



THE SPACE  
OF  
DIAGONAL HARMONICS



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$$\mathbf{E} = \sum_{i=1}^n y_i \partial_{x_i} \quad \mathbf{F} = \sum_{i=1}^n x_i \partial_{y_i} \quad \mathbf{H} = \sum_{i=1}^n y_i \partial_{y_i} - \sum_{i=1}^n x_i \partial_{x_i}$$

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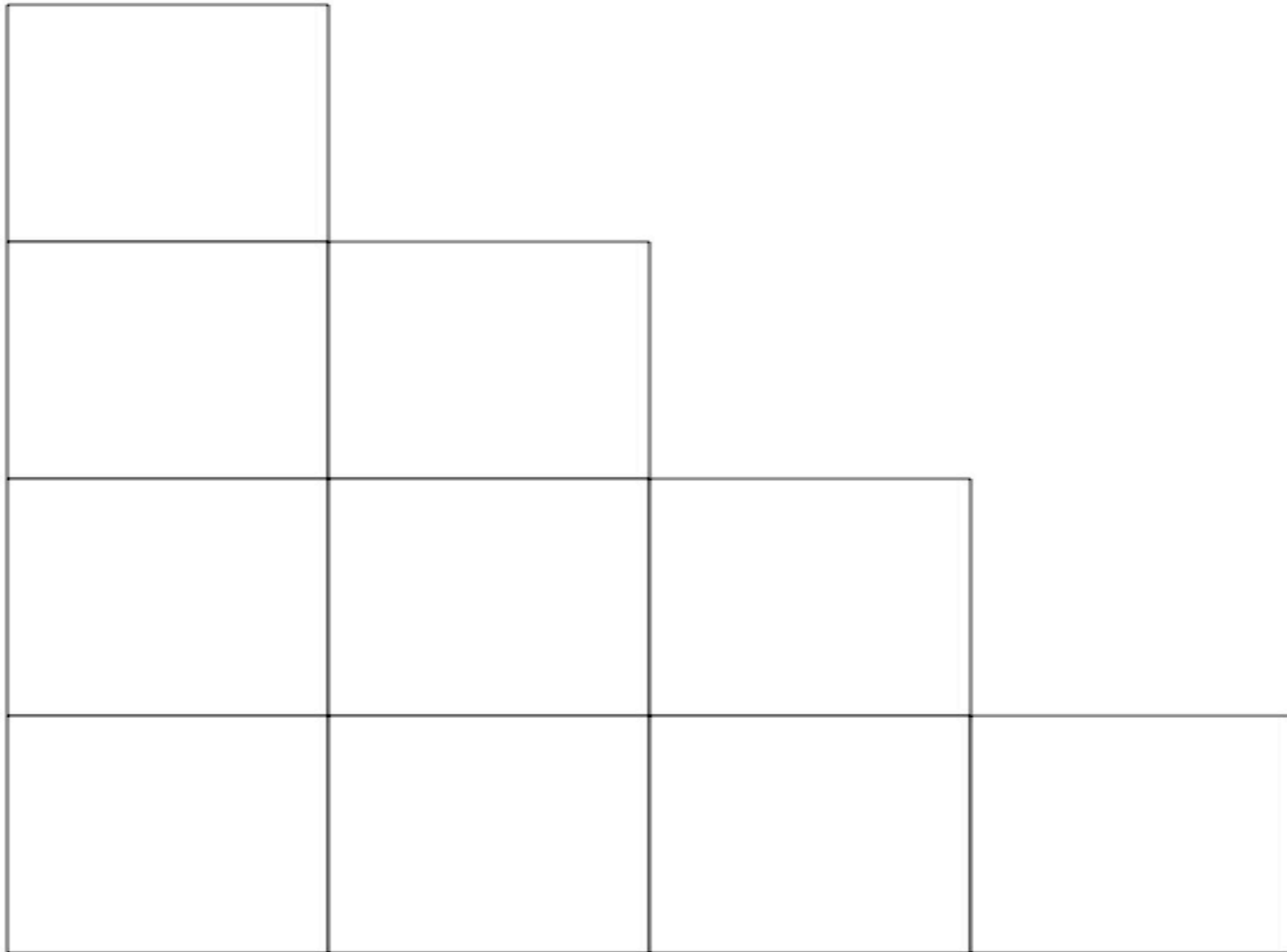
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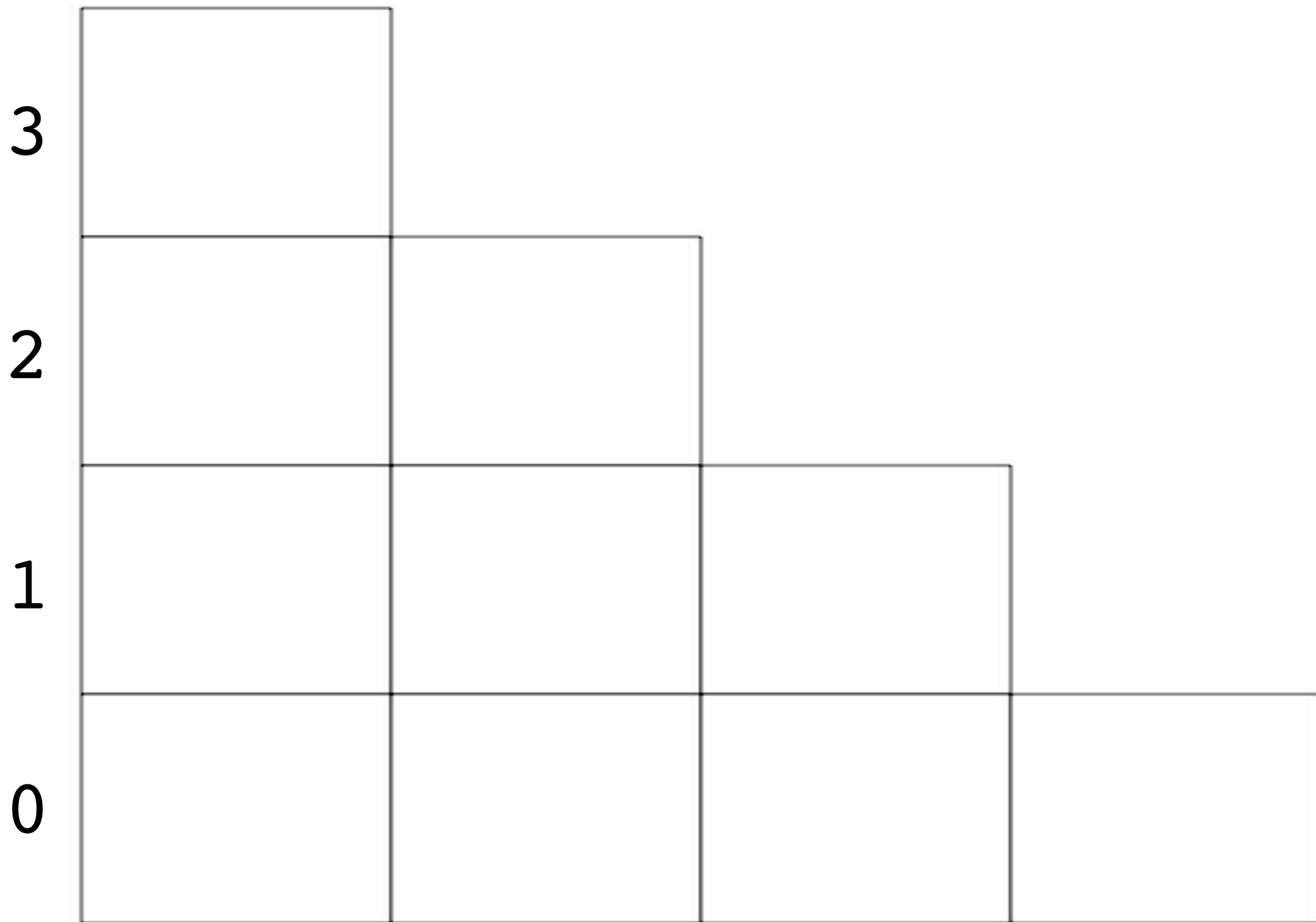


# The Diagonal Harmonics in $x_1, x_2, x_3; y_1, y_2, y_3$

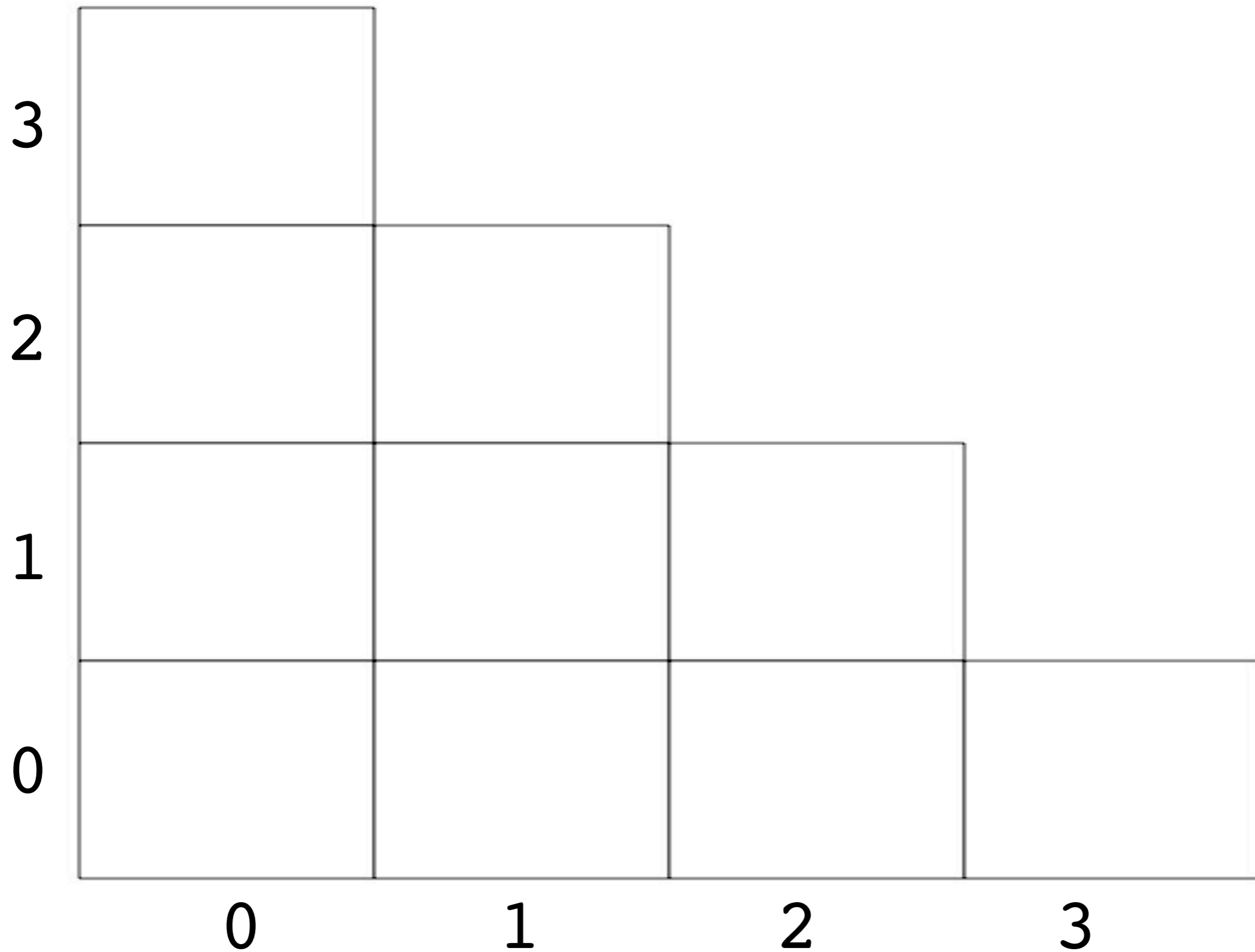
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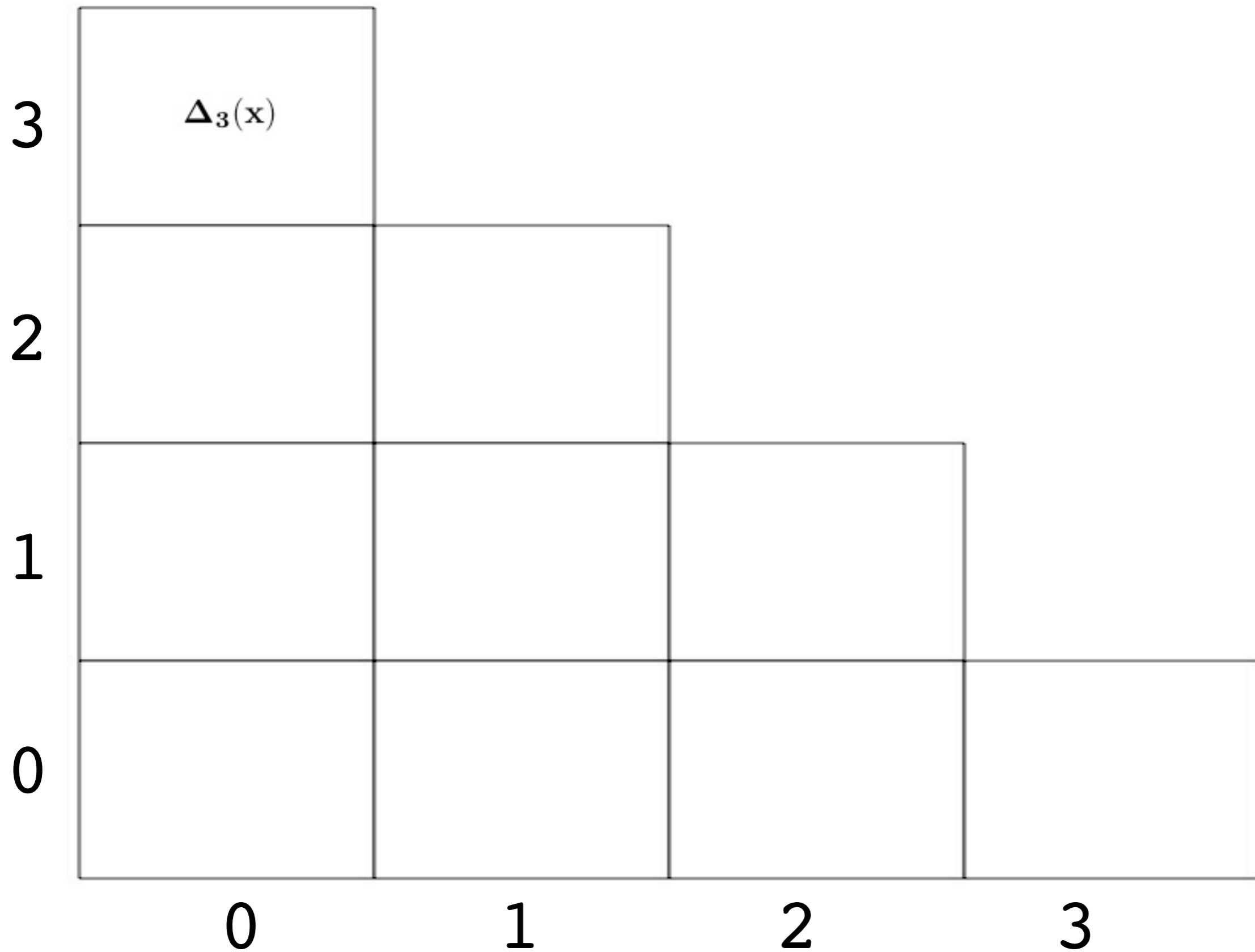
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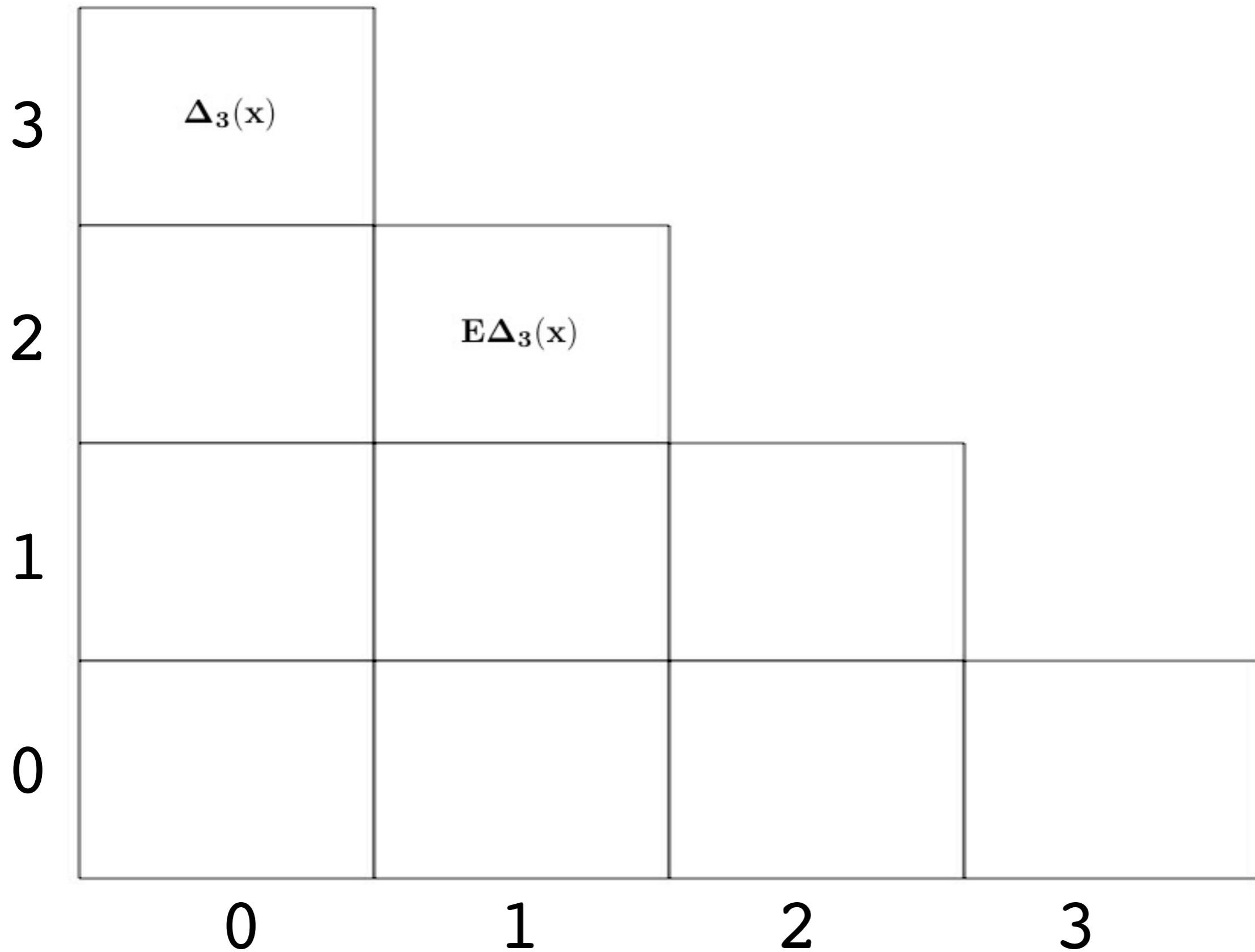
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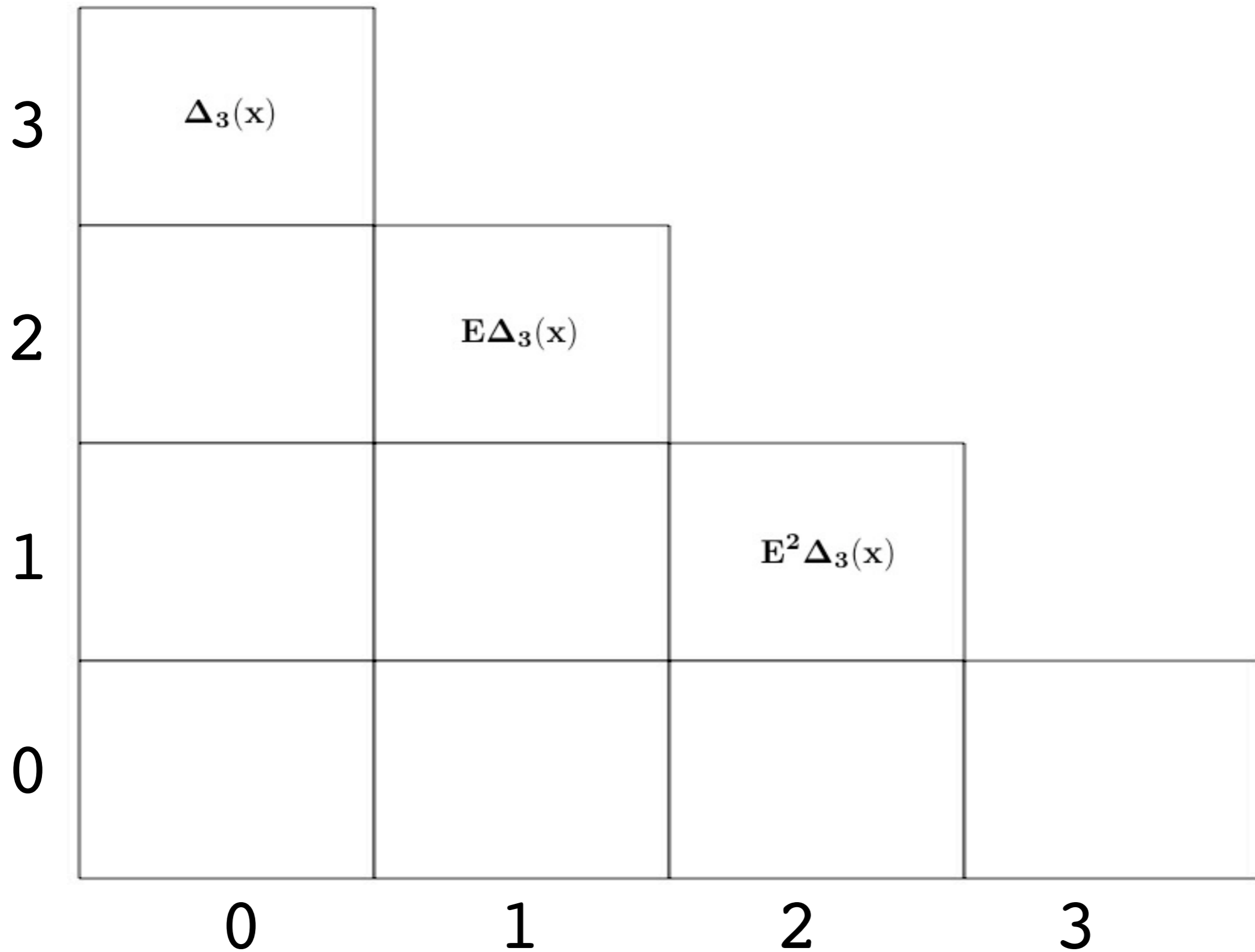
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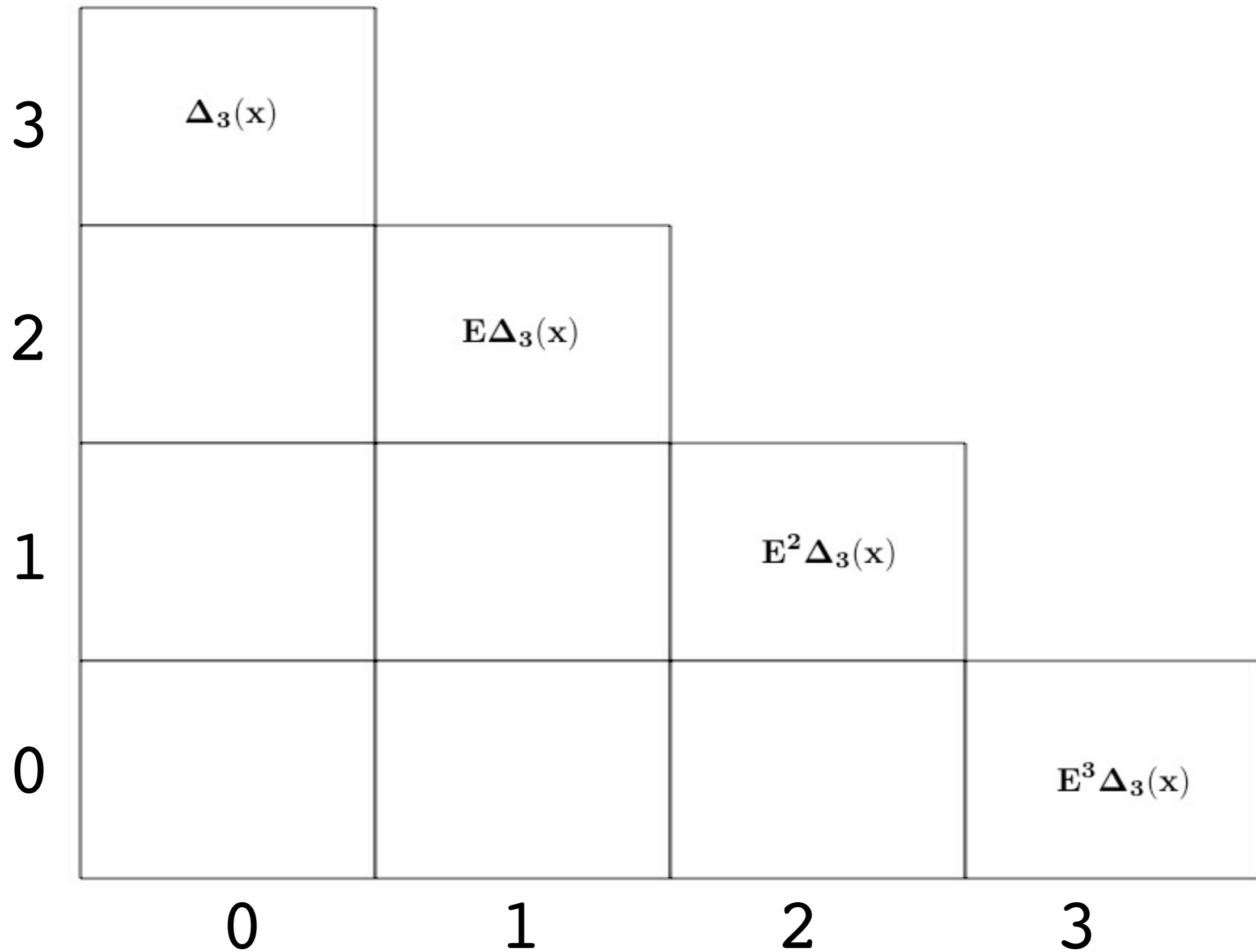
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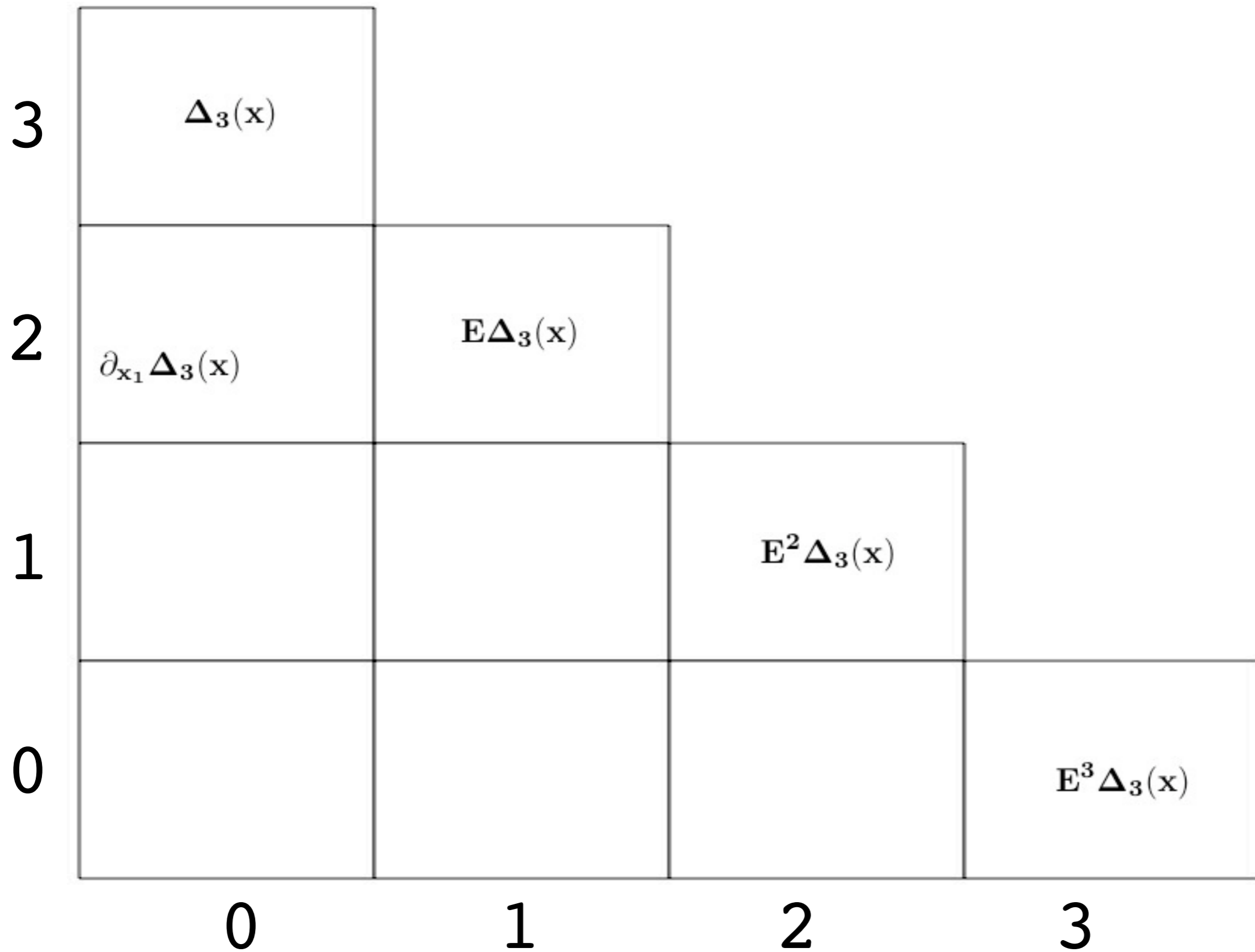


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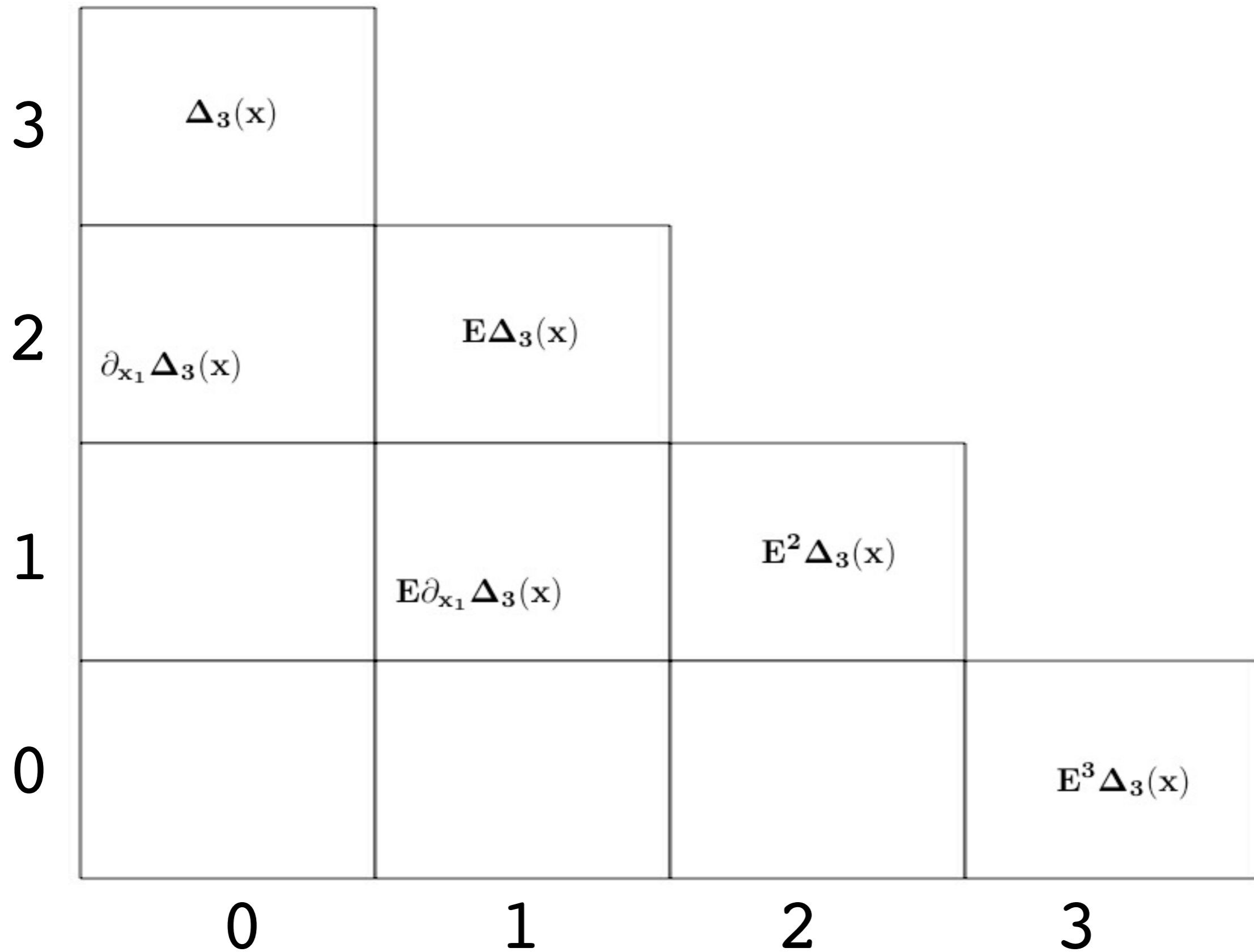




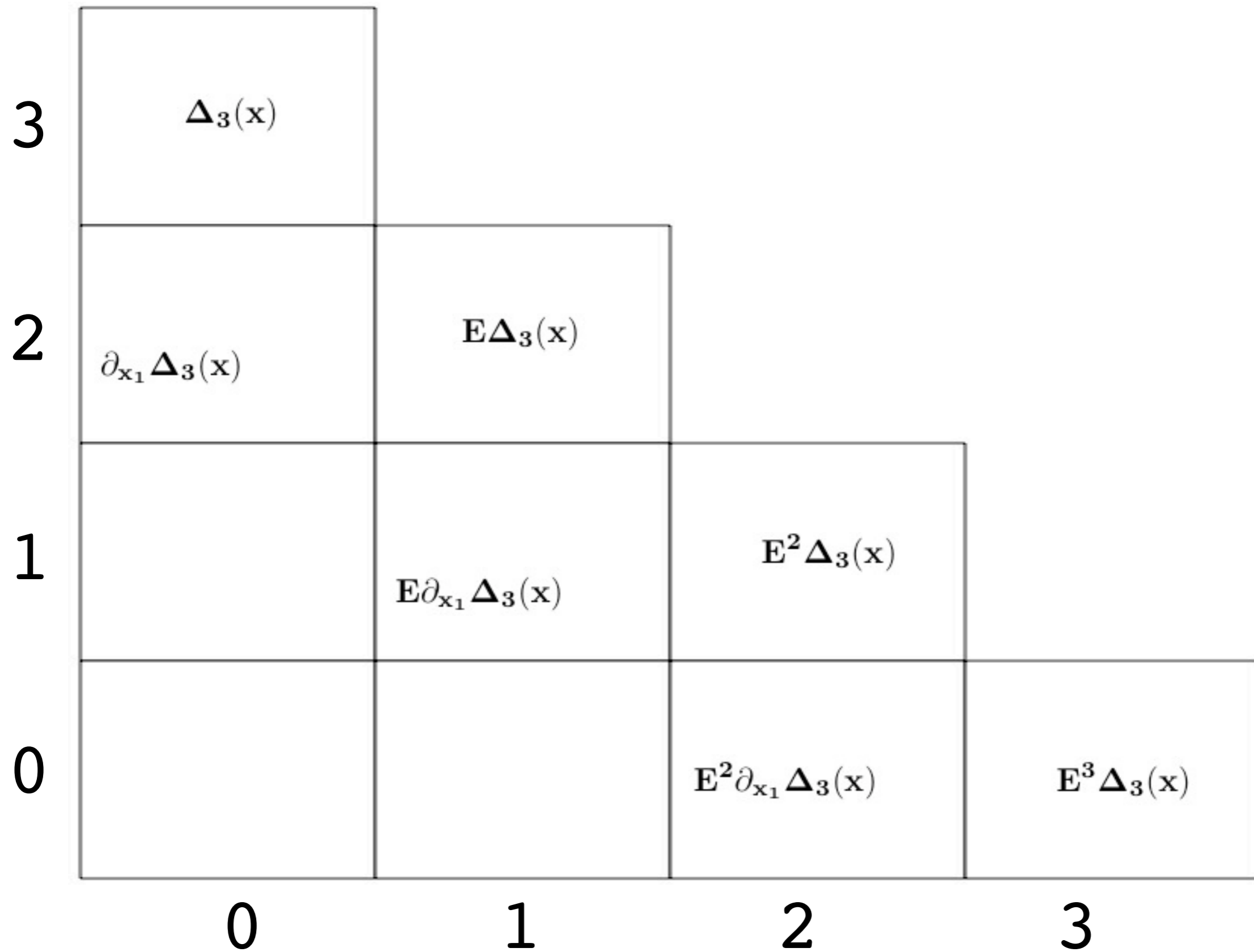
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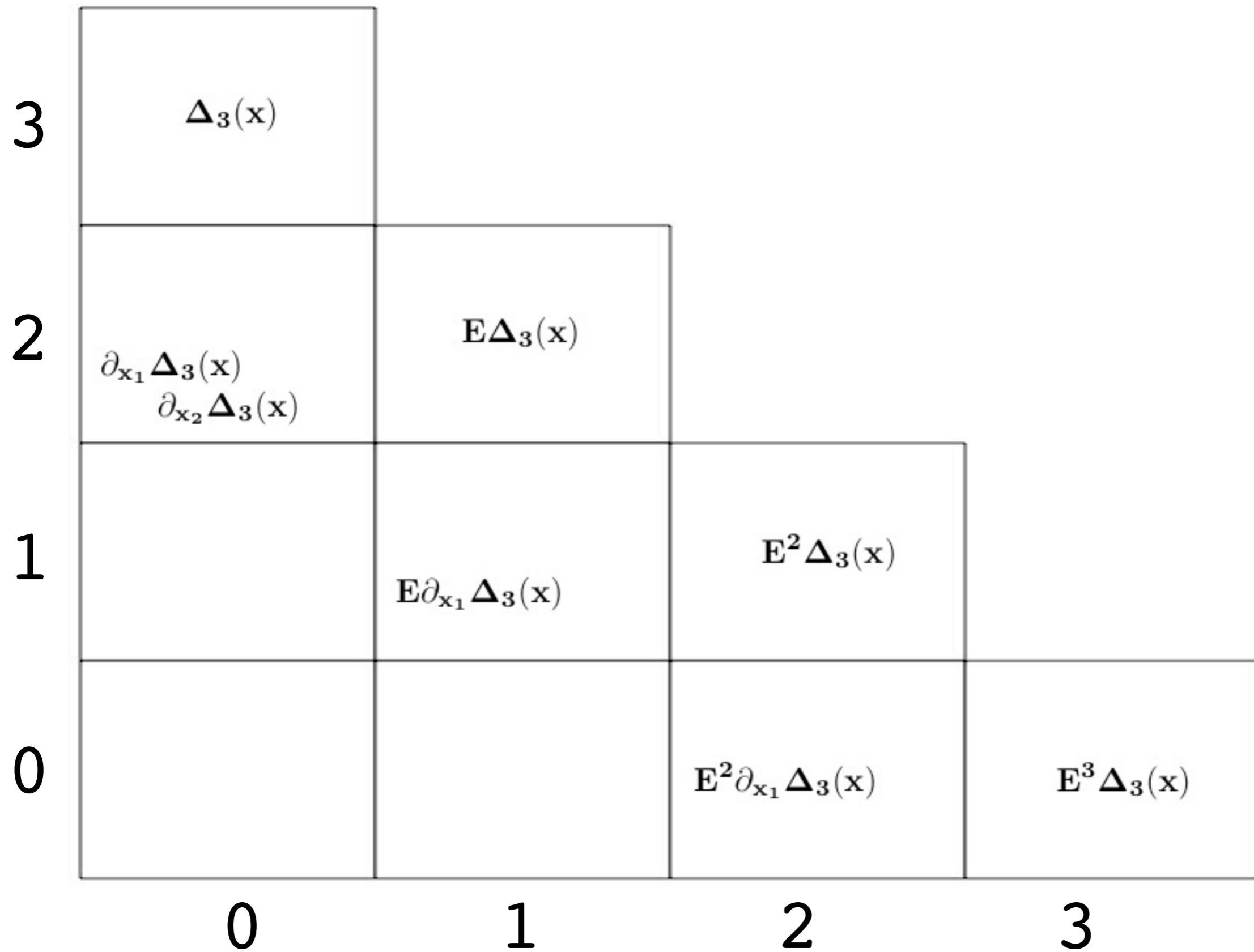
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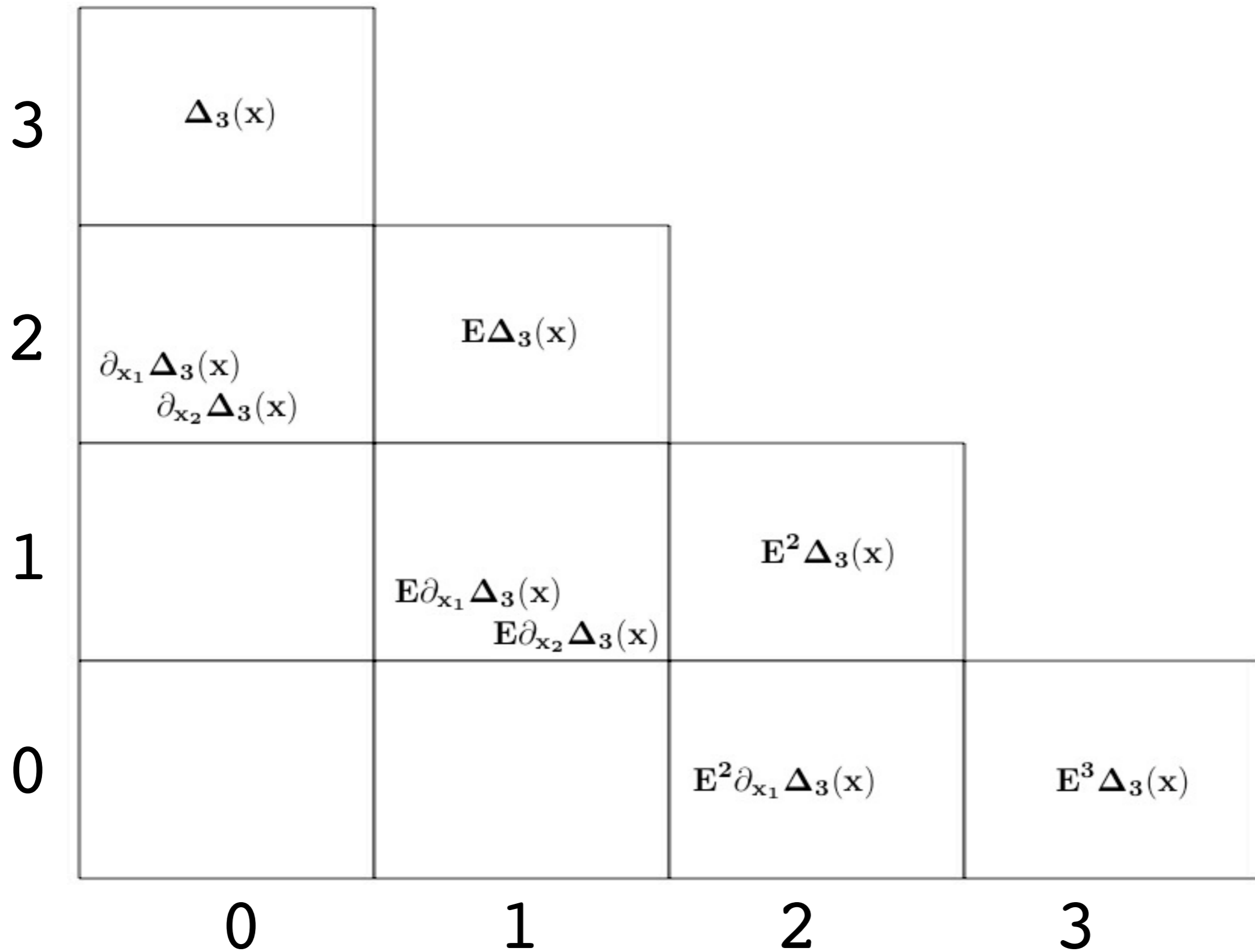
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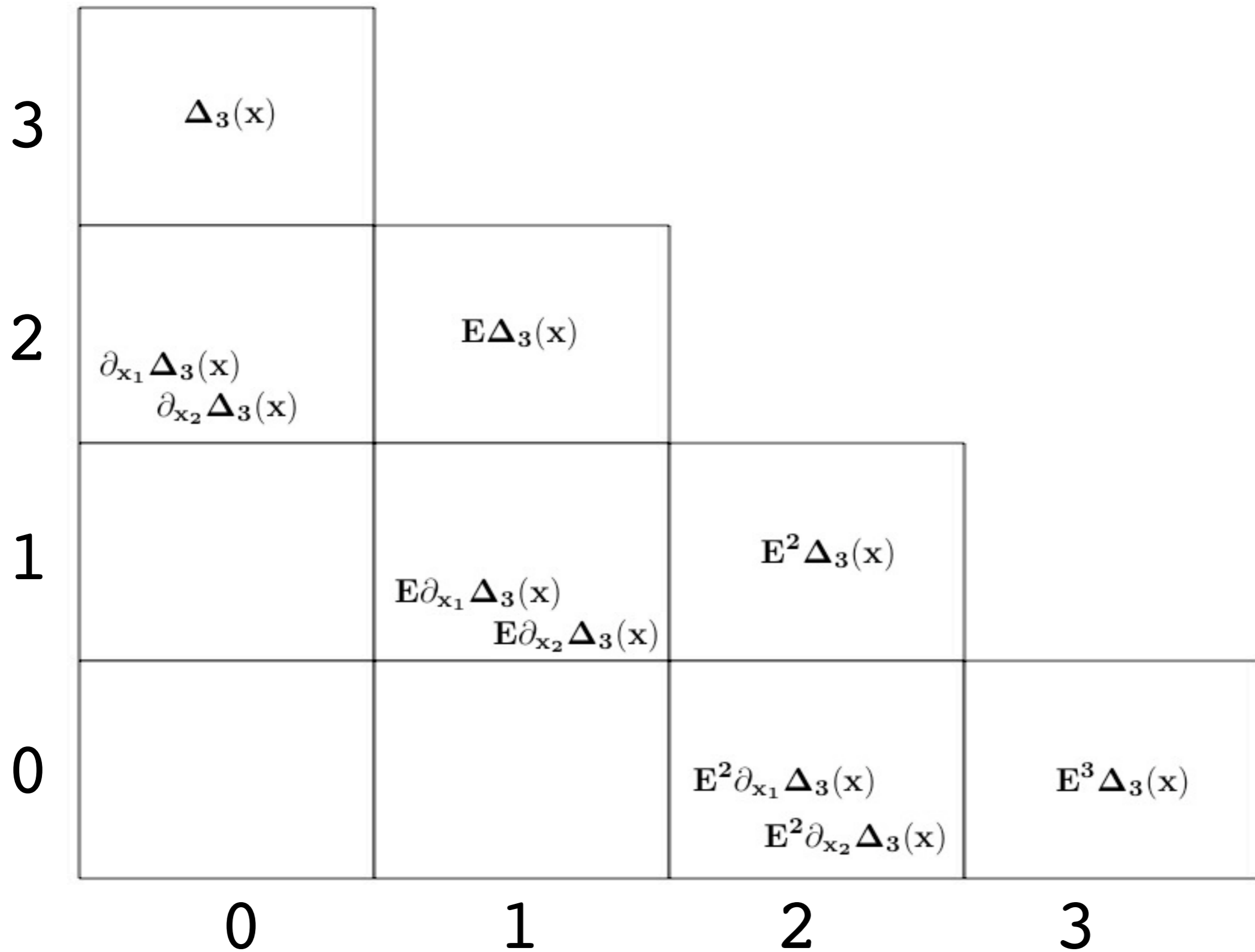
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# The Diagonal Harmonics in $x_1, x_2, x_3; y_1, y_2, y_3$

