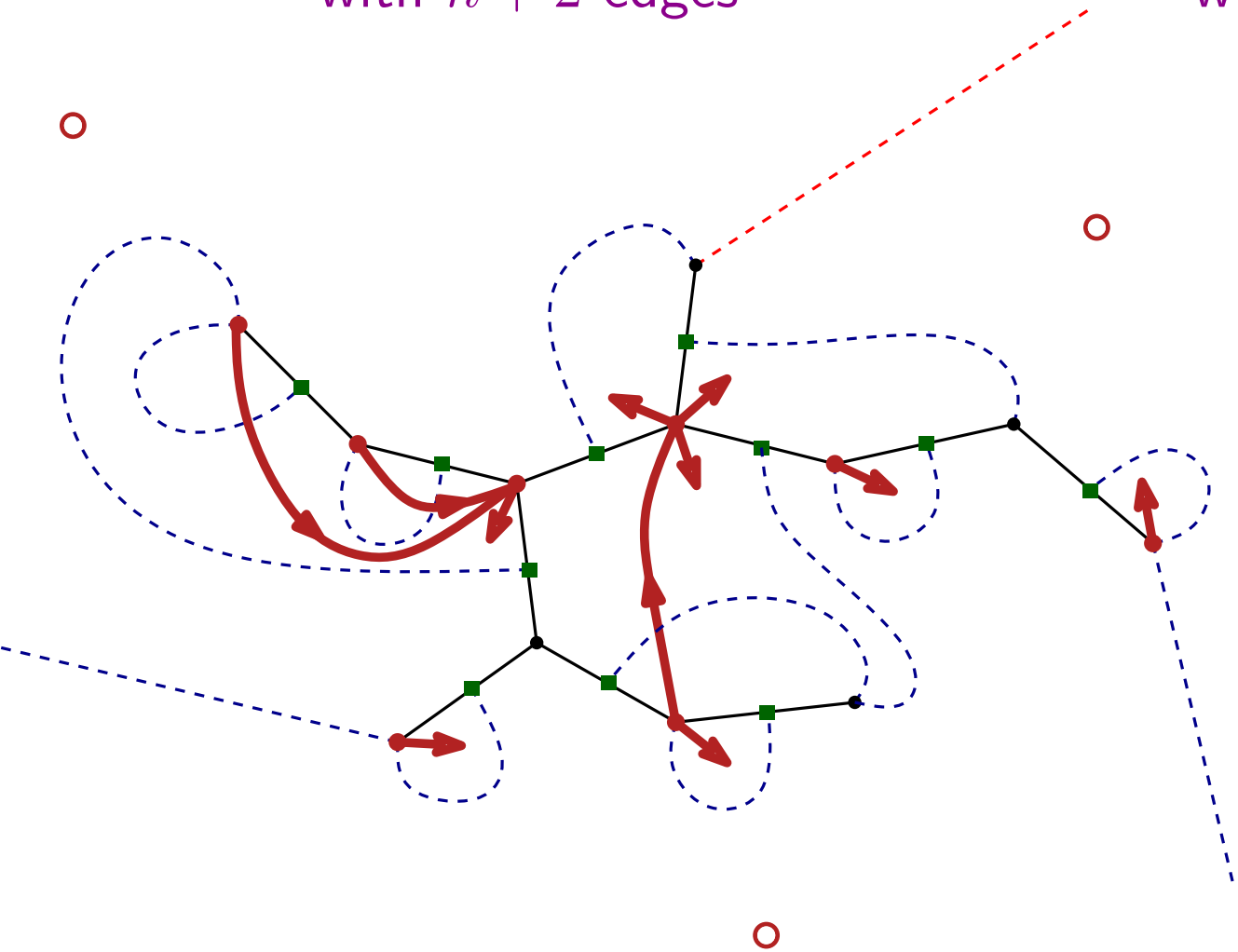

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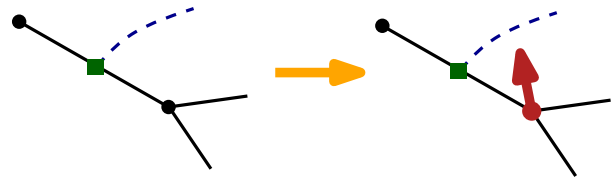

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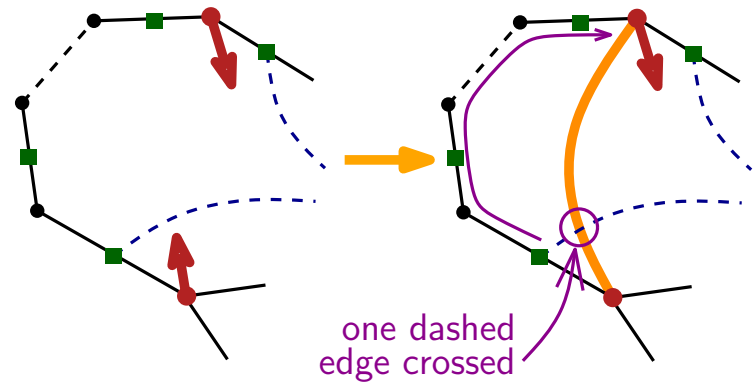

- Add 3 outer vertices and close remaining buds

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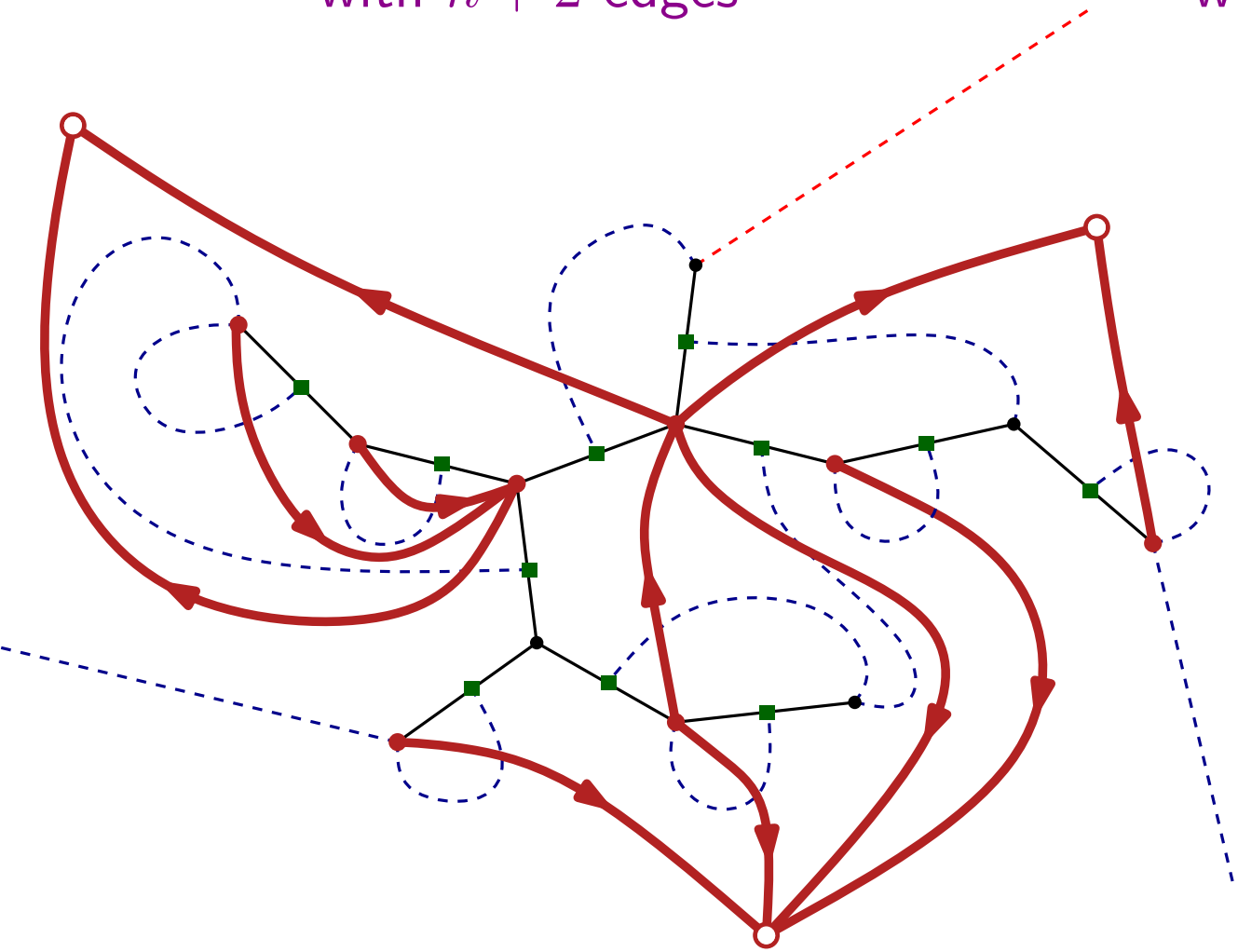

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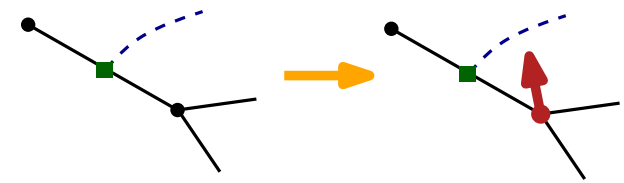

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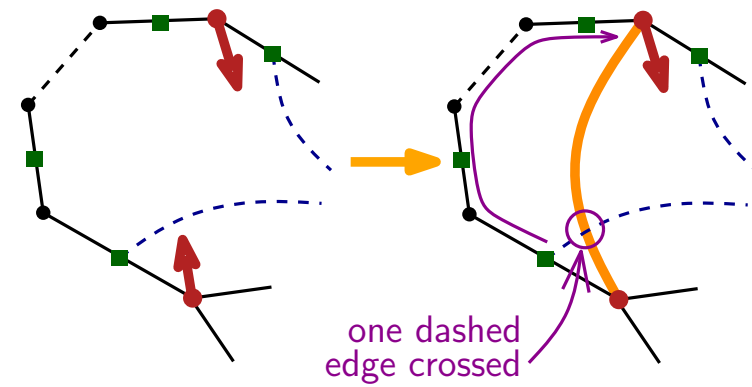

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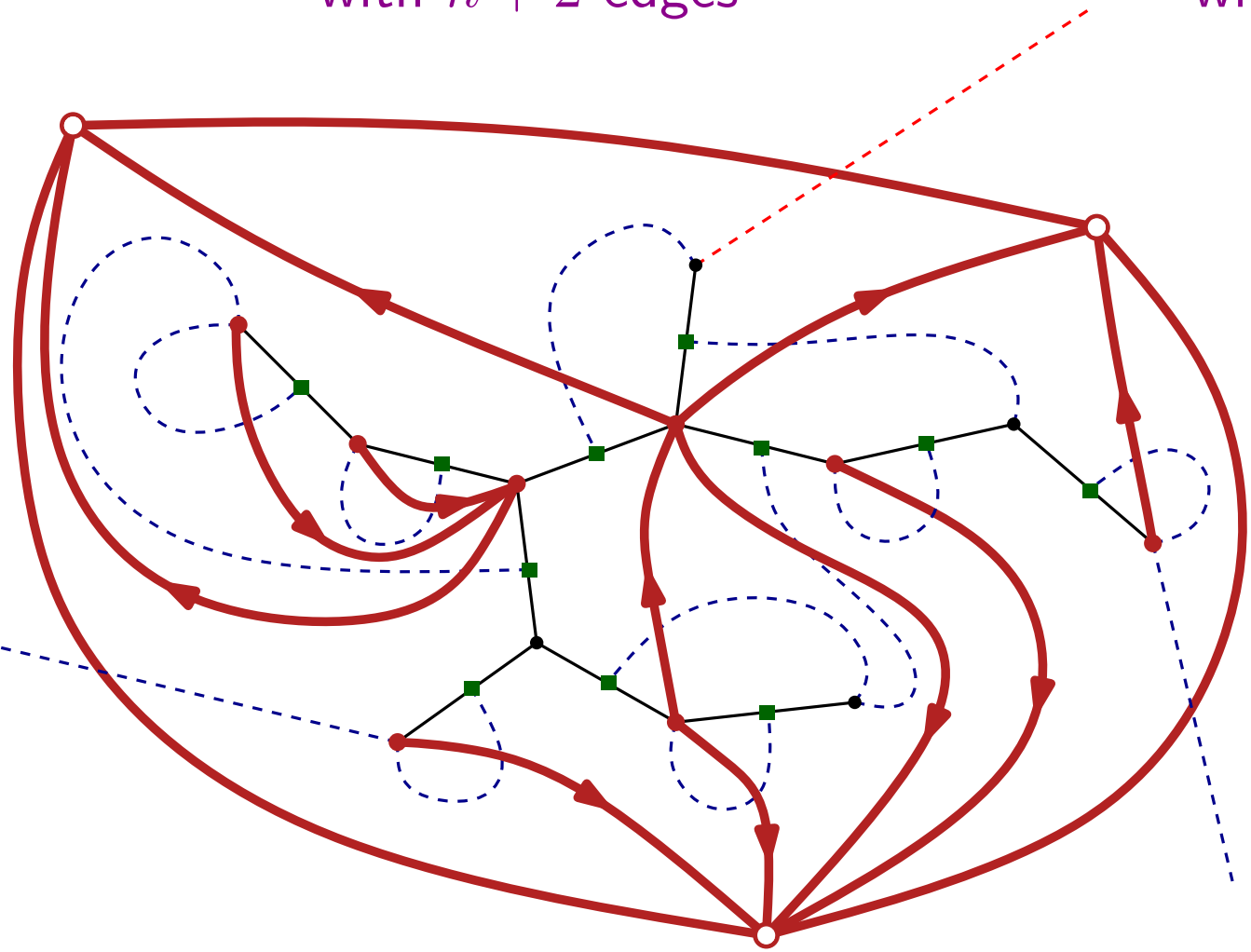

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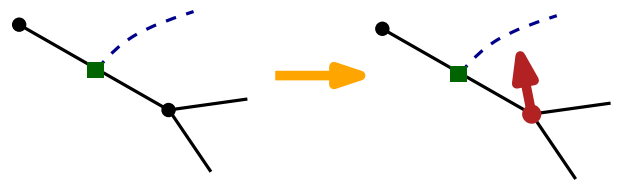

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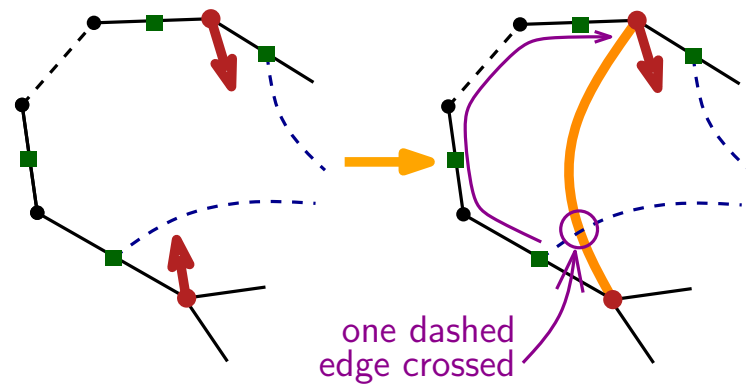

- Add 3 outer vertices and close remaining buds
- Connect outer vertices to form the outer triangle

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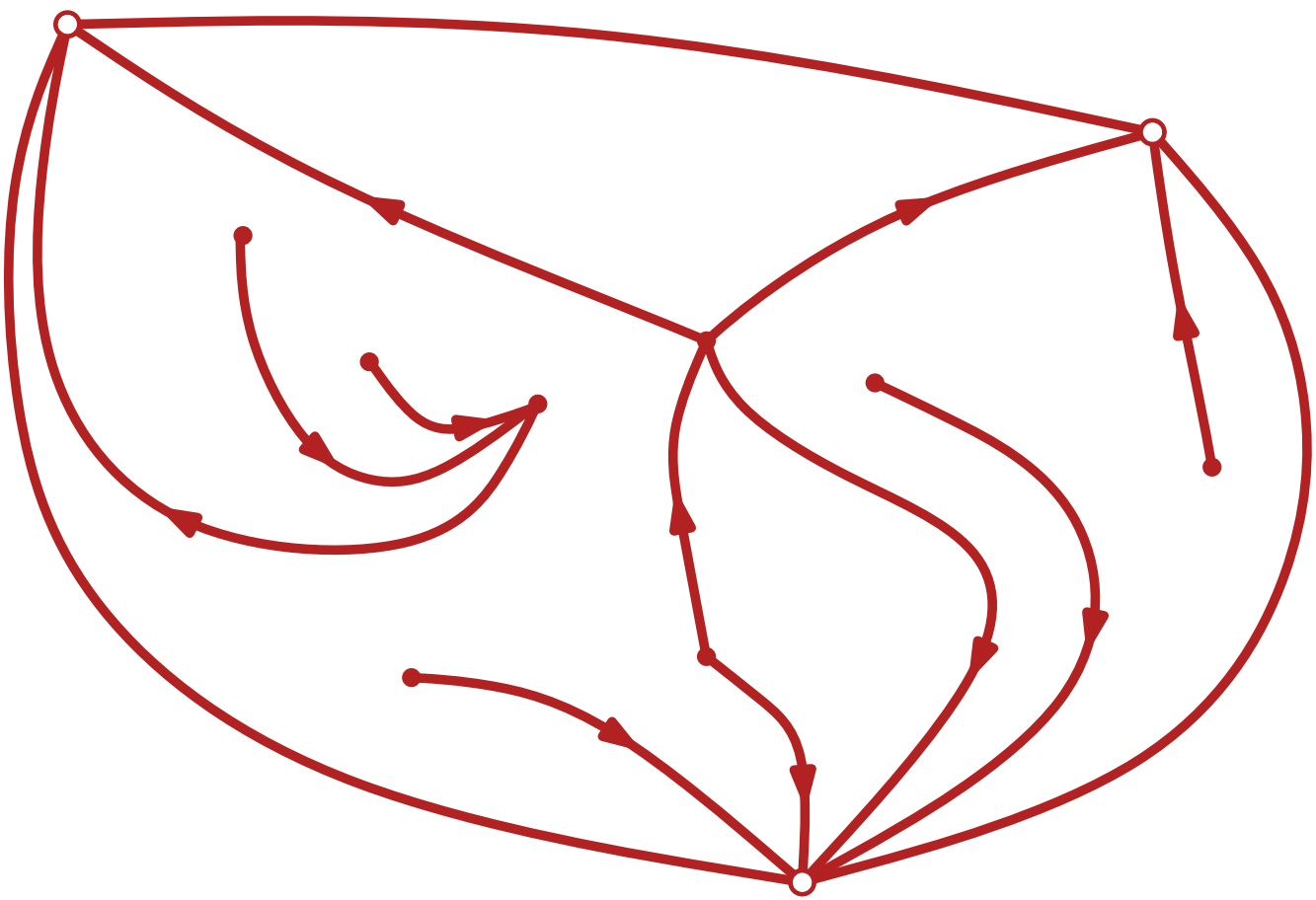

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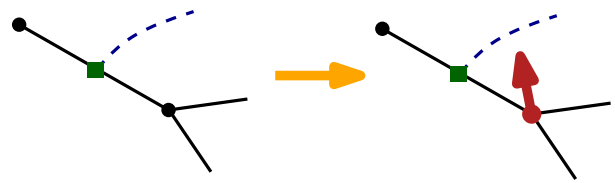

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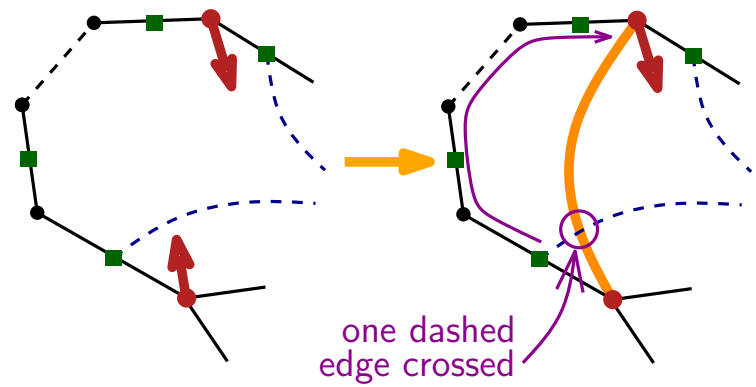

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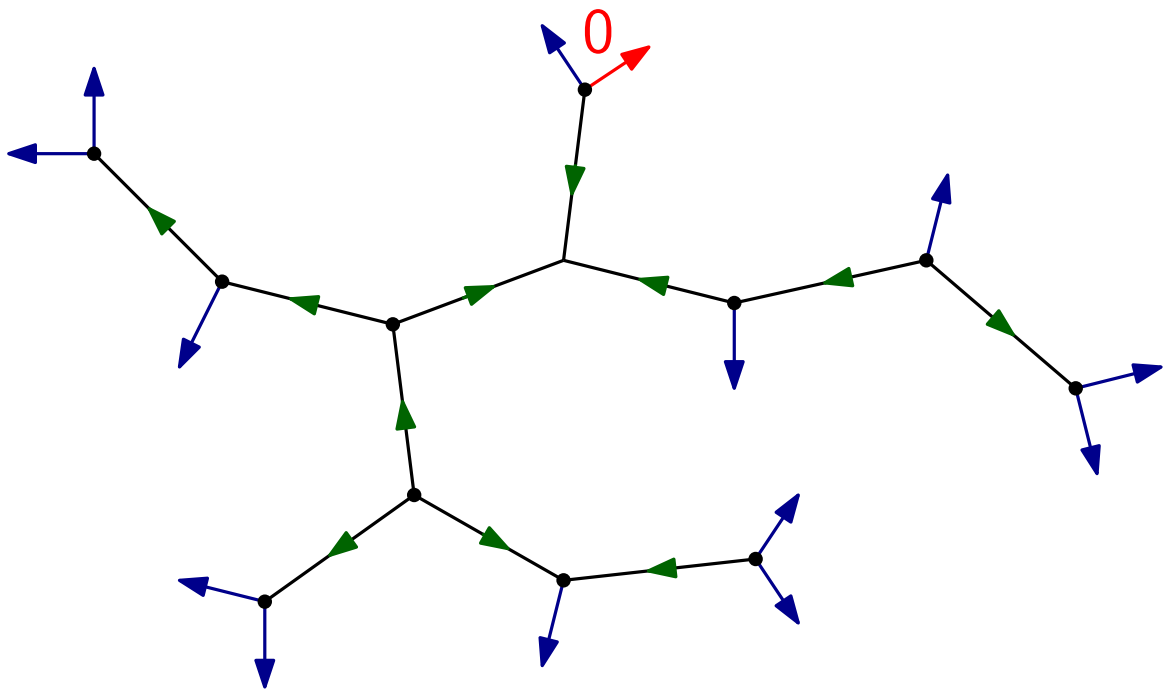
Encoding with trees: second bijection

Idea: Shorcut the bicubic maps with labels on corners

- label(root) $= 0$



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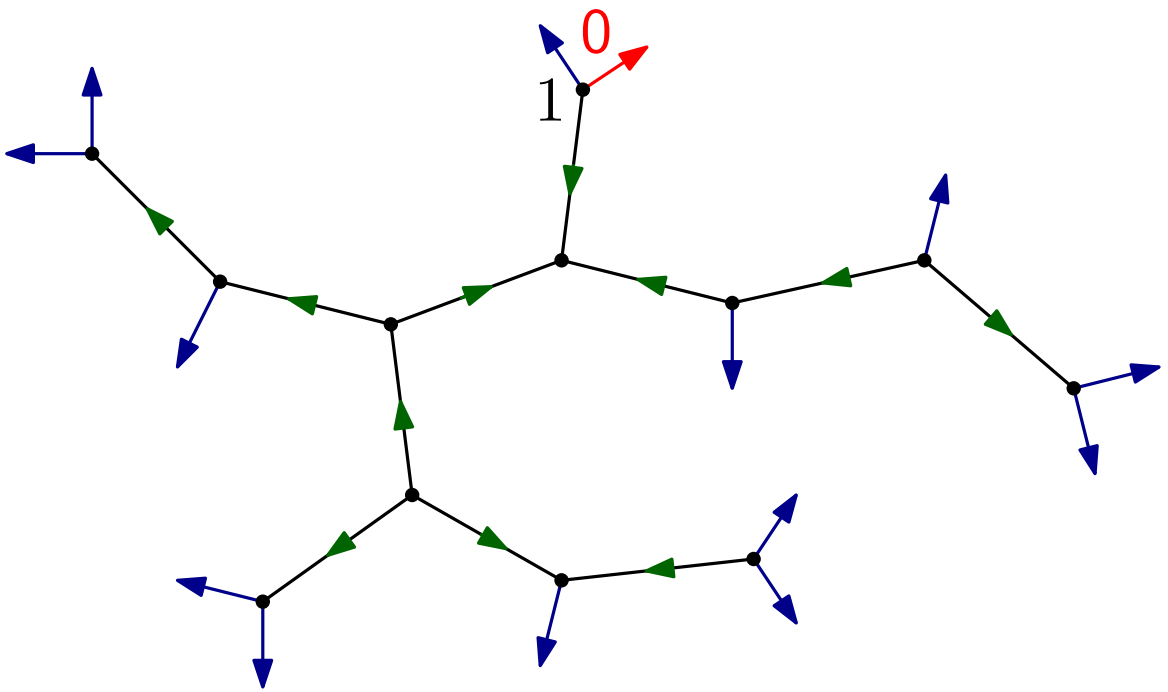

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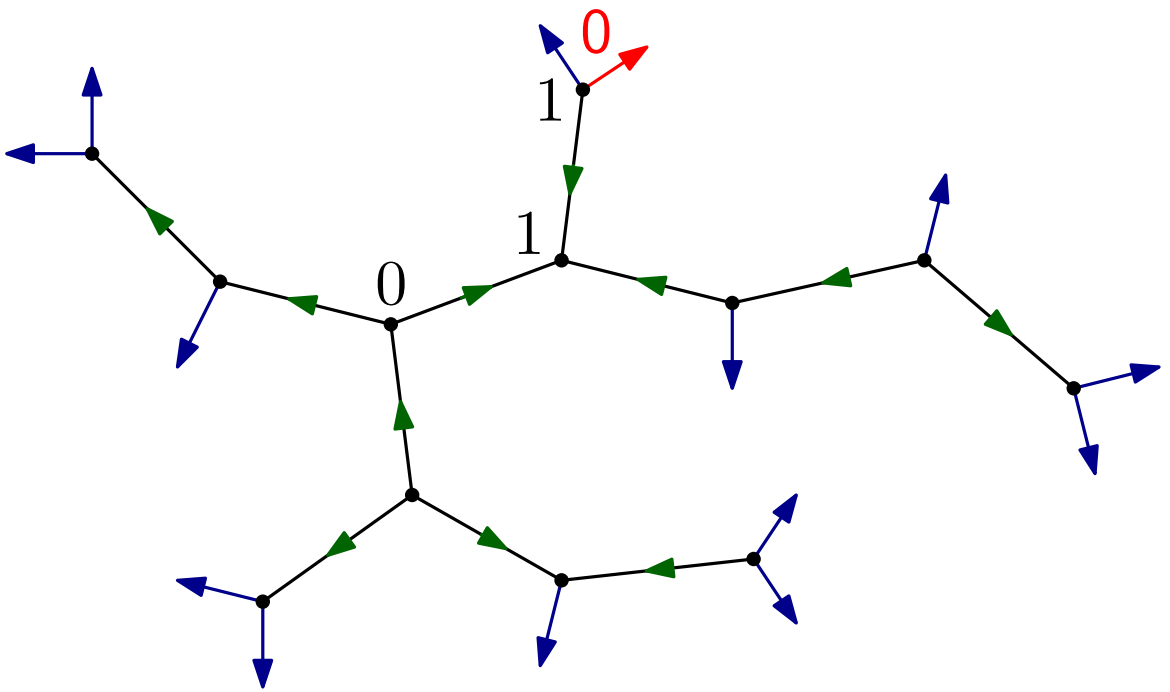