3	$\Delta_3(\mathbf{x})$			
2	$\partial_{x_1} \Delta_3(\mathbf{x})$ $\partial_{x_2} \Delta_3(\mathbf{x})$	$E\Delta_3(\mathbf{x})$		
1		$\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$ $E\partial_{x_1} \Delta_3(\mathbf{x})$ $E\partial_{x_2} \Delta_3(\mathbf{x})$	$E^2 \Delta_3(\mathbf{x})$	
0			$E^2 \partial_{x_1} \Delta_3(\mathbf{x})$ $E^2 \partial_{x_2} \Delta_3(\mathbf{x})$	$E^3 \Delta_3(\mathbf{x})$
	0	1	2	3

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3	$\Delta_3(\mathbf{x})$			
2	$\partial_{x_1} \Delta_3(\mathbf{x})$ $\partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E} \Delta_3(\mathbf{x})$		
1	$(x_1 - x_2)$	$\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$ $\mathbf{E} \partial_{x_1} \Delta_3(\mathbf{x})$ $\mathbf{E} \partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E}^2 \Delta_3(\mathbf{x})$	
0			$\mathbf{E}^2 \partial_{x_1} \Delta_3(\mathbf{x})$ $\mathbf{E}^2 \partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E}^3 \Delta_3(\mathbf{x})$
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1	$(x_1 - x_2)$	$\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$ $\mathbf{E} \partial_{x_1} \Delta_3(\mathbf{x})$ $\mathbf{E} \partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E}^2 \Delta_3(\mathbf{x})$	
0		$\mathbf{E}(x_1 - x_2)$	$\mathbf{E}^2 \partial_{x_1} \Delta_3(\mathbf{x})$ $\mathbf{E}^2 \partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E}^3 \Delta_3(\mathbf{x})$
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2	$\partial_{x_1} \Delta_3(\mathbf{x})$ $\partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E} \Delta_3(\mathbf{x})$		
1	$(x_1 - x_2)$ $(x_2 - x_3)$	$\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$ $\mathbf{E} \partial_{x_1} \Delta_3(\mathbf{x})$ $\mathbf{E} \partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E}^2 \Delta_3(\mathbf{x})$	
0		$\mathbf{E}(x_1 - x_2)$	$\mathbf{E}^2 \partial_{x_1} \Delta_3(\mathbf{x})$ $\mathbf{E}^2 \partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E}^3 \Delta_3(\mathbf{x})$
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1	$(x_1 - x_2)$ $(x_2 - x_3)$	$\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$ $\mathbf{E} \partial_{x_1} \Delta_3(\mathbf{x})$ $\mathbf{E} \partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E}^2 \Delta_3(\mathbf{x})$	
0		$\mathbf{E}(x_1 - x_2)$ $\mathbf{E}(x_2 - x_3)$	$\mathbf{E}^2 \partial_{x_1} \Delta_3(\mathbf{x})$ $\mathbf{E}^2 \partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E}^3 \Delta_3(\mathbf{x})$
	0	1	2	3

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3	$\Delta_3(x)$			
2	$\partial_{x_1} \Delta_3(x)$ $\partial_{x_2} \Delta_3(x)$	$E\Delta_3(x)$		
1	$(x_1 - x_2)$ $(x_2 - x_3)$	$\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$ $E\partial_{x_1} \Delta_3(x)$ $E\partial_{x_2} \Delta_3(x)$	$E^2 \Delta_3(x)$	
0	1	$E(x_1 - x_2)$ $E(x_2 - x_3)$	$E^2 \partial_{x_1} \Delta_3(x)$ $E^2 \partial_{x_2} \Delta_3(x)$	$E^3 \Delta_3(x)$
	0	1	2	3

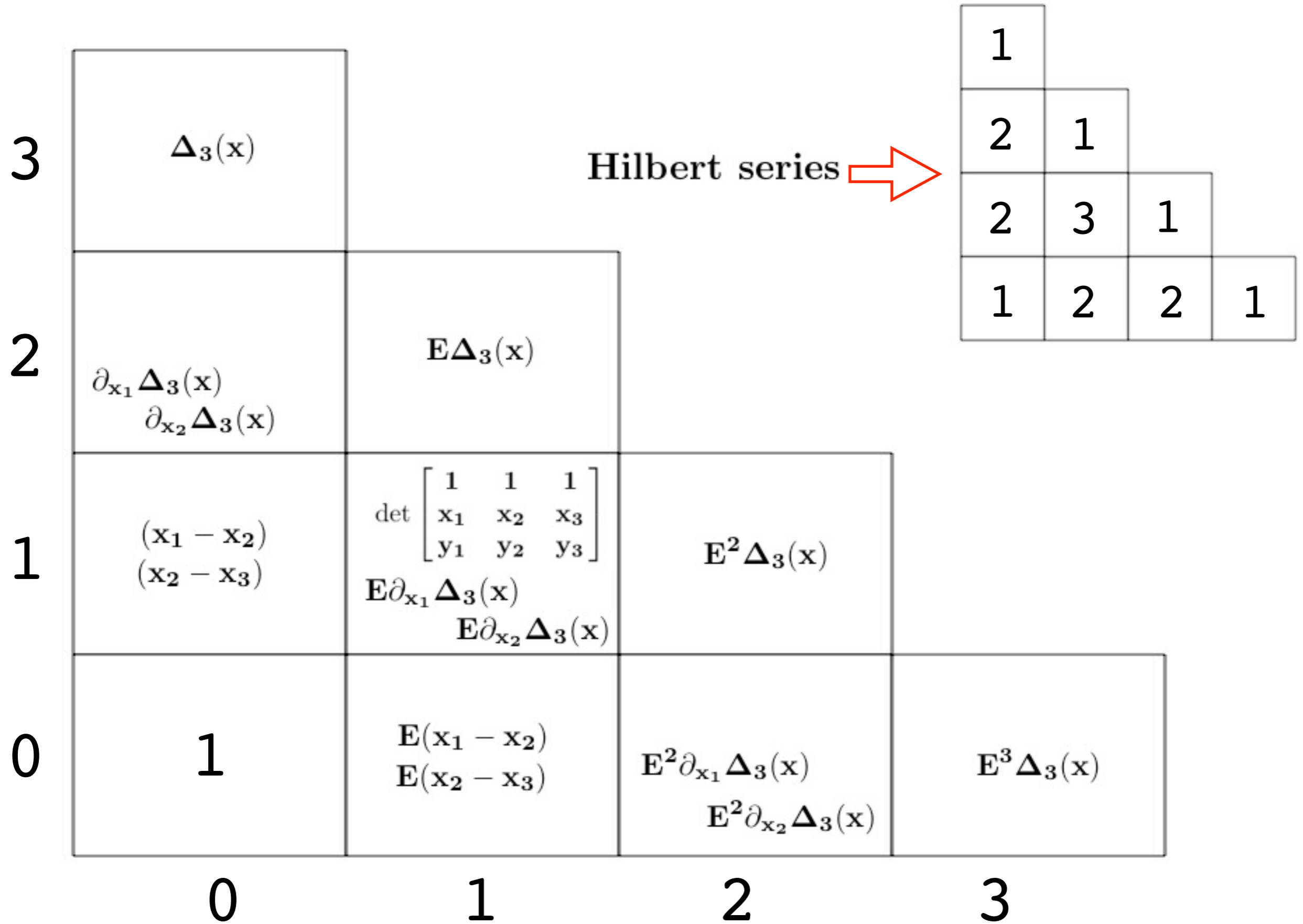
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1	$(x_1 - x_2)$ $(x_2 - x_3)$	$\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$ $\mathbf{E} \partial_{x_1} \Delta_3(\mathbf{x})$ $\mathbf{E} \partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E}^2 \Delta_3(\mathbf{x})$	
0	1	$\mathbf{E}(x_1 - x_2)$ $\mathbf{E}(x_2 - x_3)$	$\mathbf{E}^2 \partial_{x_1} \Delta_3(\mathbf{x})$ $\mathbf{E}^2 \partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E}^3 \Delta_3(\mathbf{x})$
	0	1	2	3

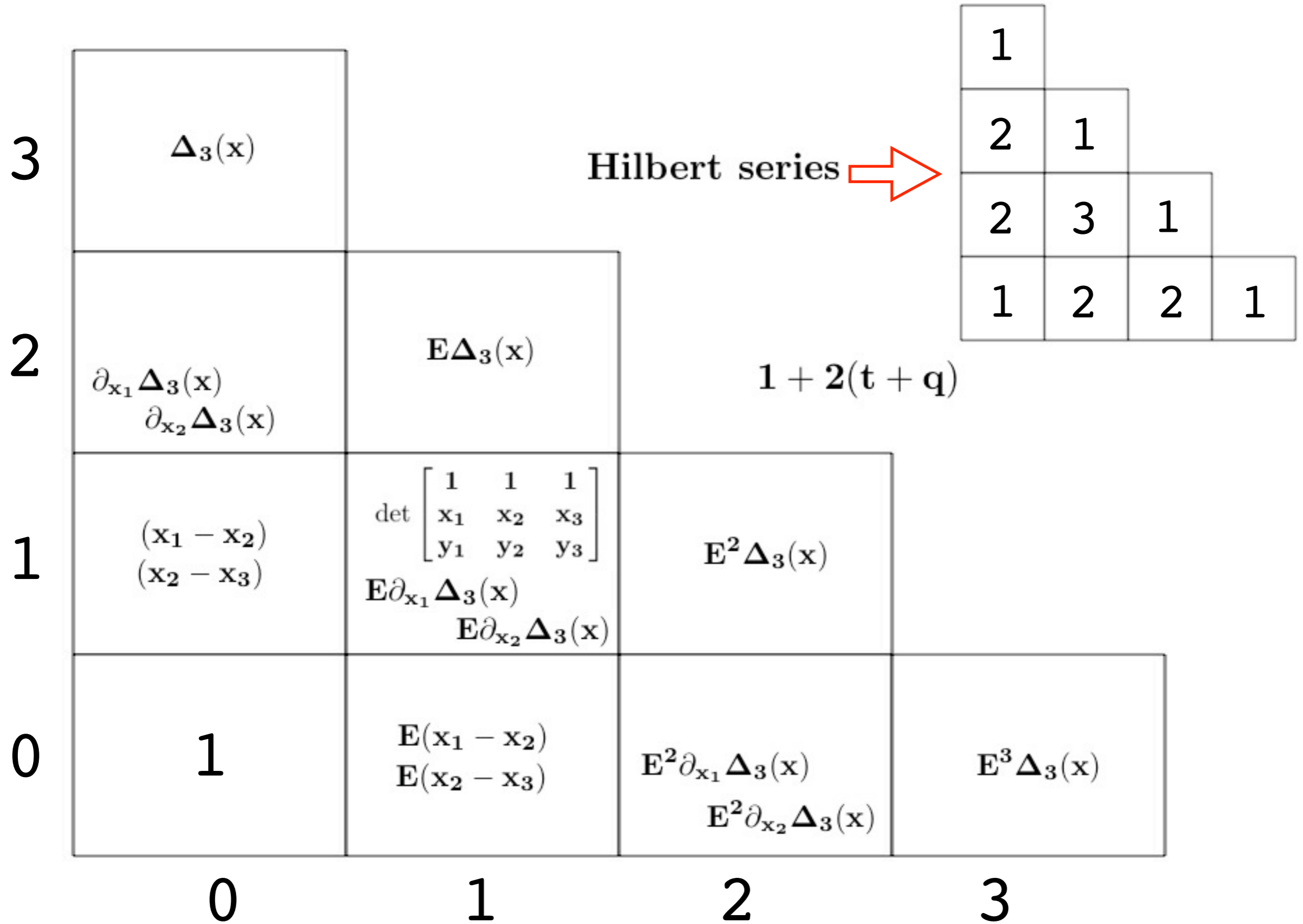
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0	1	$\mathbf{E}(x_1 - x_2)$ $\mathbf{E}(x_2 - x_3)$	$\mathbf{E}^2 \partial_{x_1} \Delta_3(\mathbf{x})$ $\mathbf{E}^2 \partial_{x_2} \Delta_3(\mathbf{x})$	$\mathbf{E}^3 \Delta_3(\mathbf{x})$
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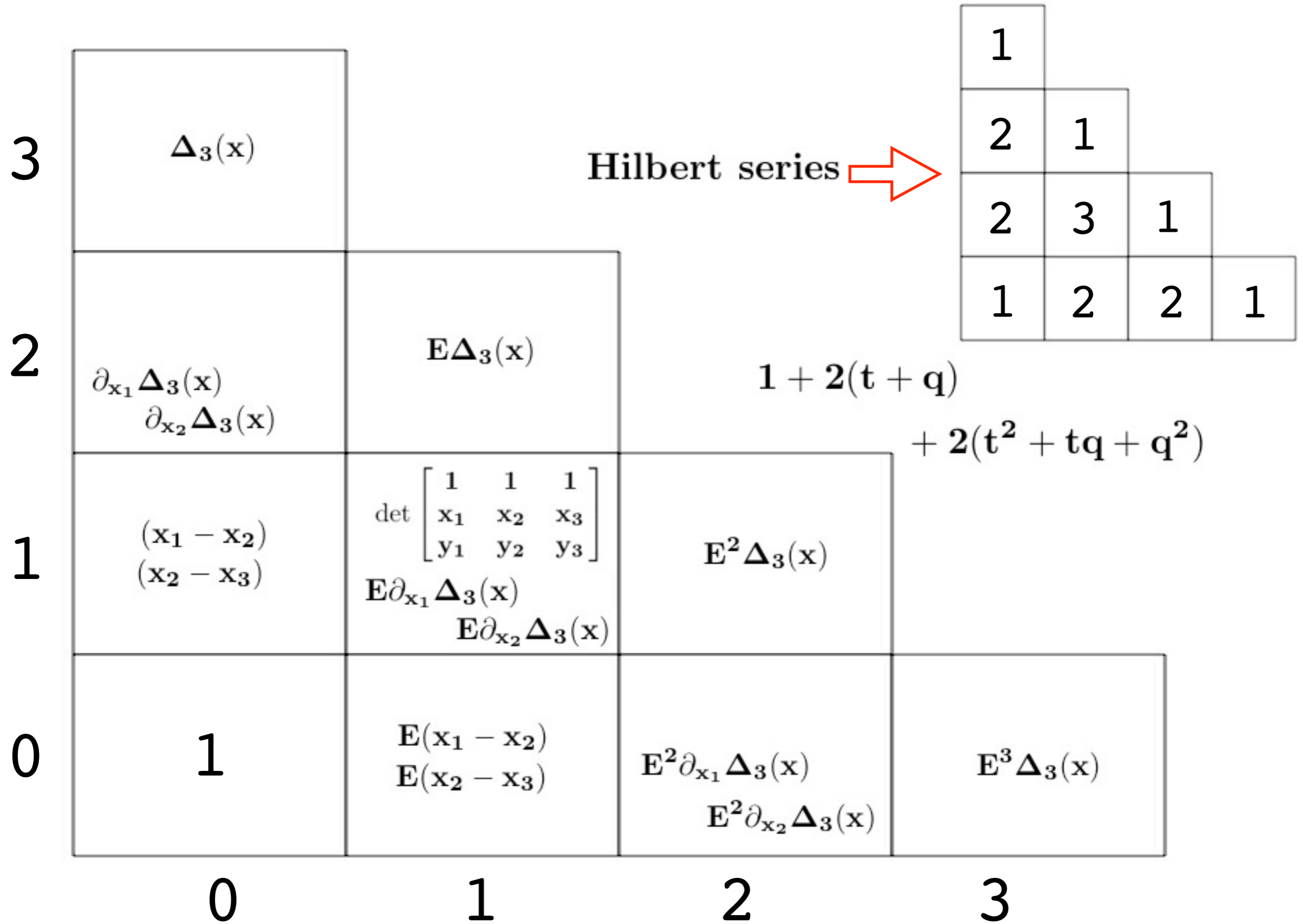


# The Diagonal Harmonics in $x_1, x_2, x_3; y_1, y_2, y_3$

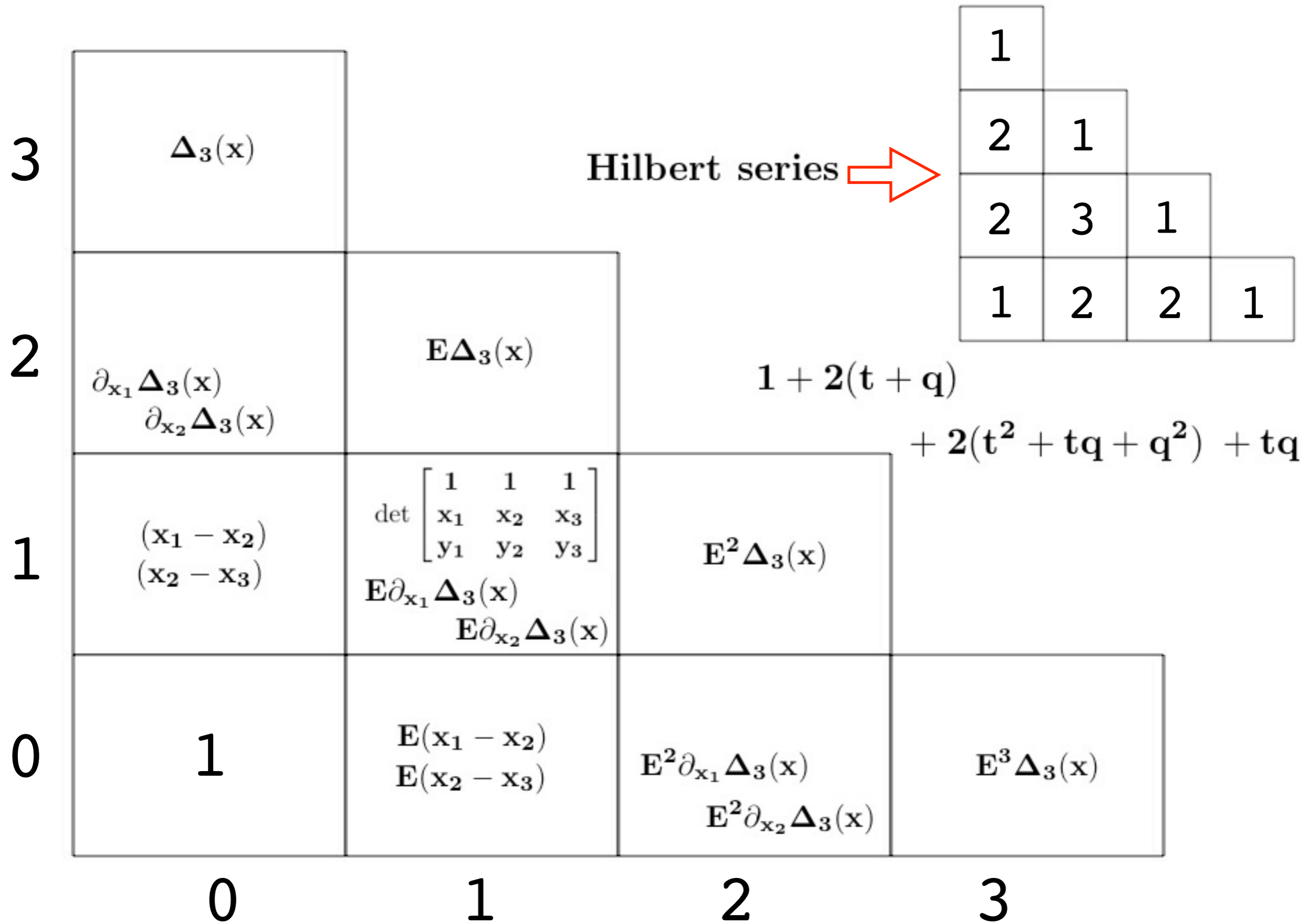




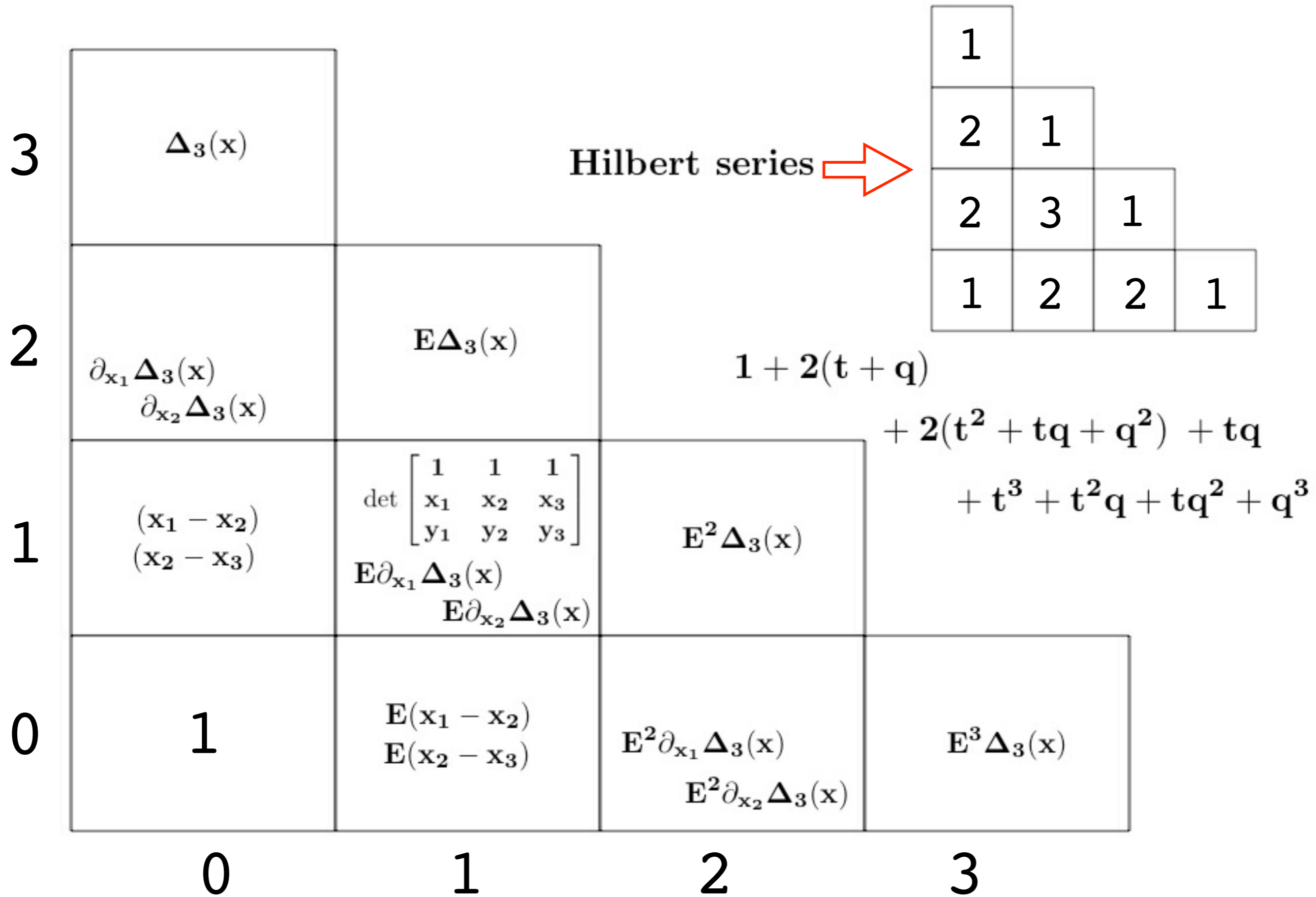
# The Diagonal Harmonics in $x_1, x_2, x_3; y_1, y_2, y_3$



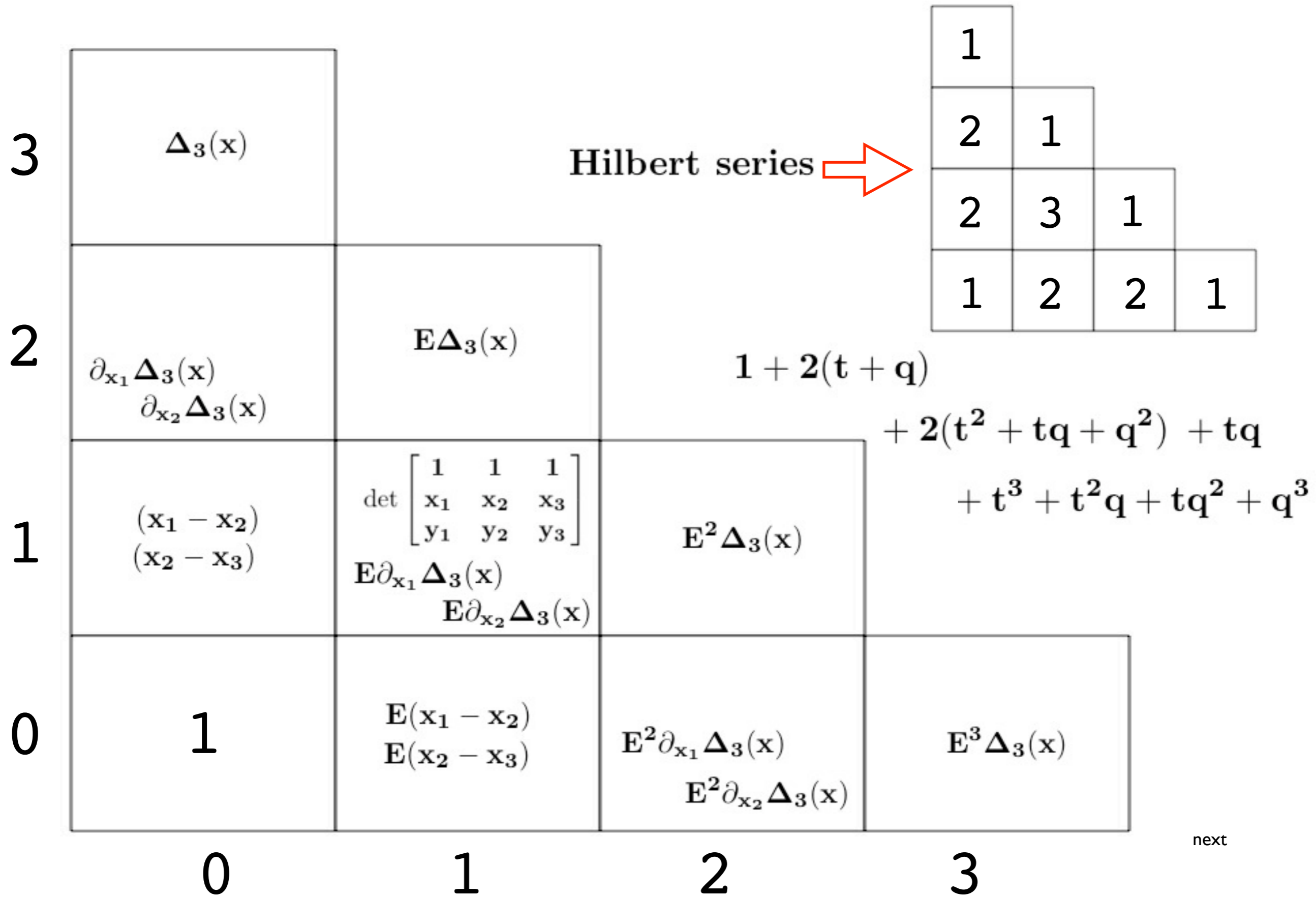
# The Diagonal Harmonics in $x_1, x_2, x_3; y_1, y_2, y_3$



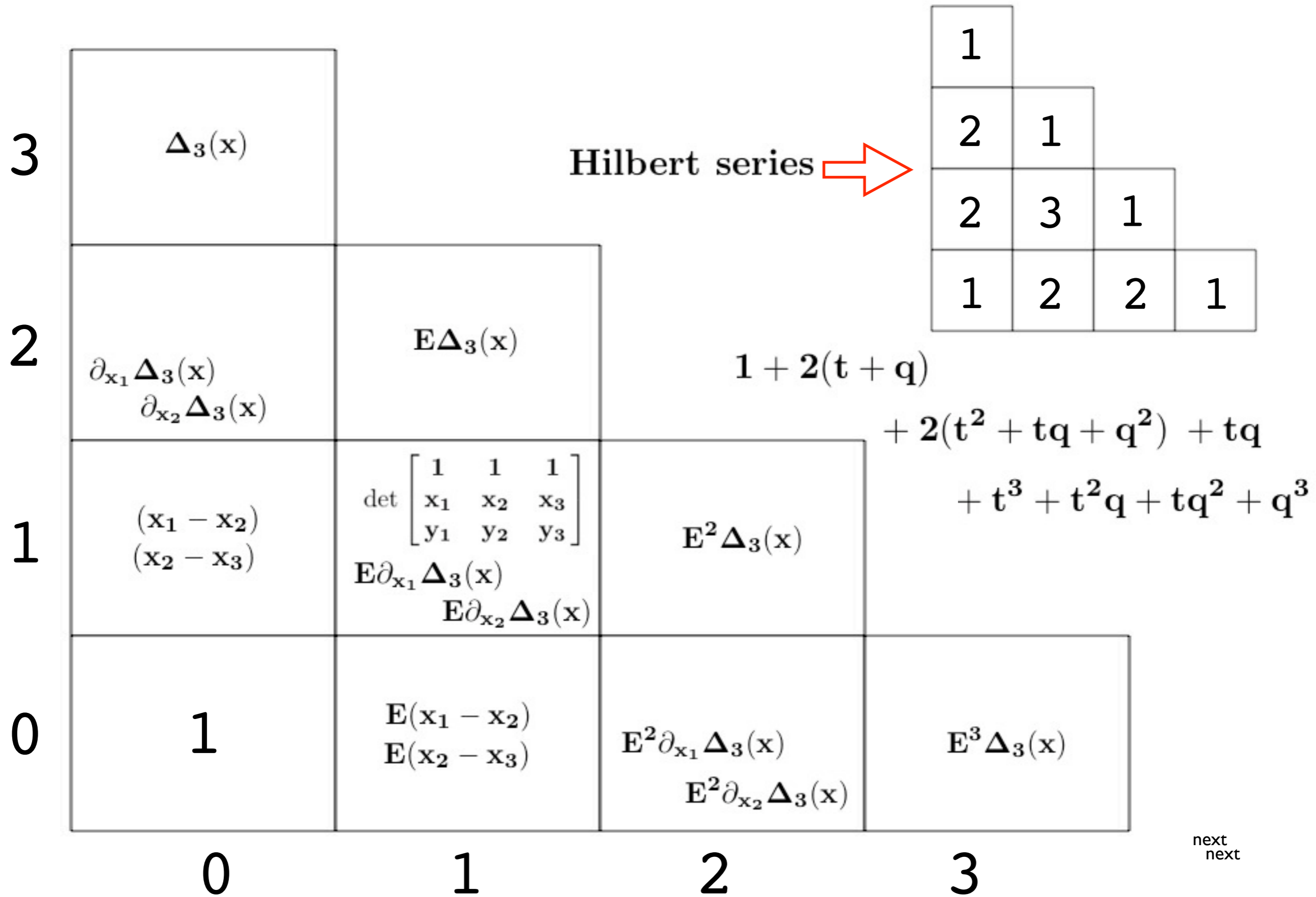
# The Diagonal Harmonics in $x_1, x_2, x_3; y_1, y_2, y_3$



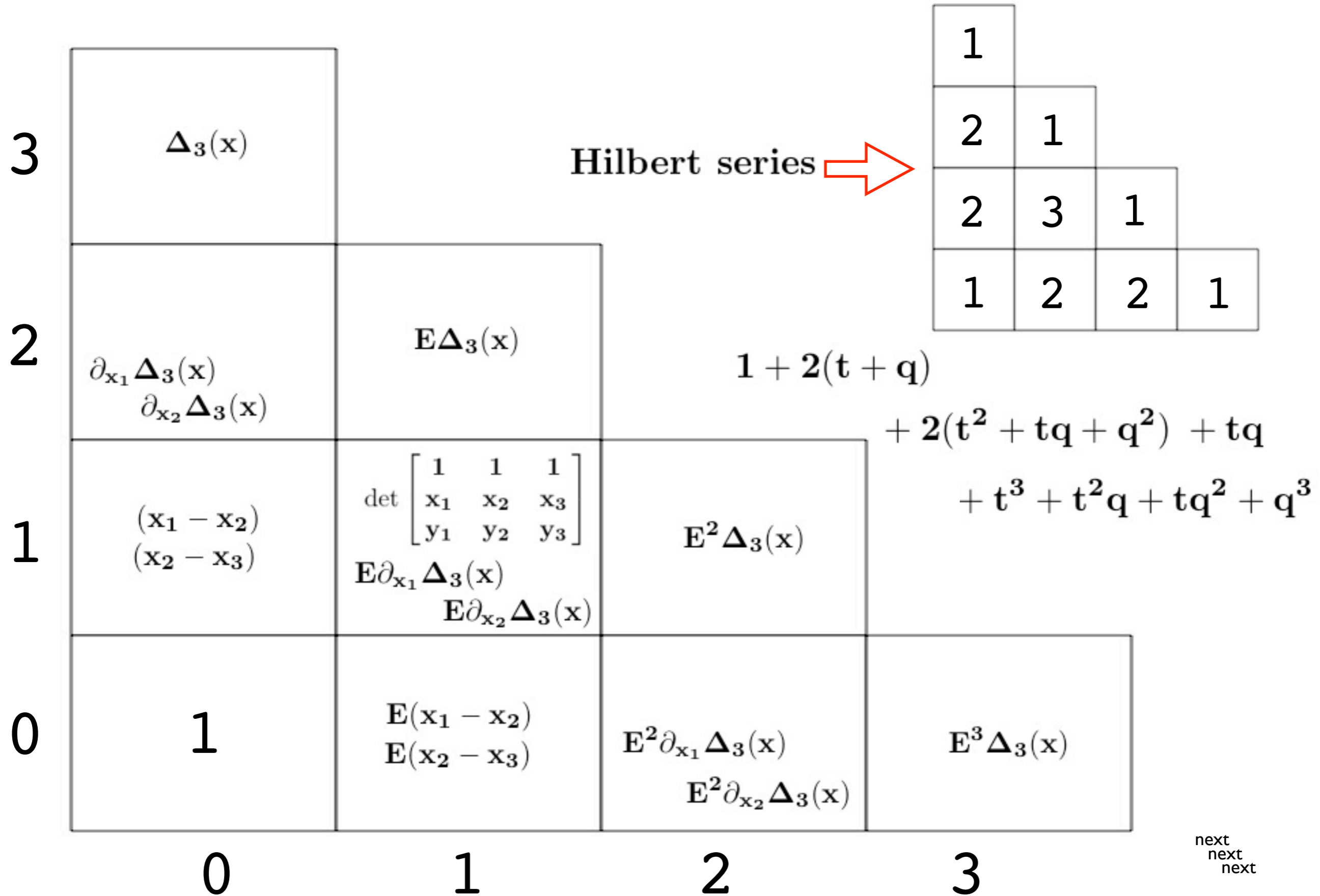
# The Diagonal Harmonics in $x_1, x_2, x_3; y_1, y_2, y_3$



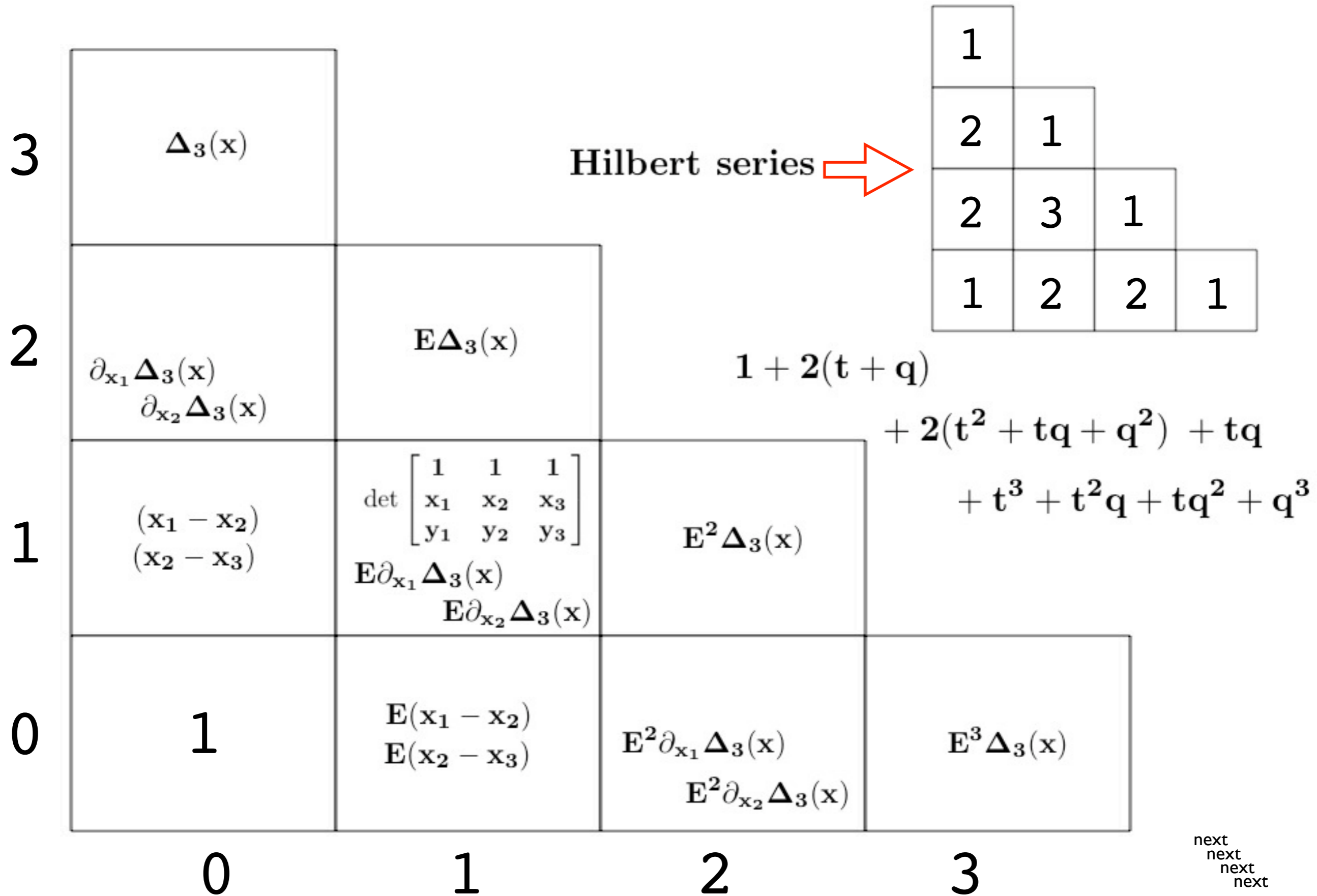
# The Diagonal Harmonics in $x_1, x_2, x_3; y_1, y_2, y_3$