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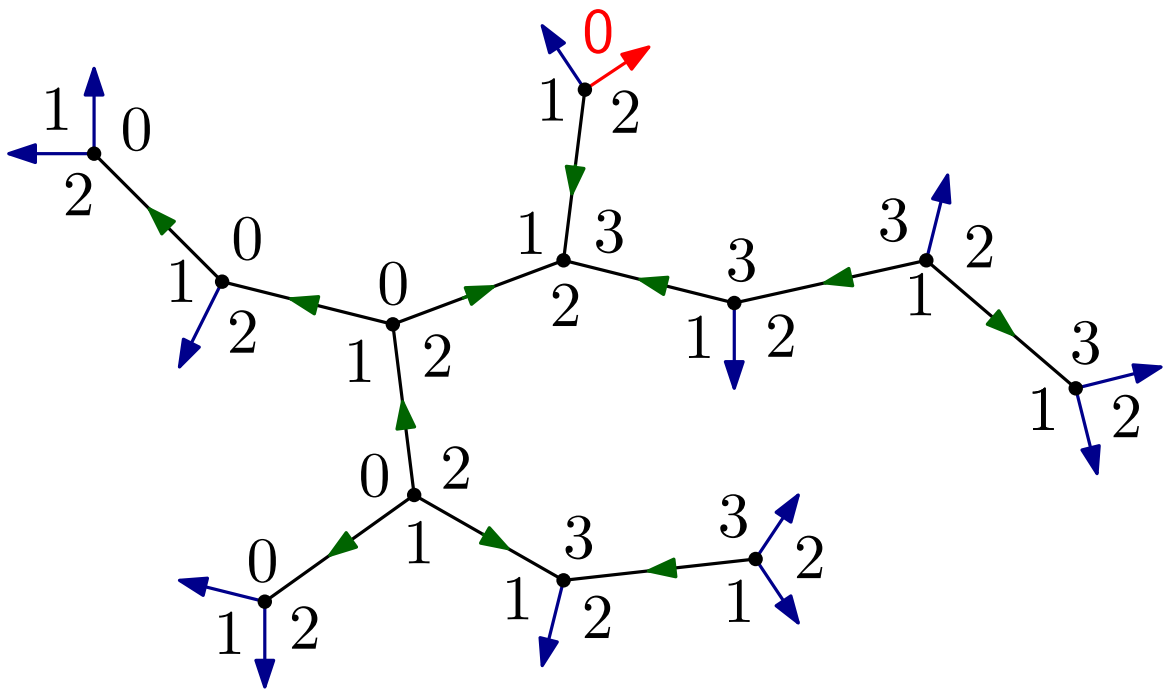
$\approx$ depth of the face in the bicubic map



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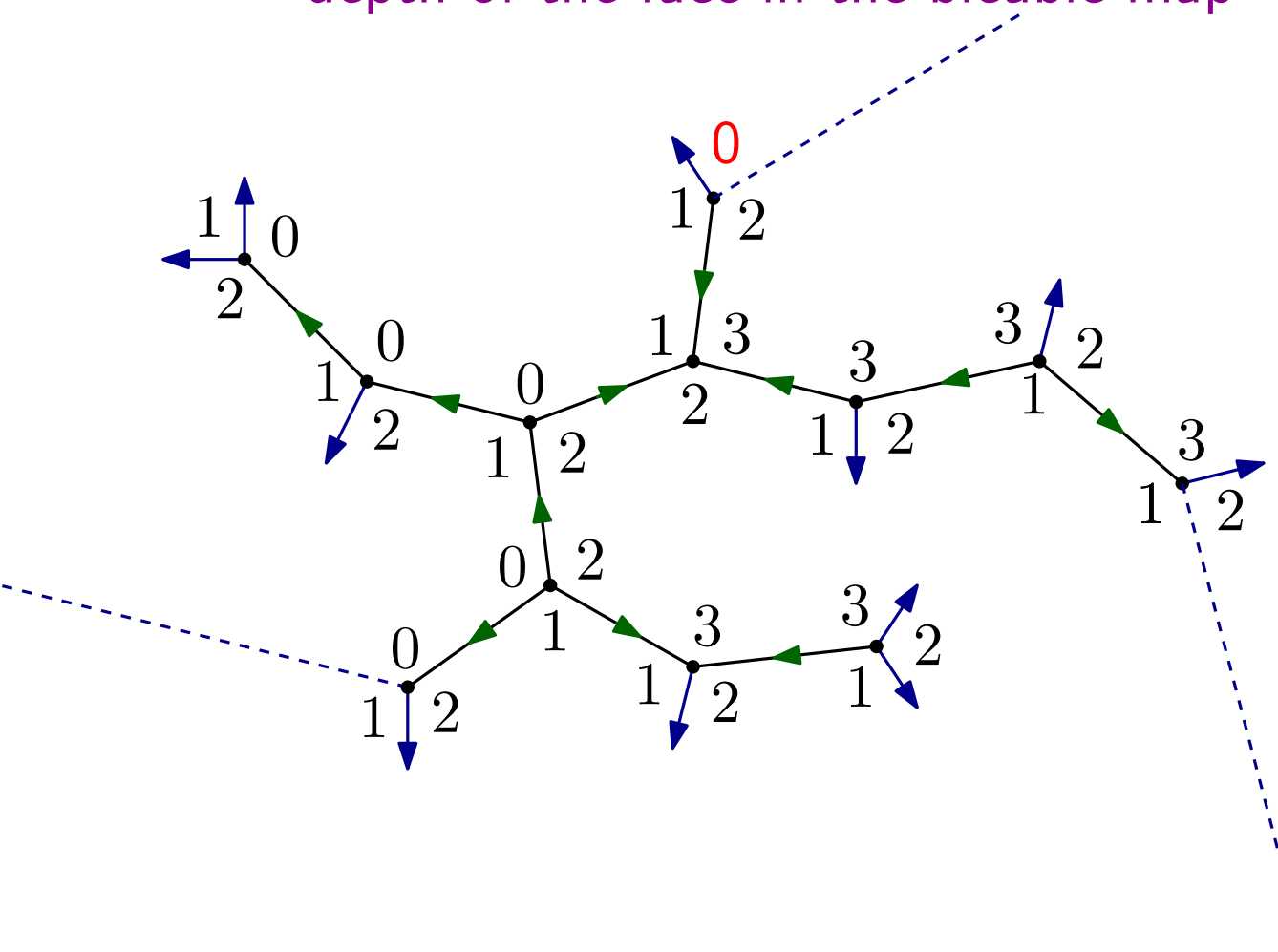
$\approx$ depth of the face in the bicubic map

- Unmatched leaves = last 0, 1 and 2 corners



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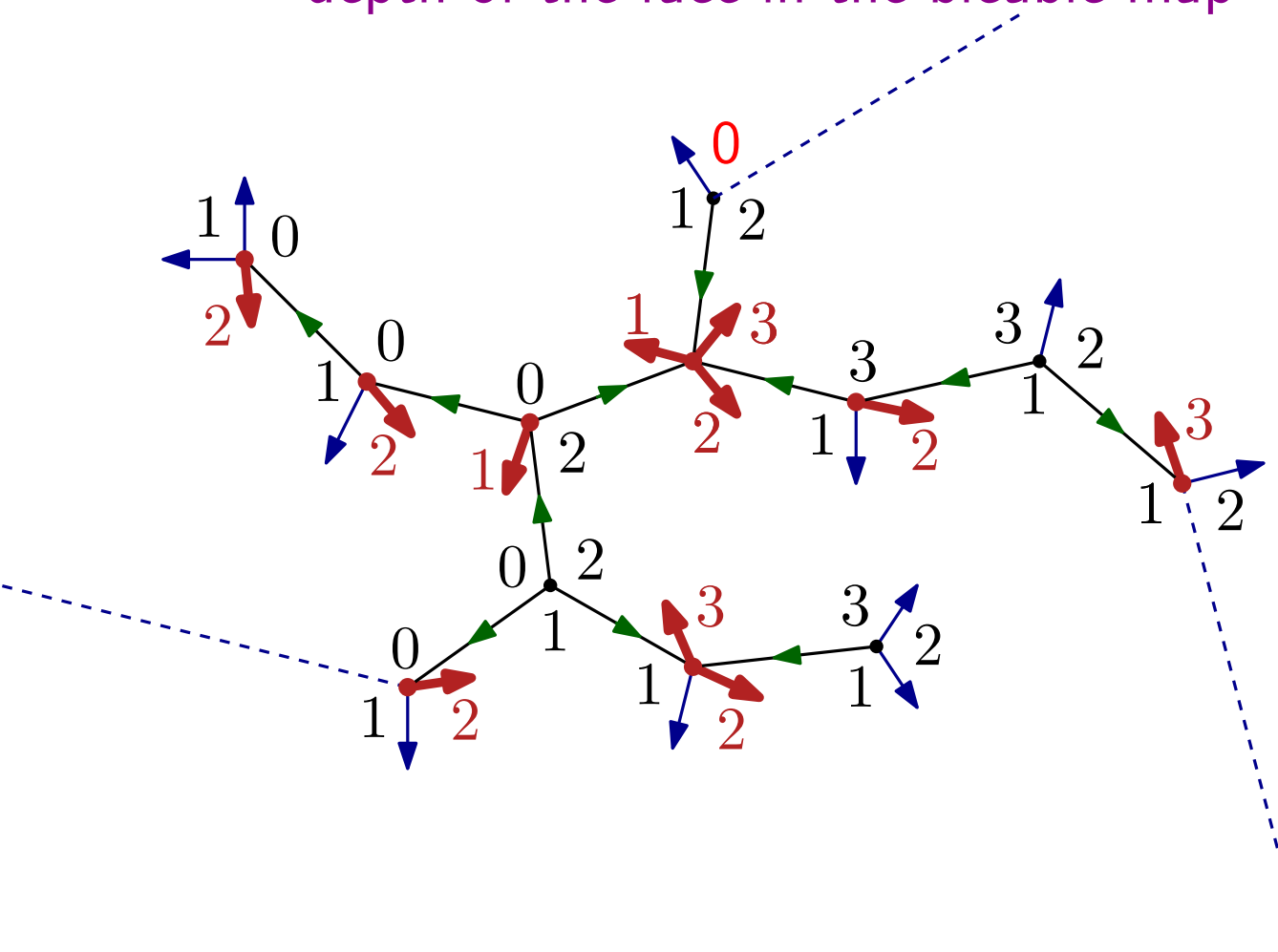

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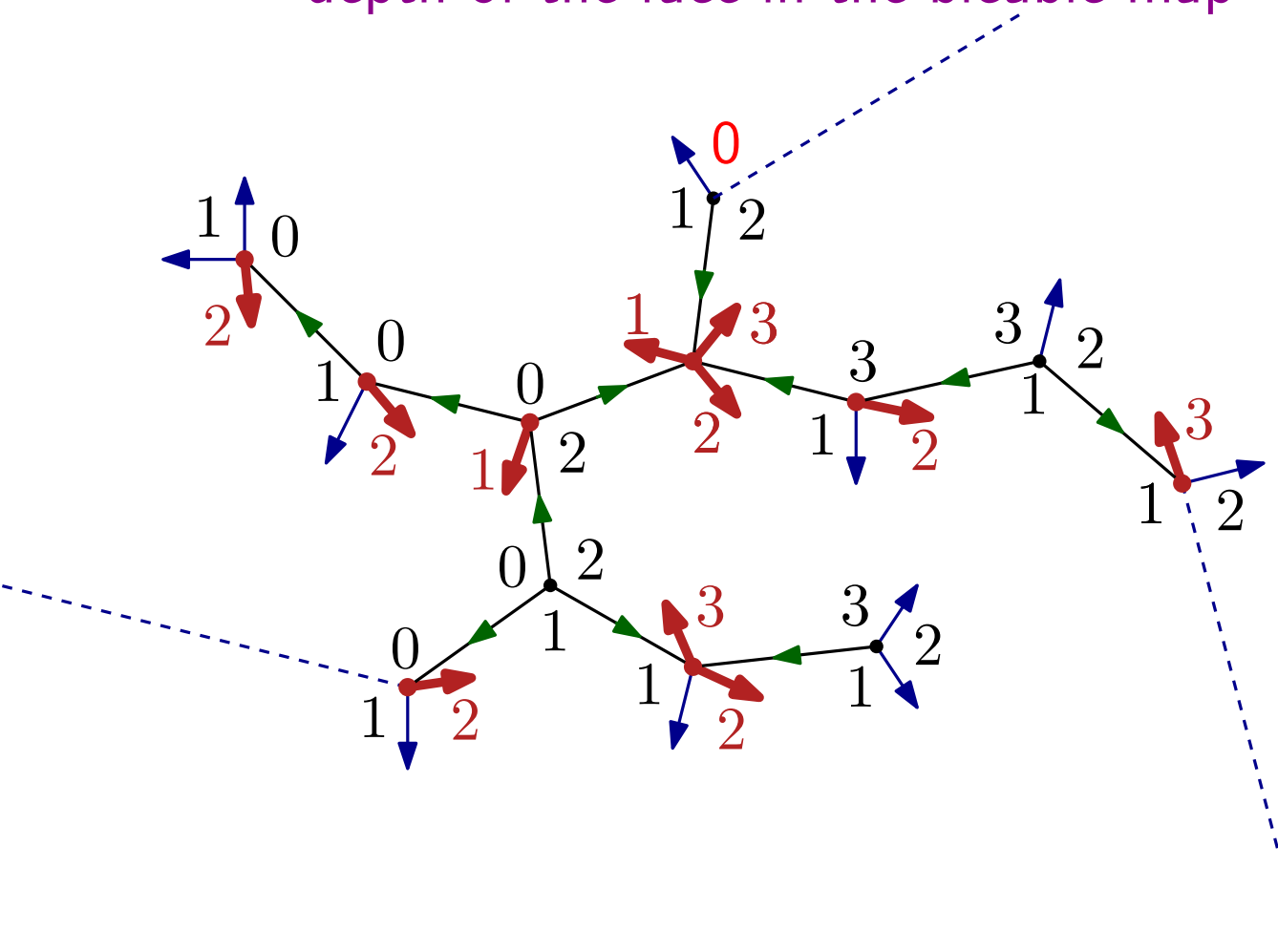

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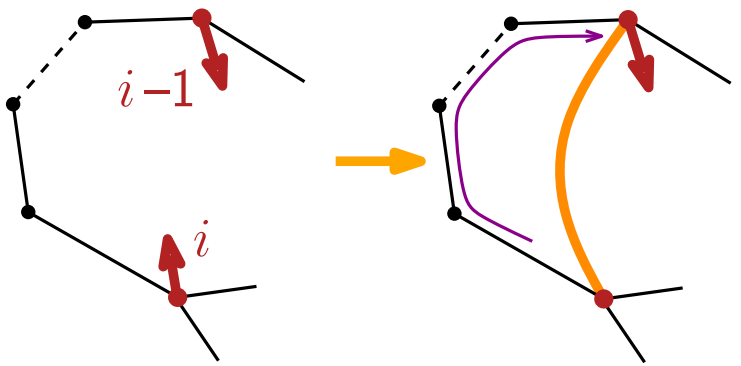

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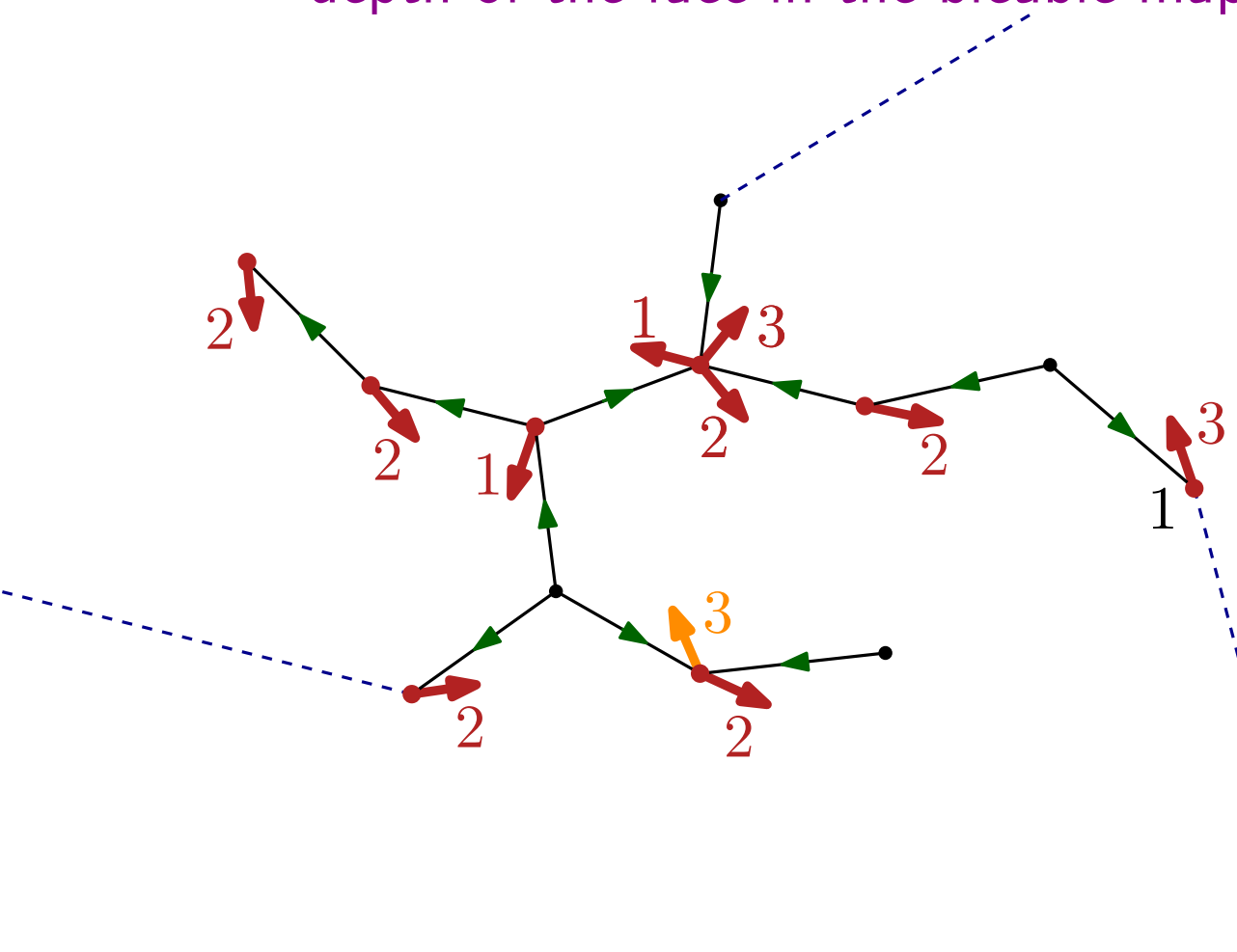

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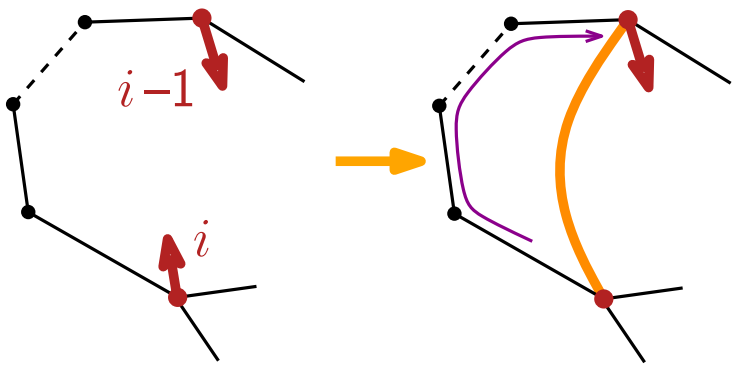

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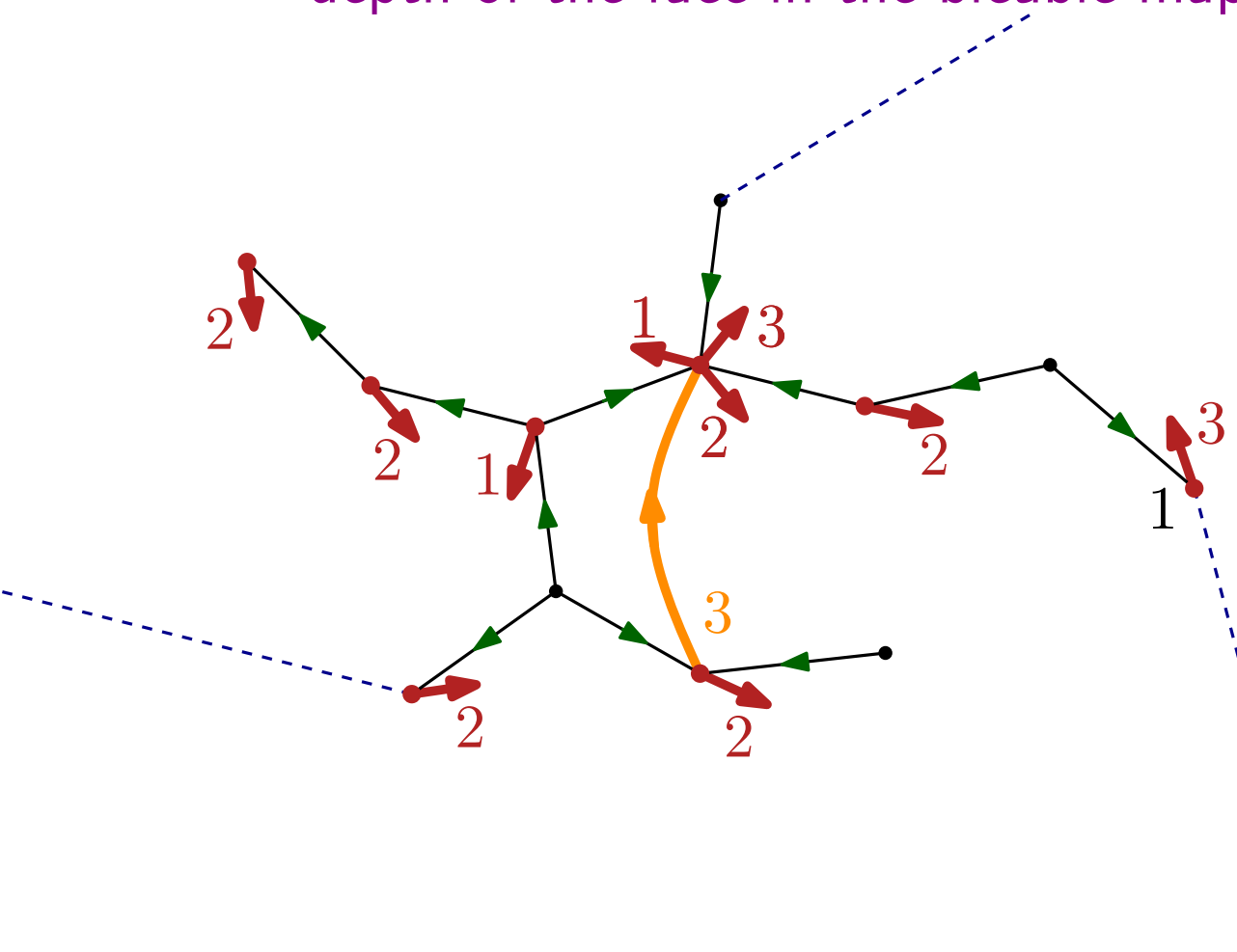

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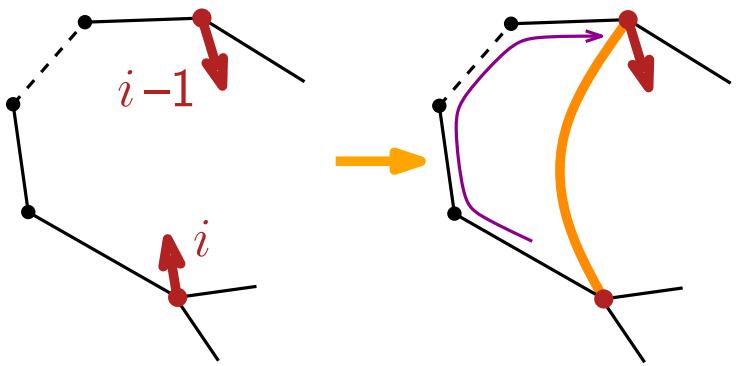

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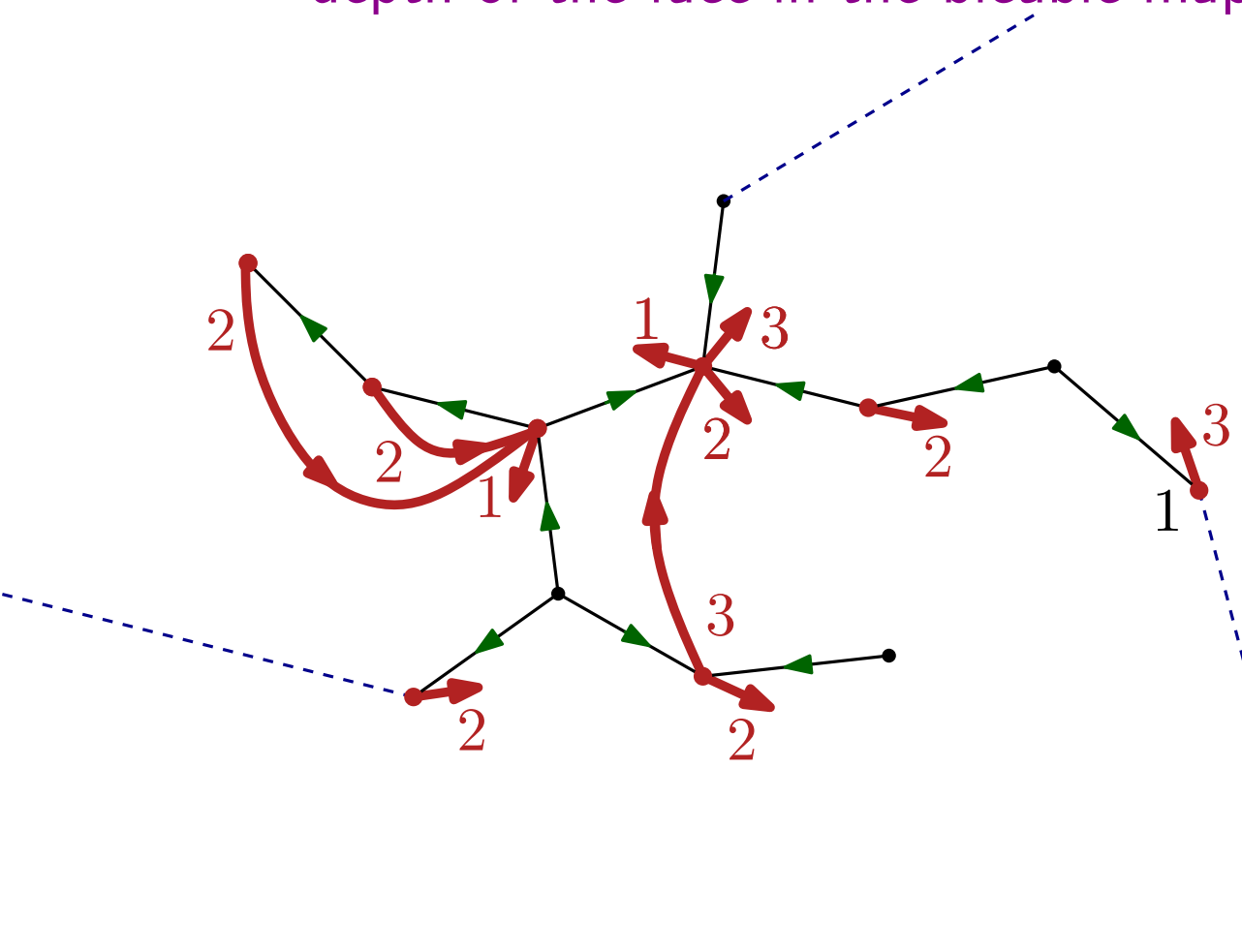