# The Diagonal Harmonics in $x_1, x_2, x_3; y_1, y_2, y_3$



# The Diagonal Harmonics in $x_1, x_2, x_3; y_1, y_2, y_3$

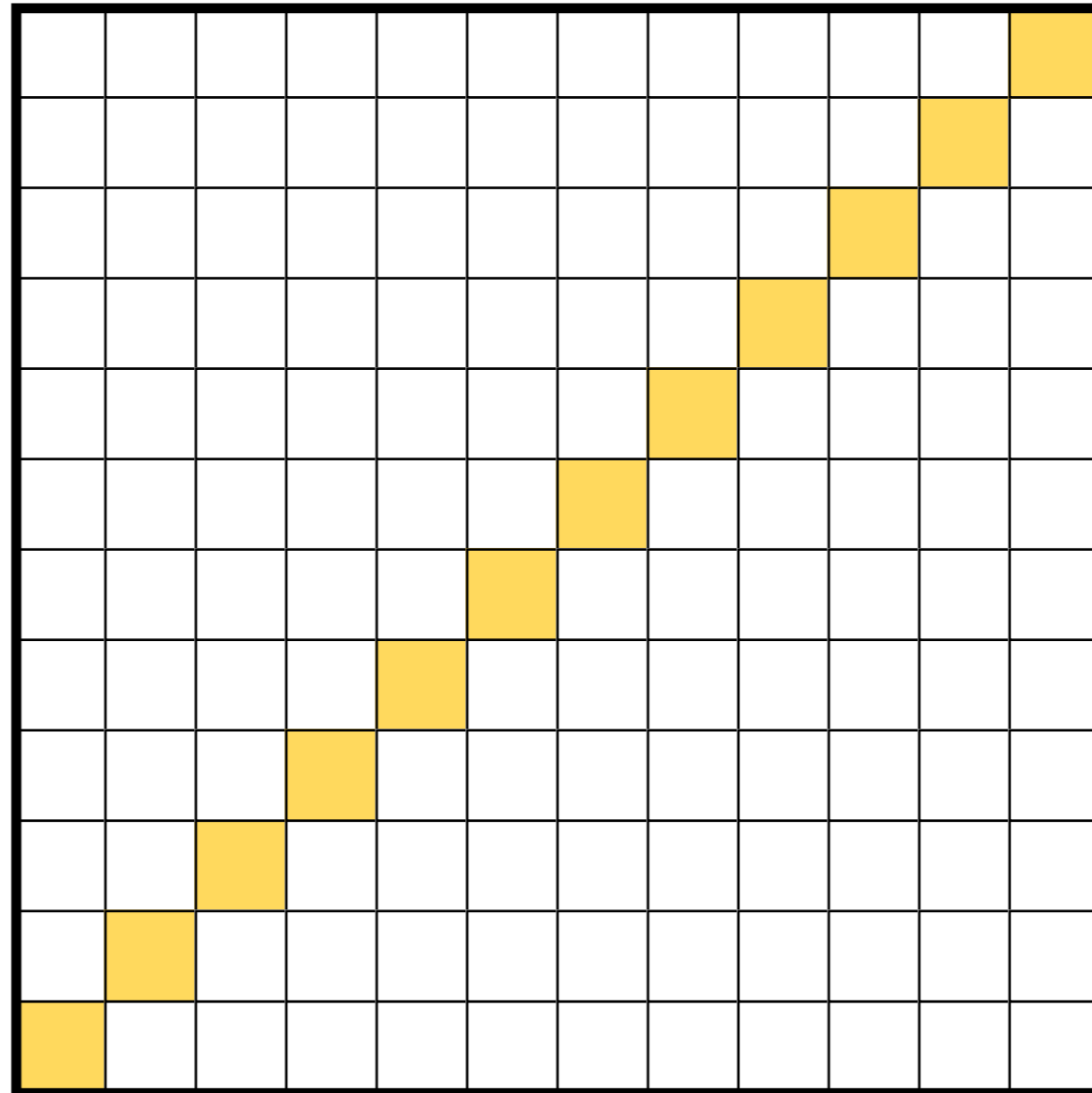




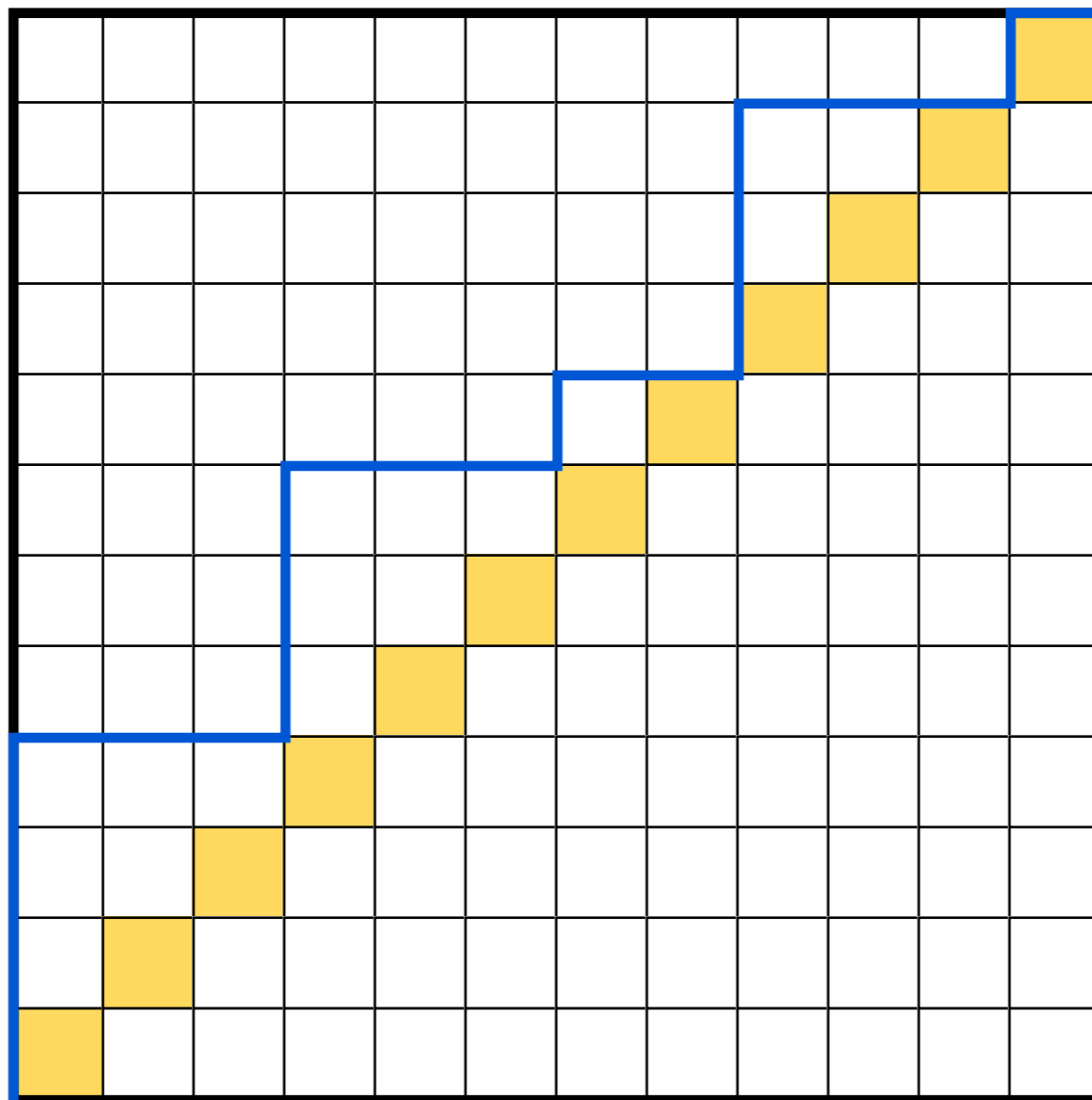


# A Parking Function

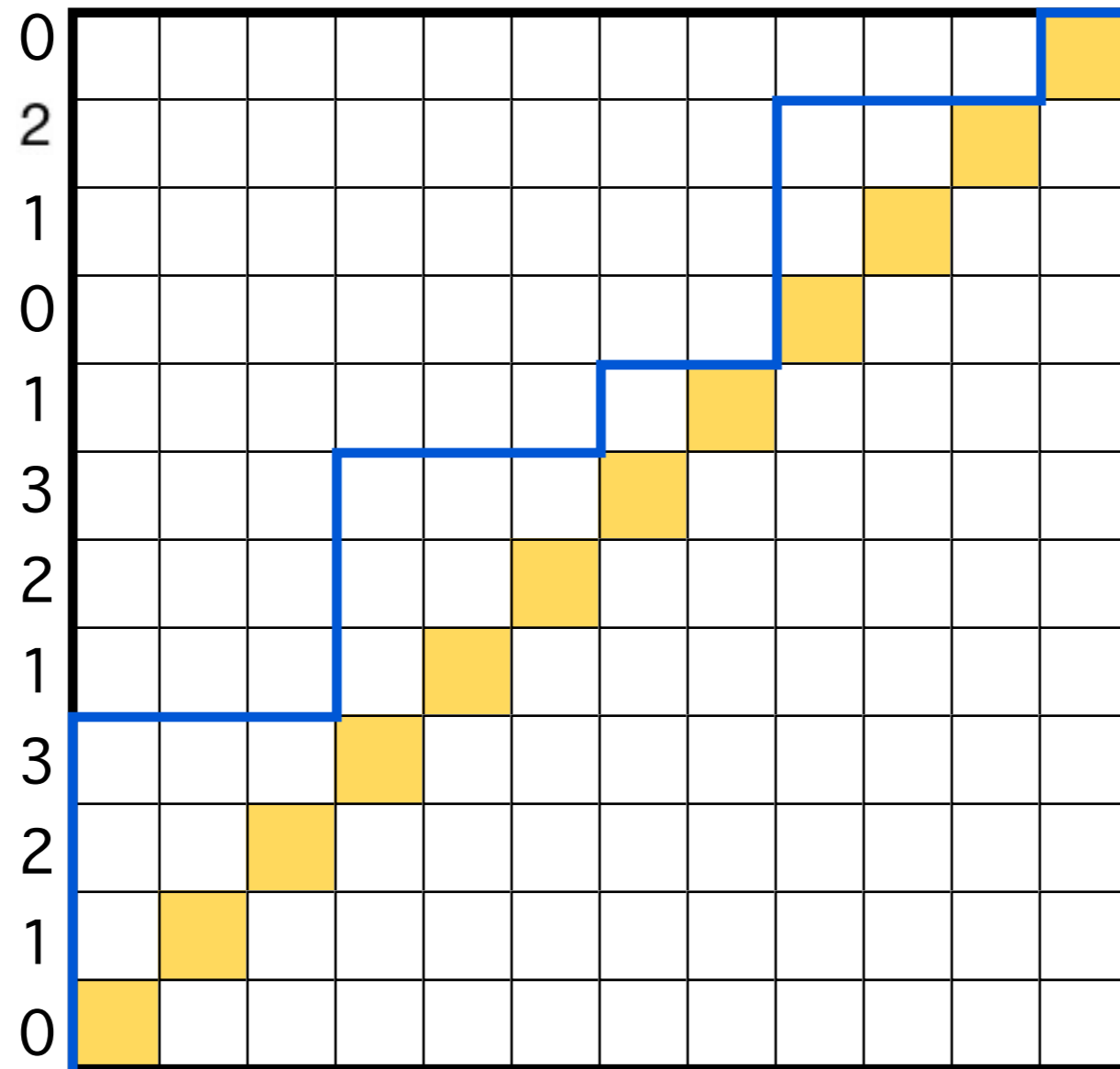
# A Parking Function



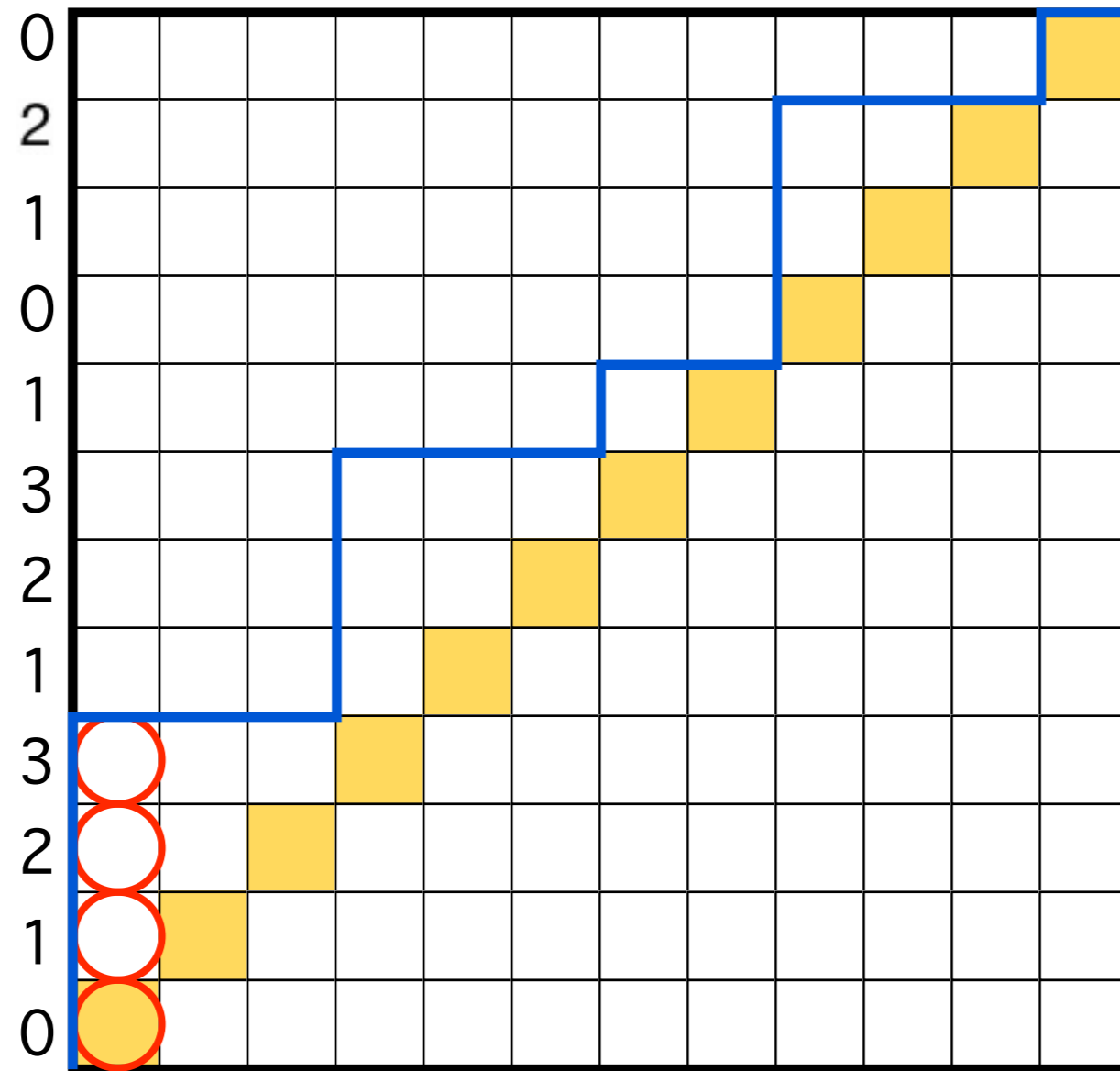
# A Parking Function



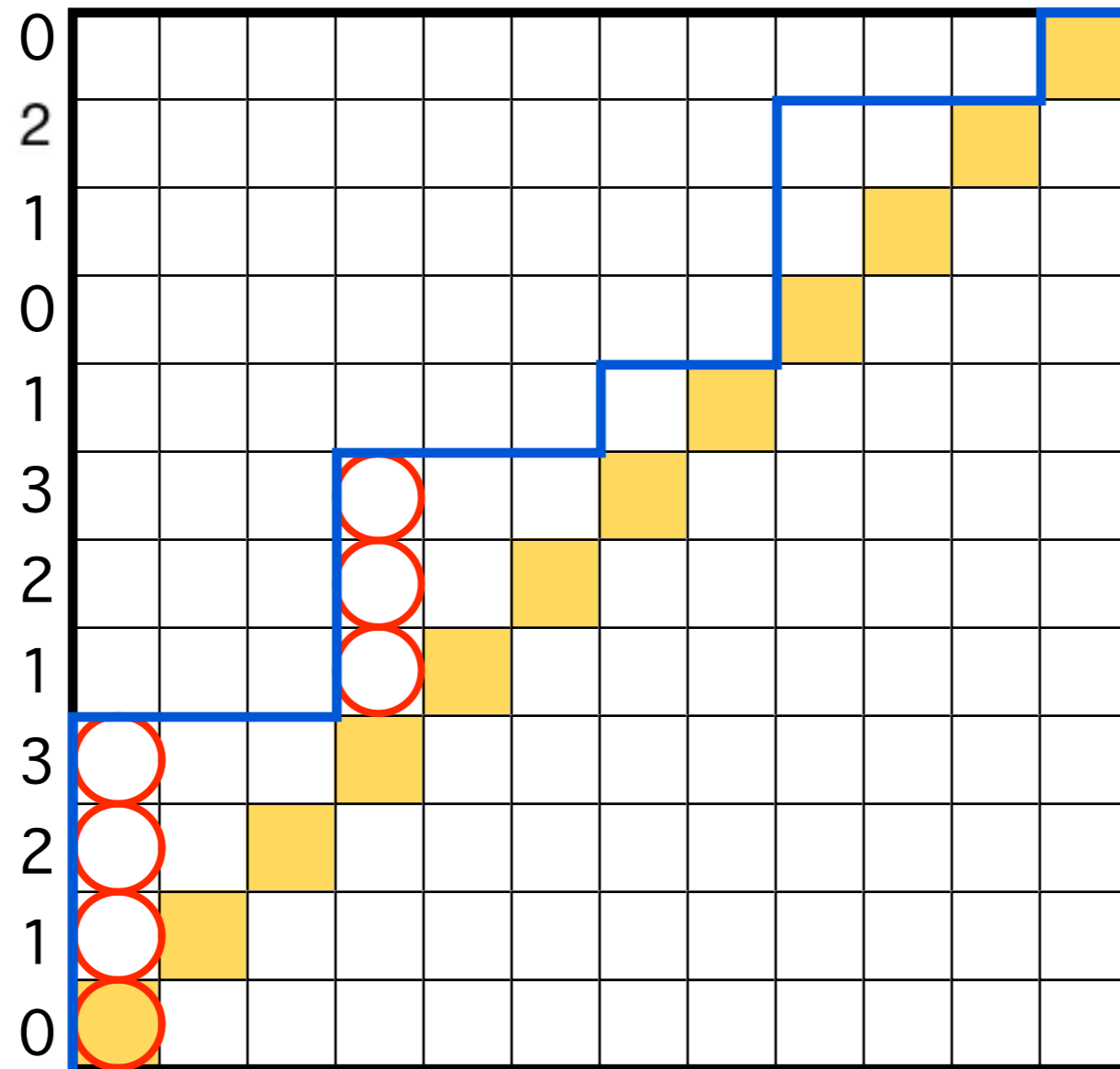
# A Parking Function



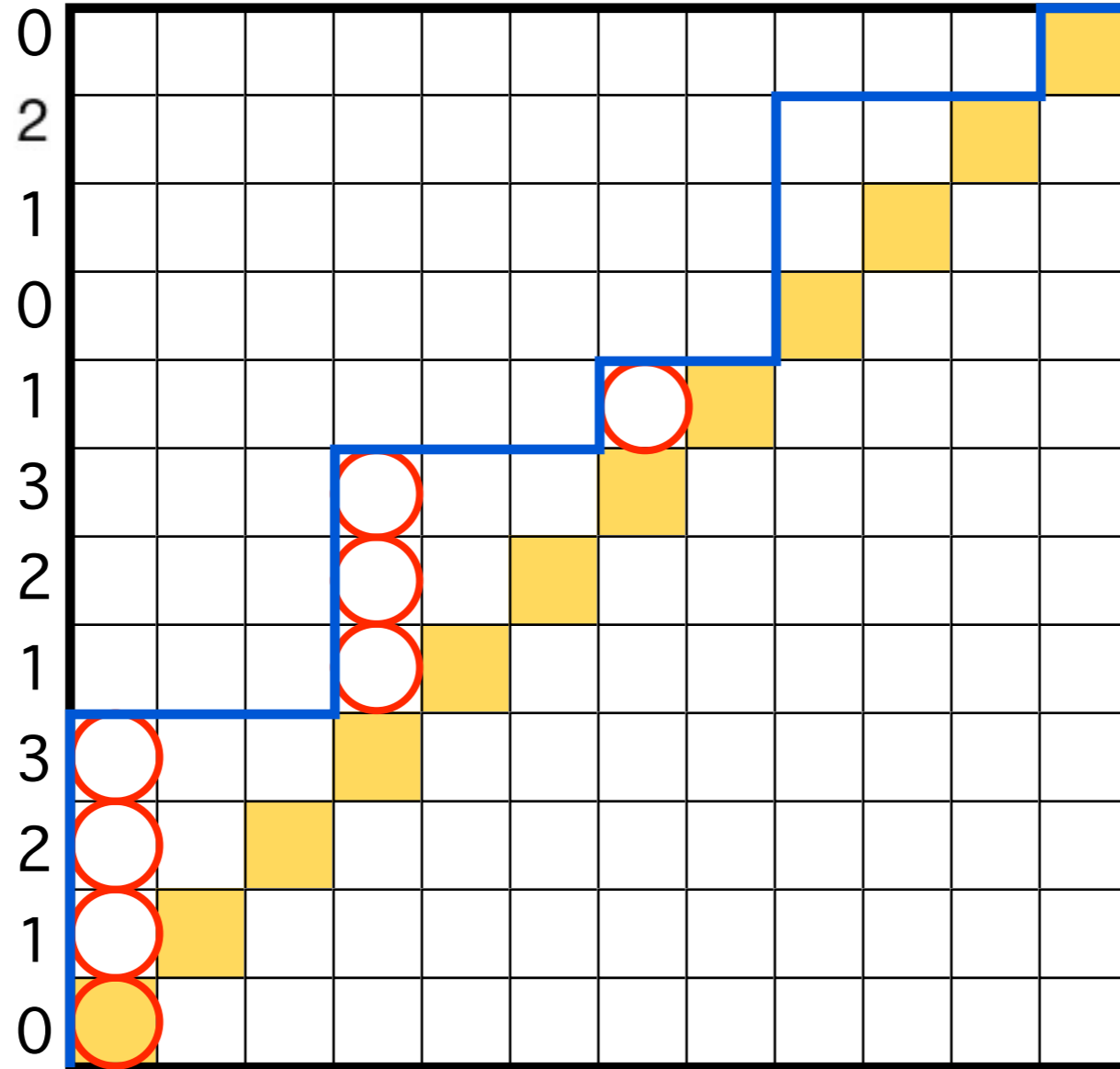
# A Parking Function



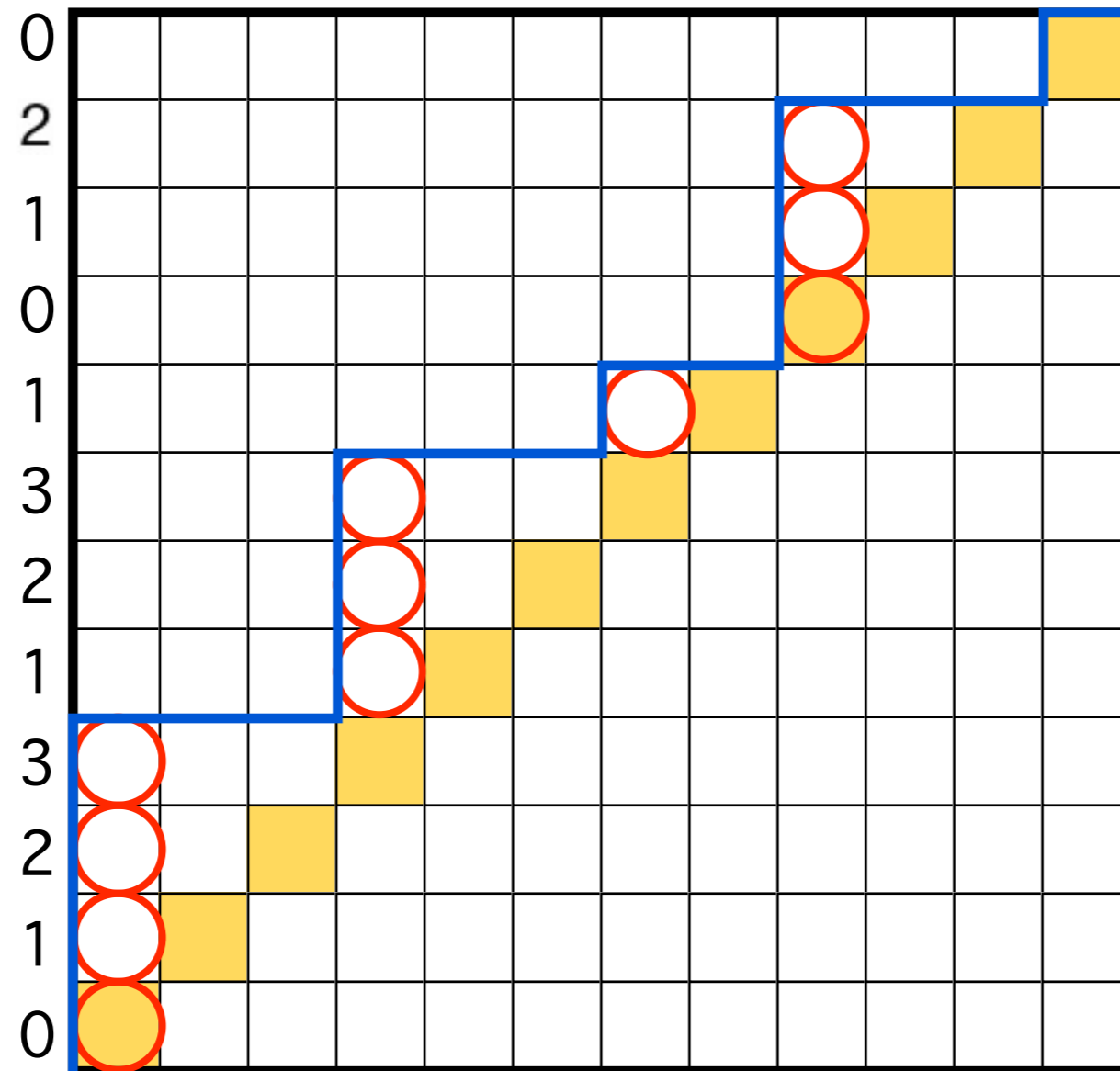
# A Parking Function



# A Parking Function

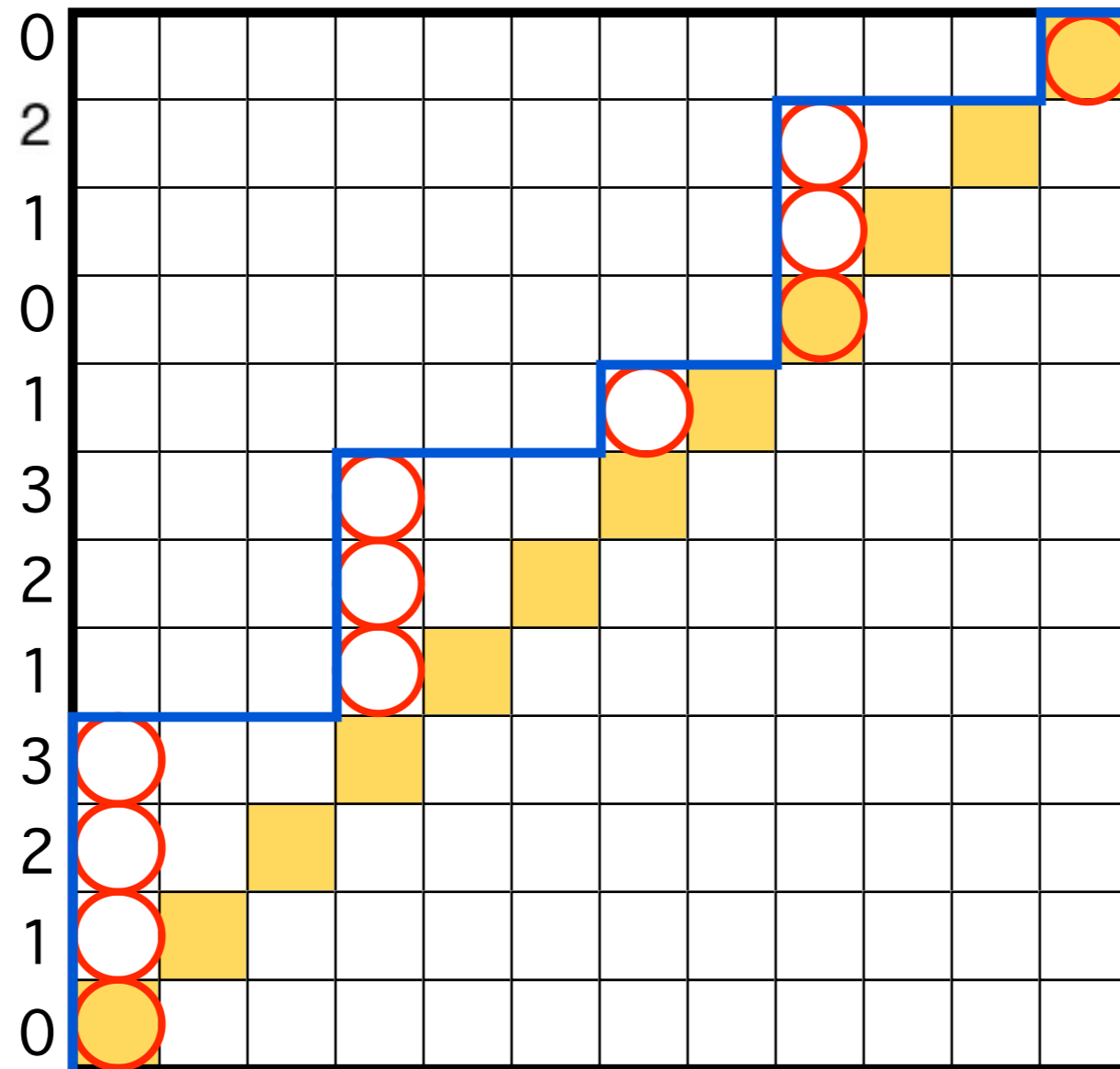


# A Parking Function

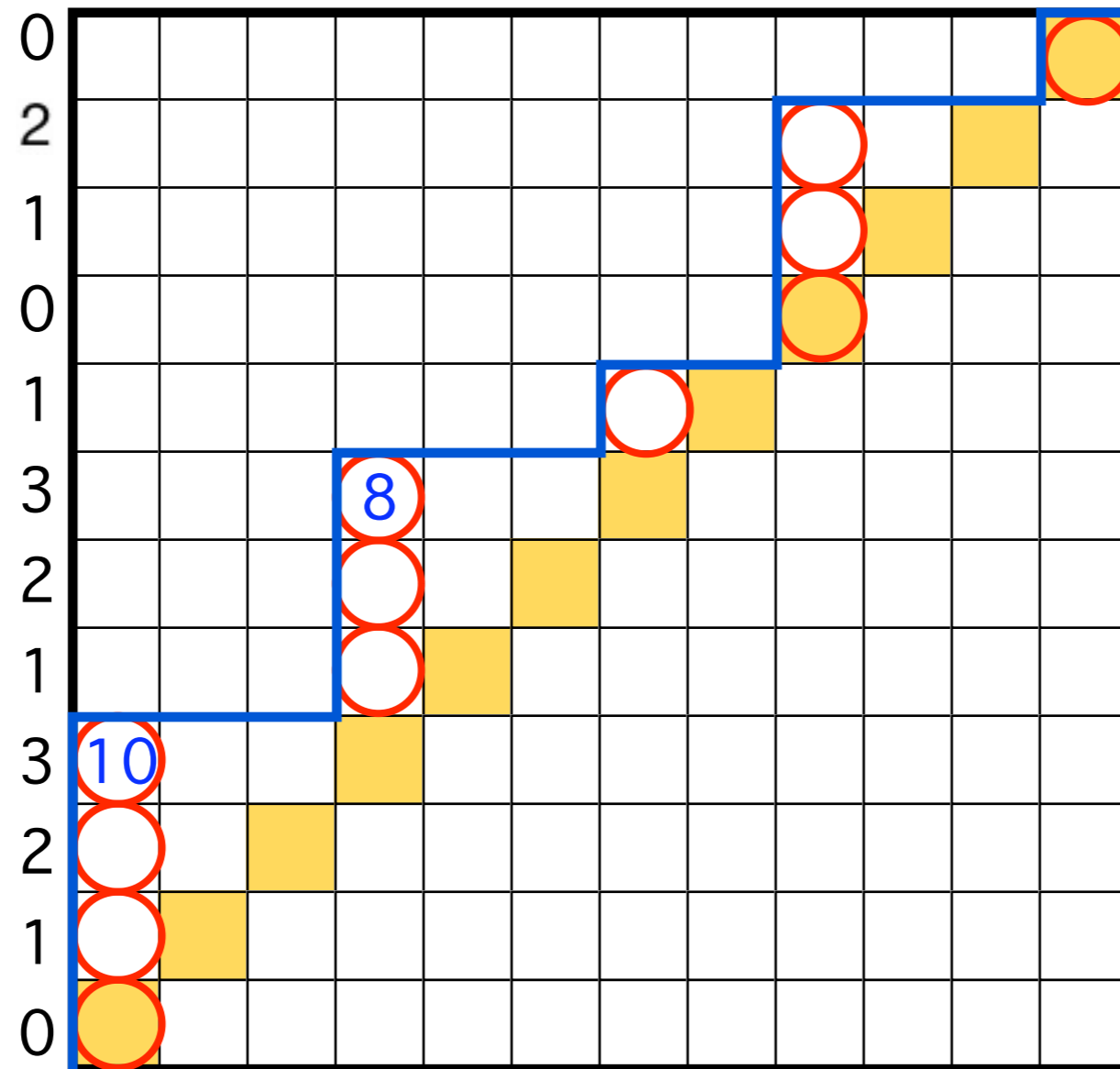




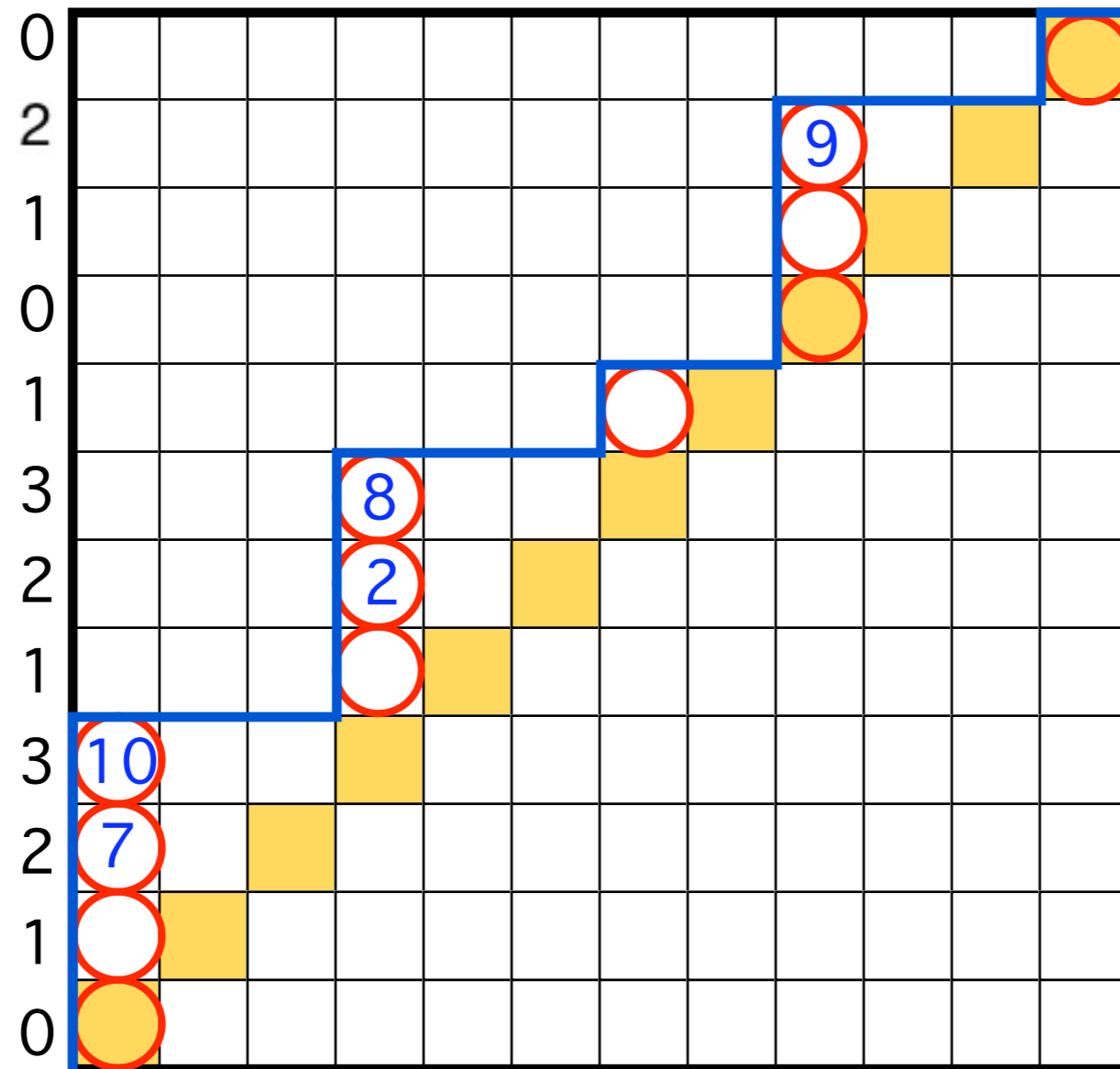
# A Parking Function



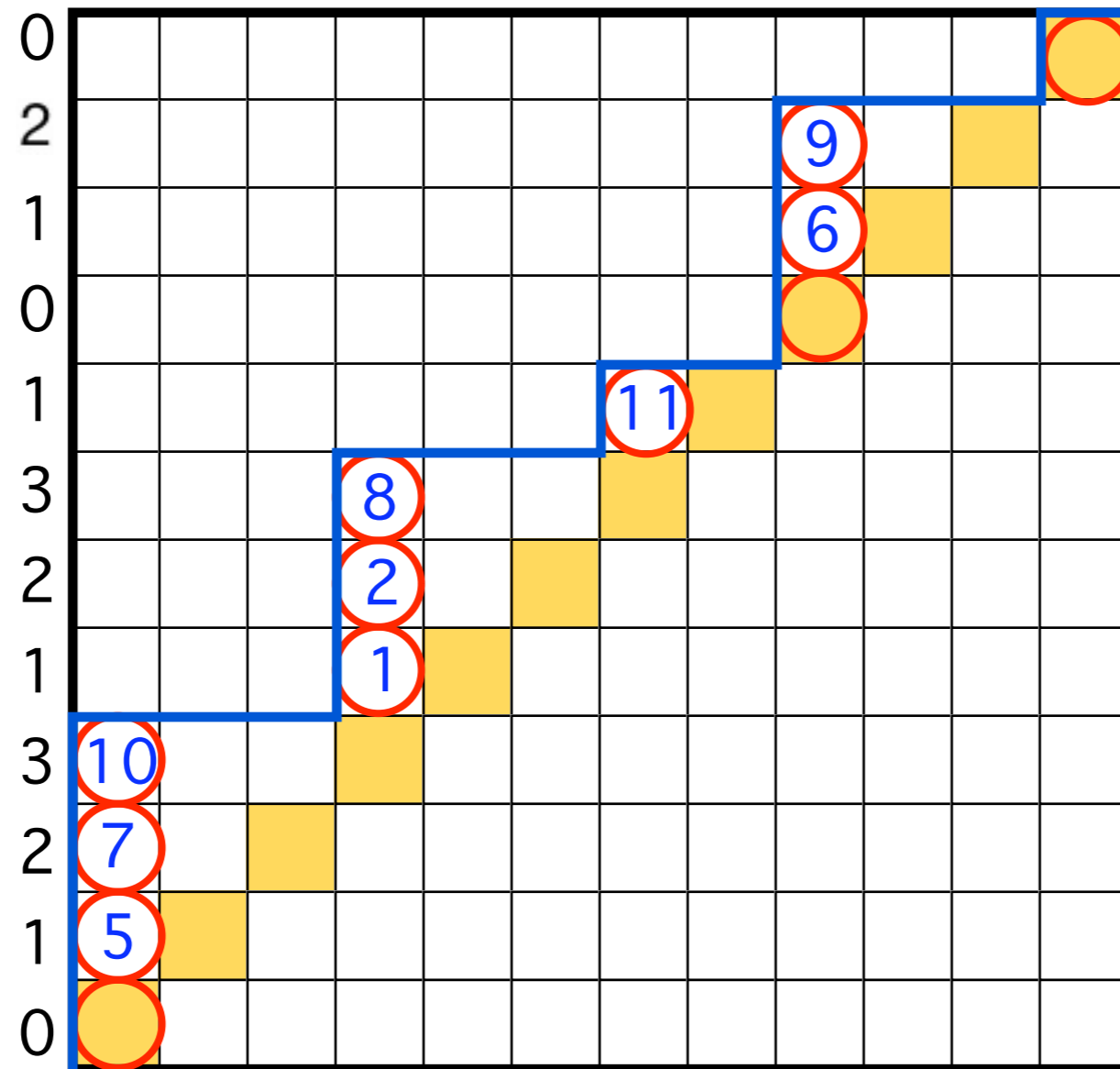
# A Parking Function



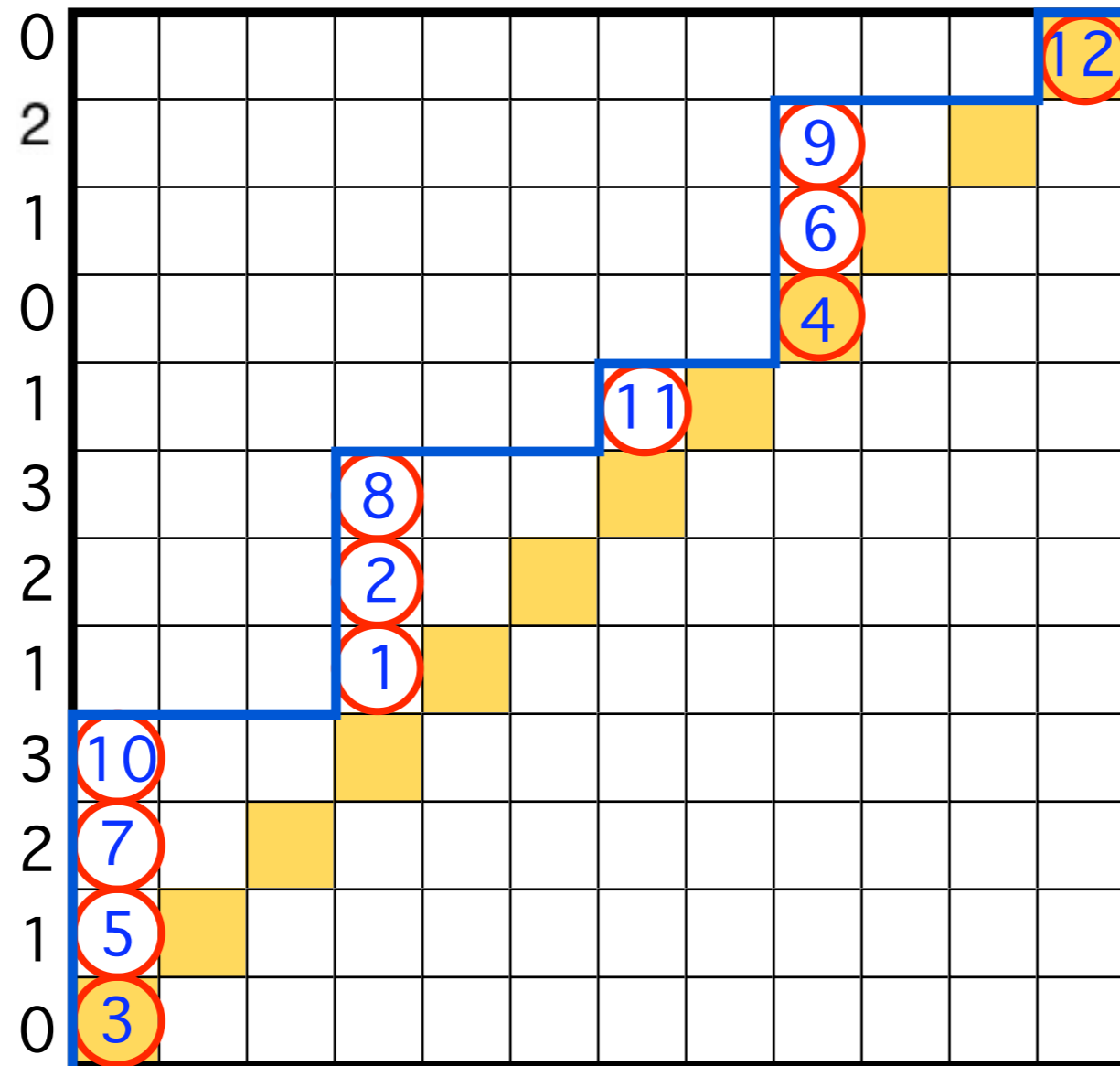
# A Parking Function



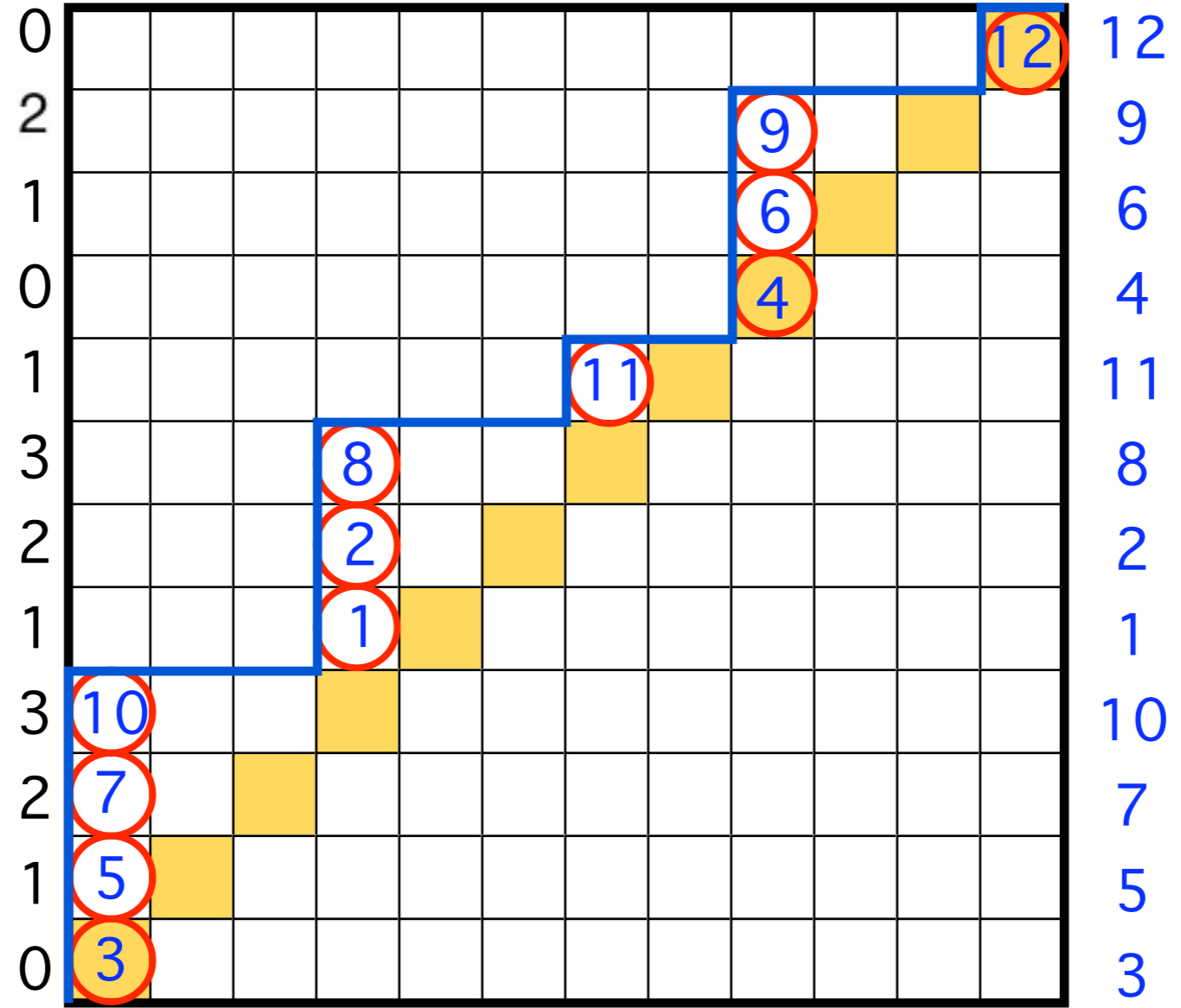
# A Parking Function



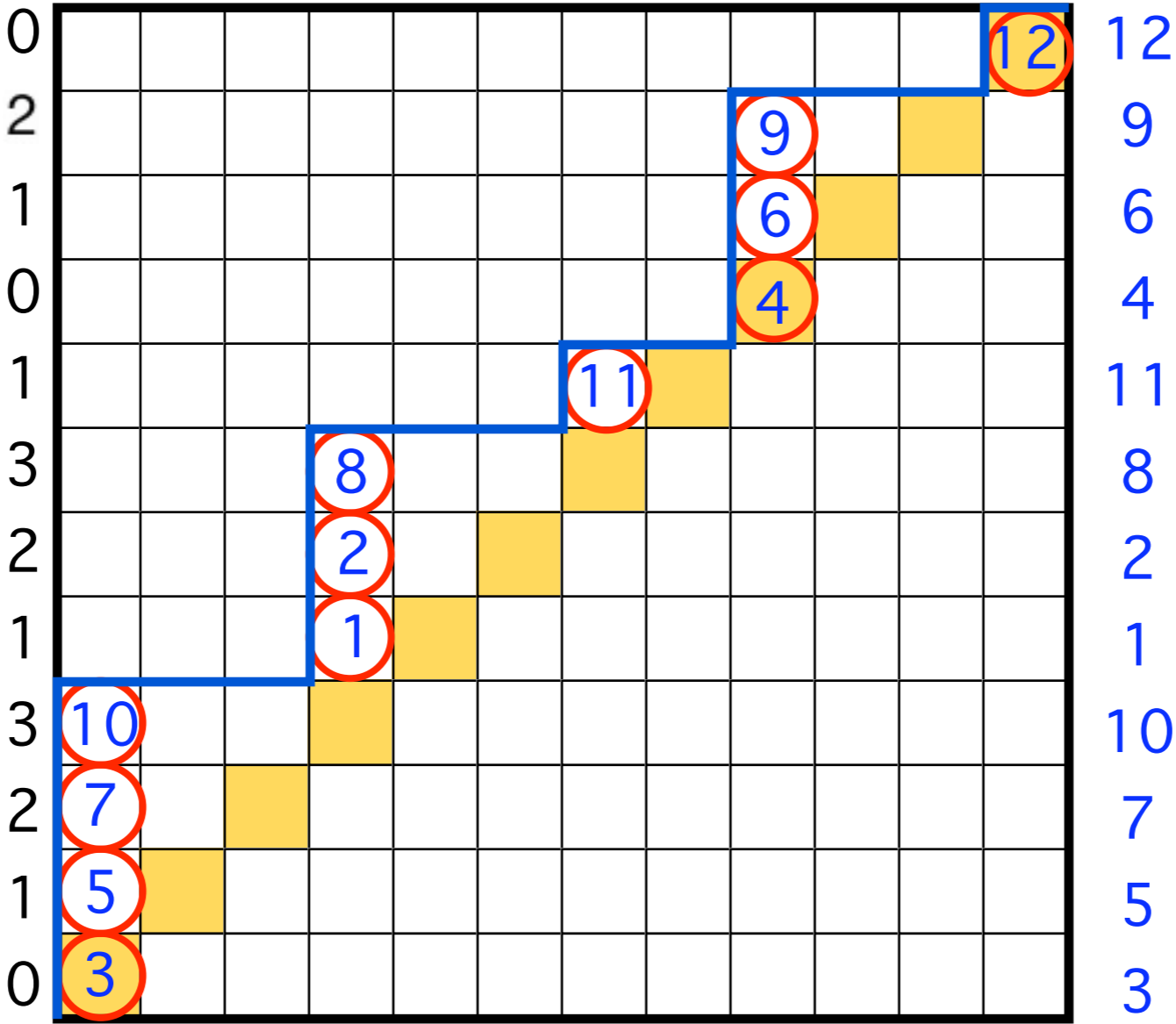
# A Parking Function



# A Parking Function

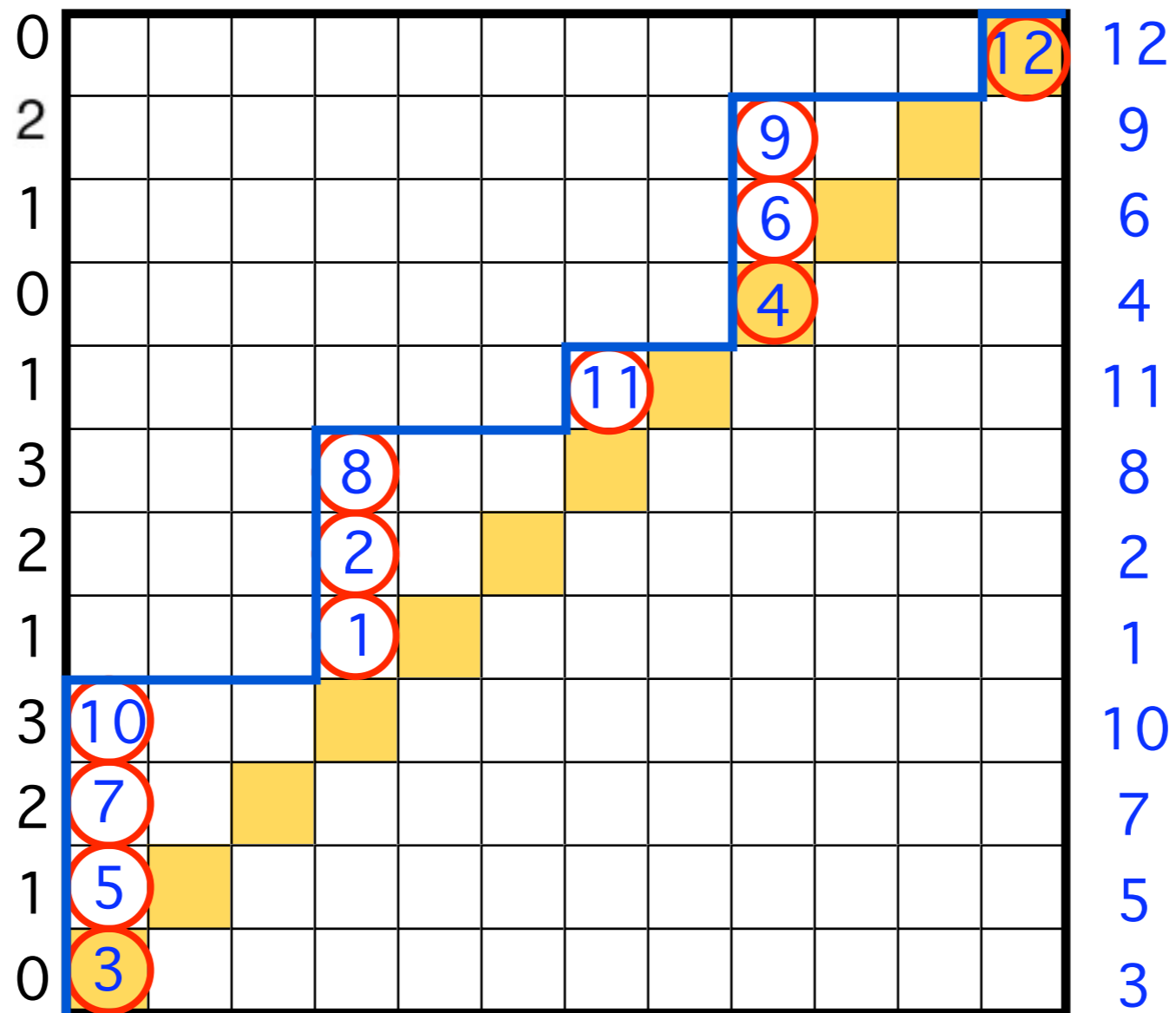


# A Parking Function



3	5	7	10	1	2	8	11	4	6	9	12
0	1	2	3	1	2	3	1	0	1	2	0

# A Parking Function



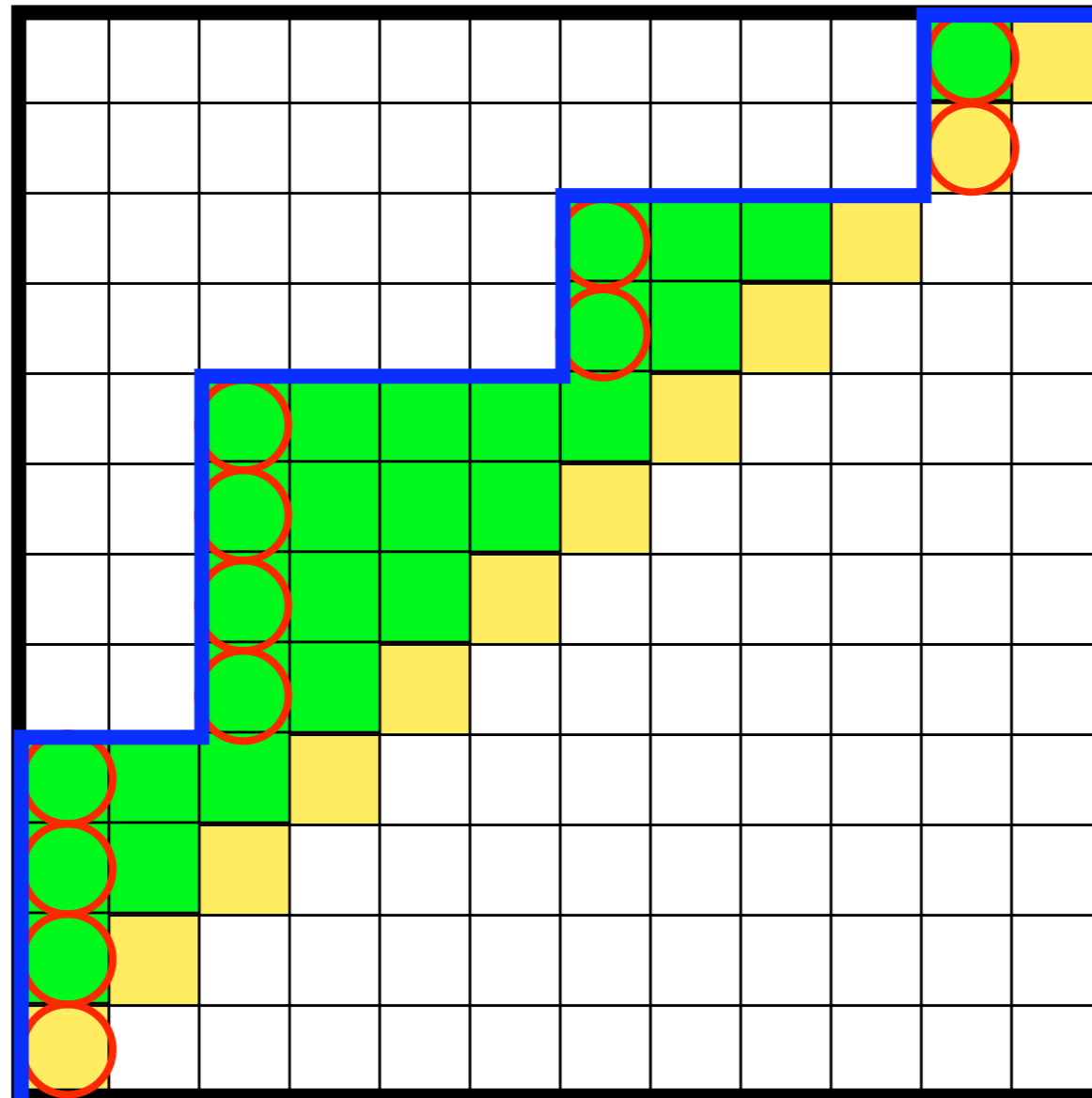
3	5	7	10	1	2	8	11	4	6	9	12
0	1	2	3	1	2	3	1	0	1	2	0



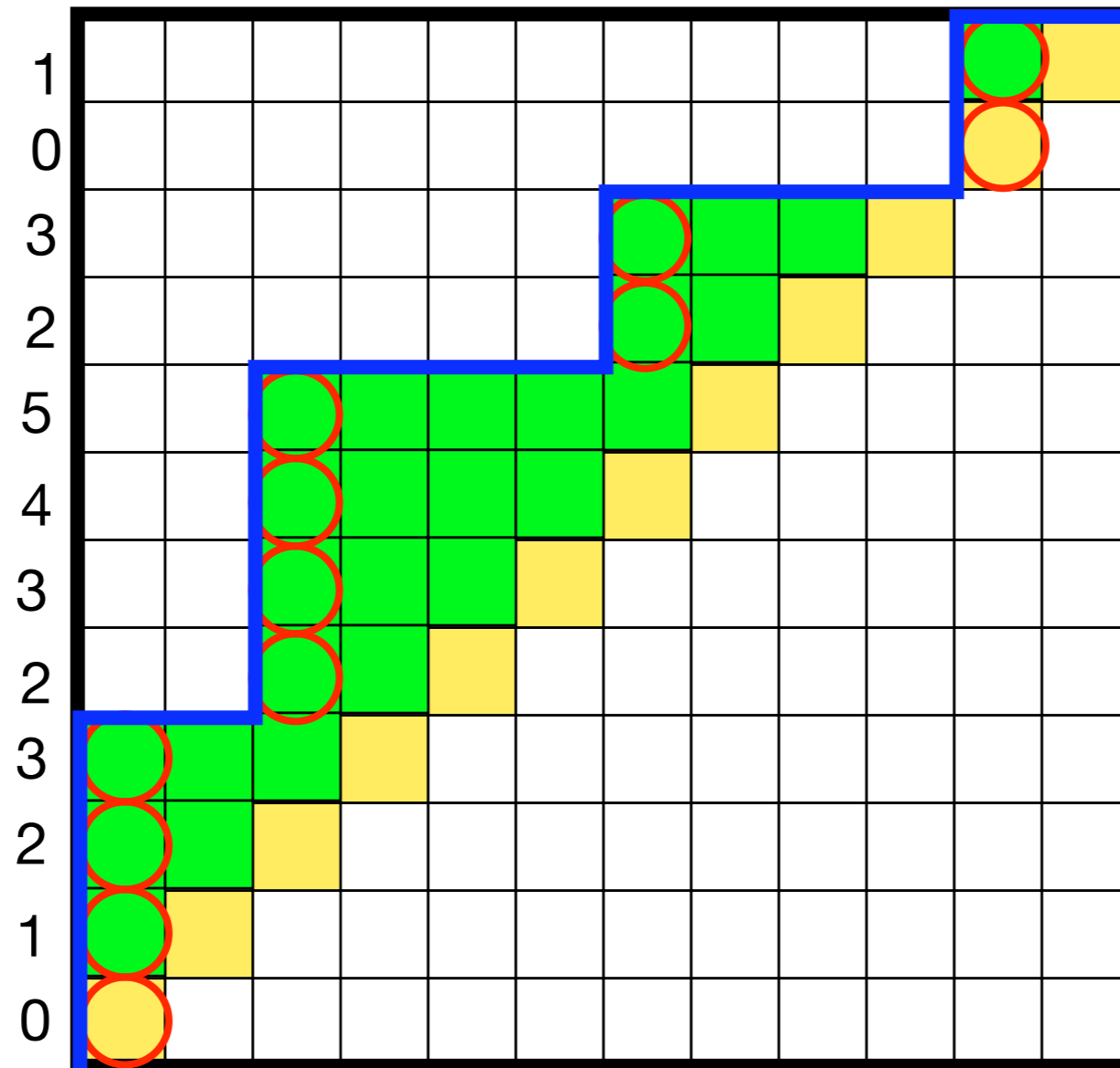


# The Area of a Parking Function

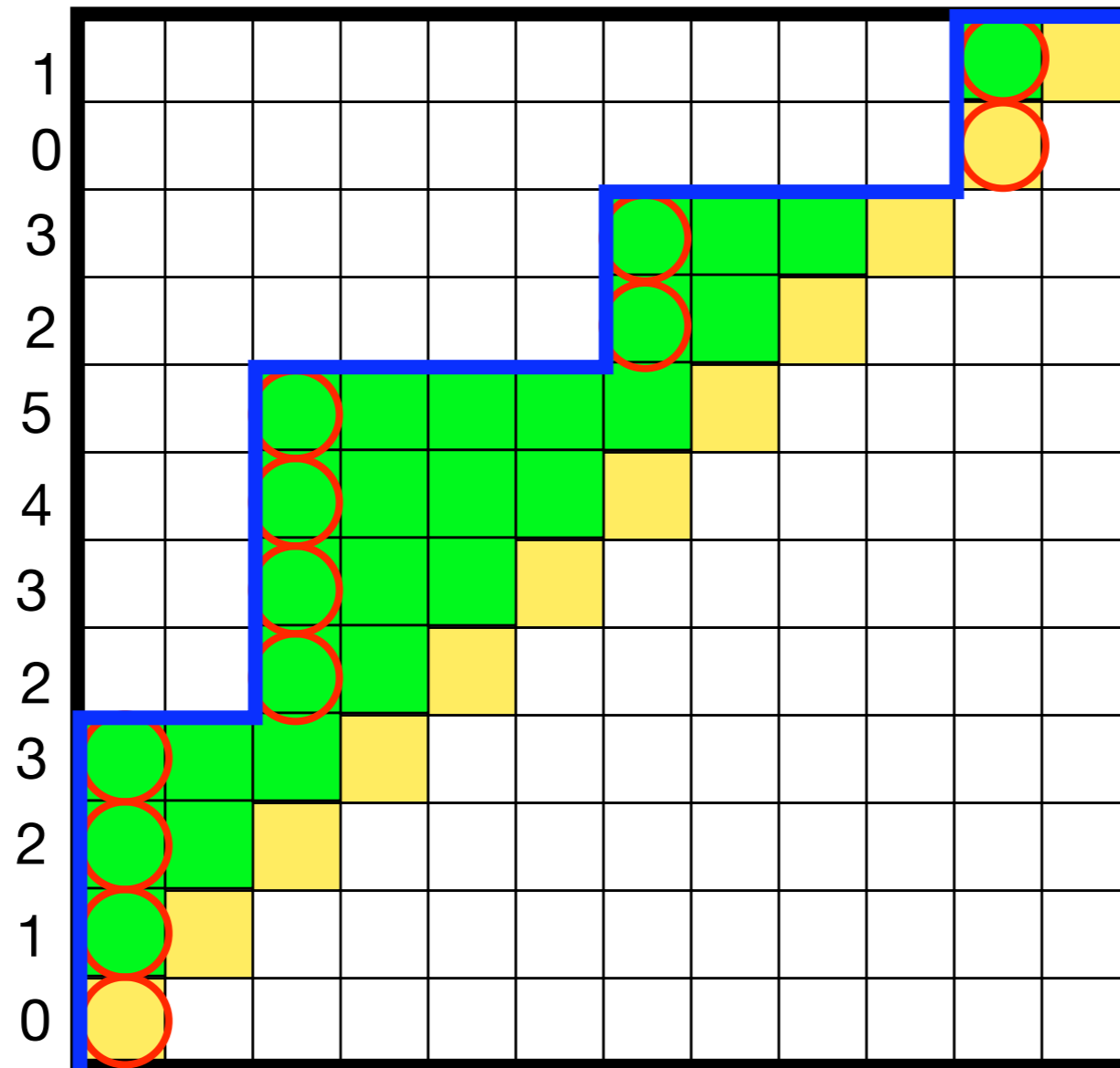
# The Area of a Parking Function



# The Area of a Parking Function

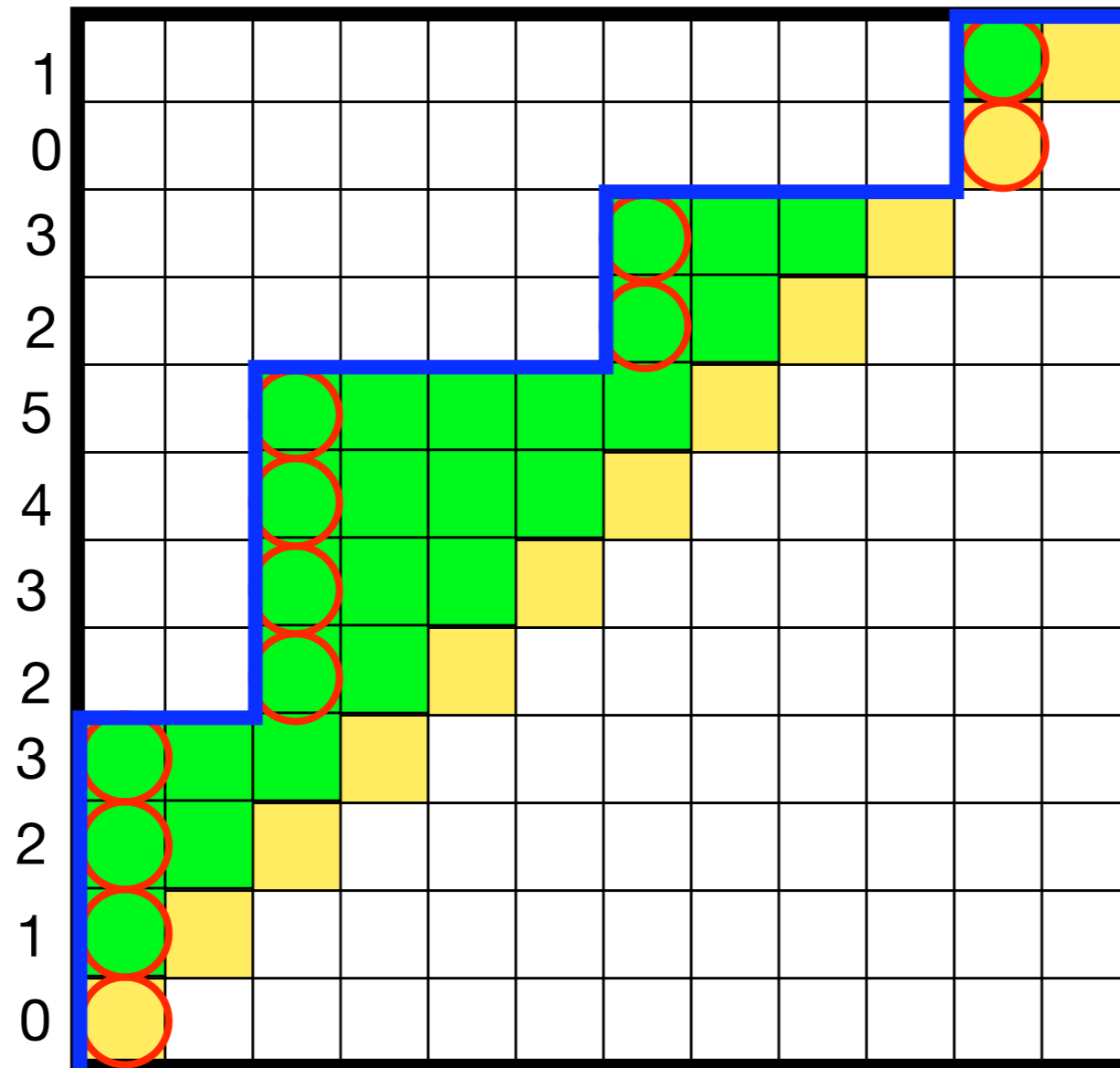


# The Area of a Parking Function



$$U = (u_1, u_2, \dots, u_n)$$

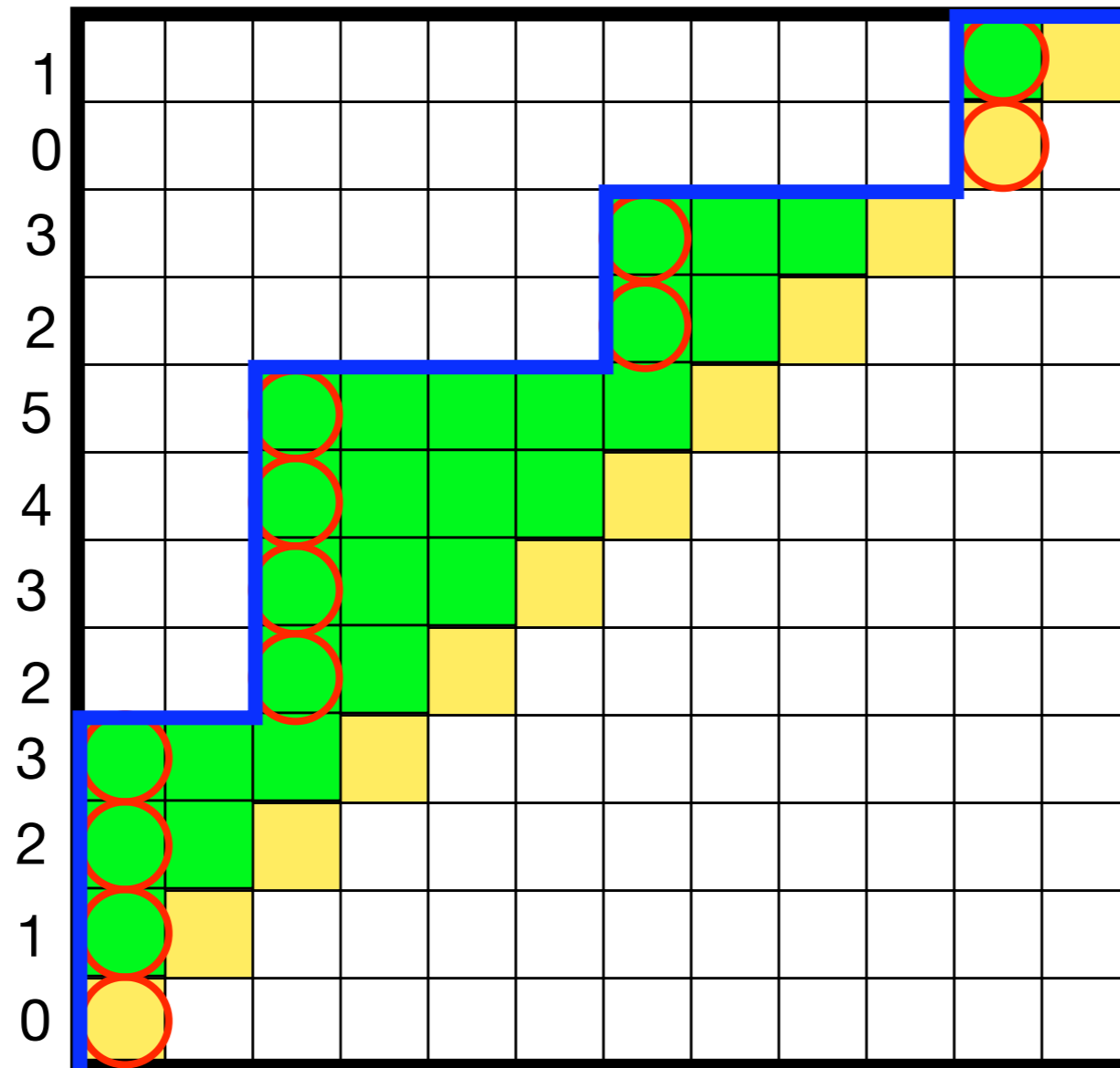
# The Area of a Parking Function



$$U = (u_1, u_2, \dots, u_n)$$

$$u_1 = 0$$

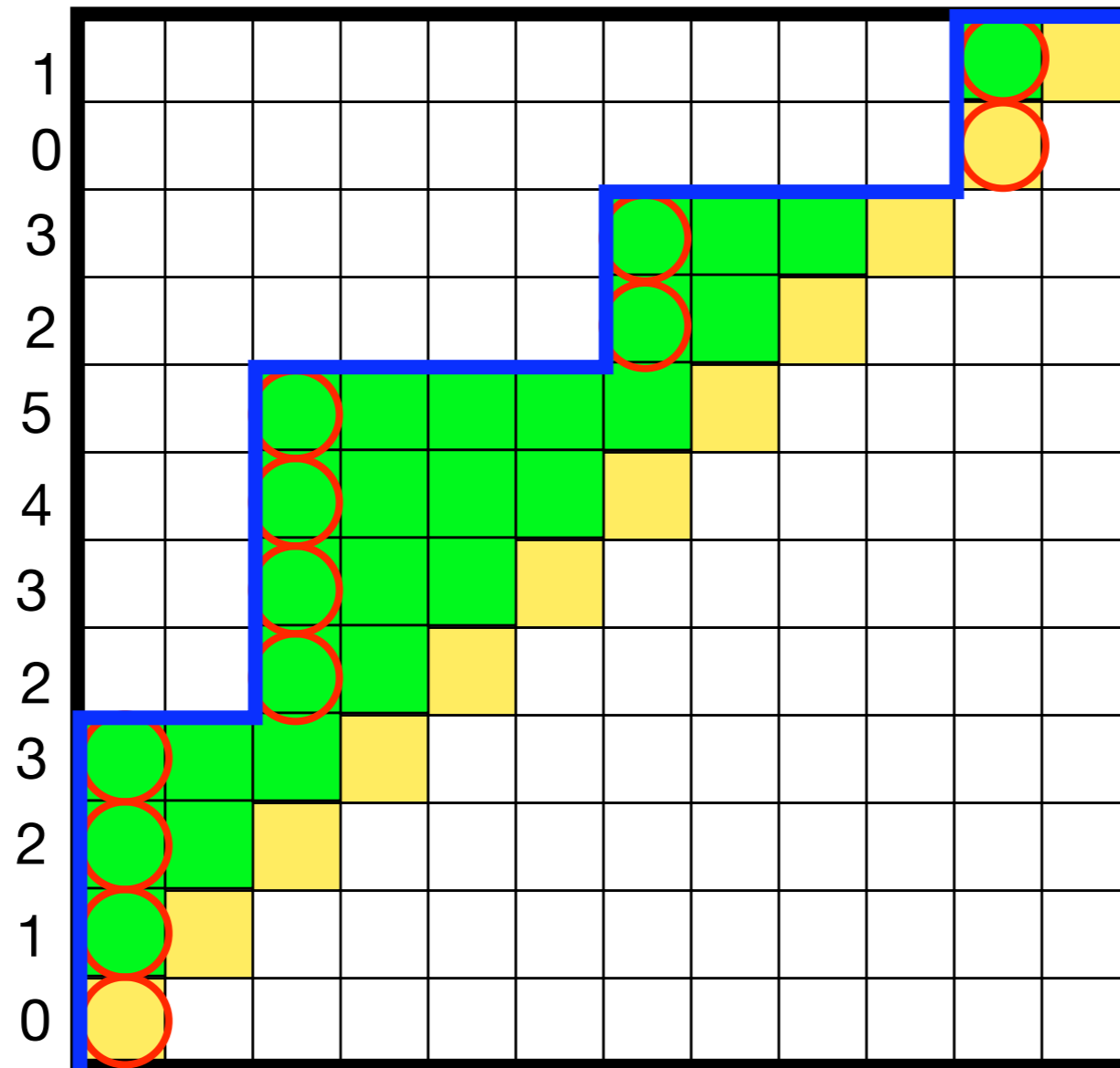
# The Area of a Parking Function



$$U = (u_1, u_2, \dots, u_n)$$

$$u_1 = 0 \qquad u_i \leq u_{i-1} + 1$$

# The Area of a Parking Function



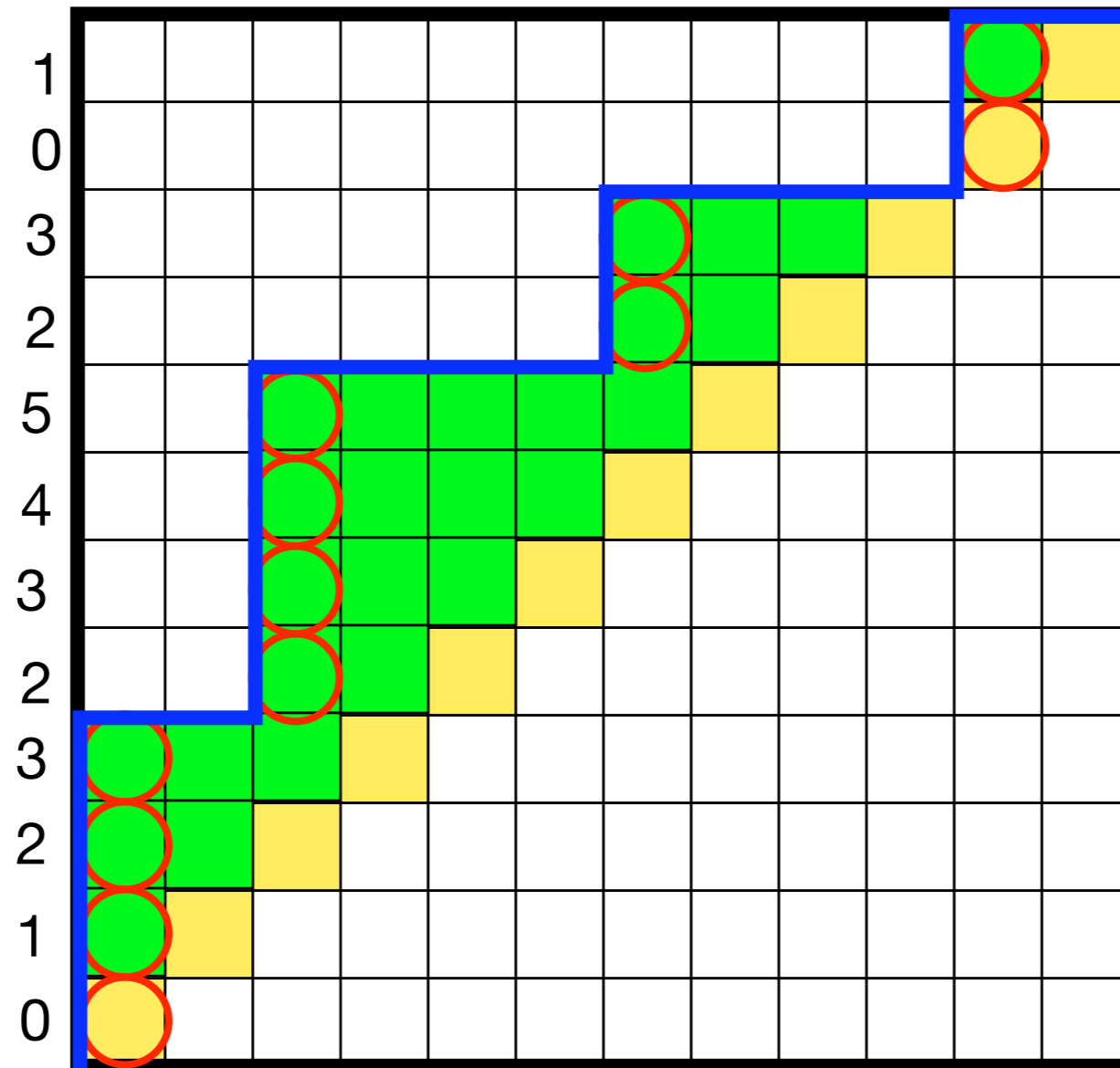
$$U = (u_1, u_2, \dots, u_n)$$

$$u_1 = 0 \quad u_i \leq u_{i-1} + 1$$

$$area(D) = u_1 + u_2 + \dots + u_n$$



# The Area of a Parking Function



$$U = (u_1, u_2, \dots, u_n)$$

$$u_1 = 0 \qquad u_i \leq u_{i-1} + 1$$

$$area(D) = u_1 + u_2 + \dots + u_n$$



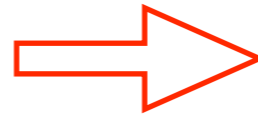
# Preference Functions

# Preference Functions

## Preference Function

# Preference Functions

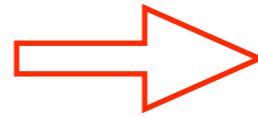
Preference Function



# Preference Functions

Preference Function

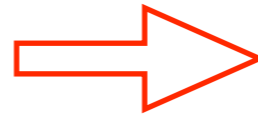
Tableau



# Preference Functions

Preference Function

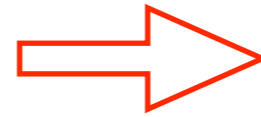
Tableau




# Preference Functions

Preference Function

Tableau



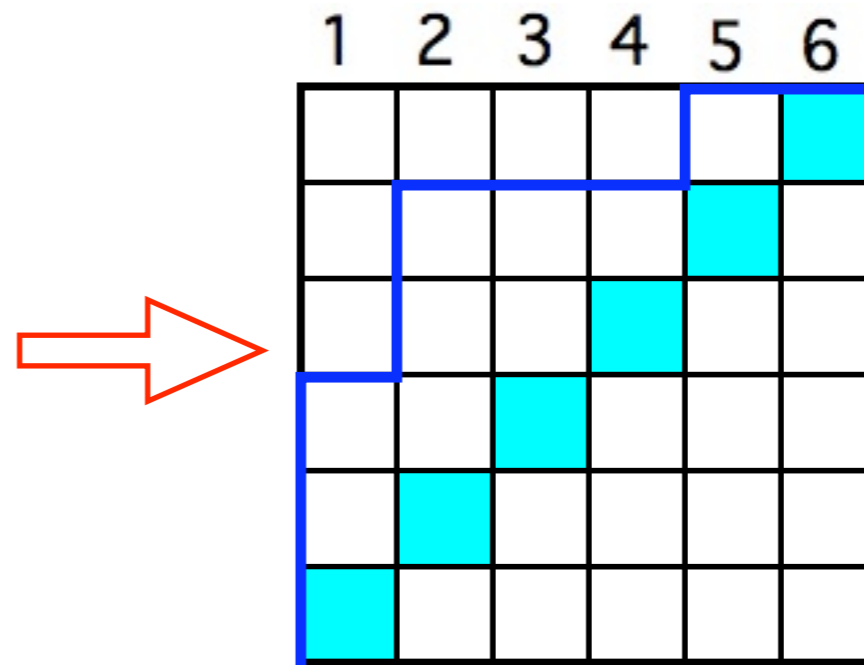

A 6x6 grid representing a Tableau. The grid is mostly empty, but contains several cyan-colored cells and a blue path. The cyan cells are located at (row, column) coordinates: (1,6), (2,5), (3,4), (4,3), (5,2), and (6,1). A blue path is drawn along the left and top edges of the grid, starting from the bottom-left corner and ending at the top-right corner, following the perimeter of the cyan cells.



# Preference Functions

Preference Function

Tableau



# Preference Functions

Preference Function

1  
5

Tableau

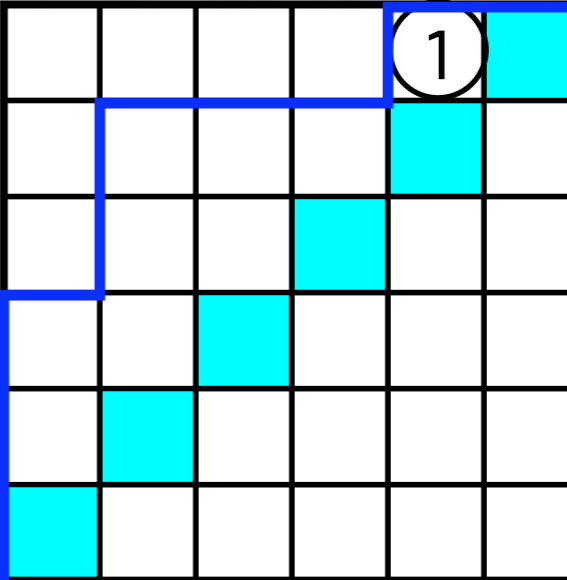
	1	2	3	4	5	6

# Preference Functions

Preference Function

1  
5

Tableau




	1	2	3	4	5	6
1					1	
2						
3						
4						
5						
6						

# Preference Functions

Preference Function

1 2  
5 2

Tableau




	1	2	3	4	5	6
					1	
1						
2						
5						

# Preference Functions

Preference Function

1 2  
5 2

Tableau




	1	2	3	4	5	6
1					1	
2						
3		2				
4						
5						
6						

# Preference Functions

Preference Function

1 2 3  
5 2 1

Tableau




	1	2	3	4	5	6
1					1	
2						
3		2				
4						
5						
6						

# Preference Functions

Preference Function

1 2 3  
5 2 1

Tableau

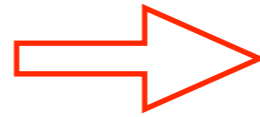


	1	2	3	4	5	6
1					1	
2						
3		2				
4						
5						
6	3					