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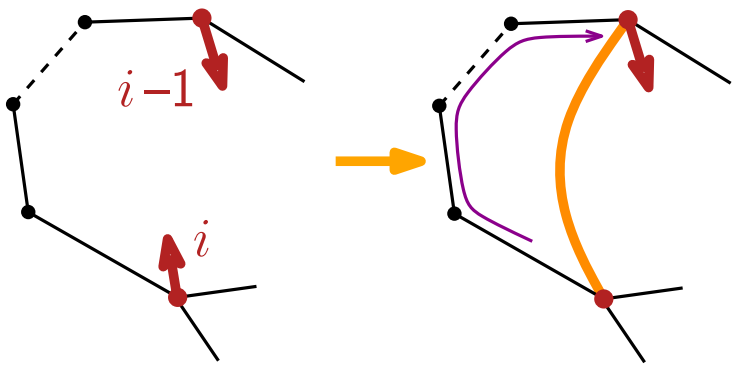

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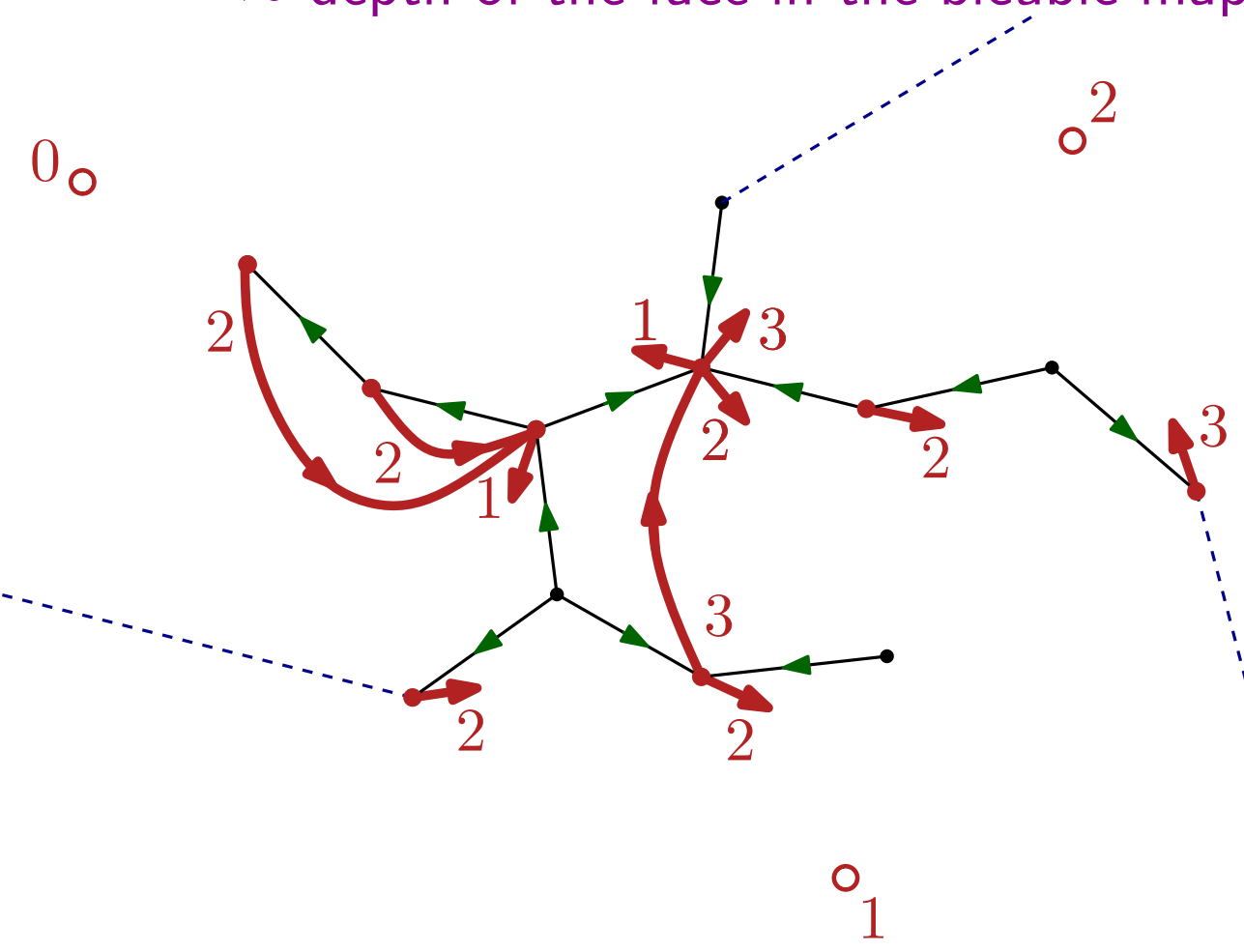

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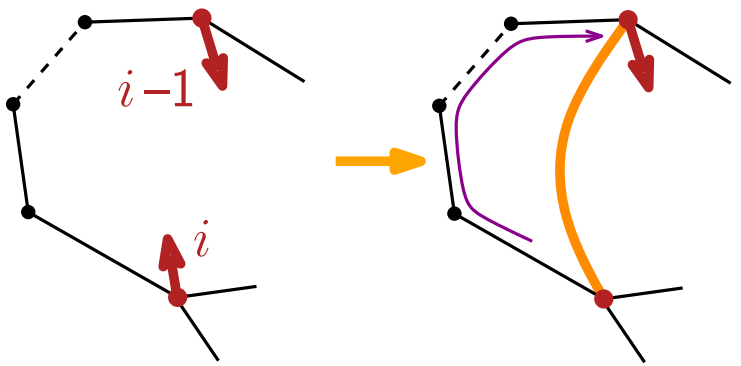

- Add 3 outer vertices and close remaining buds

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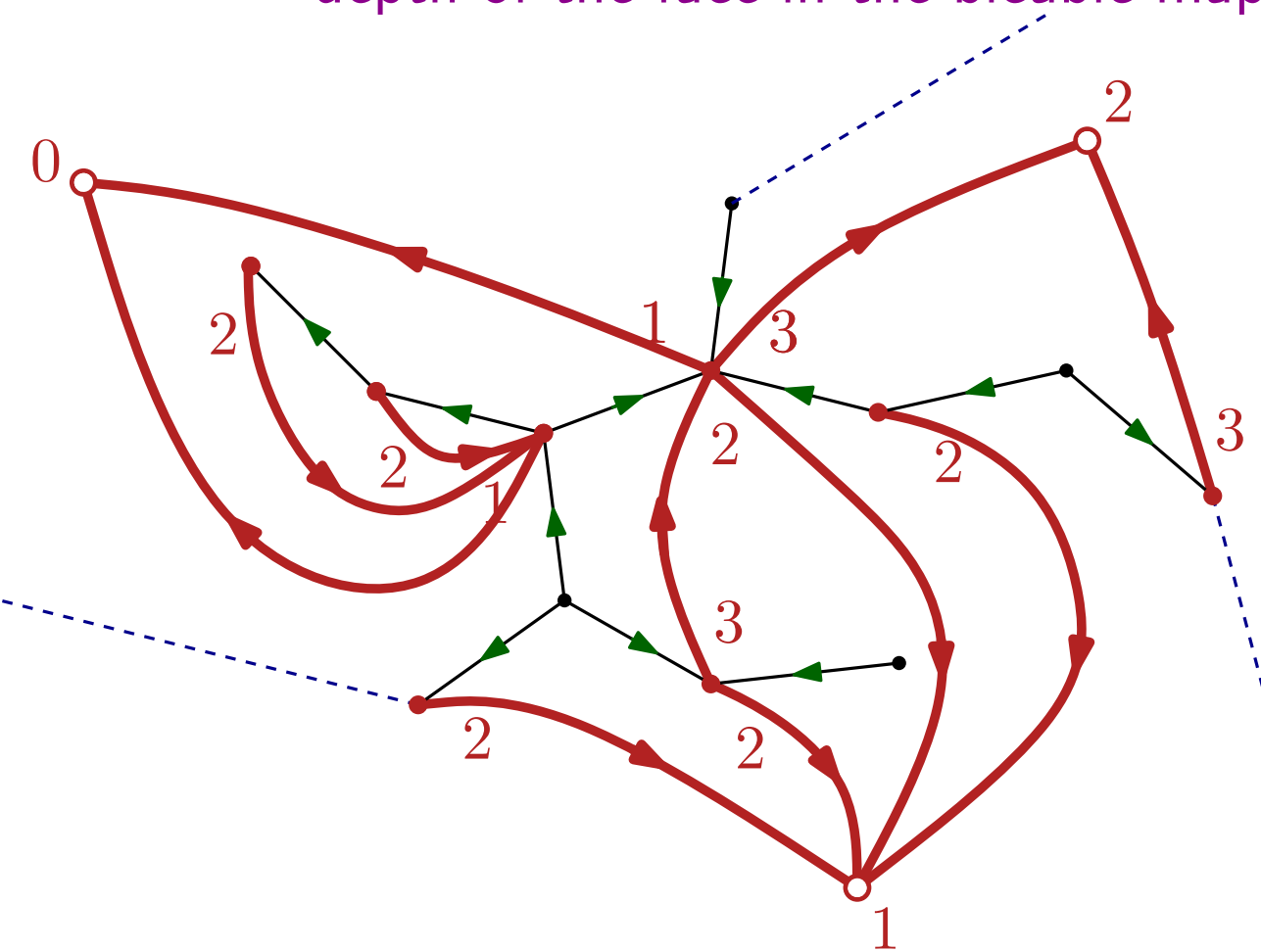

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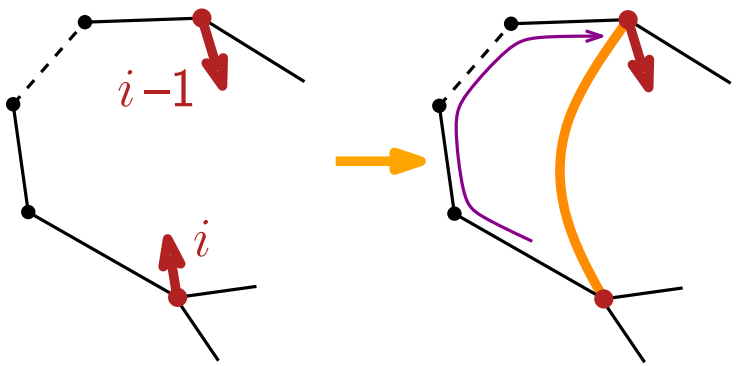

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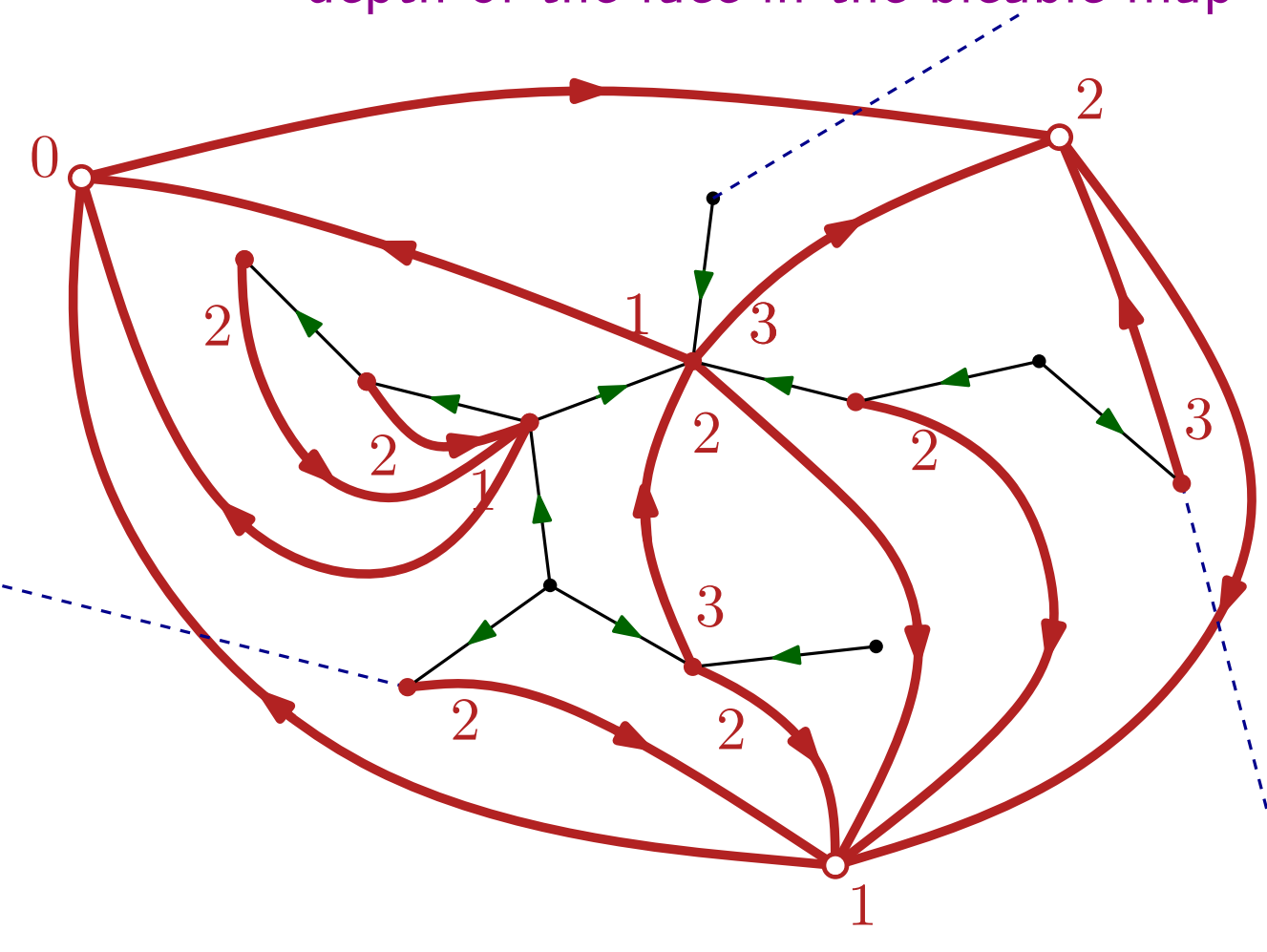