# Preference Functions

Preference Function

1	2	3	4
5	2	1	1



Tableau

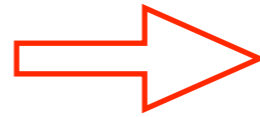
	1	2	3	4	5	6
1					1	
2						
3		2				
4						
5						
6	3					



# Preference Functions

Preference Function

1	2	3	4
5	2	1	1



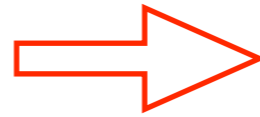
Tableau

	1	2	3	4	5	6
1					1	
2						
3		2				
4						
5	4					
6	3					

# Preference Functions

Preference Function

1	2	3	4	5
5	2	1	1	2



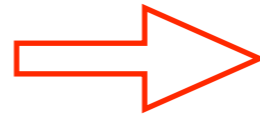
Tableau

	1	2	3	4	5	6
1					1	
2						
3		2				
4						
5	4					
6	3					

# Preference Functions

Preference Function

1	2	3	4	5
5	2	1	1	2



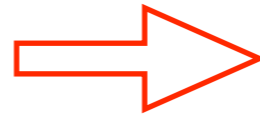
Tableau

	1	2	3	4	5	6
					1	
		5				
		2				
	4					
	3					

# Preference Functions

Preference Function

1	2	3	4	5	6
5	2	1	1	2	1



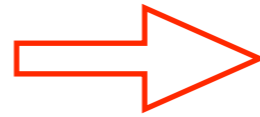
Tableau

	1	2	3	4	5	6
					1	
	5					
	2					
4						
3						

# Preference Functions

Preference Function

1	2	3	4	5	6
5	2	1	1	2	1



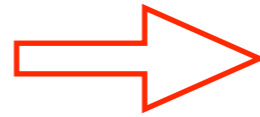
Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

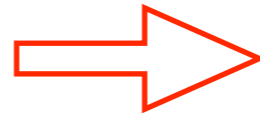
	1	2	3	4	5	6
					1	
		5				
		2				
	6					
	4					
	3					

parking and filling permutations

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

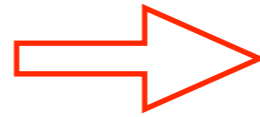
parking →

parking and filling permutations

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

parking →

1

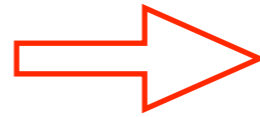
parking and filling permutations



# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

parking →

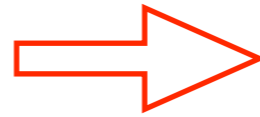
2 1

parking and filling permutations

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

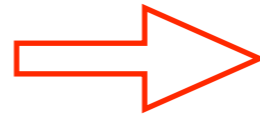
parking → 3 2 1

parking and filling permutations

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

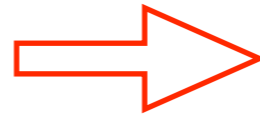
parking → 3 2 4 1

parking and filling permutations

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

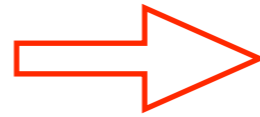
parking → 3 2 4 5 1

parking and filling permutations

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
					1	
		5				
		2				
	6					
	4					
	3					
	3	2	4	5	1	6

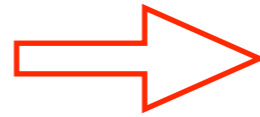
parking →

parking and filling permutations

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

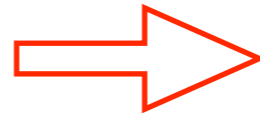
parking  $\rightarrow$  3 2 4 5 1 6 =  $\beta$

parking and filling permutations

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

parking  $\rightarrow$  3 2 4 5 1 6 =  $\beta$

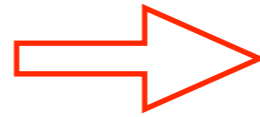
parking and filling permutations

filling  $\rightarrow$

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

parking  $\longrightarrow$  3 2 4 5 1 6 =  $\beta$

parking and filling permutations

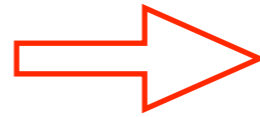
filling  $\longrightarrow$  5



# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

parking  $\rightarrow$  3 2 4 5 1 6 =  $\beta$

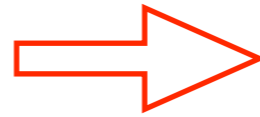
parking and filling permutations

filling  $\rightarrow$  5 2

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

parking  $\longrightarrow$  3 2 4 5 1 6 =  $\beta$

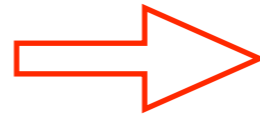
parking and filling permutations

filling  $\longrightarrow$  5 2 1

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

parking  $\longrightarrow$  3 2 4 5 1 6 =  $\beta$

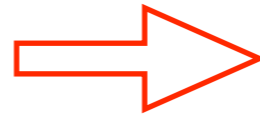
parking and filling permutations

filling  $\longrightarrow$  5 2 1 3

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

parking  $\longrightarrow$  3 2 4 5 1 6 =  $\beta$

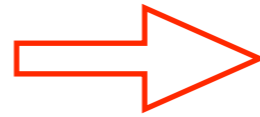
parking and filling permutations

filling  $\longrightarrow$  5 2 1 3 4

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

parking  $\longrightarrow$  3 2 4 5 1 6 =  $\beta$

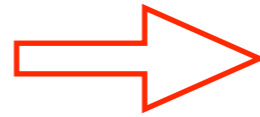
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6	3					