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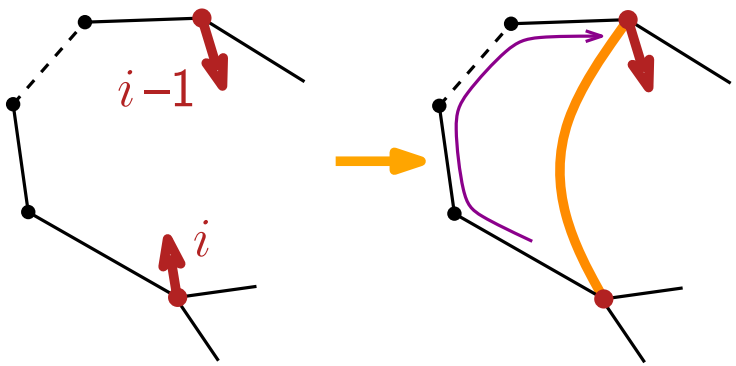

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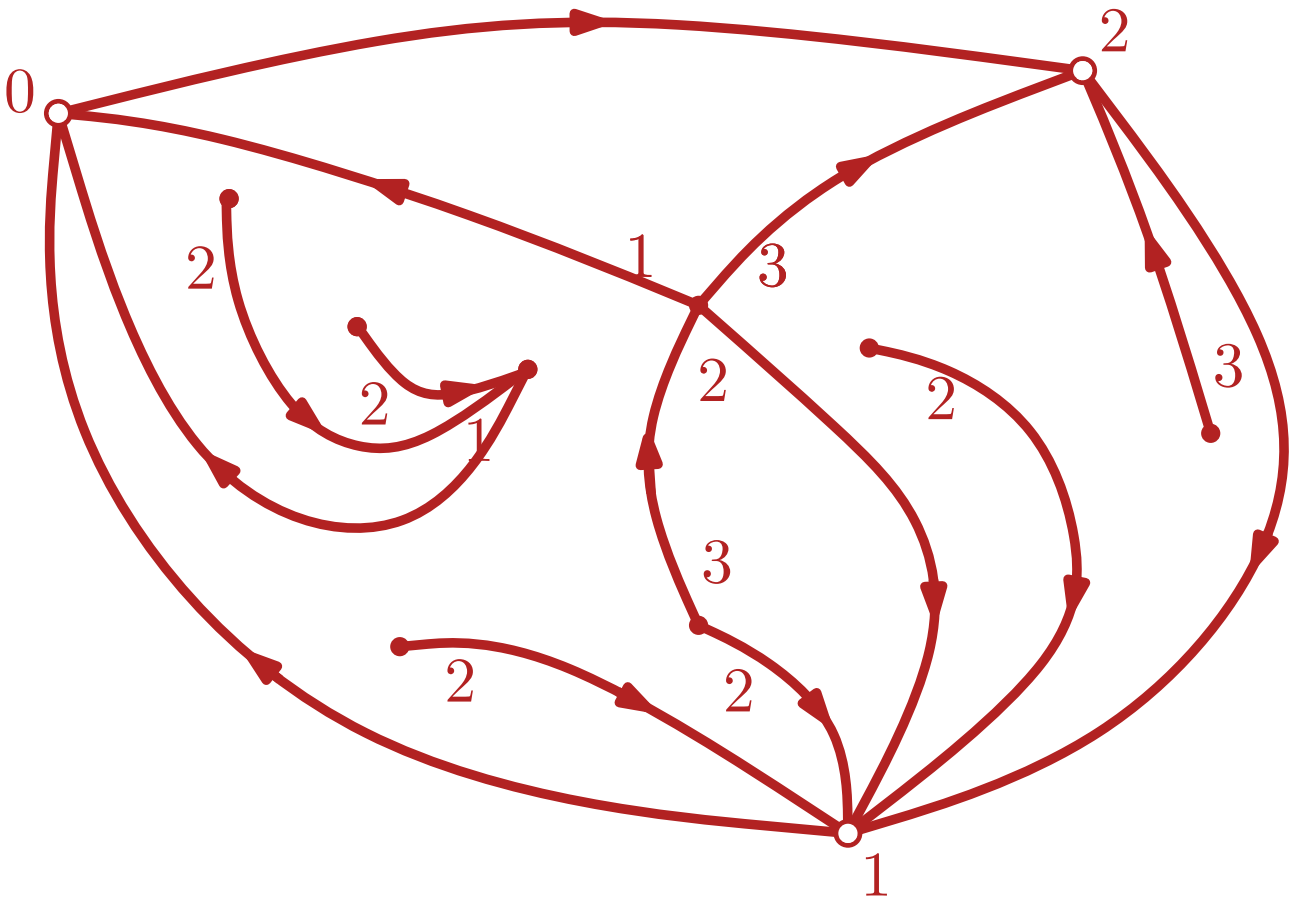
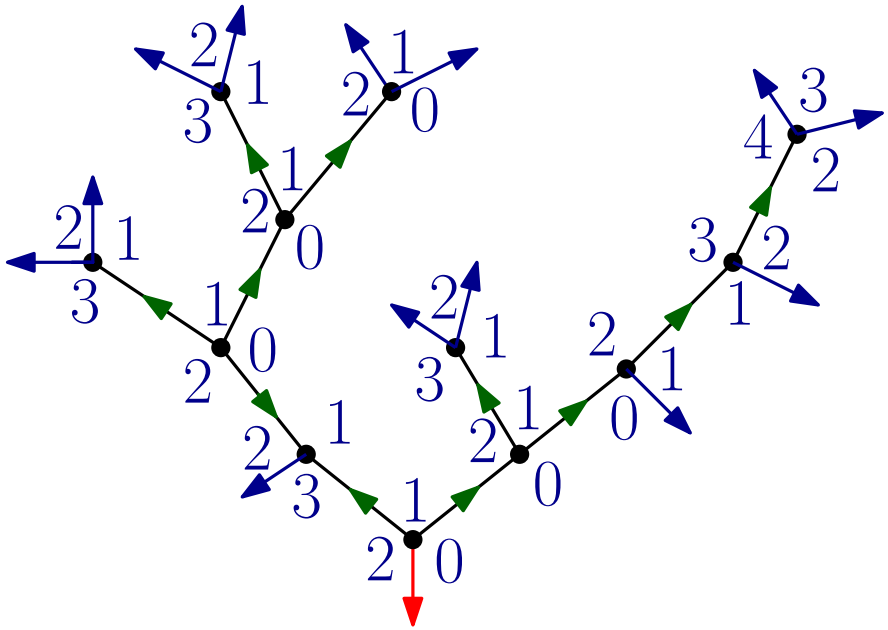
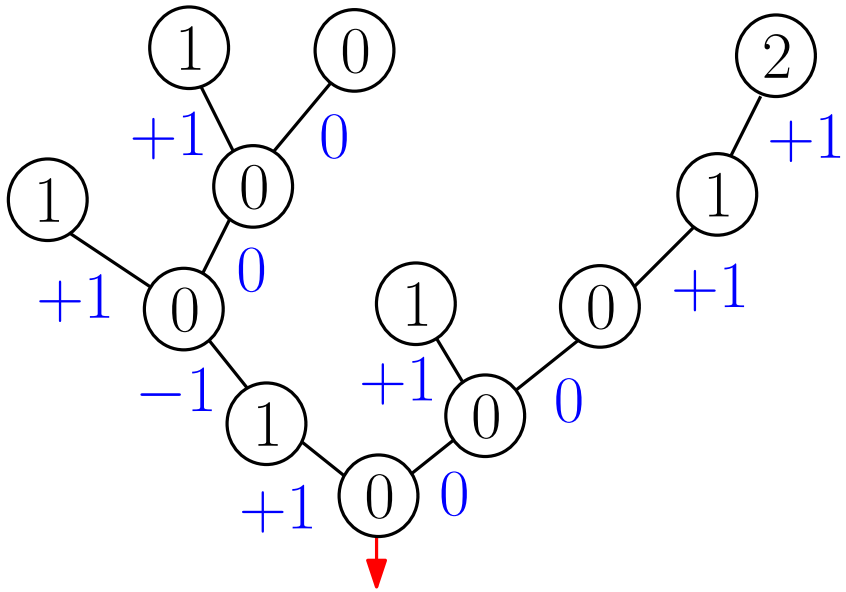


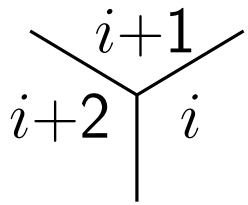
Outertriangular simple map

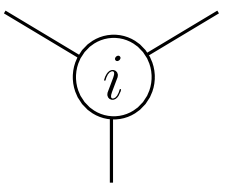
label $=$ length of RMP

From oriented binary trees to labeled trees









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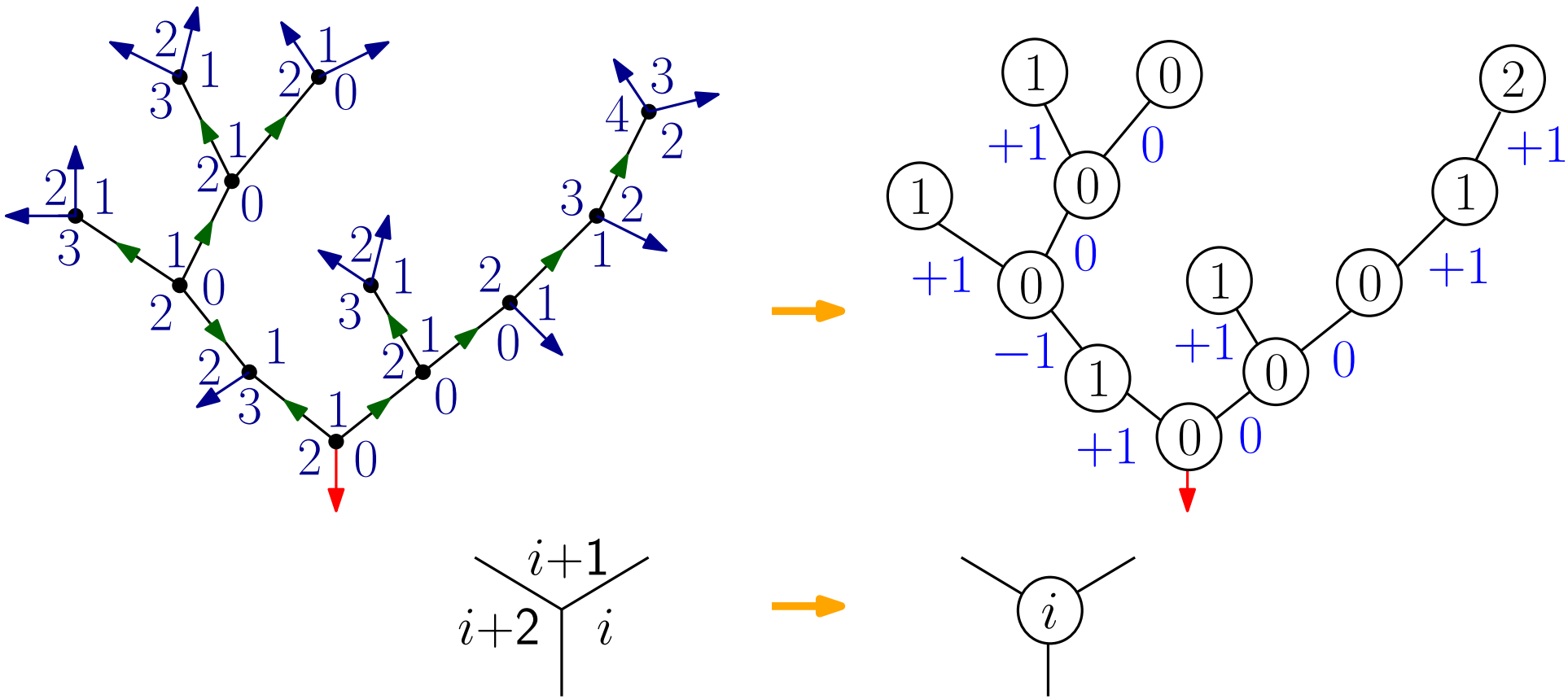
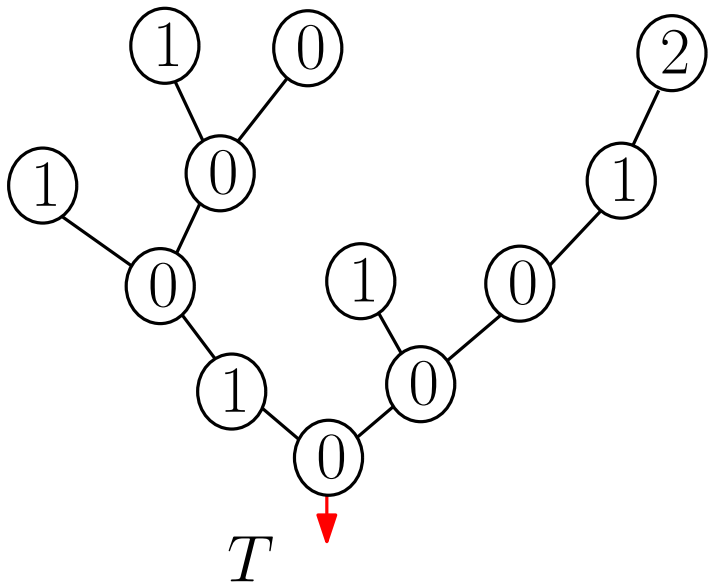