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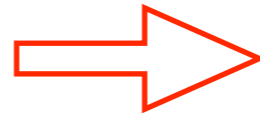
parking and filling permutations

filling  $\rightarrow$  5 2 1 3 4 6 =  $\alpha$

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1



Tableau

	1	2	3	4	5	6
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2		5				
3		2				
4	6					
5	4					
6	3					

parking  $\rightarrow$  3 2 4 5 1 6 =  $\beta$

parking and filling permutations

filling  $\rightarrow$  5 2 1 3 4 6 =  $\alpha$

$$\beta = \alpha^{-1}$$

# Preference Functions

Preference Function

1 2 3 4 5 6  
5 2 1 1 2 1

Tableau

	1	2	3	4	5	6
1					1	
2		5				
3		2				
4	6					
5	4					
6	3					

parking  $\longrightarrow$  3 2 4 5 1 6 =  $\beta$

parking and filling permutations

filling  $\longrightarrow$  5 2 1 3 4 6 =  $\alpha$

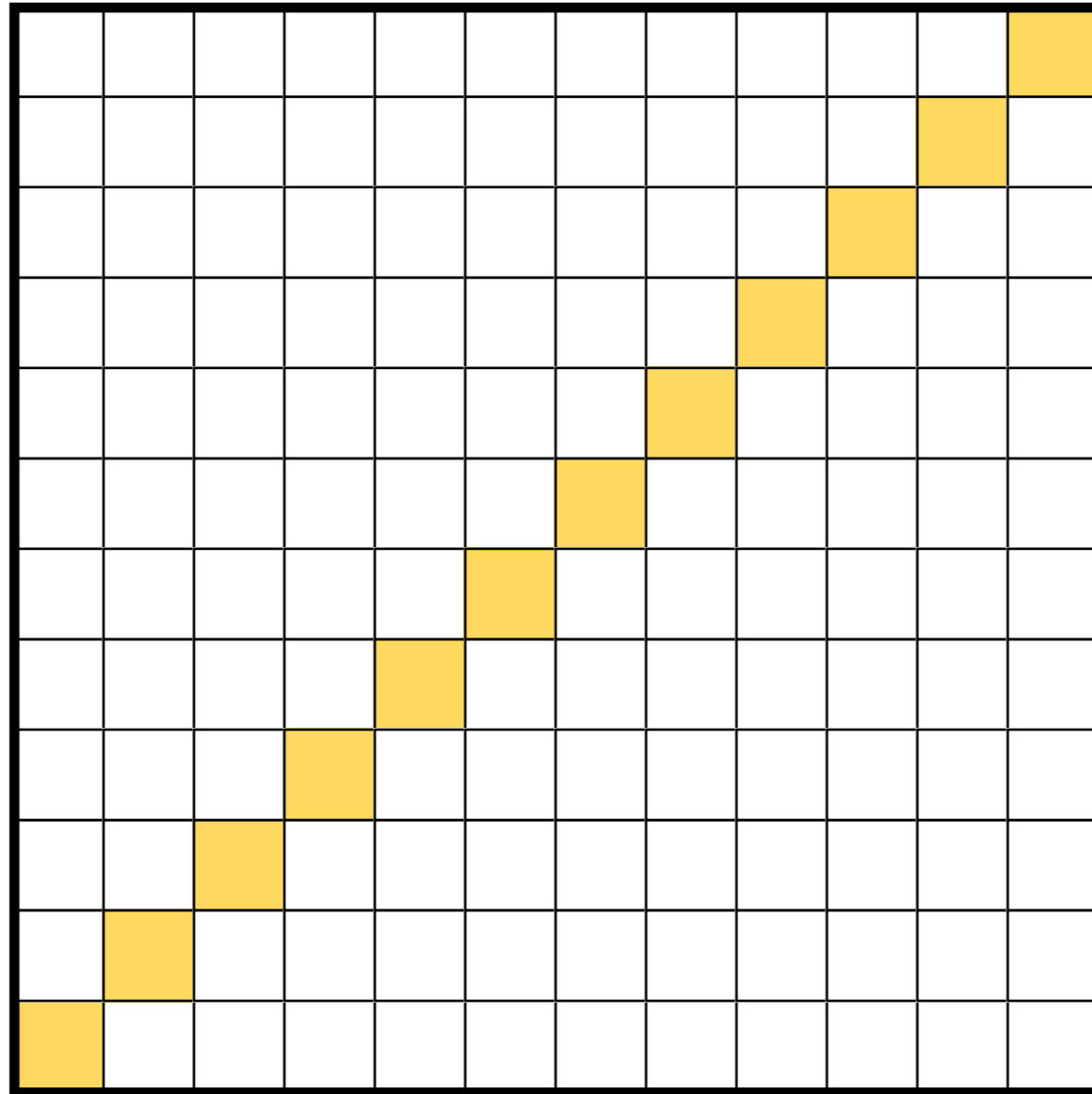
$$\beta = \alpha^{-1}$$



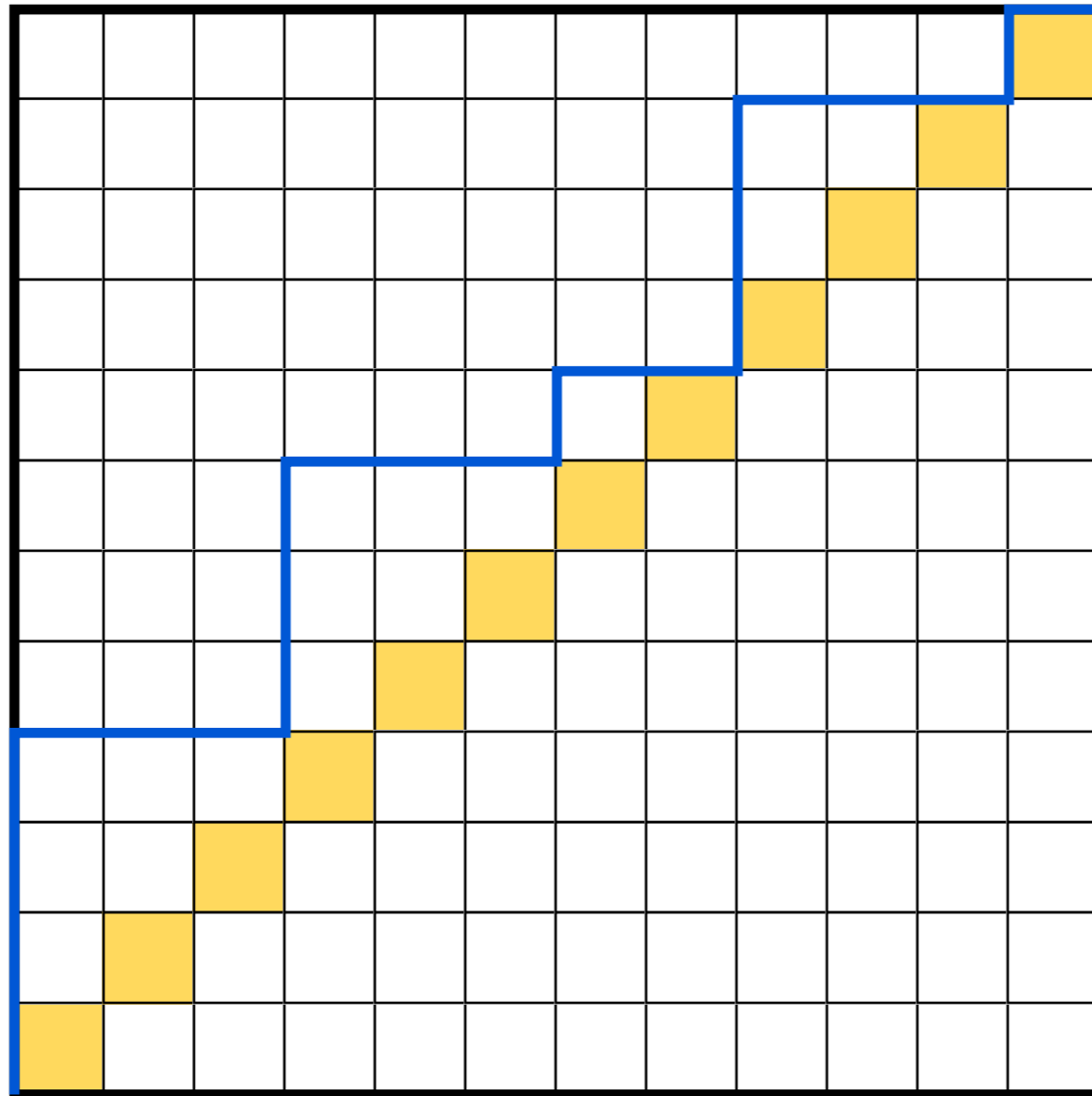


# The $d_{inv}$ statistic of a Parking Function

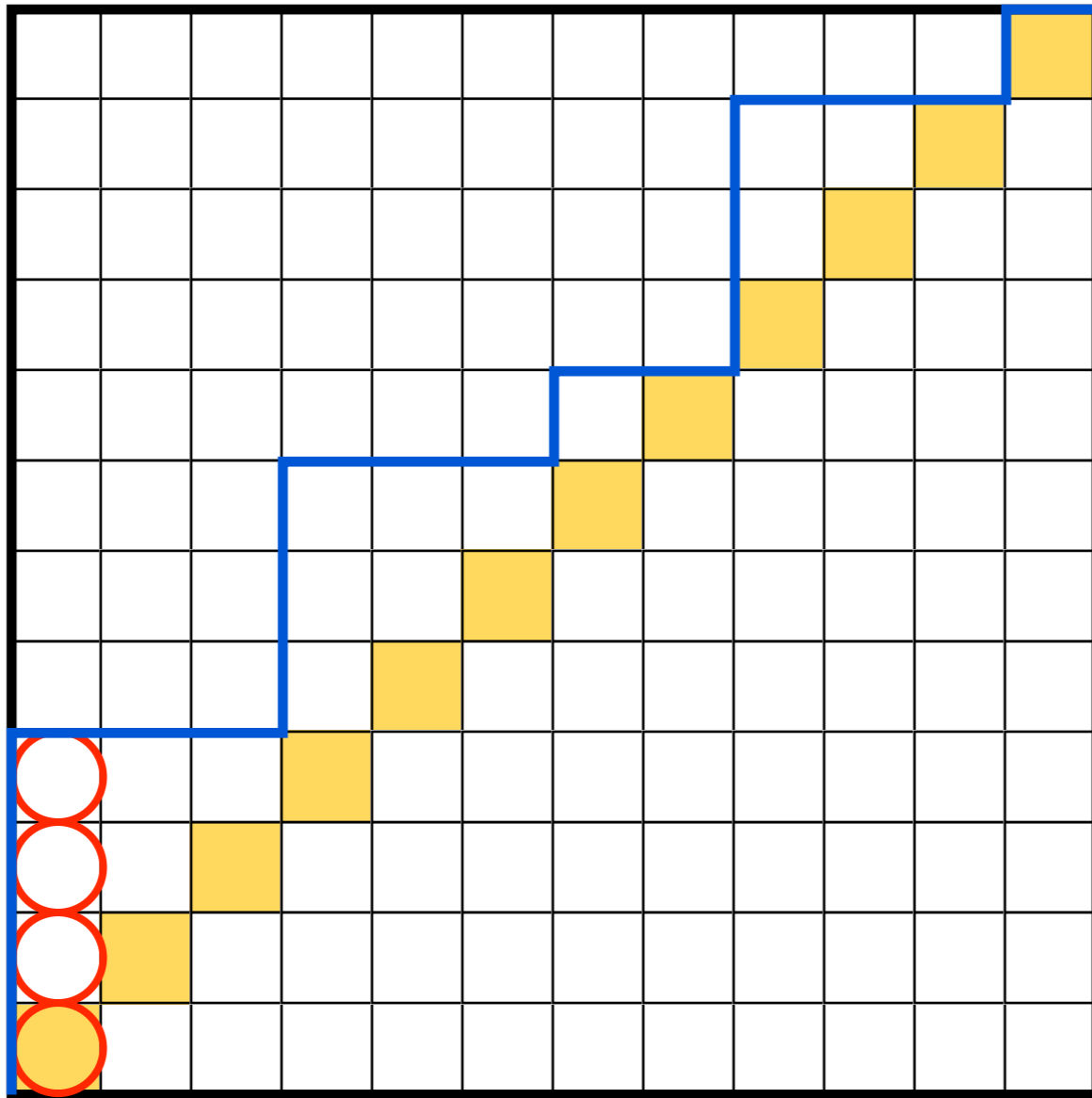
# The $d_{inv}$ statistic of a Parking Function



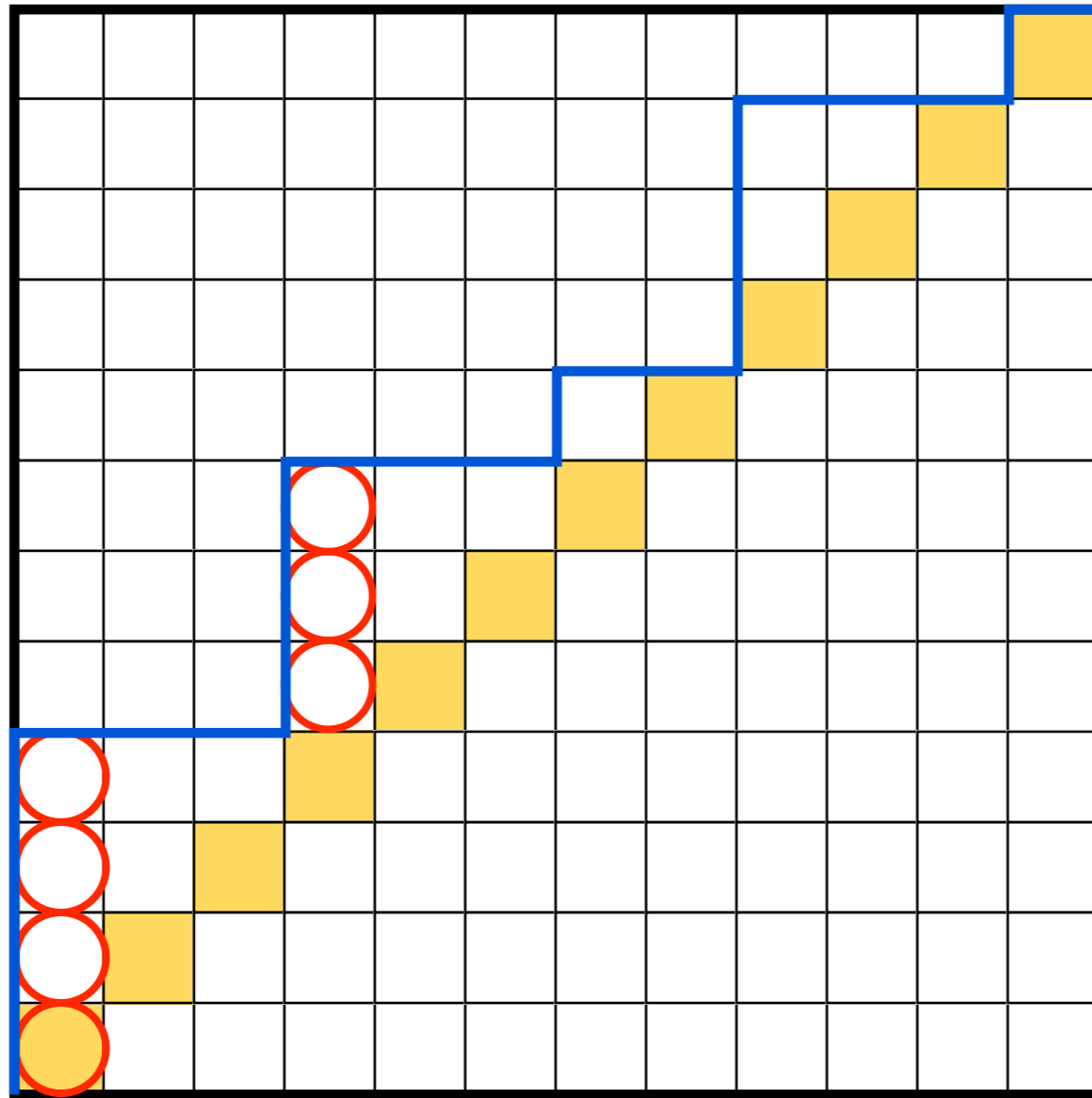
# The $d_{inv}$ statistic of a Parking Function



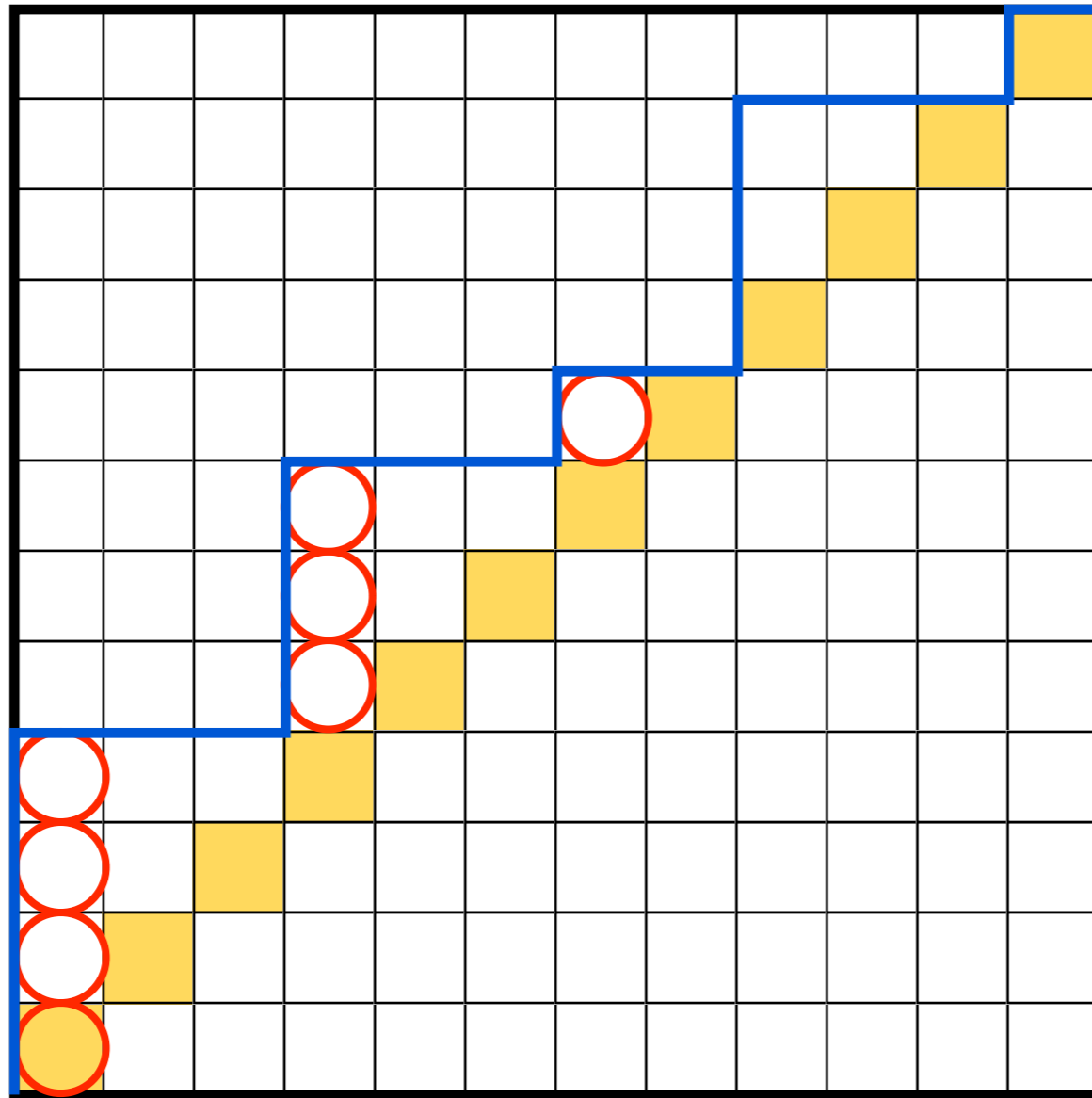
# The $d_{inv}$ statistic of a Parking Function



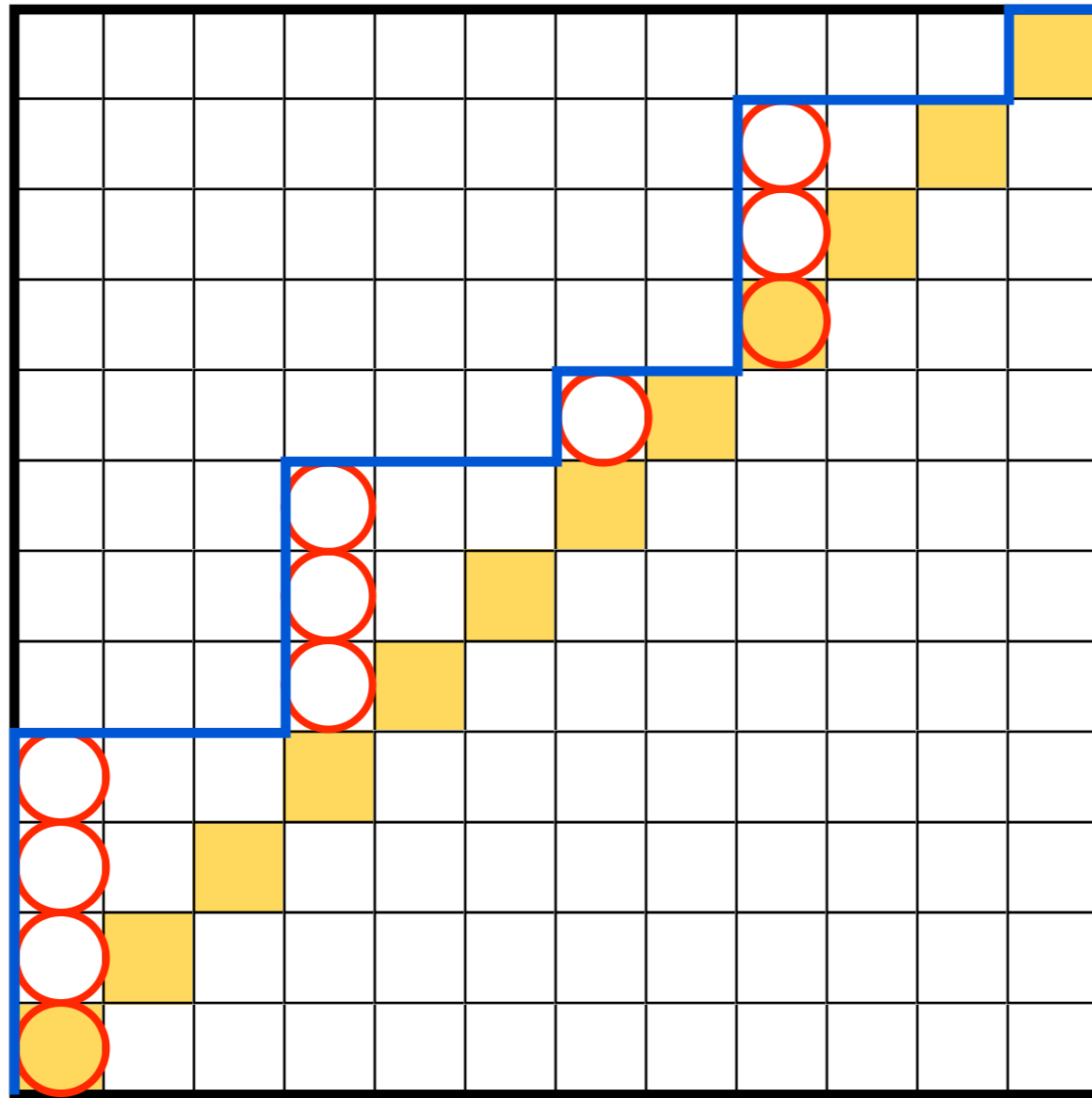
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# The $d_{inv}$ statistic of a Parking Function

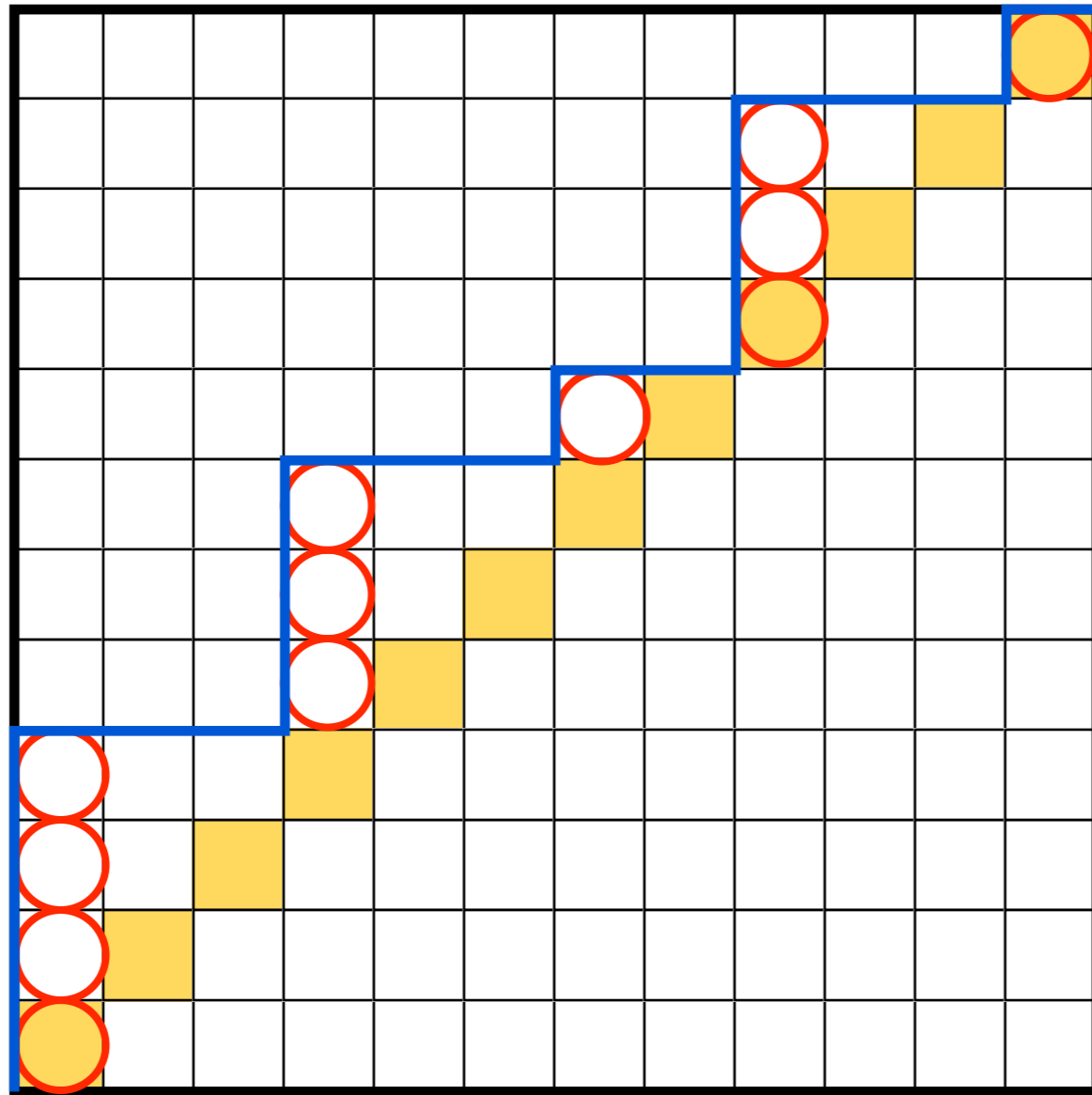


# The $d_{inv}$ statistic of a Parking Function

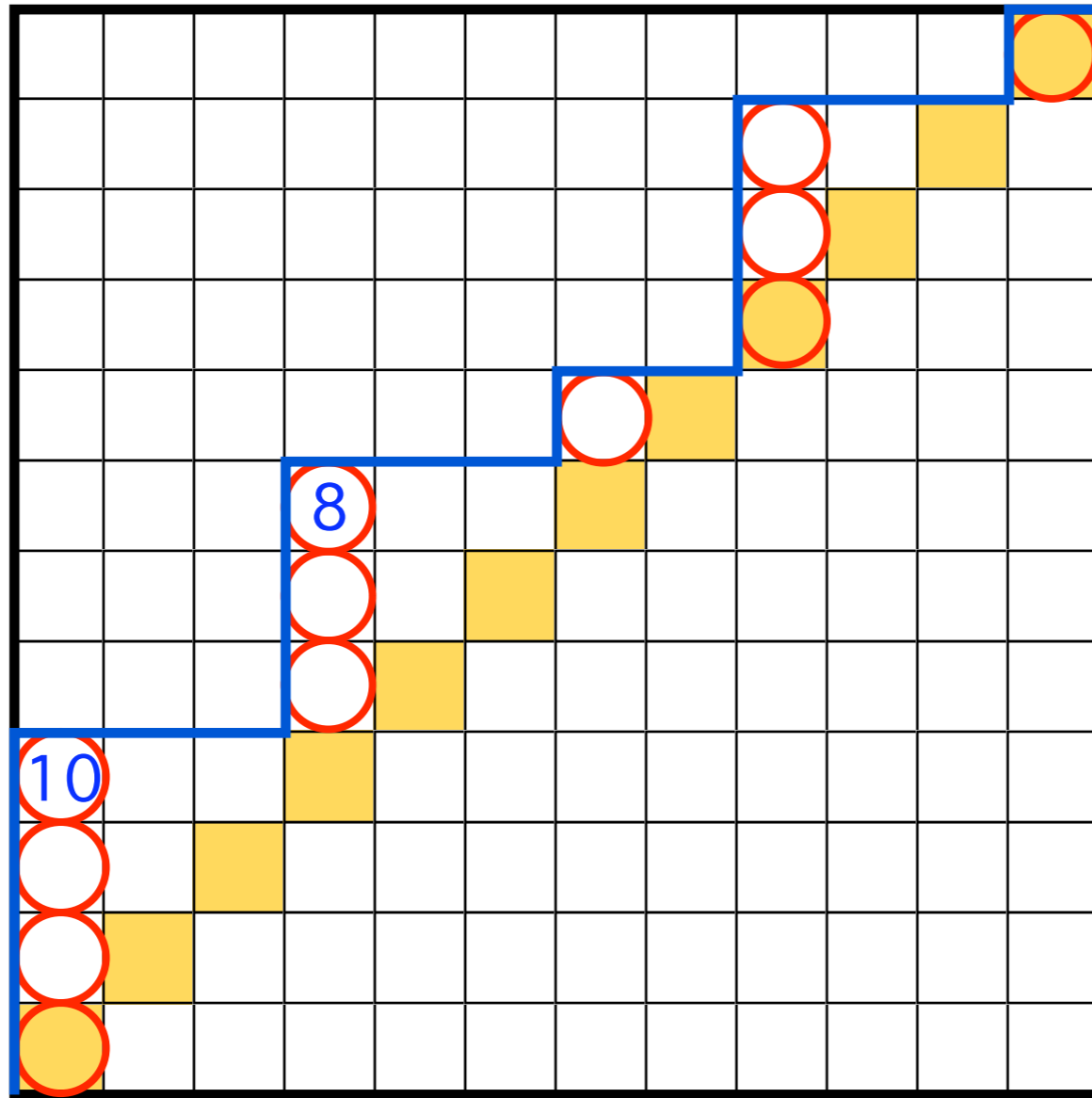




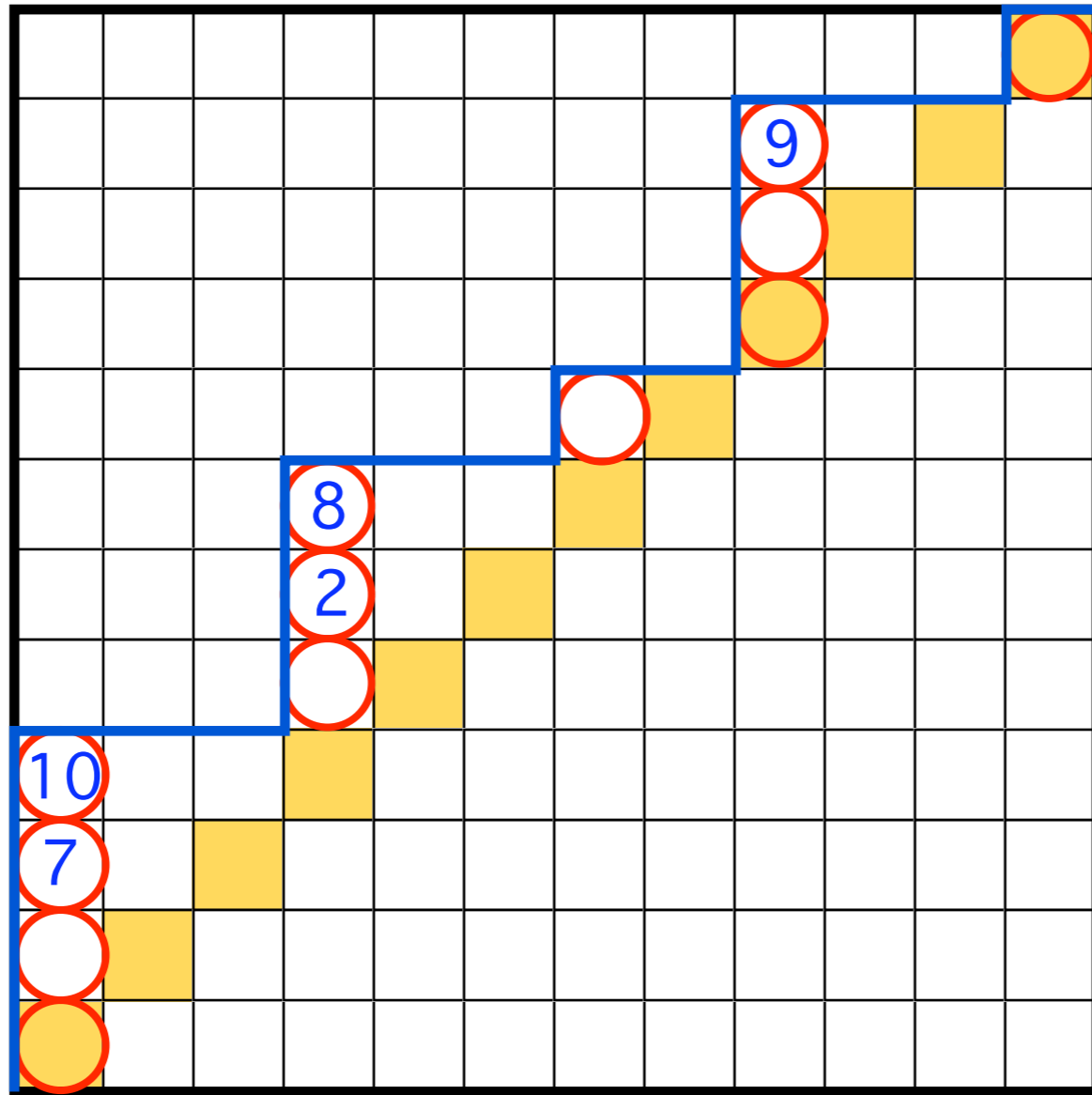
# The $d_{inv}$ statistic of a Parking Function



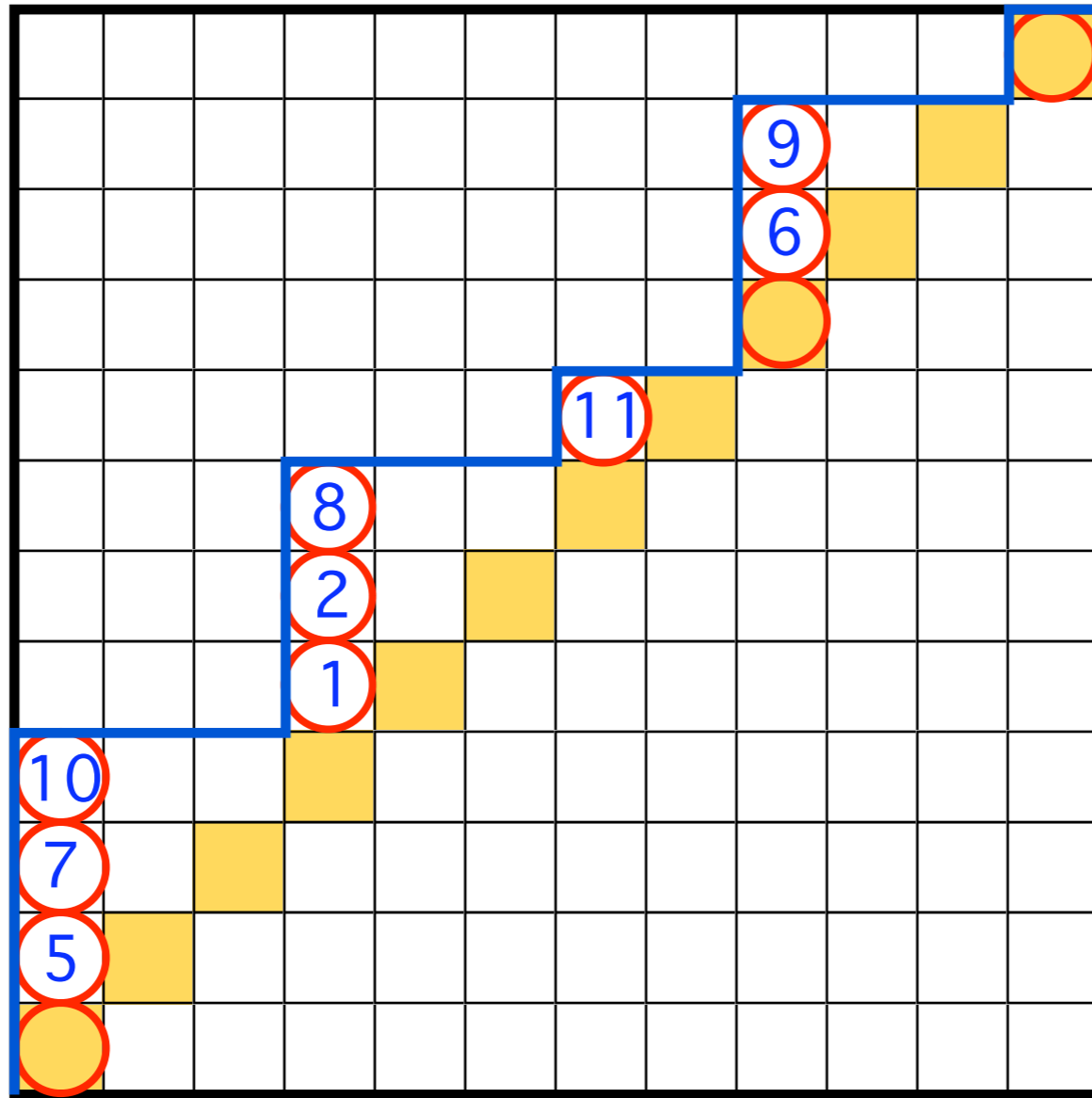
# The $d_{inv}$ statistic of a Parking Function



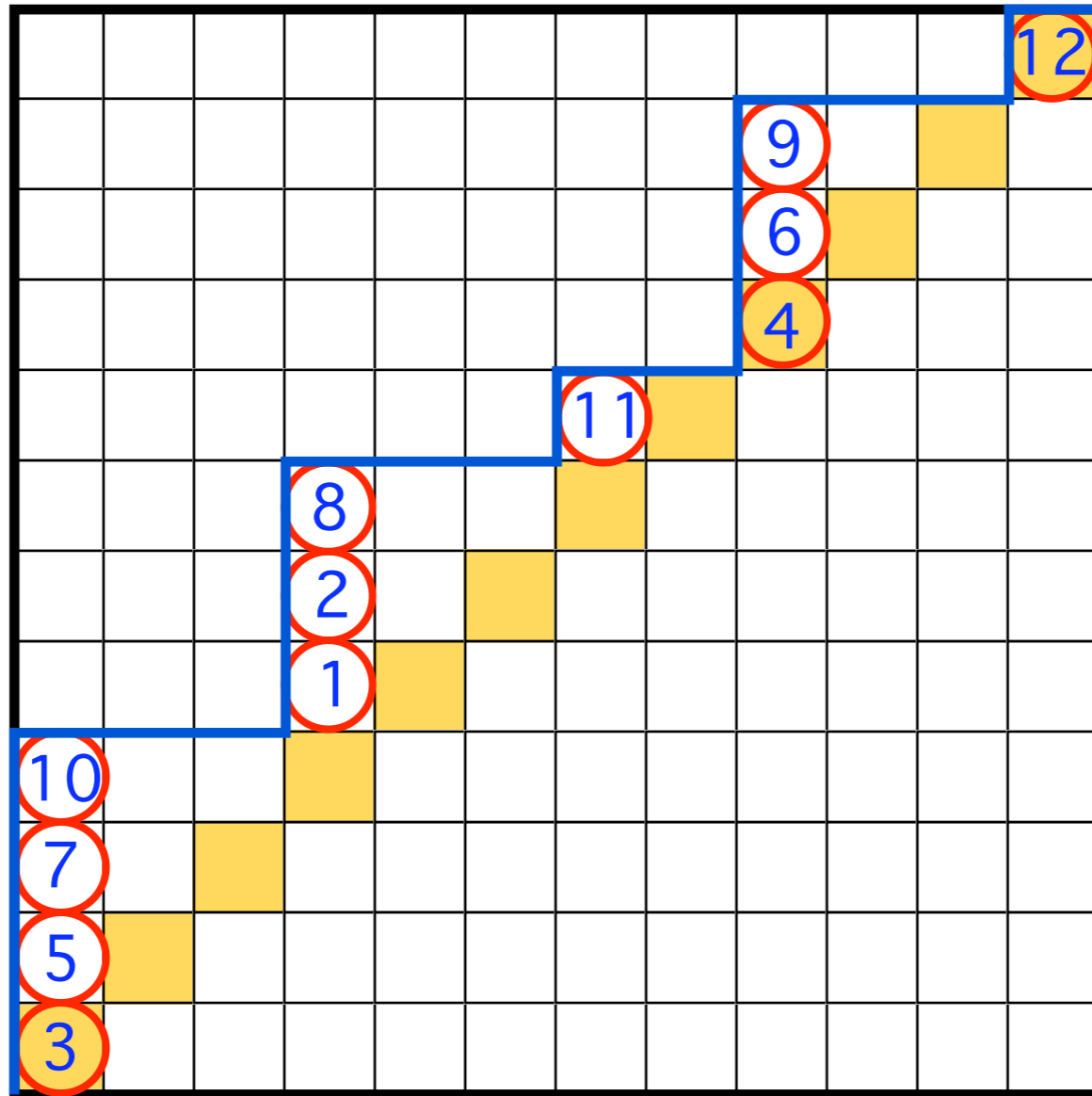
# The $d_{inv}$ statistic of a Parking Function



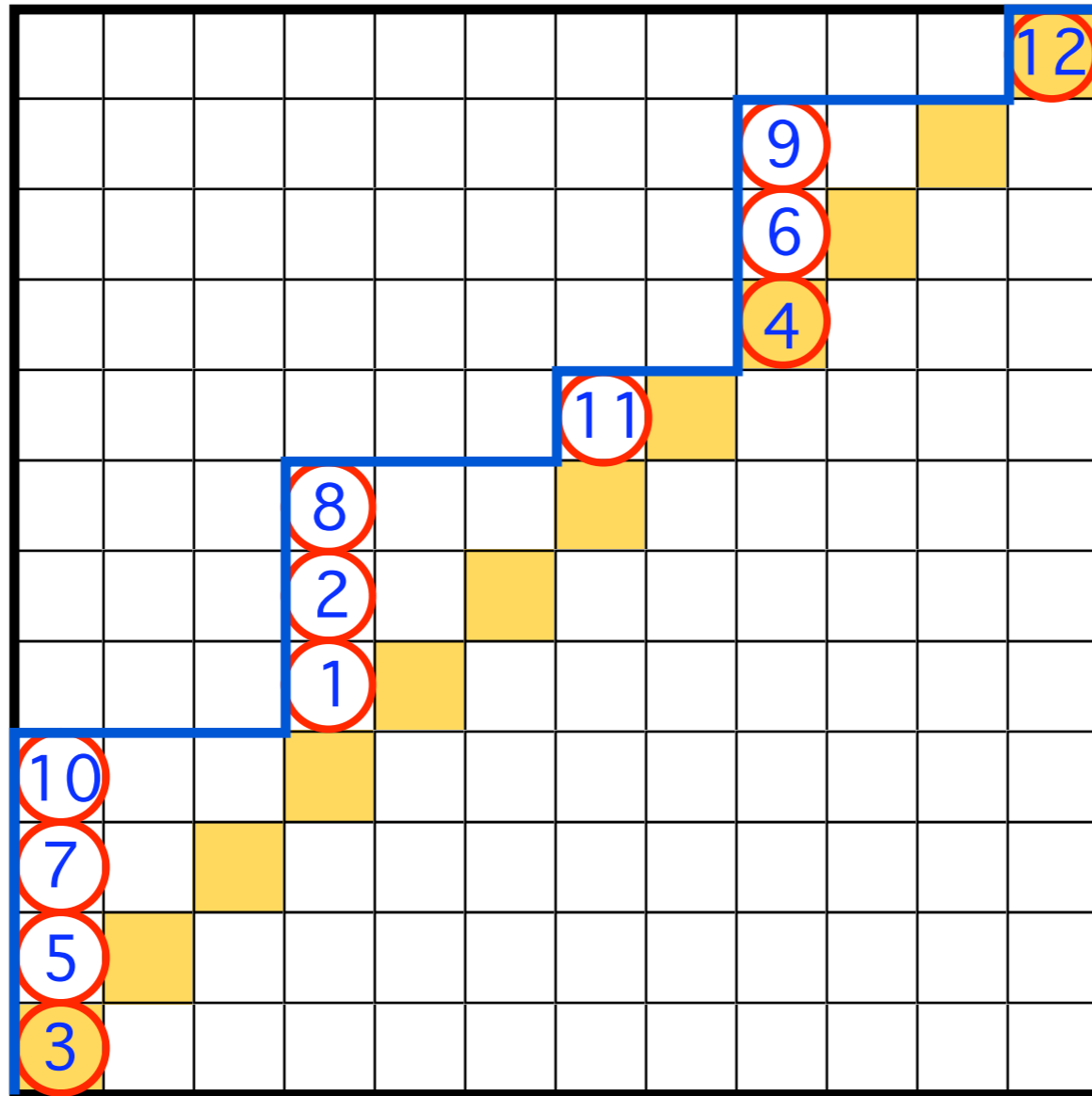
# The $d_{inv}$ statistic of a Parking Function



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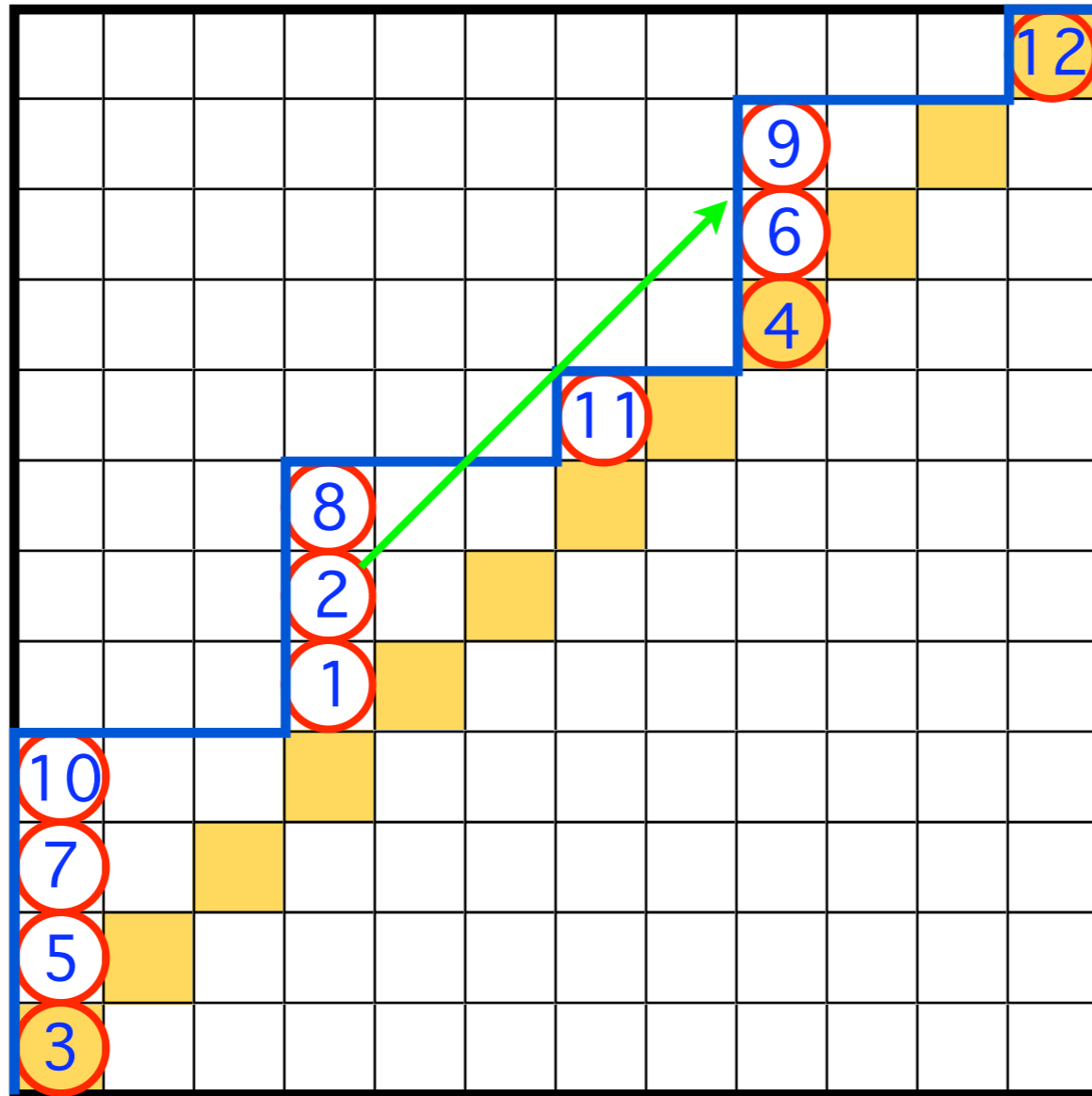


# The $d_{inv}$ statistic of a Parking Function



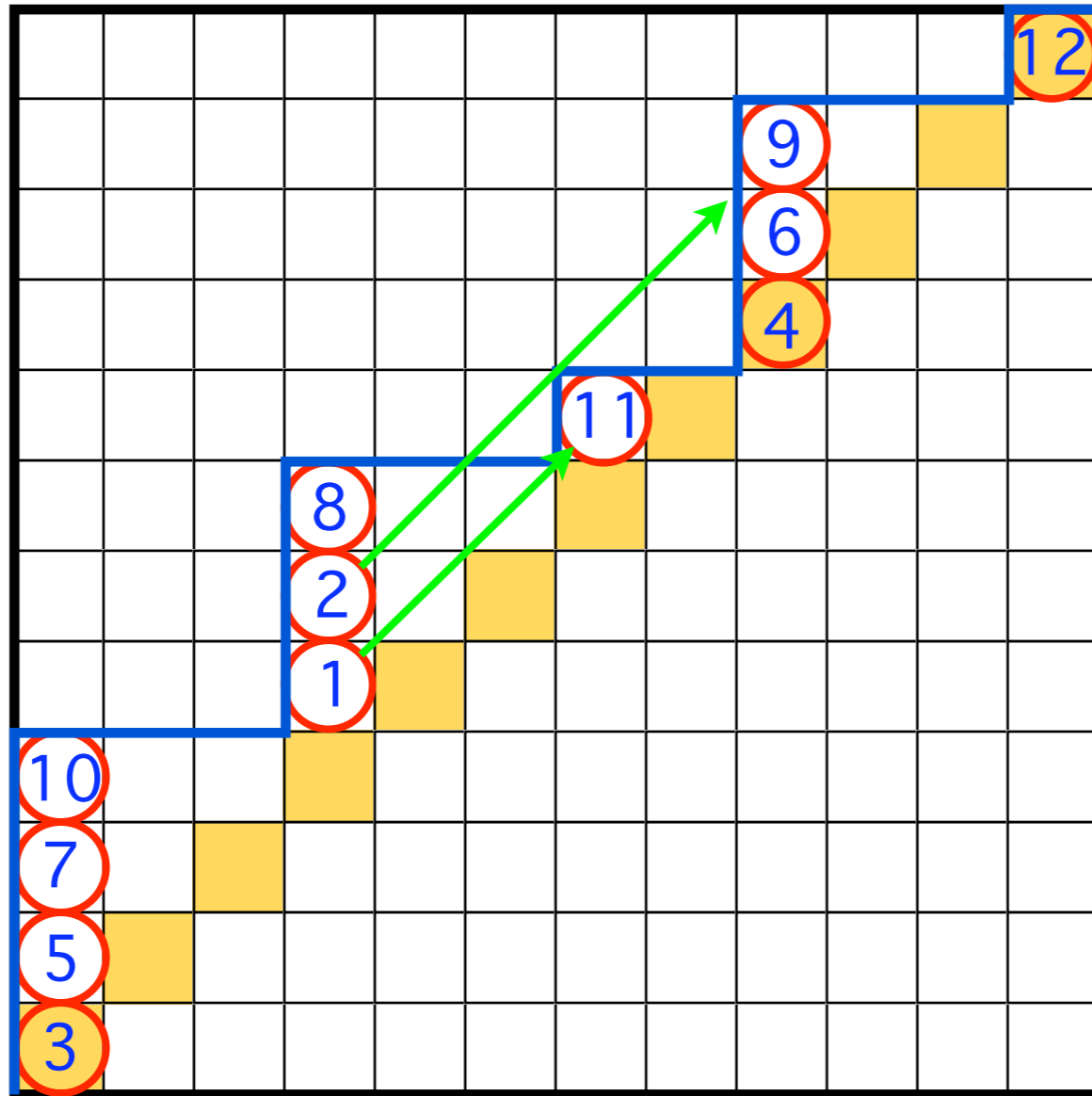
$$\text{area}(PF) = 16$$

# The $dinv$ statistic of a Parking Function



$$area(PF) = 16$$

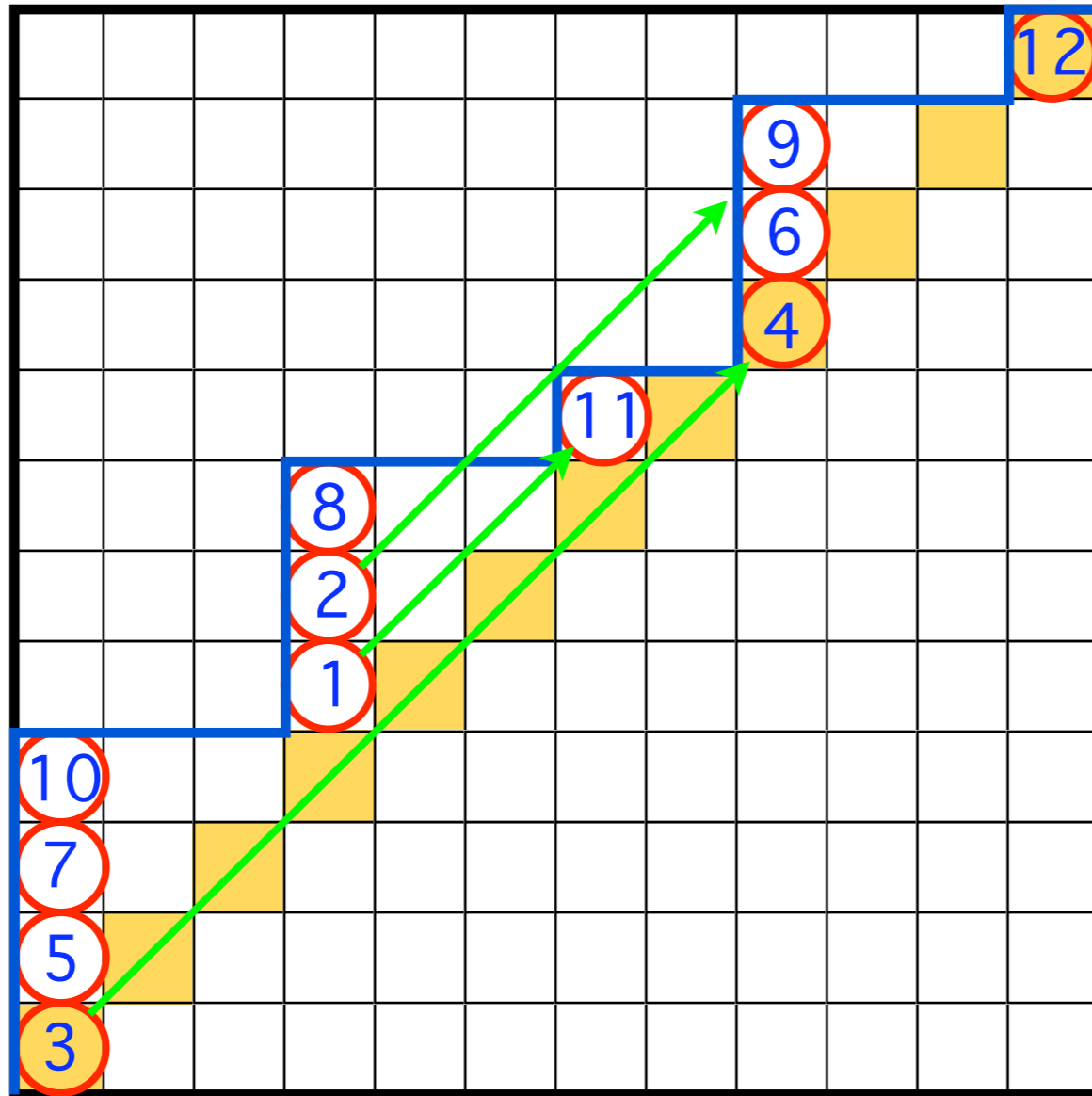
# The $d_{\text{inv}}$ statistic of a Parking Function



$$\text{area}(PF) = 16$$

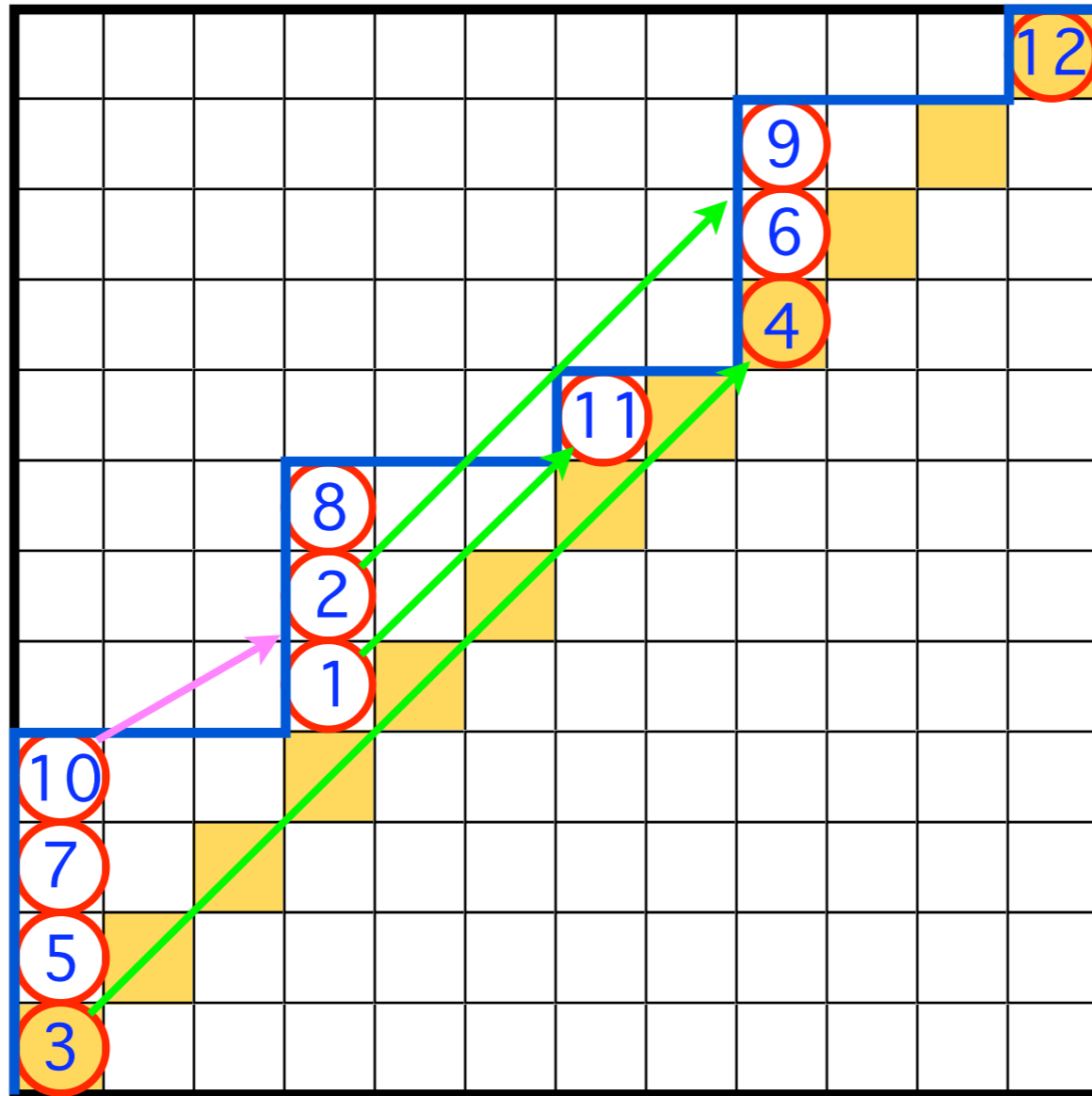


# The $dinv$ statistic of a Parking Function



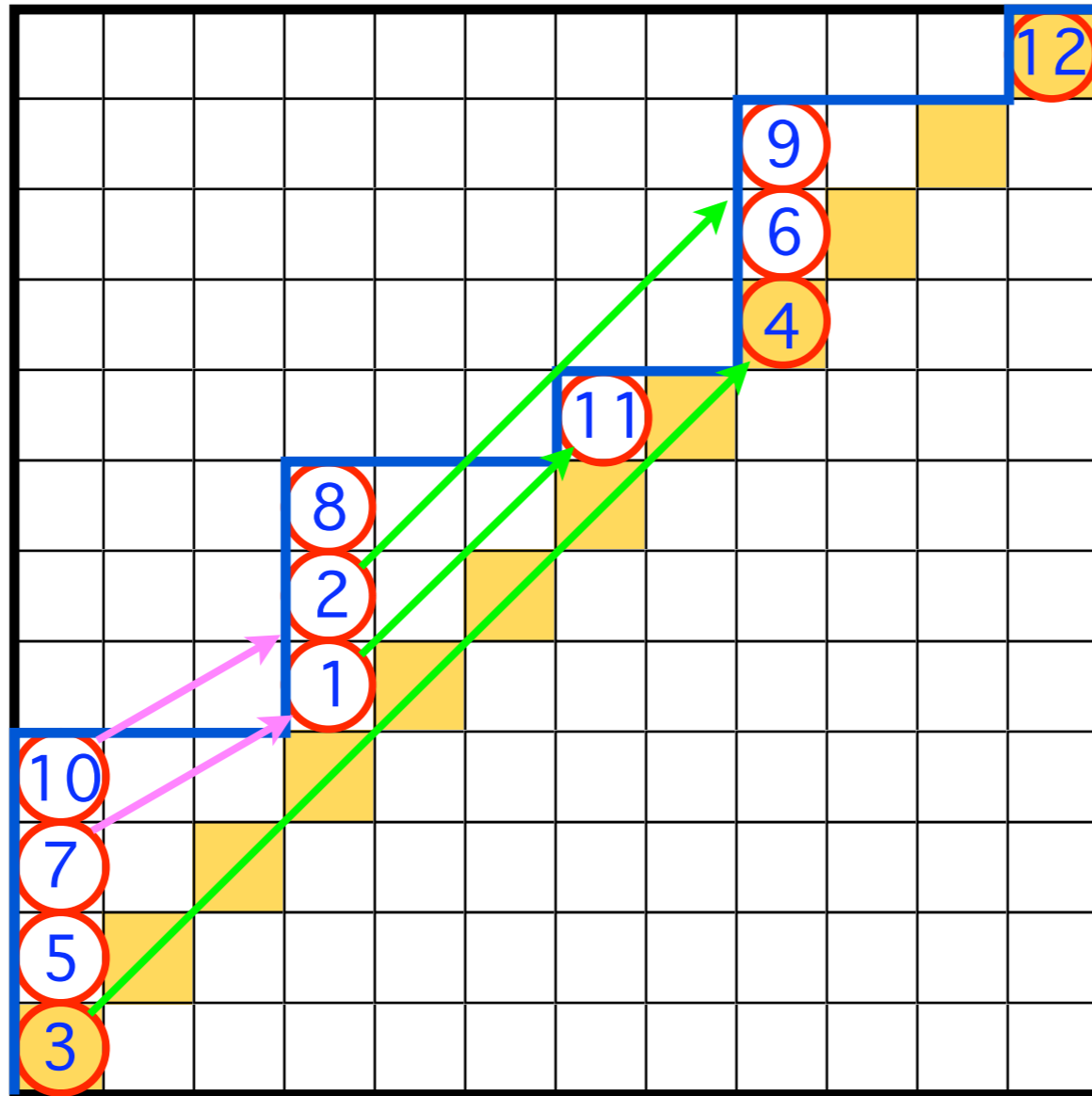
$$\text{area}(PF) = 16$$

# The *dinv* statistic of a Parking Function



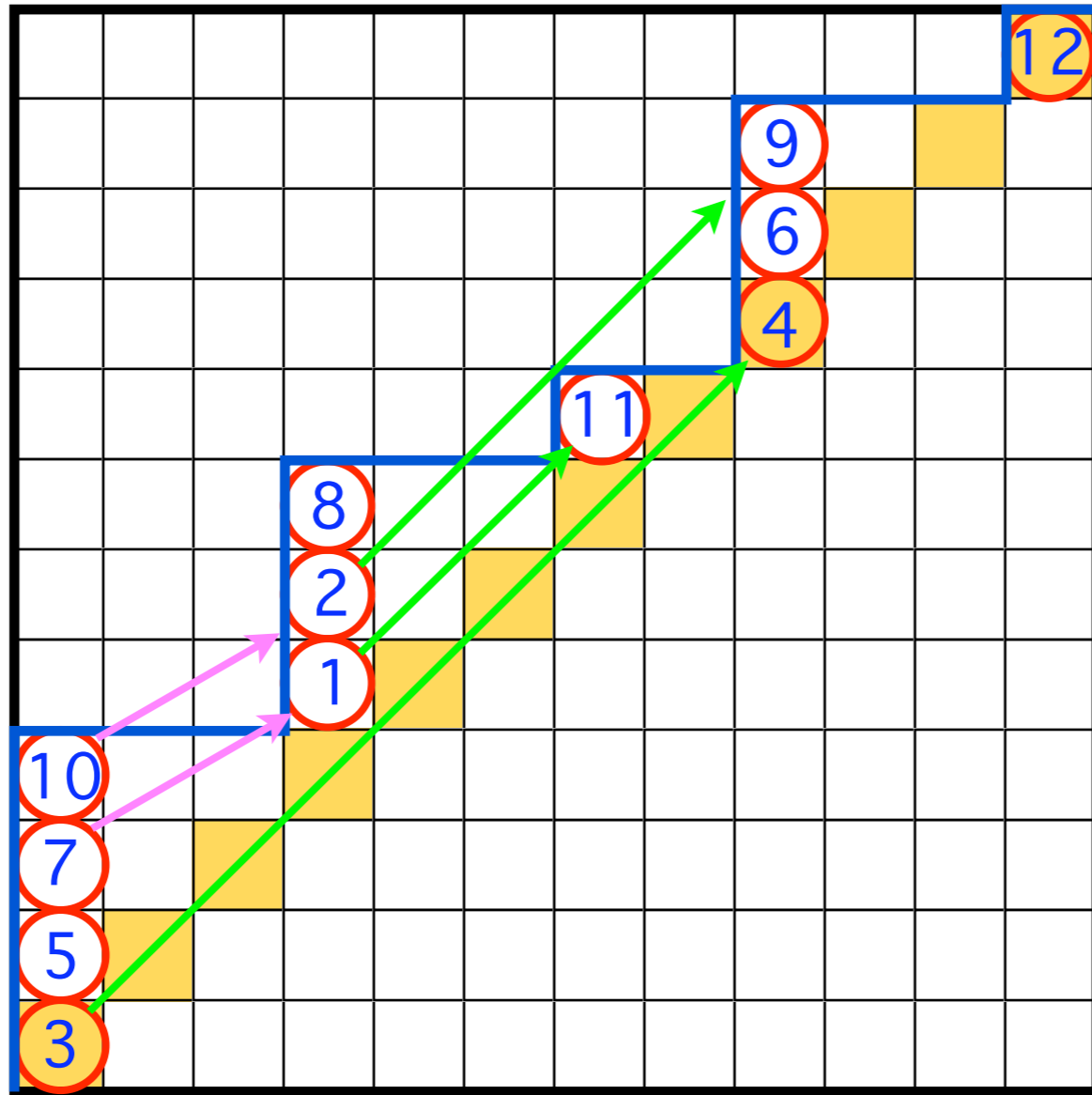
$$\text{area}(PF) = 16$$

# The *dinv* statistic of a Parking Function



$$\text{area}(PF) = 16$$

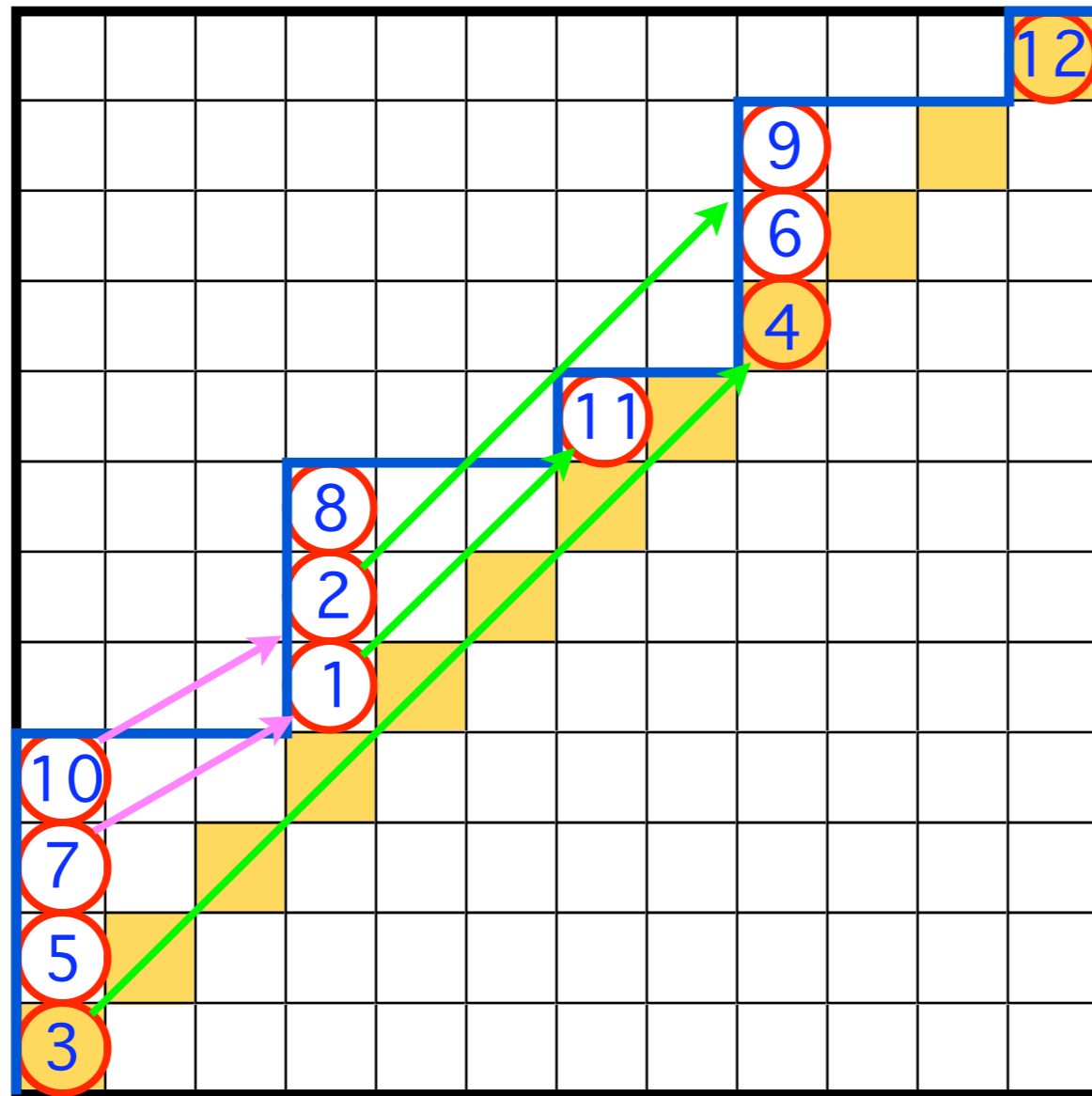
# The *dinv* statistic of a Parking Function



$$area(PF) = 16$$

$$dinv(PF) = 15$$

# The $dinv$ statistic of a Parking Function

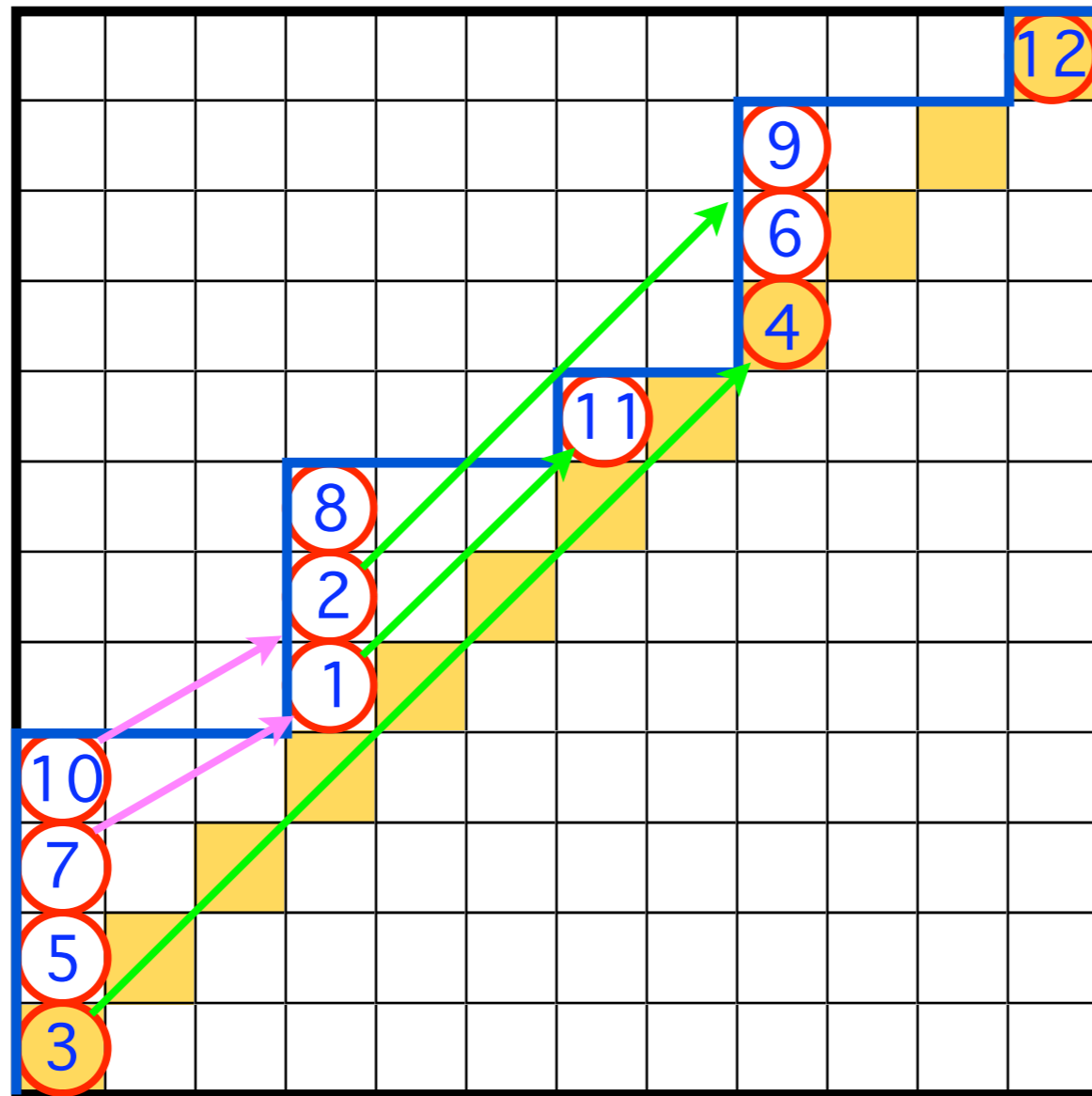


$$area(PF) = 16$$

$$dinv(PF) = 15$$

3	5	7	10	1	2	8	11	4	6	9	12
0	1	2	3	1	2	3	1	0	1	2	0

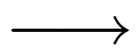
# The $dinv$ statistic of a Parking Function



$$area(PF) = 16$$

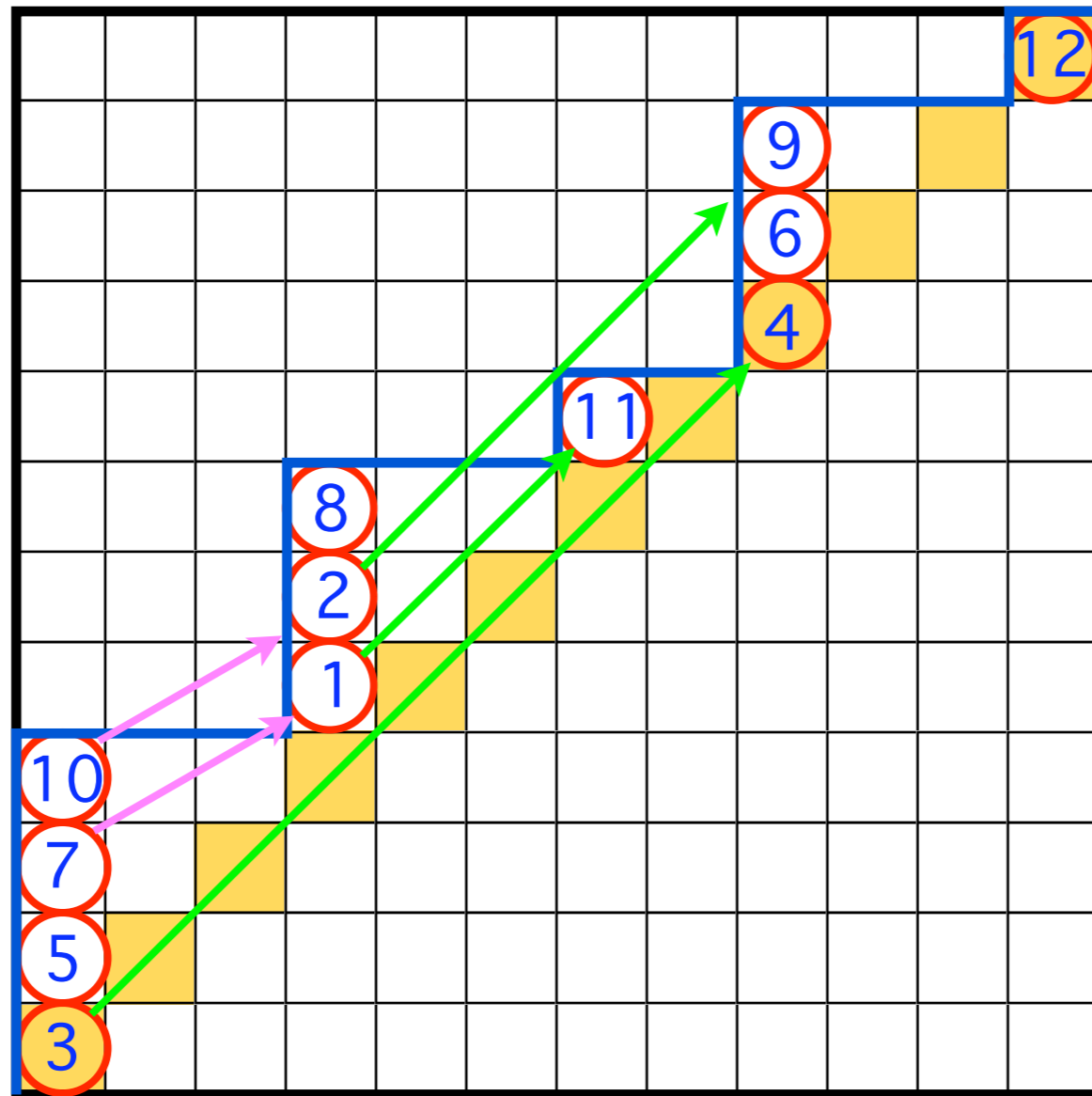
$$dinv(PF) = 15$$

$v_i$



3	5	7	10	1	2	8	11	4	6	9	12
0	1	2	3	1	2	3	1	0	1	2	0

# The $dinv$ statistic of a Parking Function

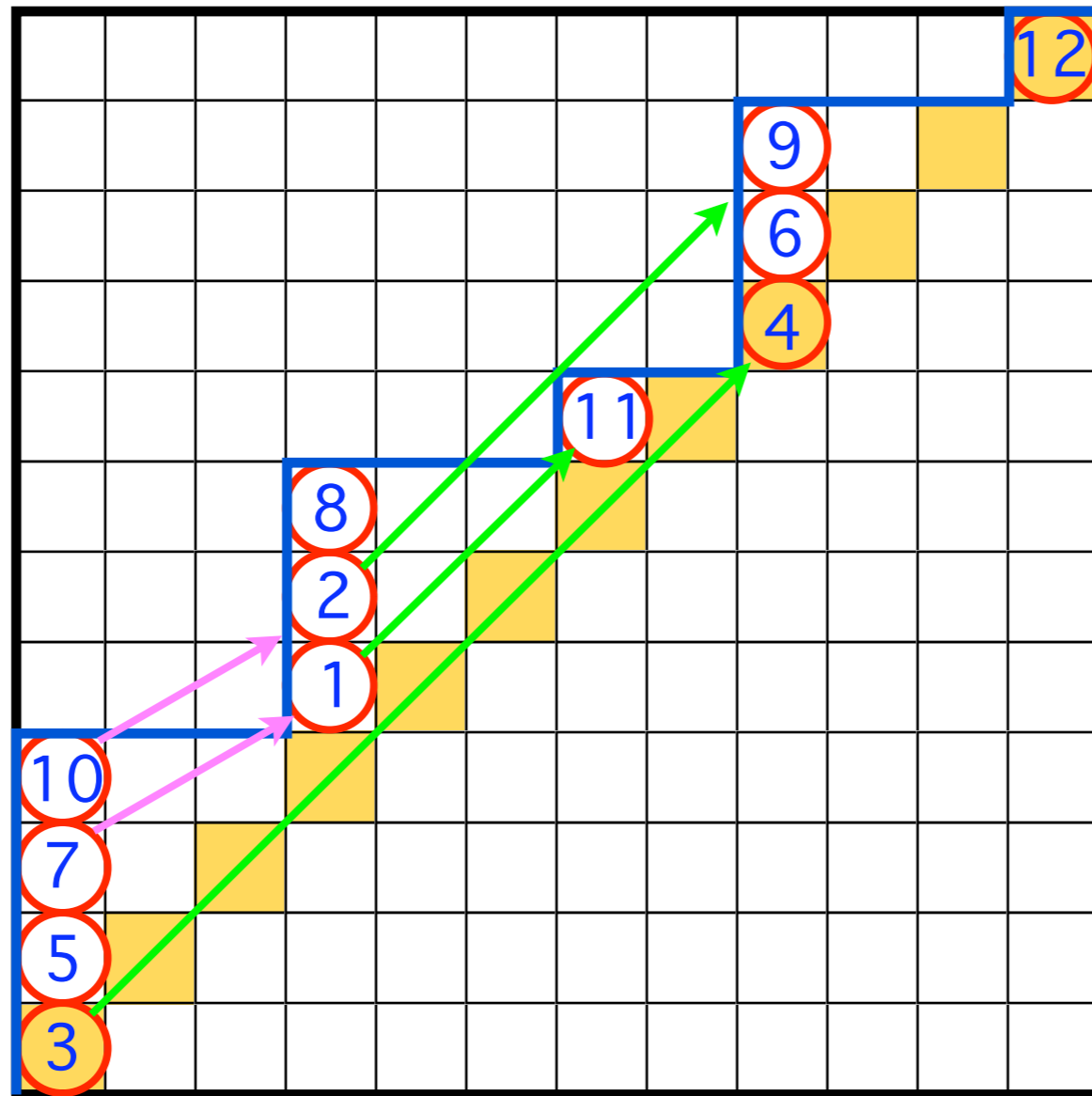


$$area(PF) = 16$$

$$dinv(PF) = 15$$

$v_i$	→	3	5	7	10	1	2	8	11	4	6	9	12
$u_i$	→	0	1	2	3	1	2	3	1	0	1	2	0

# The *dinv* statistic of a Parking Function



$$area(PF) = 16$$

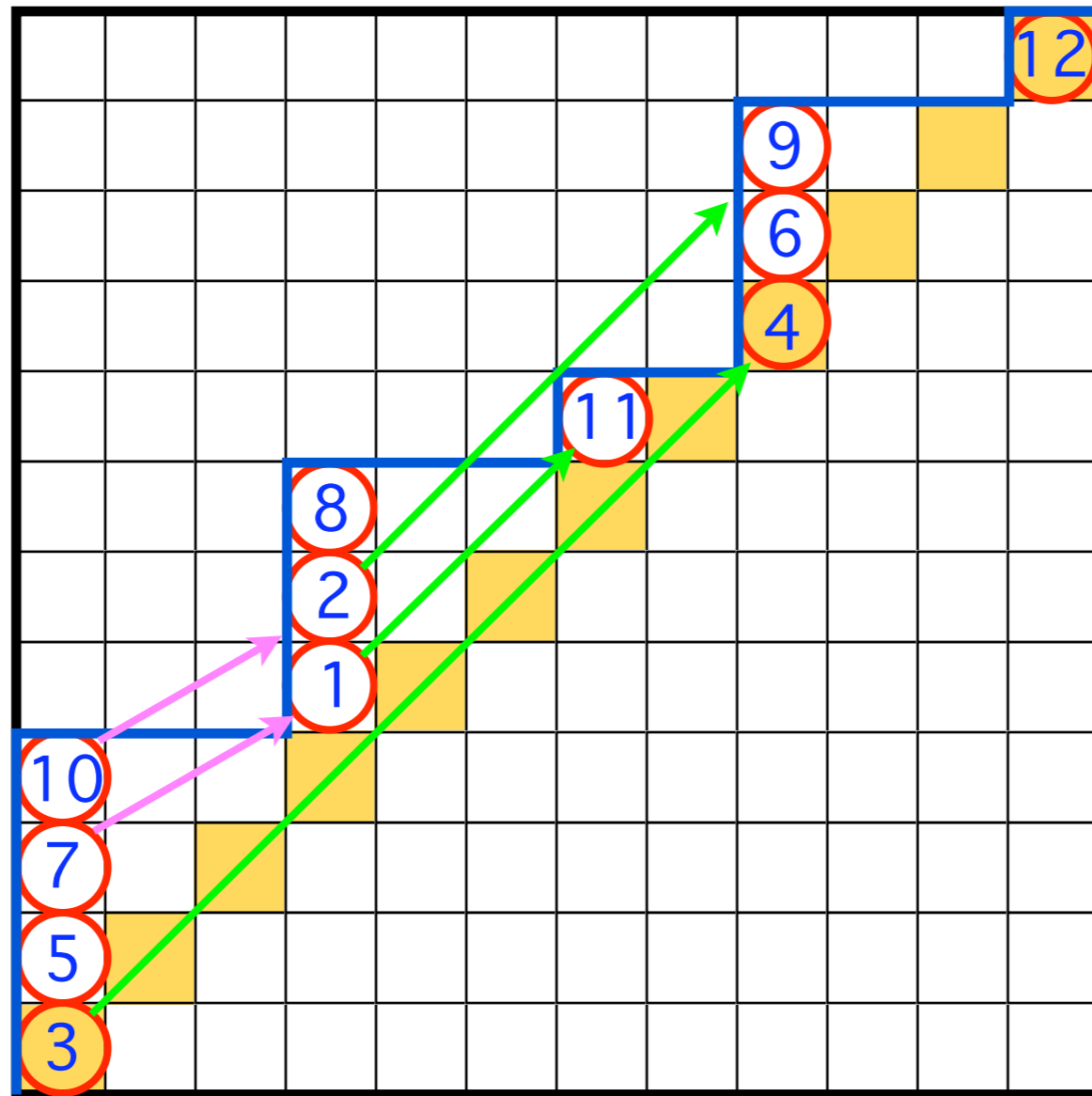
$$dinv(PF) = 15$$

$$dinv(PF) = \sum_{1 \leq i < j \leq n} \left( \chi(u_i = u_j \ \& \ v_i < v_j) + \chi(u_i = u_j + 1 \ \& \ v_i > v_j) \right)$$

$v_i$	→	3	5	7	10	1	2	8	11	4	6	9	12
$u_i$	→	0	1	2	3	1	2	3	1	0	1	2	0



# The *dinv* statistic of a Parking Function



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$v_i$	→	3	5	7	10	1	2	8	11	4	6	9	12
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next



# The Haglund-Loehr Conjecture (2002)

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The Hilbert series of Diagonal Harmonics

# The Haglund-Loehr Conjecture (2002)

The Hilbert series of Diagonal Harmonics

$$H_n(\mathbf{q}, \mathbf{t}) = \sum_{\mathbf{PF} \in \mathcal{PF}_n} \mathbf{t}^{\text{area}(\mathbf{PF})} \mathbf{q}^{\text{dinv}(\mathbf{PF})}$$

# The Haglund-Loehr Conjecture (2002)

The Hilbert series of Diagonal Harmonics

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1			
2	1		
2	3	1	
1	2	2	1

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The Hilbert series of Diagonal Harmonics

$$H_n(q, t) = \sum_{PF \in \mathcal{PF}_n} t^{\text{area}(PF)} q^{\text{dinv}(PF)}$$

1			
2	1		
2	3	1	
1	2	2	1

`mkallpark(3);`

# The Haglund-Loehr Conjecture (2002)

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$$H_n(\mathbf{q}, \mathbf{t}) = \sum_{\mathbf{PF} \in \mathcal{PF}_n} \mathbf{t}^{\text{area}(\mathbf{PF})} \mathbf{q}^{\text{dinv}(\mathbf{PF})}$$

1			
2	1		
2	3	1	
1	2	2	1

**mkallpark(3);**

$$\begin{bmatrix} \cdot & \cdot & 3 \\ \cdot & 2 & \cdot \\ 1 & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & \cdot & 2 \\ \cdot & 3 & \cdot \\ 1 & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & \cdot & 3 \\ \cdot & 1 & \cdot \\ 2 & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & 3 & \cdot \\ 2 & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & \cdot & 2 \\ \cdot & 1 & \cdot \\ 3 & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & 2 & \cdot \\ 3 & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & 3 & o \\ \cdot & 2 & \cdot \\ 1 & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & 3 & o \\ \cdot & 1 & \cdot \\ 2 & \cdot & \cdot \end{bmatrix}$$



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$$H_n(\mathbf{q}, \mathbf{t}) = \sum_{\mathbf{PF} \in \mathcal{PF}_n} \mathbf{t}^{\text{area}(\mathbf{PF})} \mathbf{q}^{\text{dinv}(\mathbf{PF})}$$

1			
2	1		
2	3	1	
1	2	2	1

**mkallpark(3);**

$$\begin{aligned}
 & \begin{bmatrix} . & . & 3 \\ . & 2 & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 2 \\ . & 3 & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 3 \\ . & 1 & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 1 \\ . & 3 & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 2 \\ . & 1 & . \\ 3 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 1 \\ . & 2 & . \\ 3 & . & . \end{bmatrix}, \begin{bmatrix} . & 3 & o \\ . & 2 & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & 3 & o \\ . & 1 & . \\ 2 & . & . \end{bmatrix} \\
 & \begin{bmatrix} . & 2 & o \\ . & 1 & . \\ 3 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 3 \\ 2 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 2 \\ 3 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 1 \\ 3 & o & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} . & 3 & o \\ 2 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & 2 & o \\ 3 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & 1 & o \\ 3 & o & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} 3 & o & o \\ 2 & o & . \\ 1 & . & . \end{bmatrix}
 \end{aligned}$$

# The Haglund-Loehr Conjecture (2002)

## The Hilbert series of Diagonal Harmonics

$$H_n(q, t) = \sum_{PF \in \mathcal{PF}_n} t^{\text{area}(PF)} q^{\text{dinv}(PF)}$$