Labeled trees = Galton-Watson trees with random displacements on edges

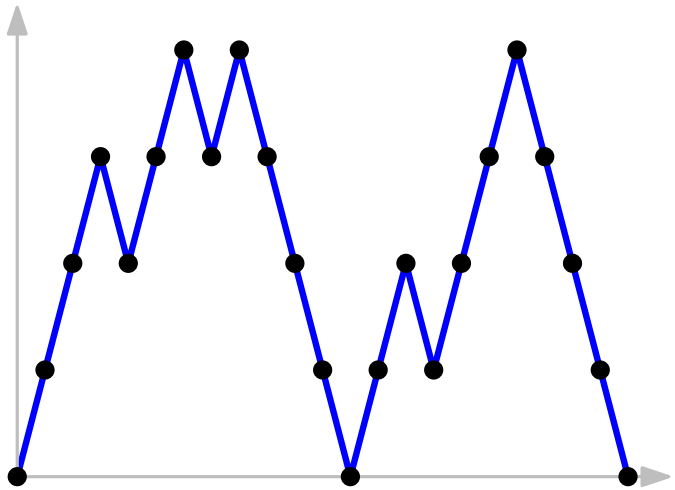
[Marckert '08] Converges to Brownian snake when labels rescaled by $(2n)^{1/4}$

Study of the limit of labeled trees

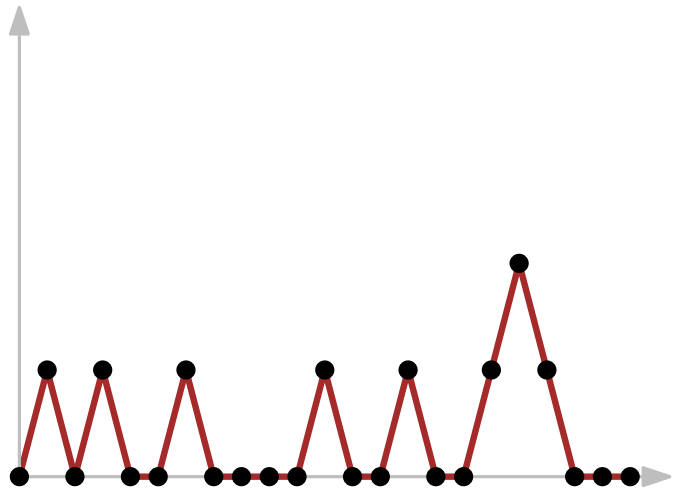
Contour and label processes of a labeled tree



T



$C^T(.)$



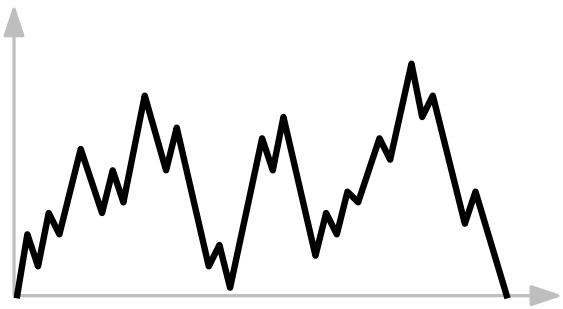
$\tilde{Z}^T(.)$

[Marckert '08]

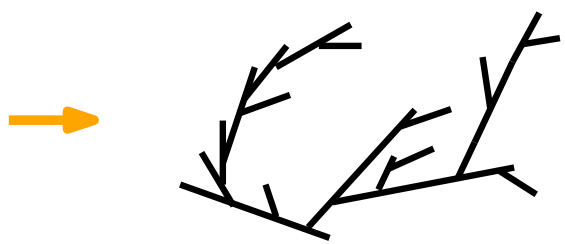
For a sequence in $\mathcal{S}_n$, and the associated labeled trees $(T_n)_{n\geq 0}$:

$$\left((8n)^{-1/2}C_n(\lfloor nt\rfloor), (2n)^{-1/4}\tilde{Z}_n(\lfloor nt\rfloor)\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} (e_t, Z_t)_{0\leq t\leq 1}$$

Brownian snake



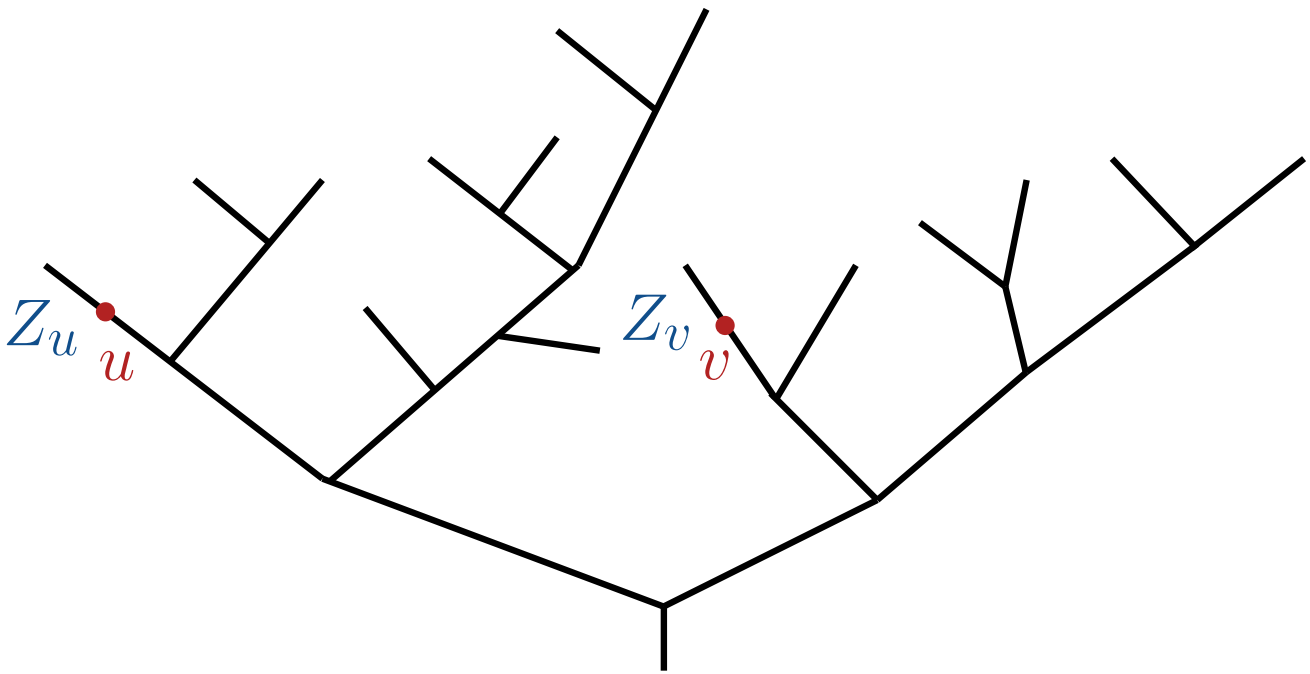
e_t: Brownian excursion



[Aldous] Brownian tree

+ Brownian motion conditioned on the tree

Interpretation of the distances



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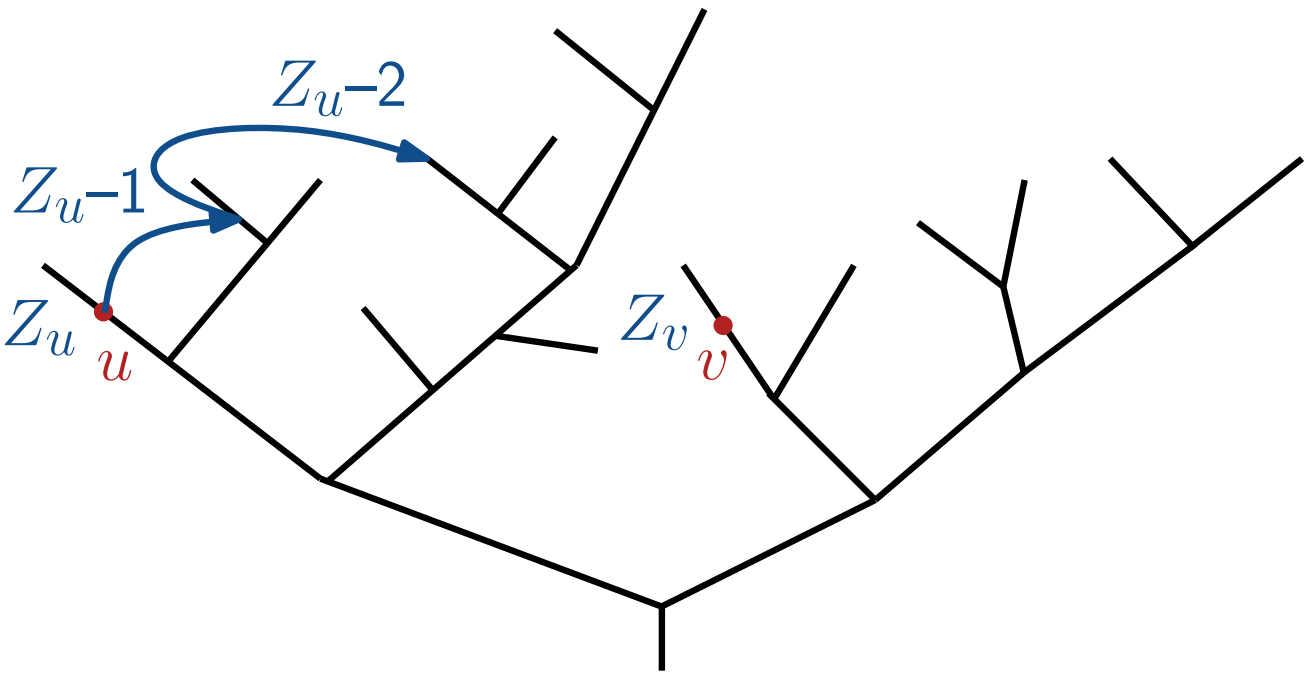

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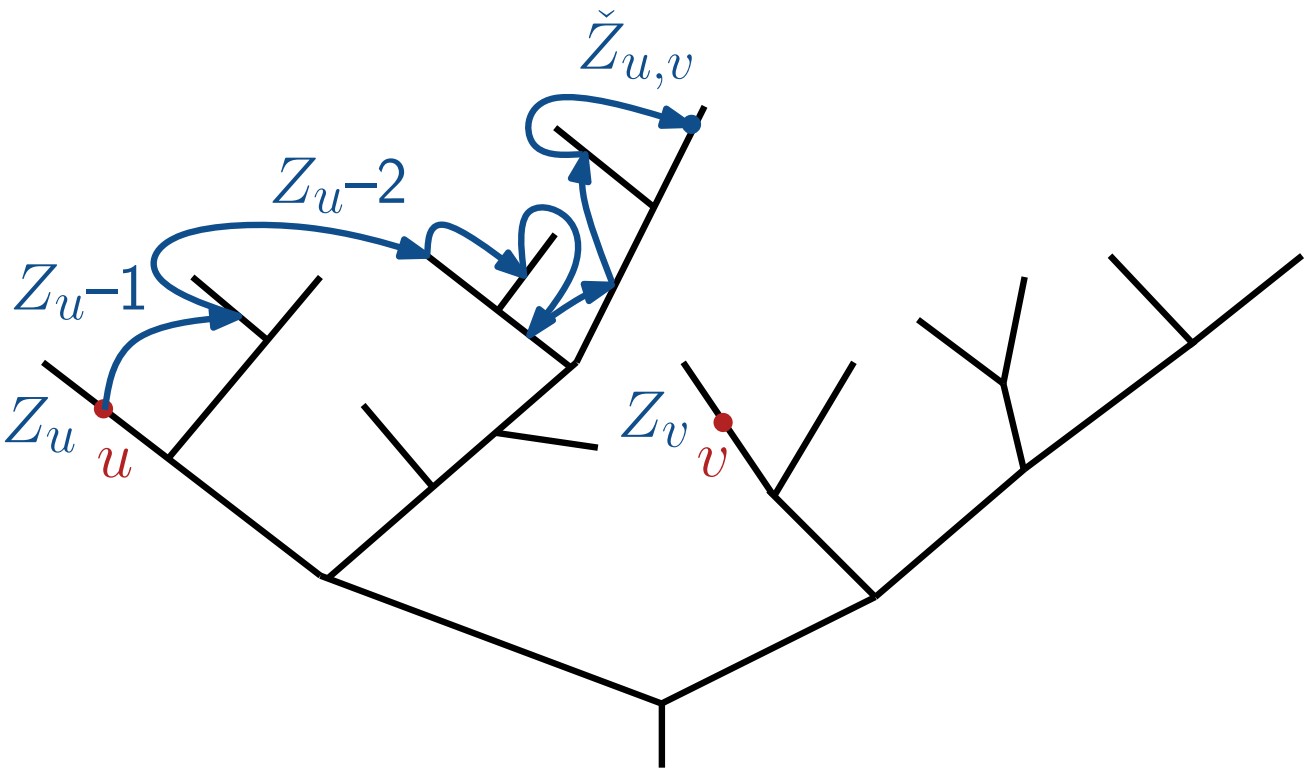