1			
2	1		
2	3	1	
1	2	2	1

**mkallpark(3);**

$\begin{bmatrix} . & . & 3 \\ . & 2 & . \\ 1 & . & . \end{bmatrix}$	$\begin{bmatrix} . & . & 2 \\ . & 3 & . \\ 1 & . & . \end{bmatrix}$	$\begin{bmatrix} . & . & 3 \\ . & 1 & . \\ 2 & . & . \end{bmatrix}$	$\begin{bmatrix} . & . & 1 \\ . & 3 & . \\ 2 & . & . \end{bmatrix}$	$\begin{bmatrix} . & . & 2 \\ . & 1 & . \\ 3 & . & . \end{bmatrix}$	$\begin{bmatrix} . & . & 1 \\ . & 2 & . \\ 3 & . & . \end{bmatrix}$	$\begin{bmatrix} . & 3 & o \\ . & 2 & . \\ 1 & . & . \end{bmatrix}$	$\begin{bmatrix} . & 3 & o \\ . & 1 & . \\ 2 & . & . \end{bmatrix}$
$\begin{bmatrix} . & 2 & o \\ . & 1 & . \\ 3 & . & . \end{bmatrix}$	$\begin{bmatrix} . & . & 3 \\ 2 & o & . \\ 1 & . & . \end{bmatrix}$	$\begin{bmatrix} . & . & 2 \\ 3 & o & . \\ 1 & . & . \end{bmatrix}$	$\begin{bmatrix} . & . & 1 \\ 3 & o & . \\ 2 & . & . \end{bmatrix}$	$\begin{bmatrix} . & 3 & o \\ 2 & o & . \\ 1 & . & . \end{bmatrix}$	$\begin{bmatrix} . & 2 & o \\ 3 & o & . \\ 1 & . & . \end{bmatrix}$	$\begin{bmatrix} . & 1 & o \\ 3 & o & . \\ 2 & . & . \end{bmatrix}$	$\begin{bmatrix} 3 & o & o \\ 2 & o & . \\ 1 & . & . \end{bmatrix}$

**stat := PF -> t^area(PF) \* q^dinv(PF);**

$$stat := PF \mapsto t^{\text{area}(PF)} q^{\text{dinv}(PF)}$$

# The Haglund-Loehr Conjecture (2002)

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$$H_n(q, t) = \sum_{PF \in \mathcal{PF}_n} t^{\text{area}(PF)} q^{\text{dinv}(PF)}$$

1			
2	1		
2	3	1	
1	2	2	1

**mkallpark(3);**

$$\begin{aligned} & \begin{bmatrix} . & . & 3 \\ . & 2 & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 2 \\ . & 3 & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 3 \\ . & 1 & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 1 \\ . & 3 & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 2 \\ . & 1 & . \\ 3 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 1 \\ . & 2 & . \\ 3 & . & . \end{bmatrix}, \begin{bmatrix} . & 3 & o \\ . & 2 & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & 3 & o \\ . & 1 & . \\ 2 & . & . \end{bmatrix} \\ & \begin{bmatrix} . & 2 & o \\ . & 1 & . \\ 3 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 3 \\ 2 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 2 \\ 3 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 1 \\ 3 & o & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} . & 3 & o \\ 2 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & 2 & o \\ 3 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & 1 & o \\ 3 & o & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} 3 & o & o \\ 2 & o & . \\ 1 & . & . \end{bmatrix} \end{aligned}$$

**stat:=PF->t^area(PF)\*q^dinv(PF);**

$$stat := PF \mapsto t^{\text{area}(PF)} q^{\text{dinv}(PF)}$$

**convert(map(stat,PF3),`+`);;**

$$q^3 + q^2 t + q t^2 + t^3 + 2 q^2 + 3 q t + 2 t^2 + 2 q + 2 t + 1$$

# The Haglund-Loehr Conjecture (2002)

## The Hilbert series of Diagonal Harmonics

$$H_n(q, t) = \sum_{PF \in \mathcal{PF}_n} t^{\text{area}(PF)} q^{\text{dinv}(PF)}$$

1			
2	1		
2	3	1	
1	2	2	1

**mkallpark(3);**

$$\begin{bmatrix} . & . & 3 \\ . & 2 & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 2 \\ . & 3 & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 3 \\ . & 1 & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 1 \\ . & 3 & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 2 \\ . & 1 & . \\ 3 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 1 \\ . & 2 & . \\ 3 & . & . \end{bmatrix}, \begin{bmatrix} . & 3 & o \\ . & 2 & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & 3 & o \\ . & 1 & . \\ 2 & . & . \end{bmatrix}$$

$$\begin{bmatrix} . & 2 & o \\ . & 1 & . \\ 3 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 3 \\ 2 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 2 \\ 3 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 1 \\ 3 & o & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} . & 3 & o \\ 2 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & 2 & o \\ 3 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & 1 & o \\ 3 & o & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} 3 & o & o \\ 2 & o & . \\ 1 & . & . \end{bmatrix}$$

**stat := PF -> t^area(PF) \* q^dinv(PF);**

$$stat := PF \mapsto t^{\text{area}(PF)} q^{\text{dinv}(PF)}$$

**convert(map(stat, PF3), `+`);;**

$$q^3 + q^2 t + q t^2 + t^3 + 2 q^2 + 3 q t + 2 t^2 + 2 q + 2 t + 1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

# The Haglund-Loehr Conjecture (2002)

## The Hilbert series of Diagonal Harmonics

$$H_n(q, t) = \sum_{PF \in \mathcal{PF}_n} t^{\text{area}(PF)} q^{\text{dinv}(PF)}$$

1			
2	1		
2	3	1	
1	2	2	1

**mkallpark(3);**

$$\begin{bmatrix} . & . & 3 \\ . & 2 & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 2 \\ . & 3 & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 3 \\ . & 1 & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 1 \\ . & 3 & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 2 \\ . & 1 & . \\ 3 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 1 \\ . & 2 & . \\ 3 & . & . \end{bmatrix}, \begin{bmatrix} . & 3 & o \\ . & 2 & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & 3 & o \\ . & 1 & . \\ 2 & . & . \end{bmatrix}$$

$$\begin{bmatrix} . & 2 & o \\ . & 1 & . \\ 3 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 3 \\ 2 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 2 \\ 3 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & . & 1 \\ 3 & o & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} . & 3 & o \\ 2 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & 2 & o \\ 3 & o & . \\ 1 & . & . \end{bmatrix}, \begin{bmatrix} . & 1 & o \\ 3 & o & . \\ 2 & . & . \end{bmatrix}, \begin{bmatrix} 3 & o & o \\ 2 & o & . \\ 1 & . & . \end{bmatrix}$$

**stat := PF -> t^area(PF) \* q^dinv(PF);**

$$stat := PF \mapsto t^{\text{area}(PF)} q^{\text{dinv}(PF)}$$

**convert(map(stat, PF3), `+`);;**

$$q^3 + q^2 t + q t^2 + t^3 + 2 q^2 + 3 q t + 2 t^2 + 2 q + 2 t + 1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

next



# Tesler Matrices with hook weights

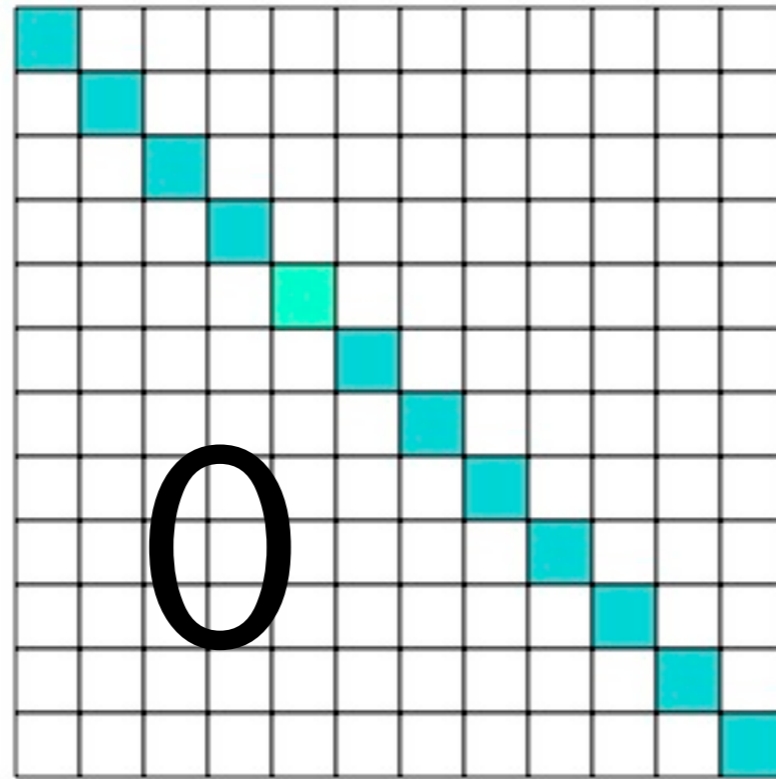
# Tesler Matrices with hook weights

$$y_1, y_2, y_3, \dots, y_n$$



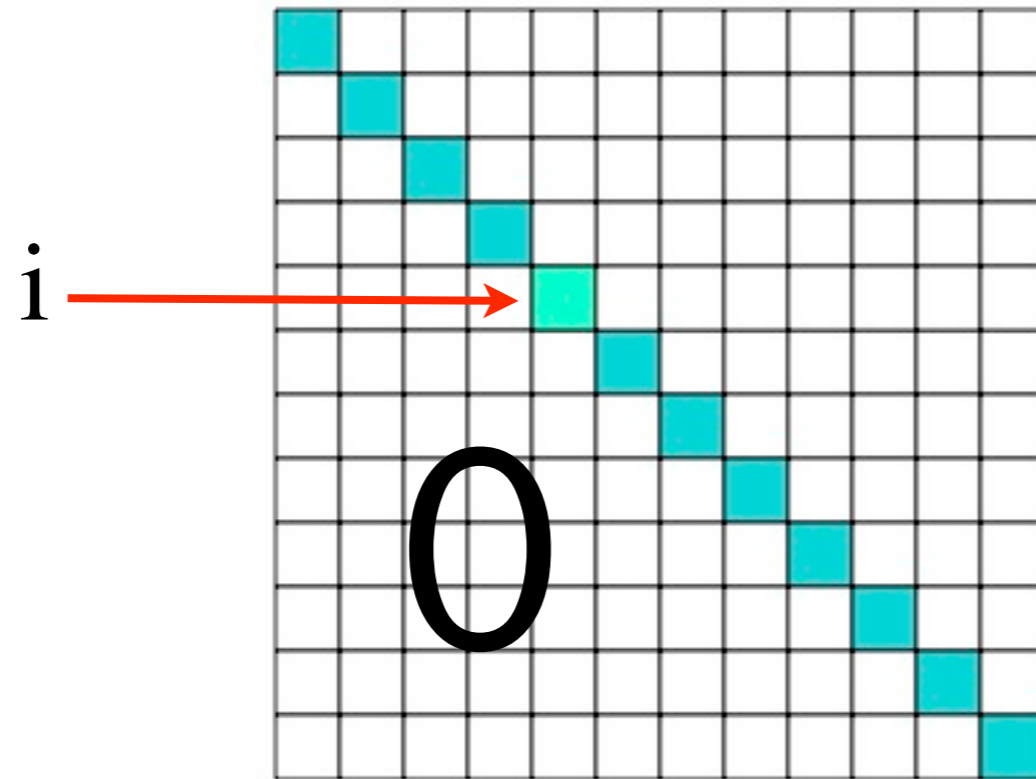
# Tesler Matrices with hook weights

$$y_1, y_2, y_3, \dots, y_n$$



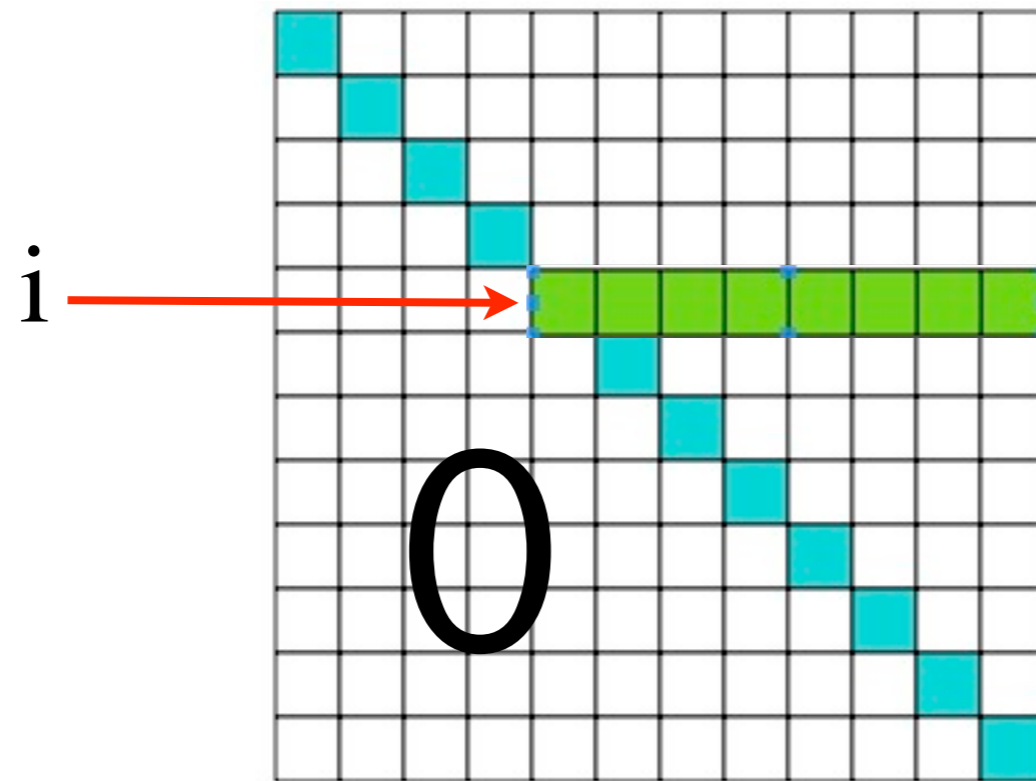
# Tesler Matrices with hook weights

$$y_1, y_2, y_3, \dots, y_n$$



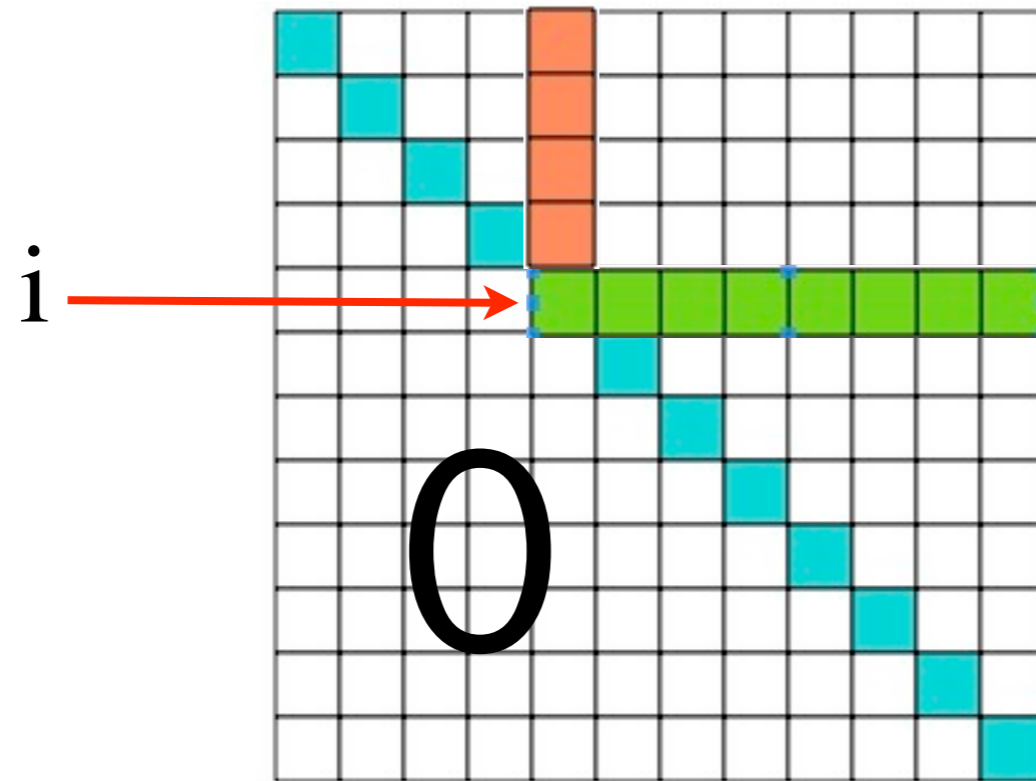
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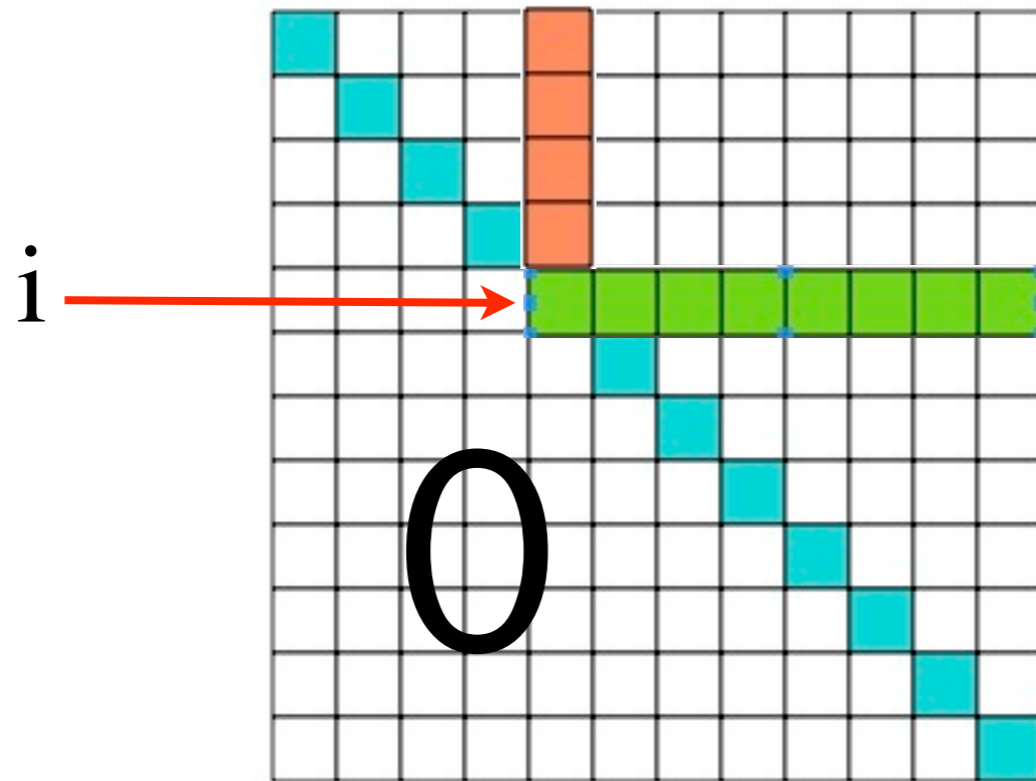
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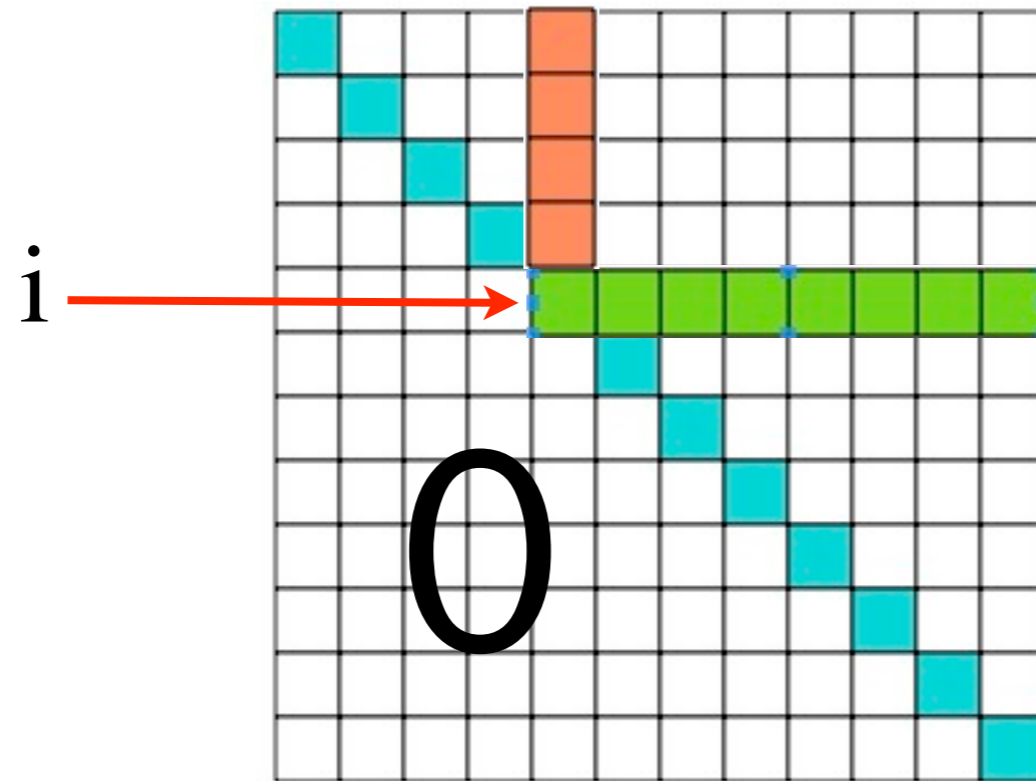
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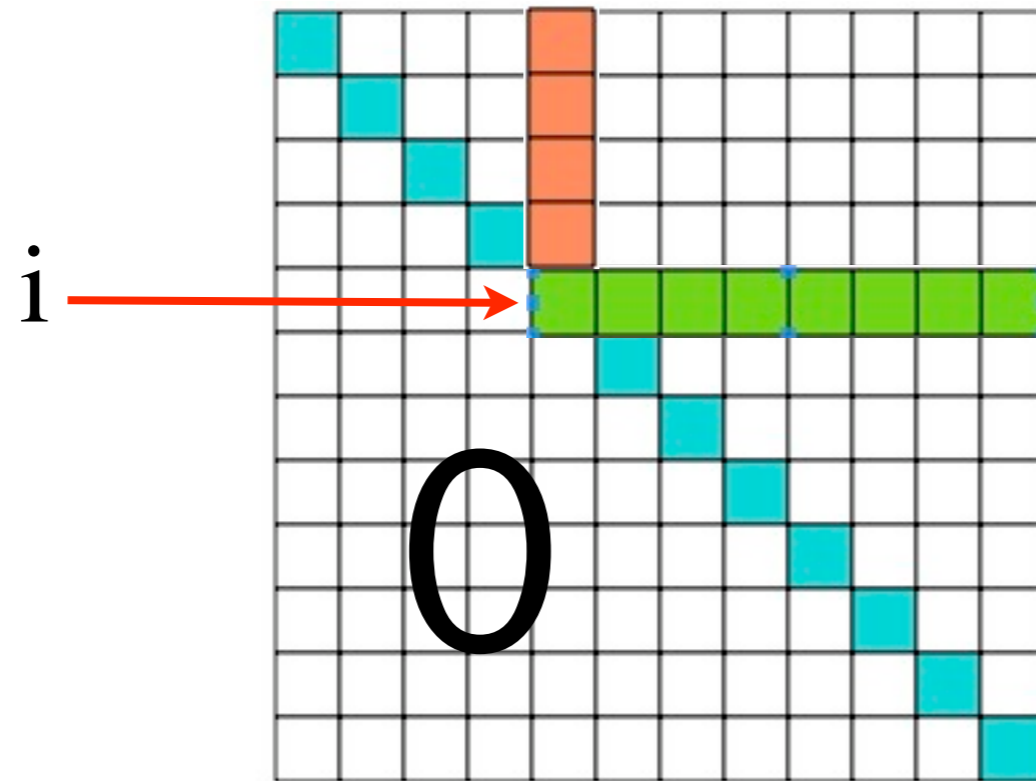
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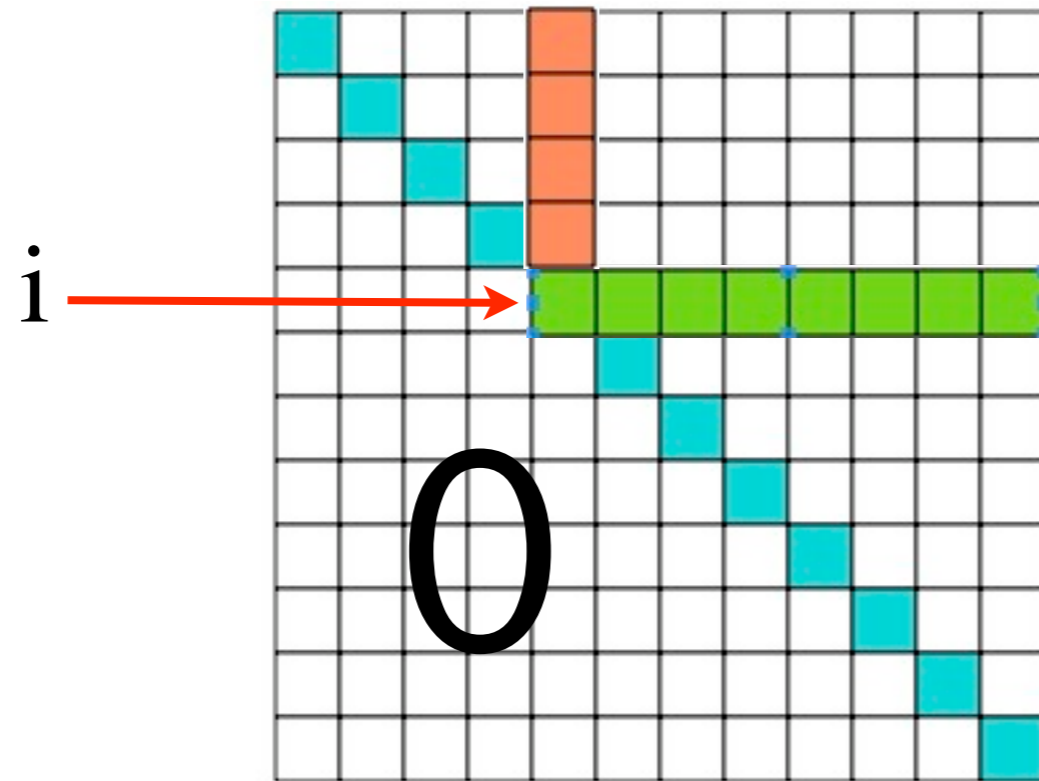
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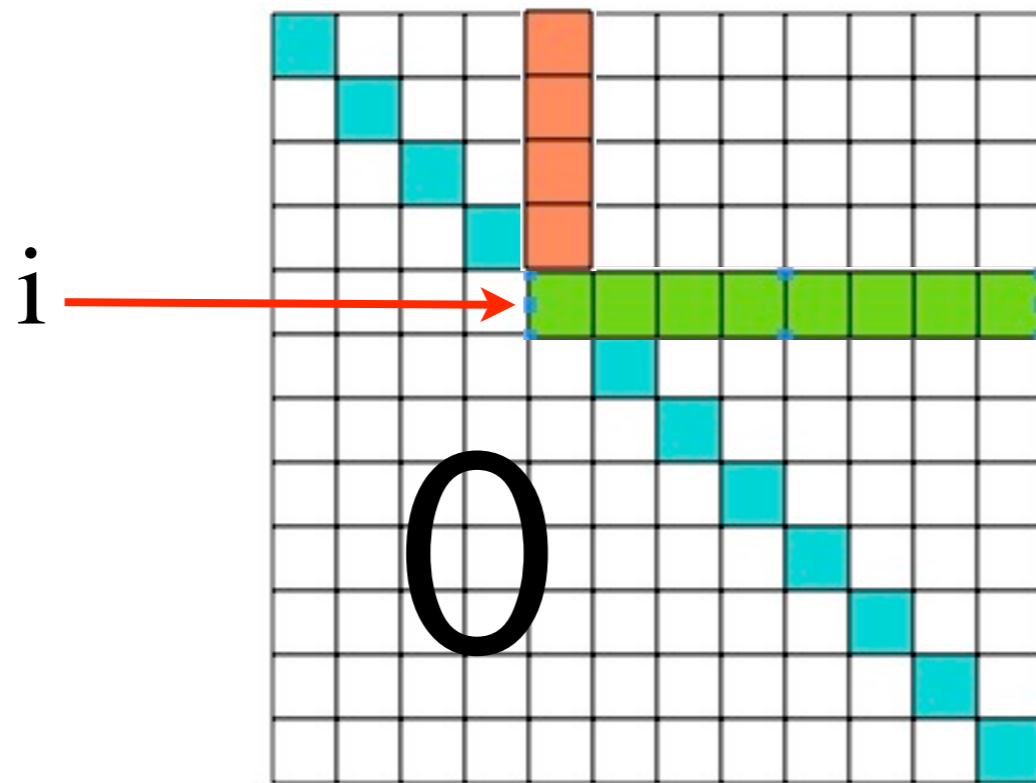


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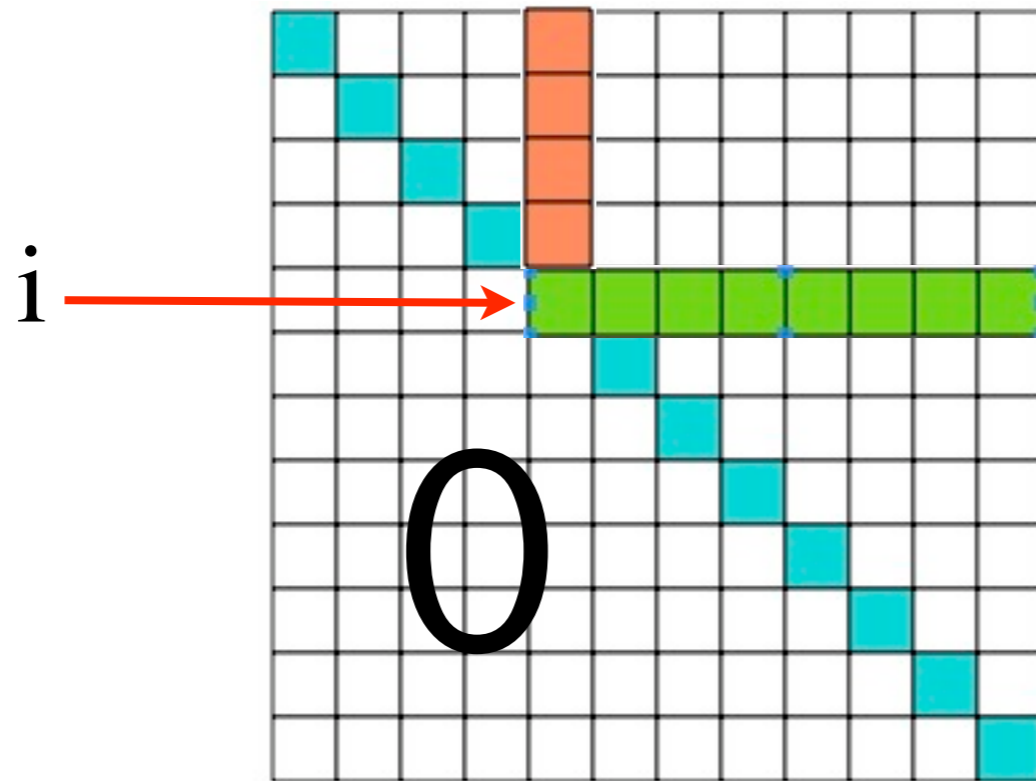


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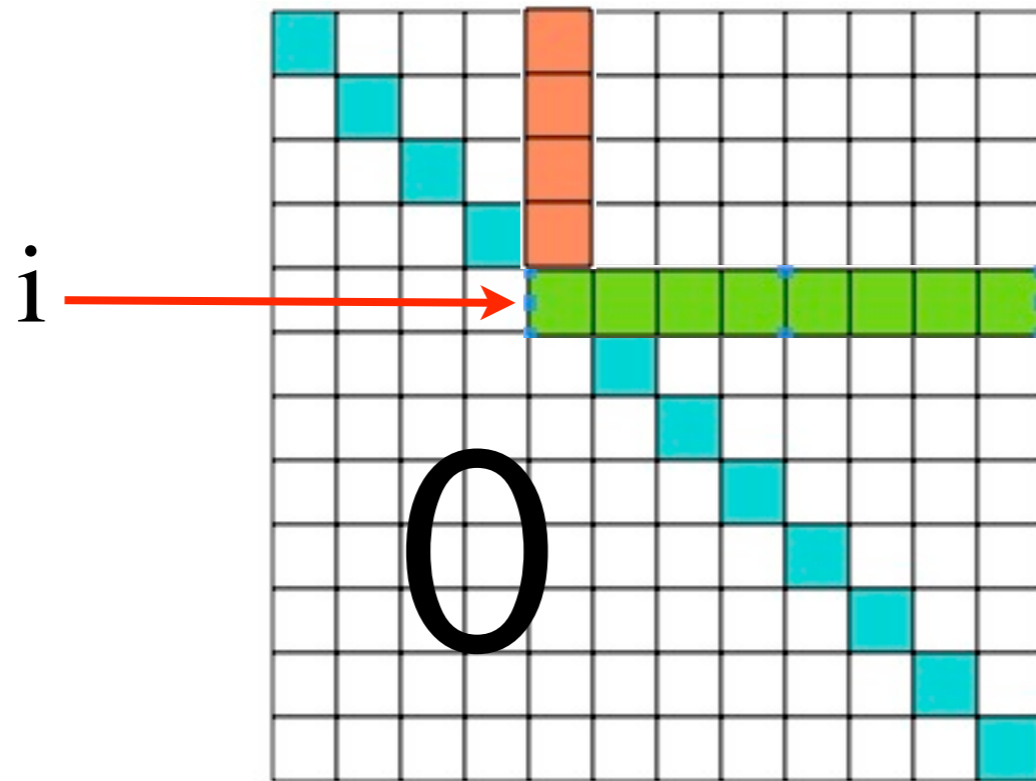
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# Haglunds Discovery

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**Theorem**

# Haglund's Discovery

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next



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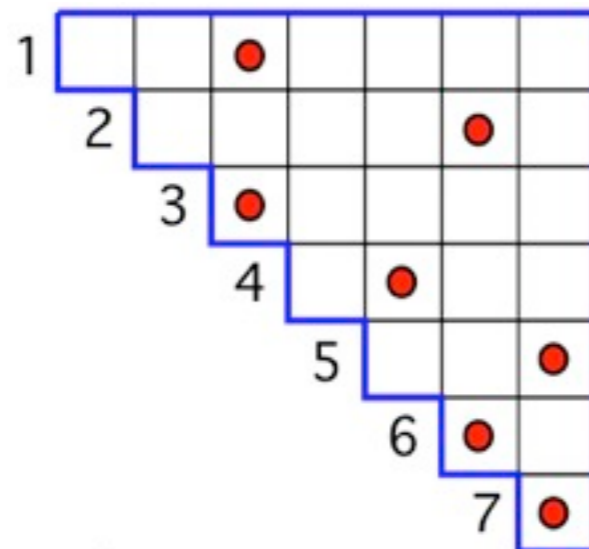
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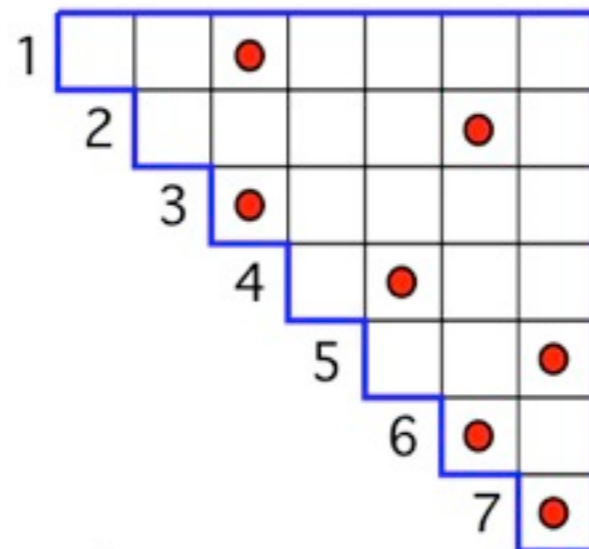
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(1) place  $u_1$  in row 1 and column  $i_1$ .



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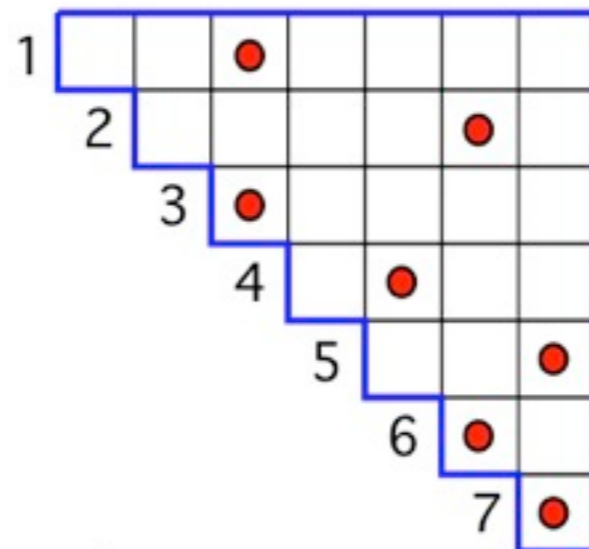
*For any given choice of corner weights  $u_1, u_2, \dots, u_n$*

*there are  $n!$  Permutational Tesler matrices*

## Proof

Given a triangular sequence  $1 \leq i_s \leq s$  (for  $1 \leq s \leq n$ )