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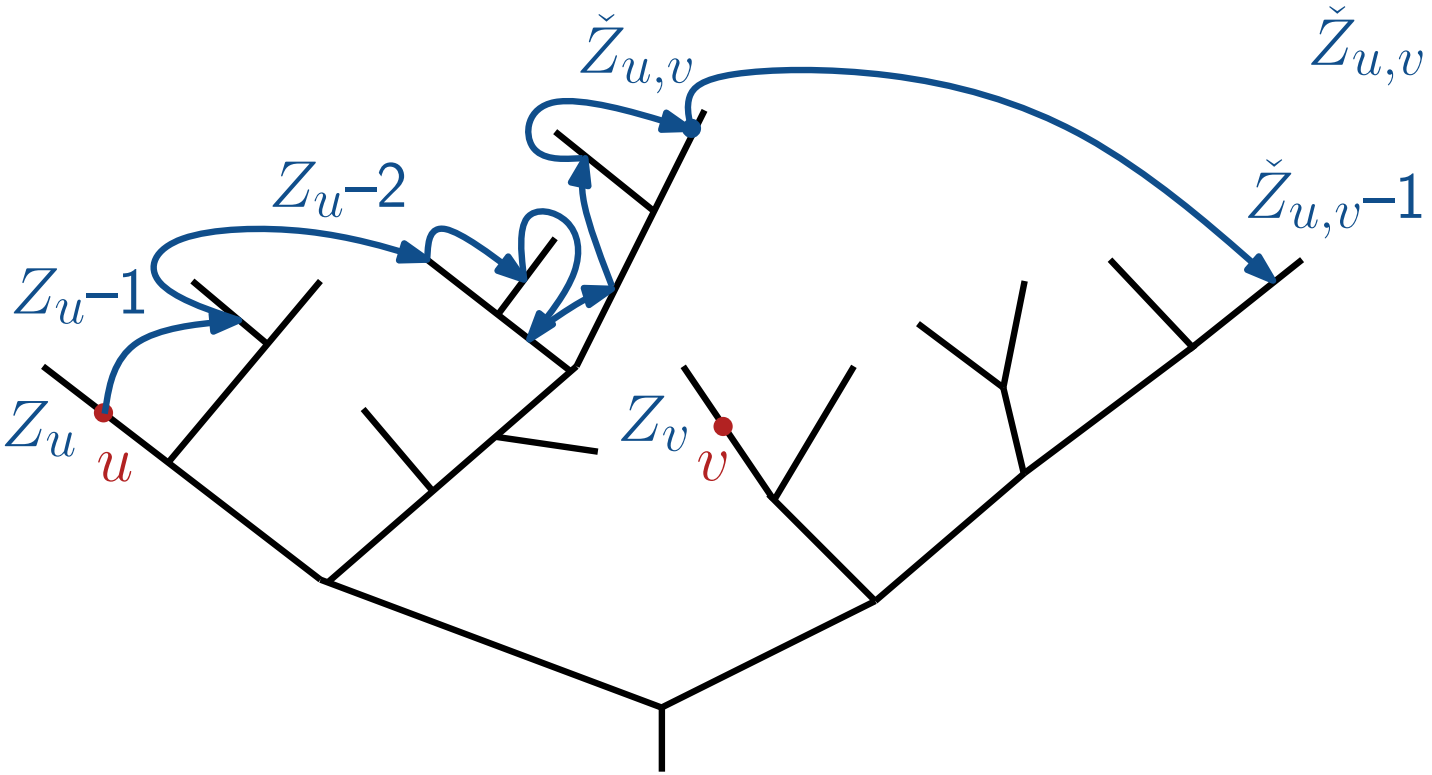

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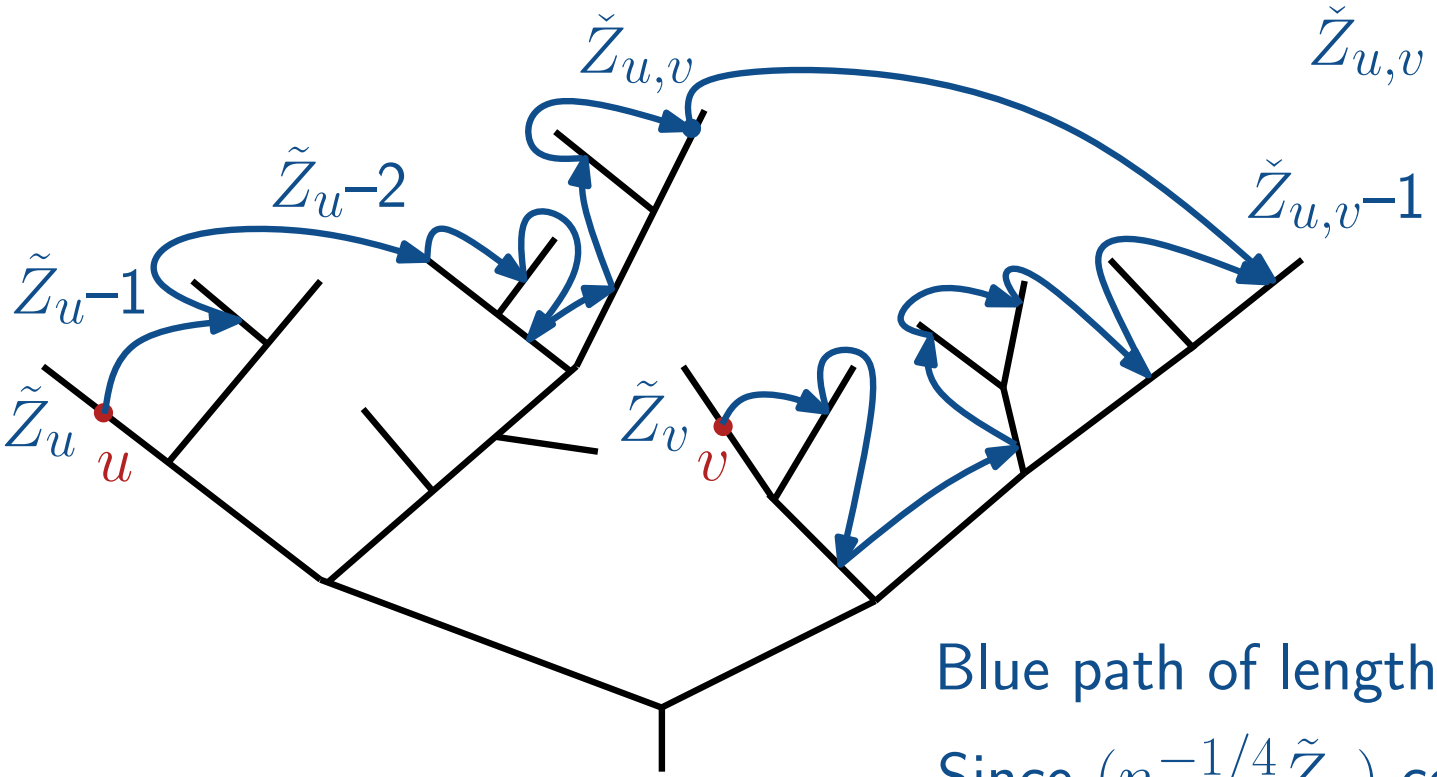

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Blue path of length: $\tilde{Z}_u + \tilde{Z}_v - 2\check{Z}_{u,v} + 2$

Since $(n^{-1/4}\tilde{Z}_n)$ converges, $(d_n)_{n\ge 0}$ tight

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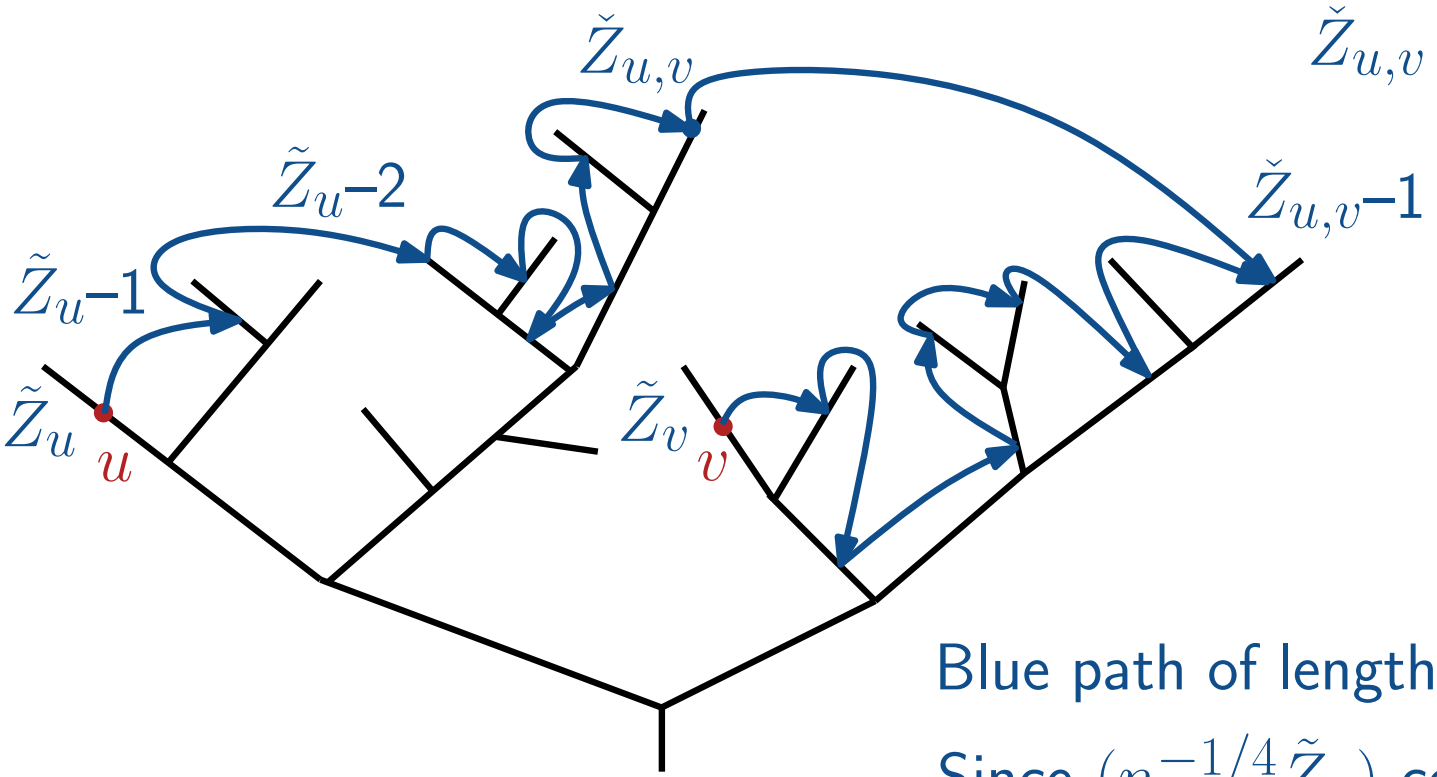

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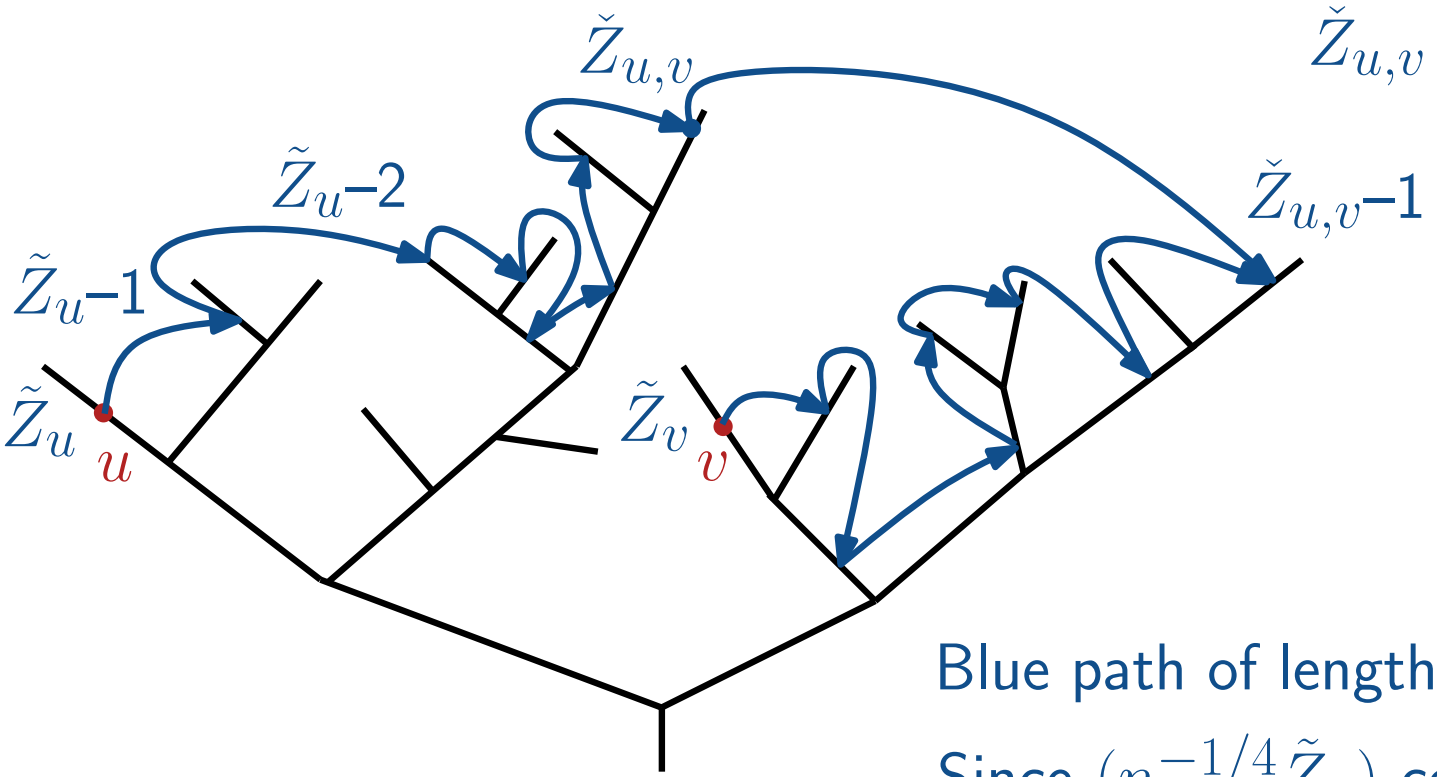

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$D^\circ(s,t) = Z_s + Z_t - 2\max\{\inf_{s\le u\le t} Z_u, \inf_{t\le u\le s} Z_v\}$

$D^\star(a,b) = \inf\left\{\Sigma_{i=0}^{k-1} D^\circ(a_i, a_{i+1}) : k \ge 1, a = a_1, \dots, a_k = b\right\}$

Thank you!