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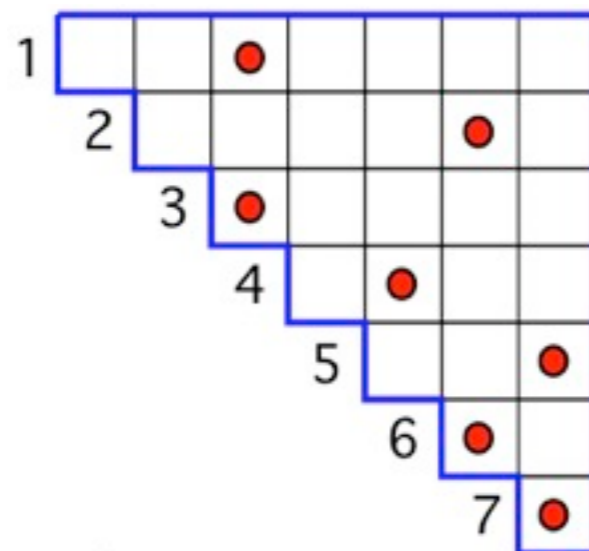
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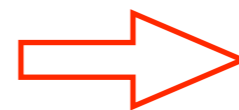
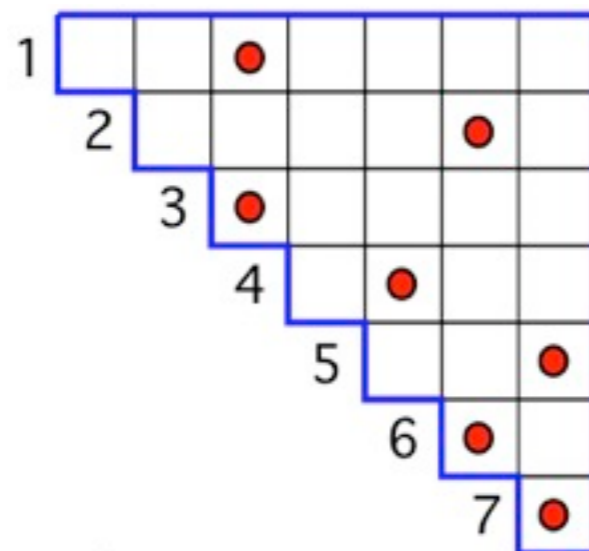
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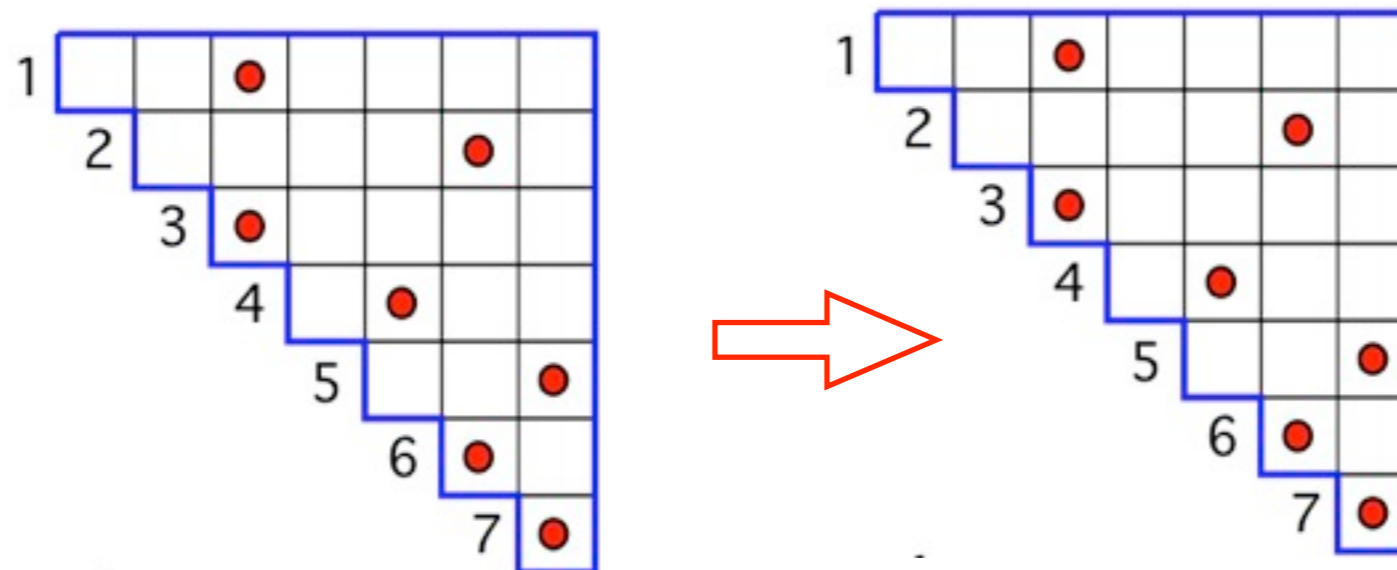
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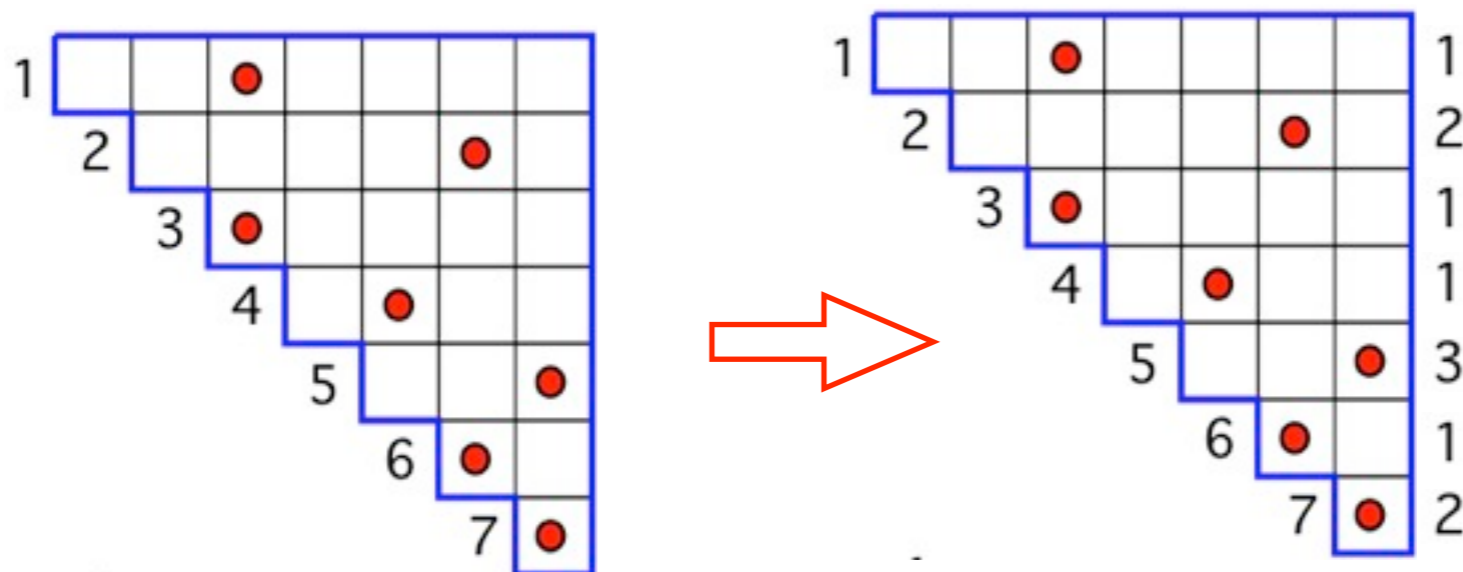
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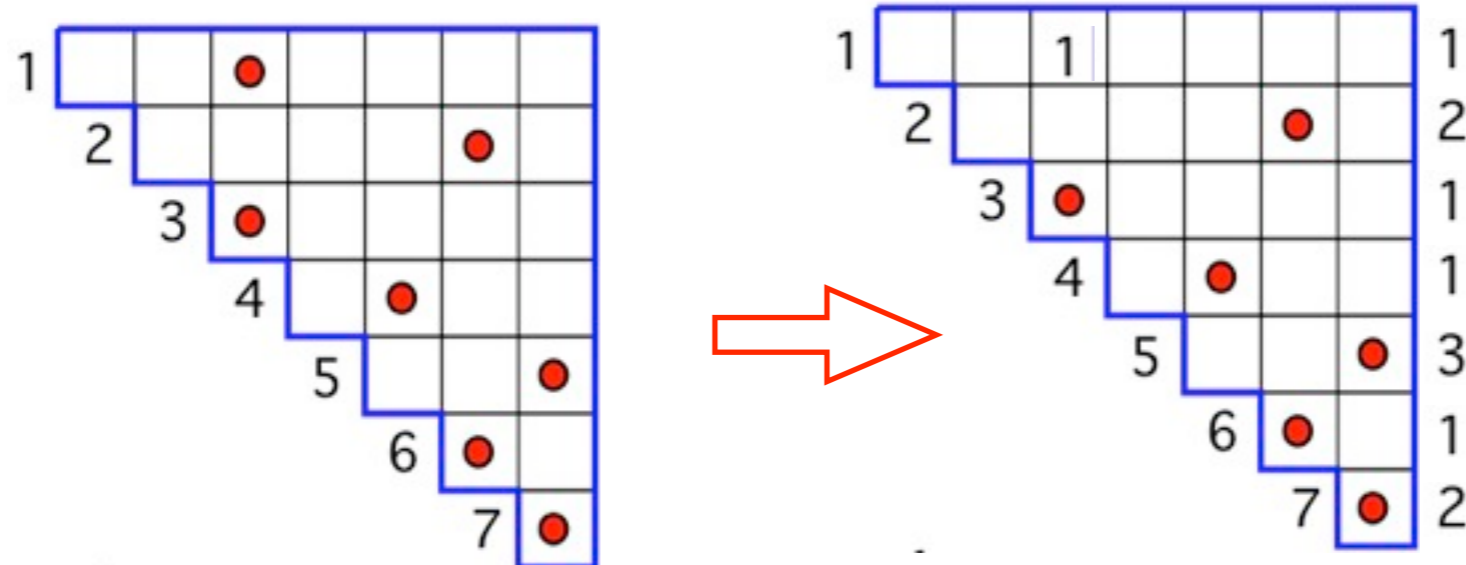
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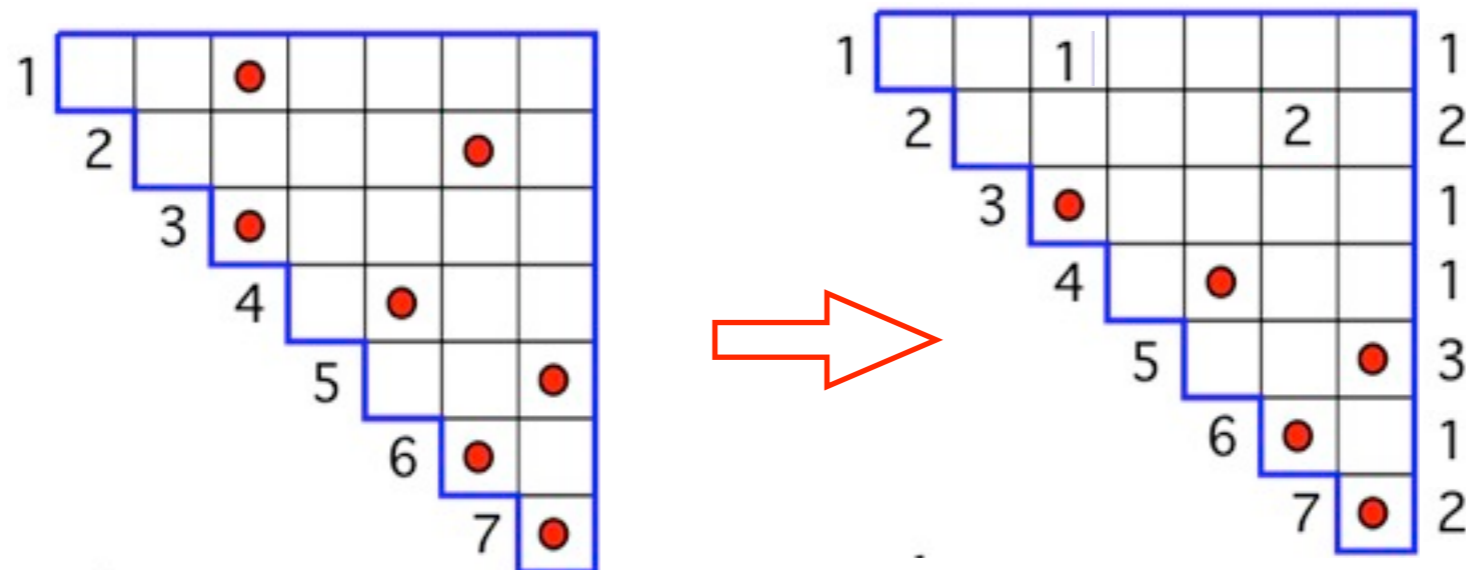
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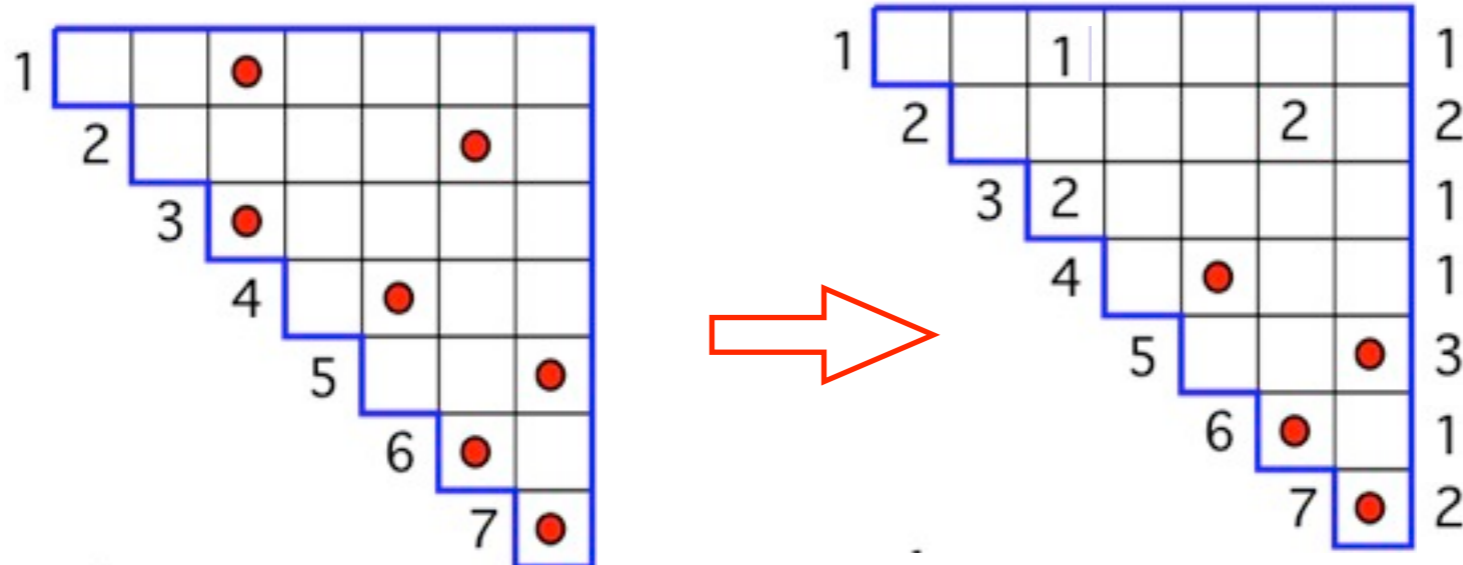
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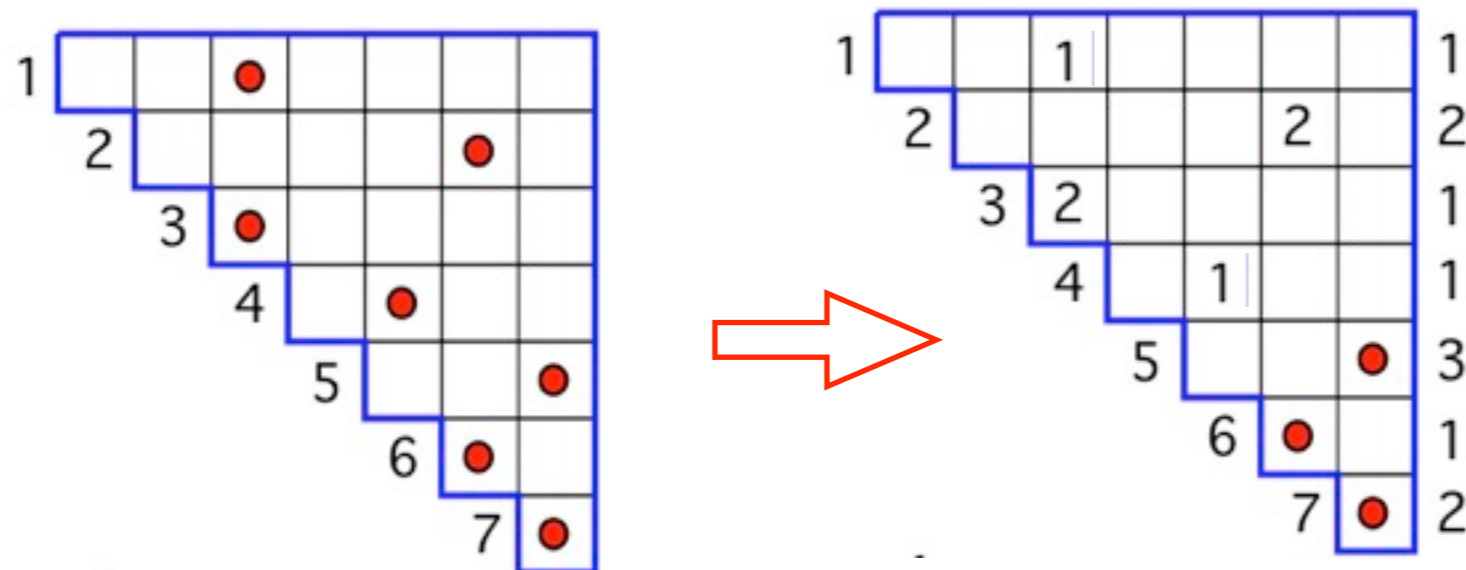
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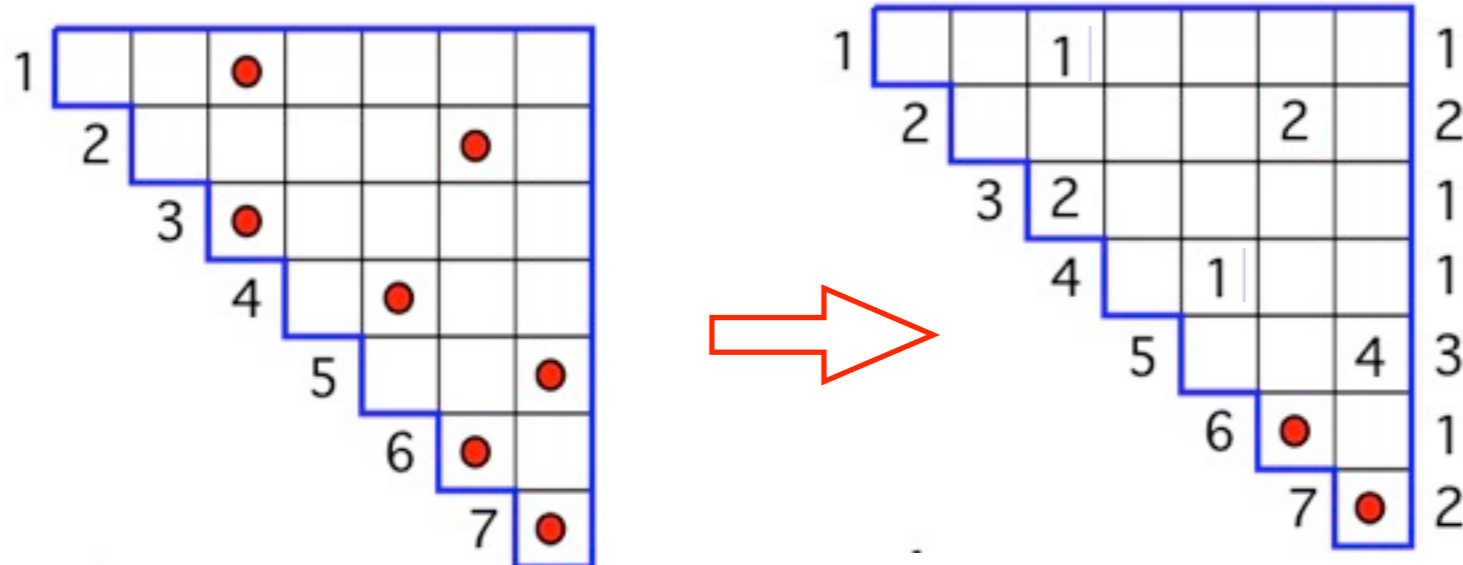
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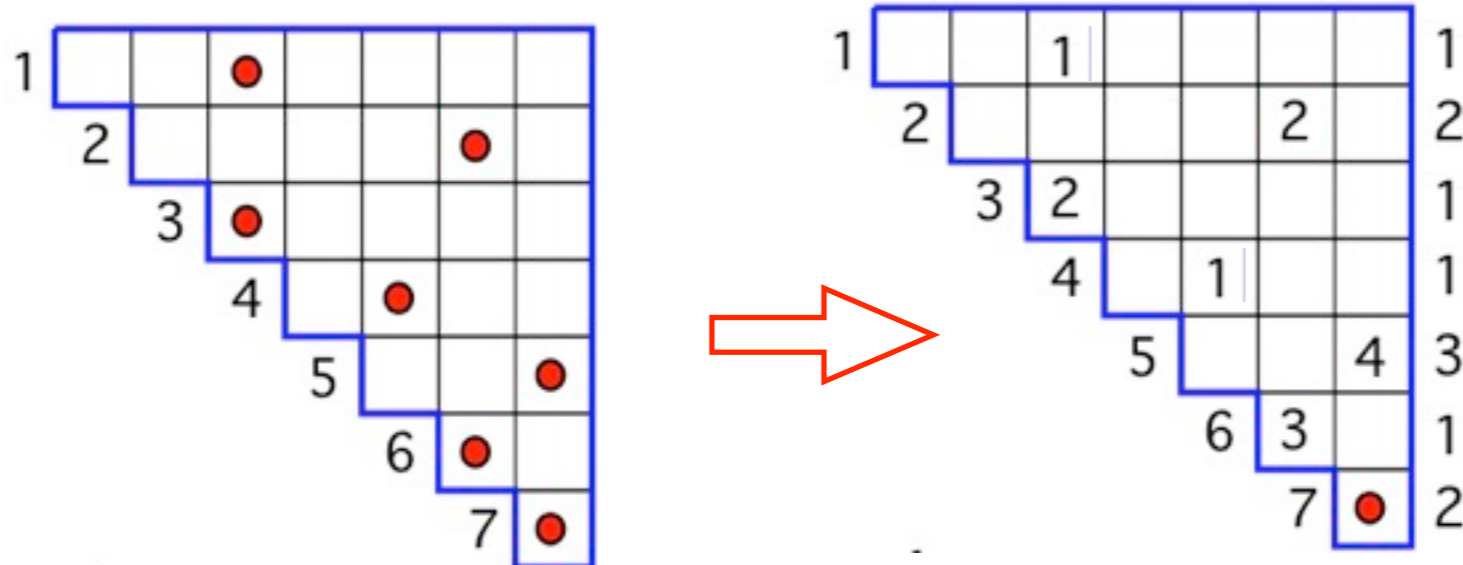
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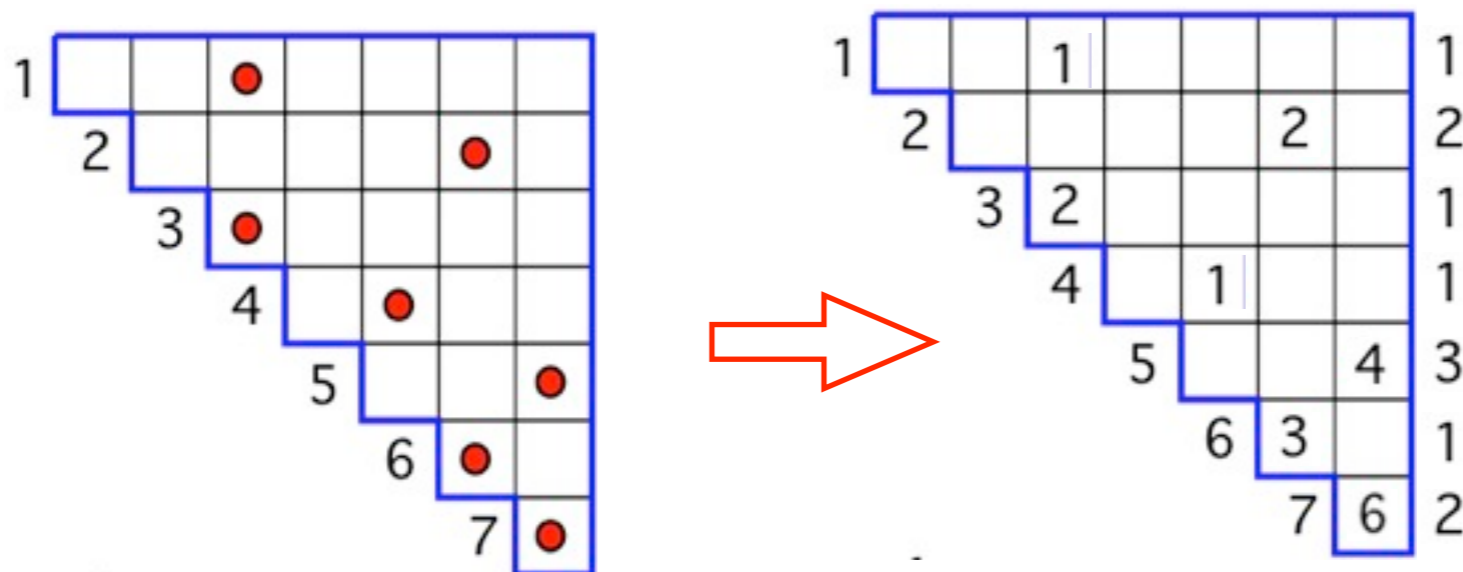
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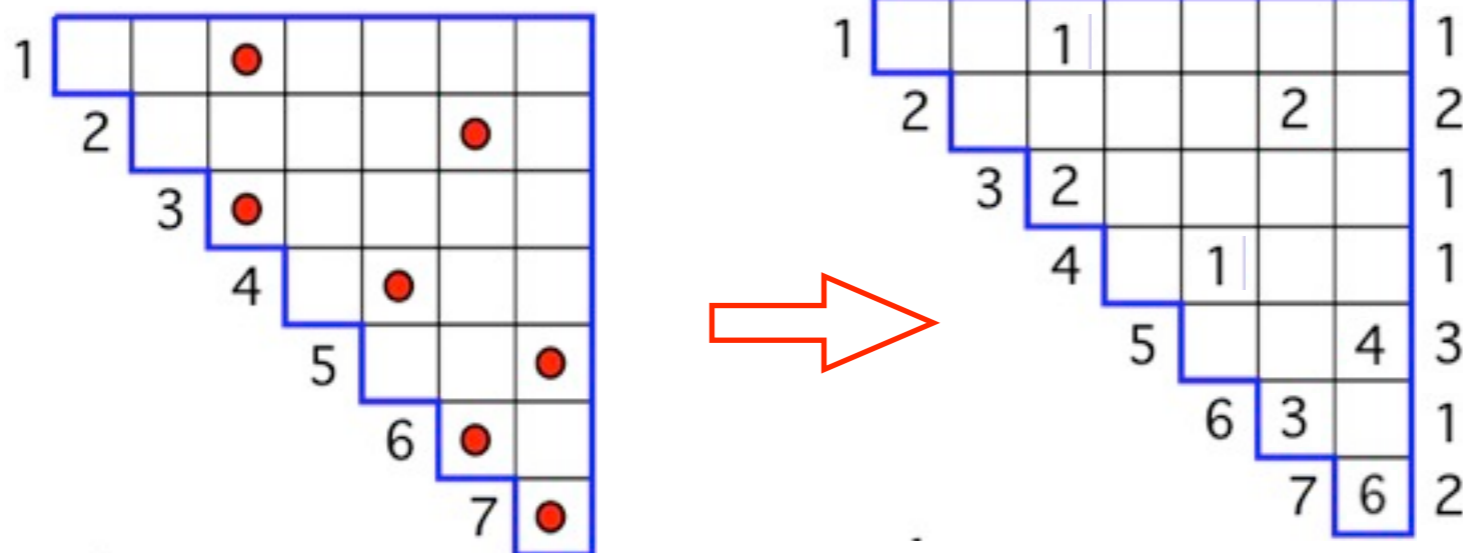
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*Tesler matrices with hook weights  $y_1, y_2, y_3$*

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$$\begin{bmatrix} 0 & y_1 & 0 \\ 0 & 0 & y_1 + y_2 \\ 0 & 0 & y_1 + y_2 + y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & y_1 \\ 0 & y_2 & 0 \\ 0 & 0 & y_1 + y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & y_1 \\ 0 & 0 & y_2 \\ 0 & 0 & y_1 + y_2 + y_3 \end{bmatrix} \begin{bmatrix} 0 & y_1 & 0 \\ 0 & y_1 + y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix} \begin{bmatrix} y_1 & 0 & 0 \\ 0 & 0 & y_2 \\ 0 & 0 & y_2 + y_3 \end{bmatrix} \begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix}$$

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map(evamat, yMA3);
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$$y_1 (y_1 + 3 y_2) (y_1 + y_2 + 2 y_3)$$

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**Theorem** (D. Armstrong-B. Sagan)

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The sum of the weights the permutational Tesler matrices with hook weights  $y_1, y_2, \dots, y_n$  is

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$$y_1 (y_1 + ny_2) (y_1 + y_2 + (n-1)y_3) \cdots (y_1 + y_2 + \cdots + y_{n-1} + 2y_n)$$

# Permutational Tesler matrices

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$$\begin{bmatrix} 0 & y_1 & 0 \\ 0 & 0 & y_1 + y_2 \\ 0 & 0 & y_1 + y_2 + y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & y_1 \\ 0 & y_2 & 0 \\ 0 & 0 & y_1 + y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & y_1 \\ 0 & 0 & y_2 \\ 0 & 0 & y_1 + y_2 + y_3 \end{bmatrix} \begin{bmatrix} 0 & y_1 & 0 \\ 0 & y_1 + y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix} \begin{bmatrix} y_1 & 0 & 0 \\ 0 & 0 & y_2 \\ 0 & 0 & y_2 + y_3 \end{bmatrix} \begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix}$$

**map(evamat, yMA3);**

$y_1 (y_1 + y_2) (y_1 + y_2 + y_3), y_1 y_2 (y_1 + y_3), y_1 y_2 (y_1 + y_2 + y_3), y_1 (y_1 + y_2) y_3, y_1 y_2 (y_2 + y_3), y_1 y_2 y_3]$

**factor(convert(%, `+`));**

$$y_1 (y_1 + 3y_2) (y_1 + y_2 + 2y_3)$$

**Theorem** (D. Armstrong-B. Sagan)

The sum of the weights the permutational Tesler matrices with hook weights  $y_1, y_2, \dots, y_n$  is

$$y_1 (y_1 + ny_2) (y_1 + y_2 + (n-1)y_3) \cdots (y_1 + y_2 + \cdots + y_{n-1} + 2y_n)$$

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The **tail** and the **target** of an **entry** of a permutation



# The tail and the target of an entry of a permutation

target = smallest larger element to the right

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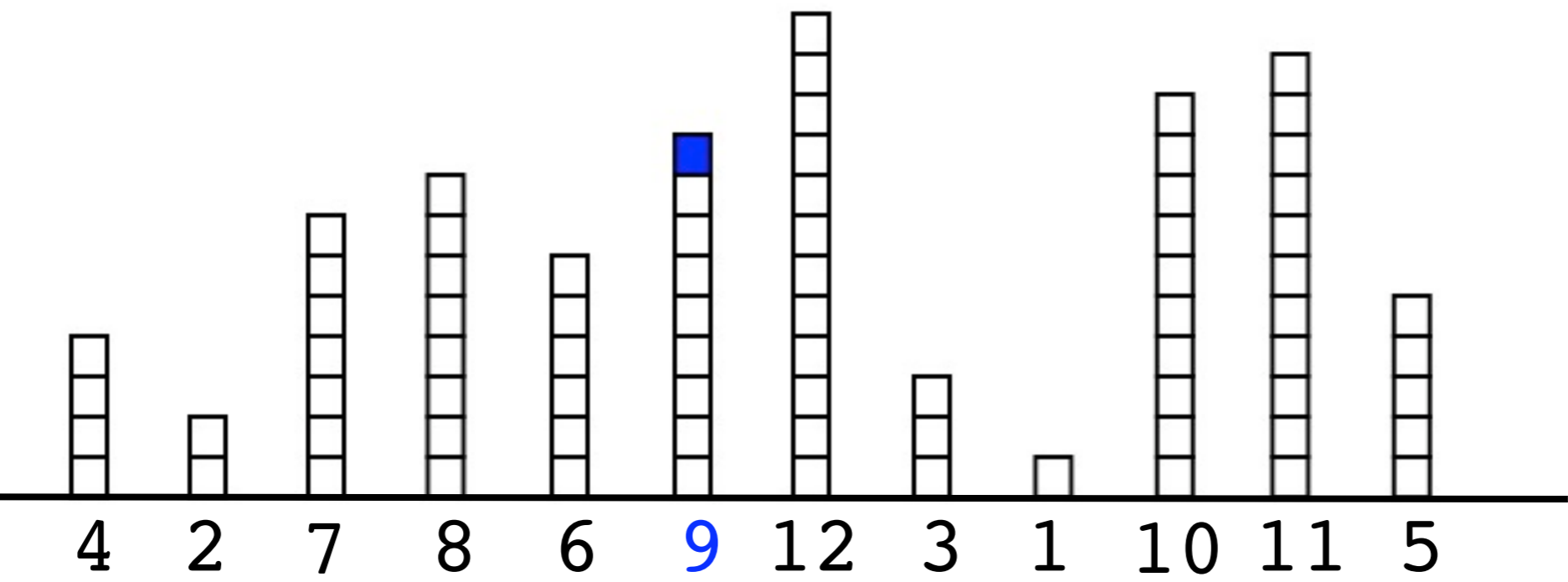
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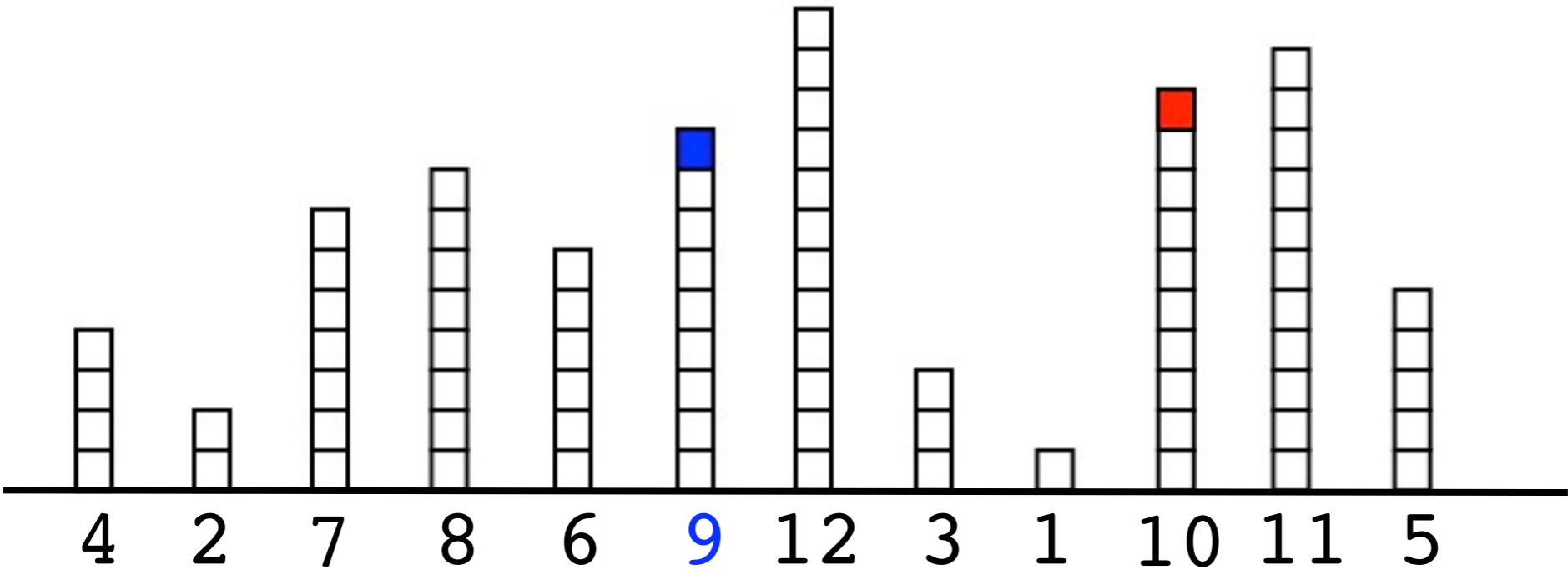
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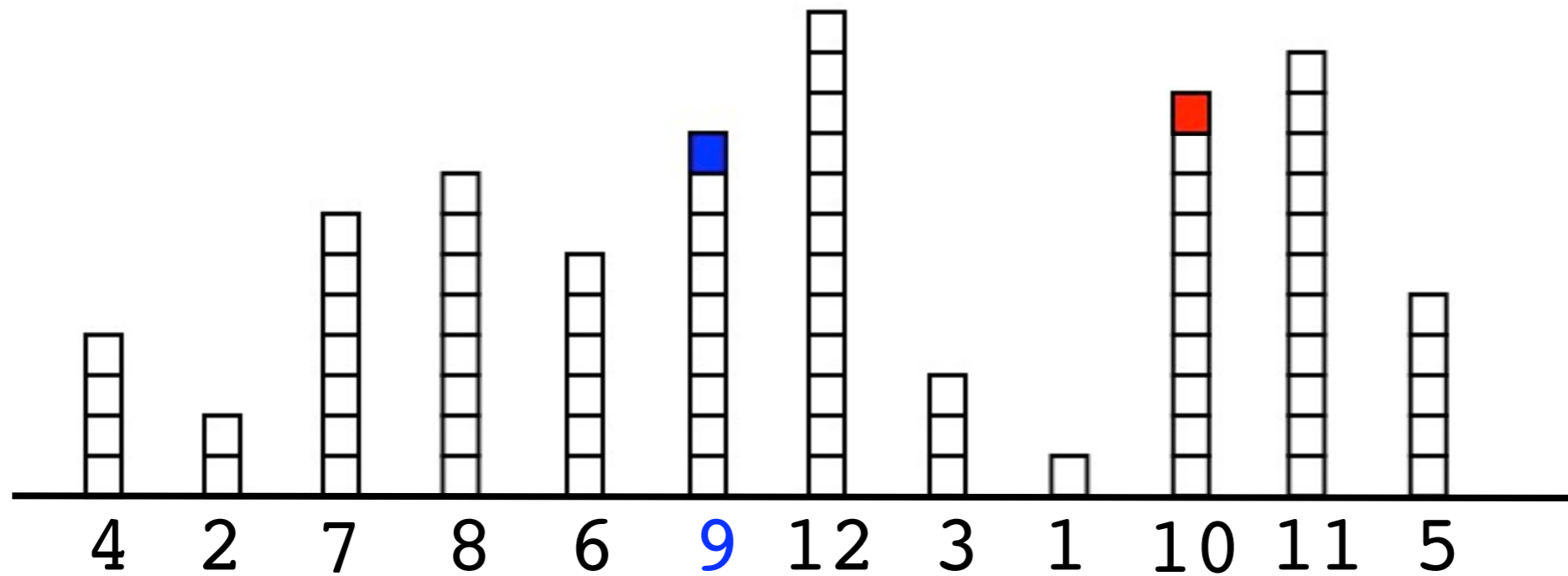


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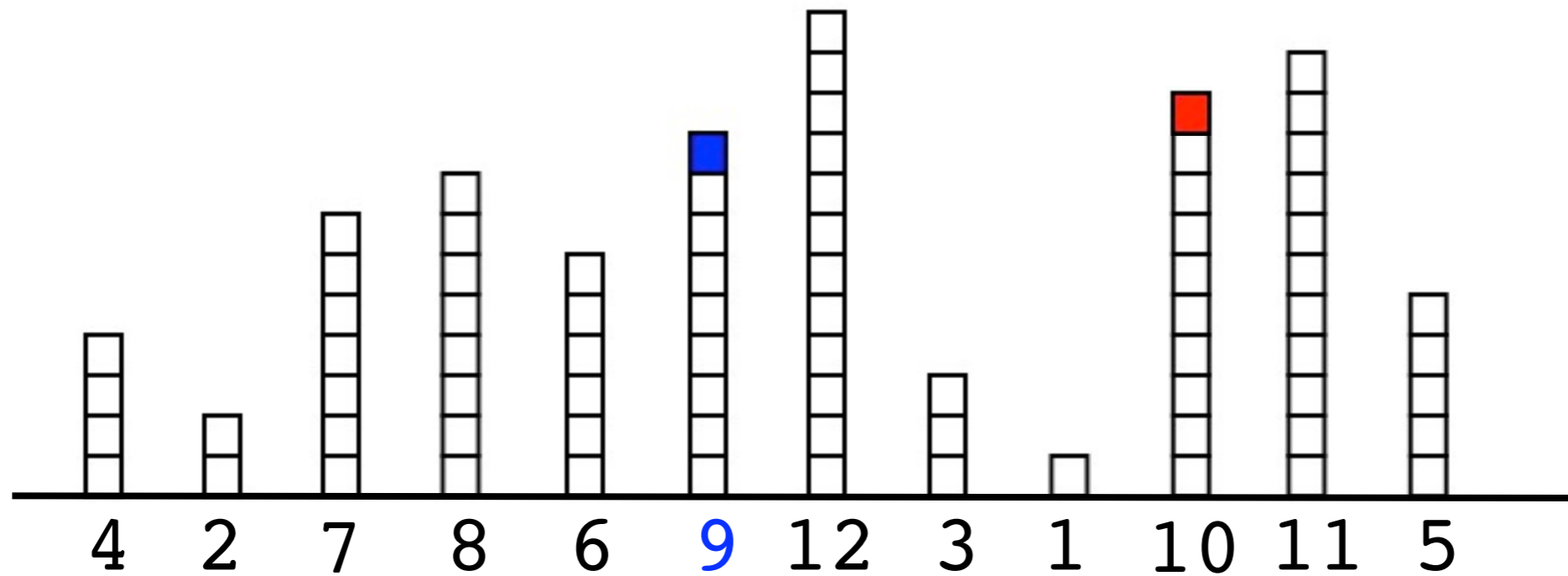


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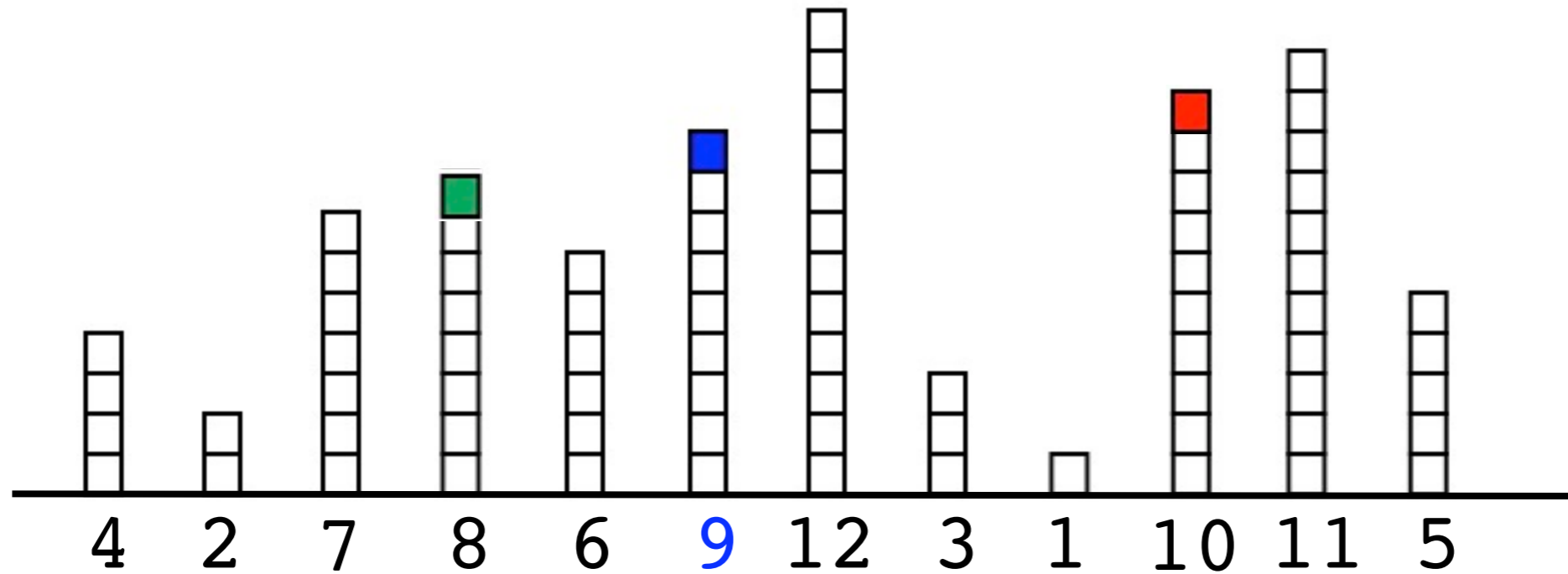
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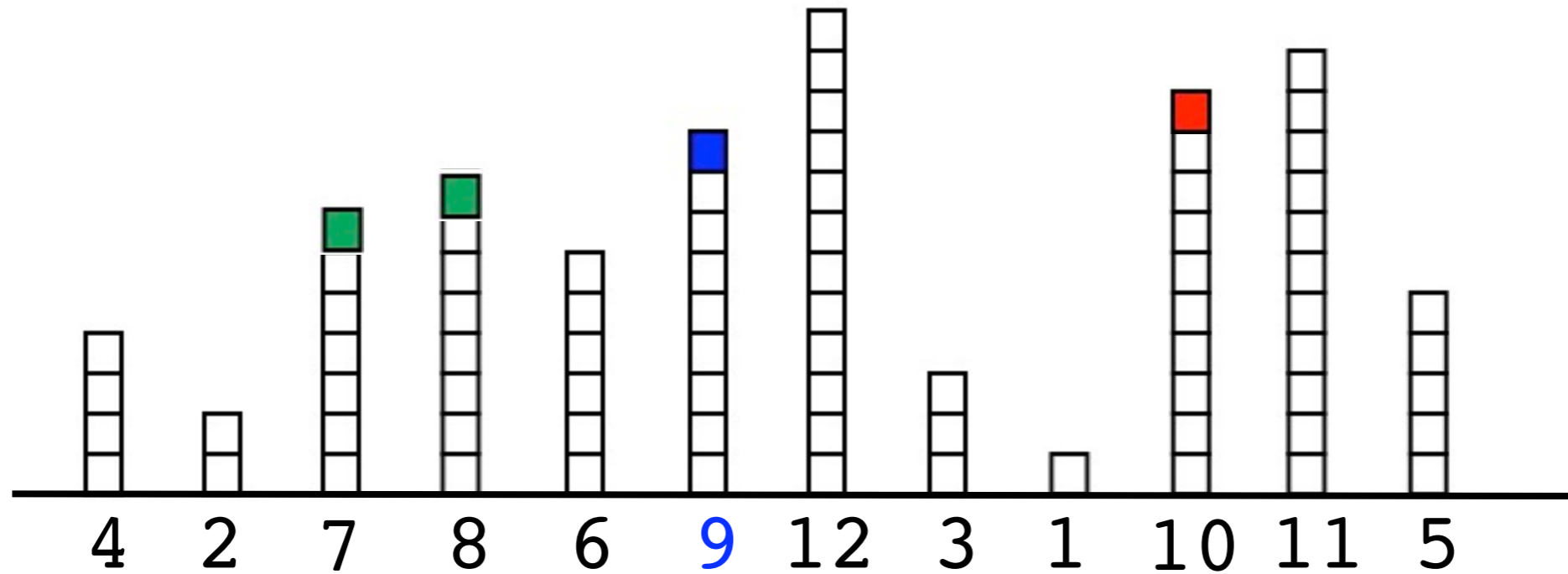
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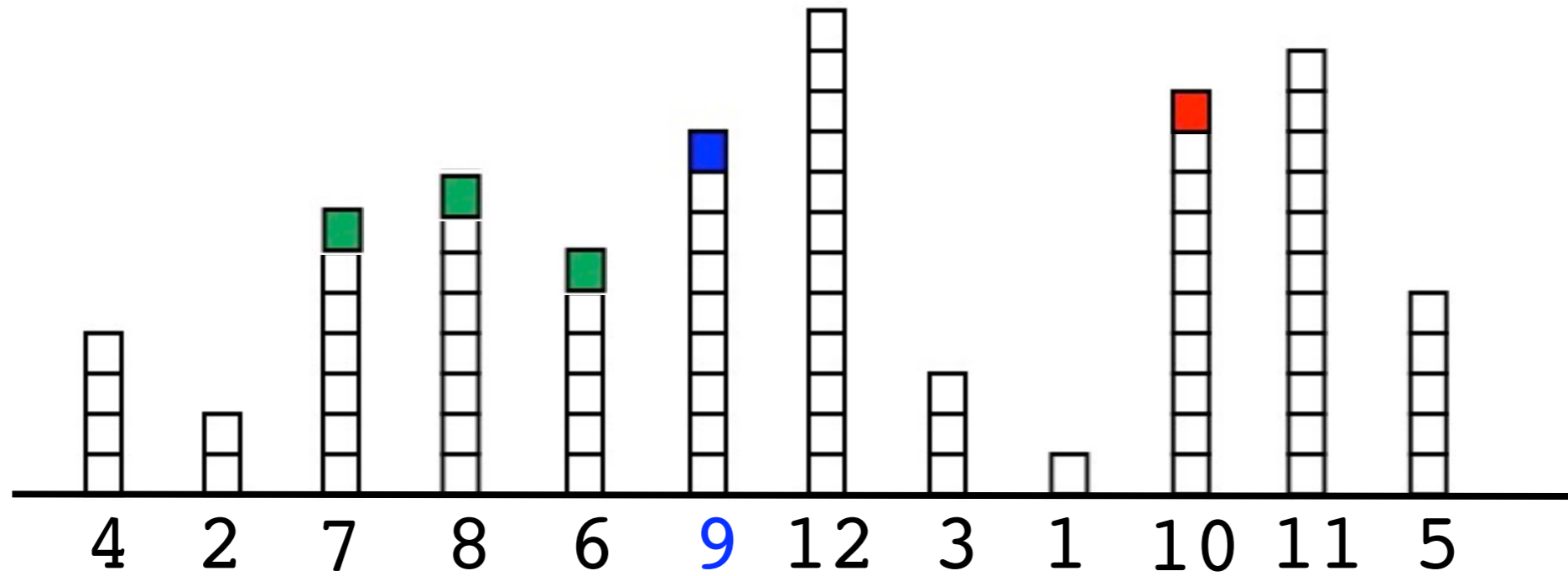
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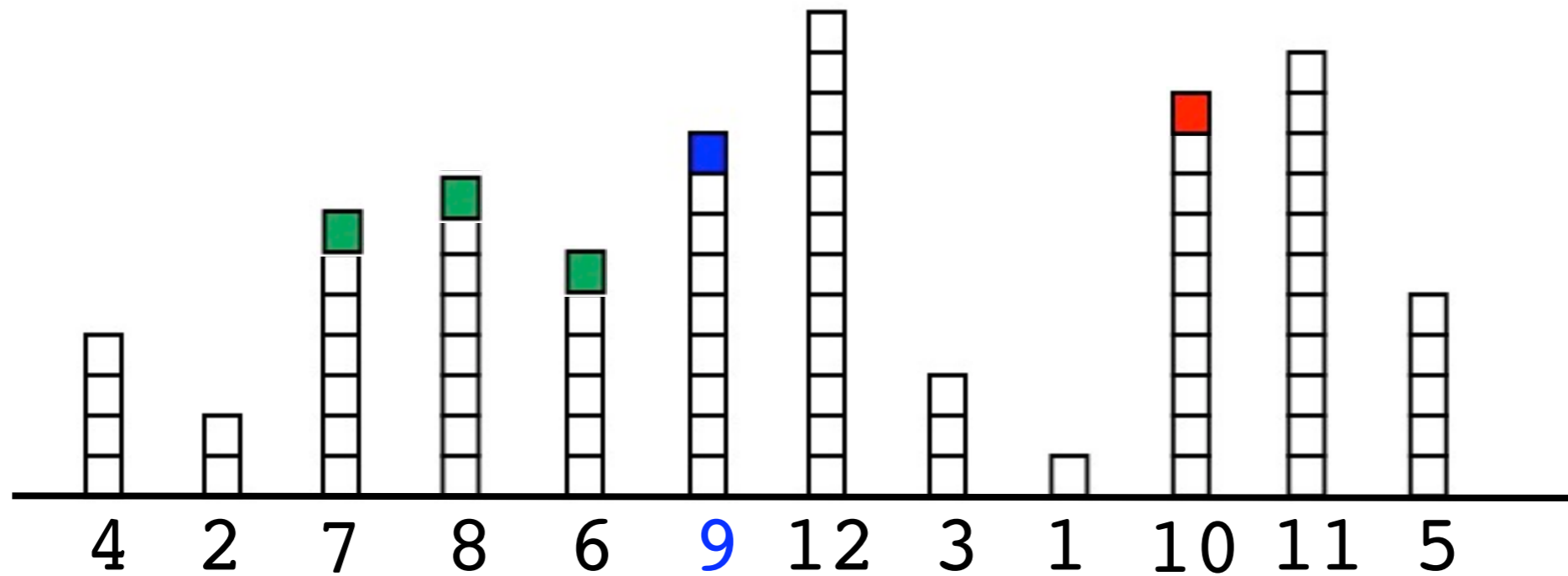
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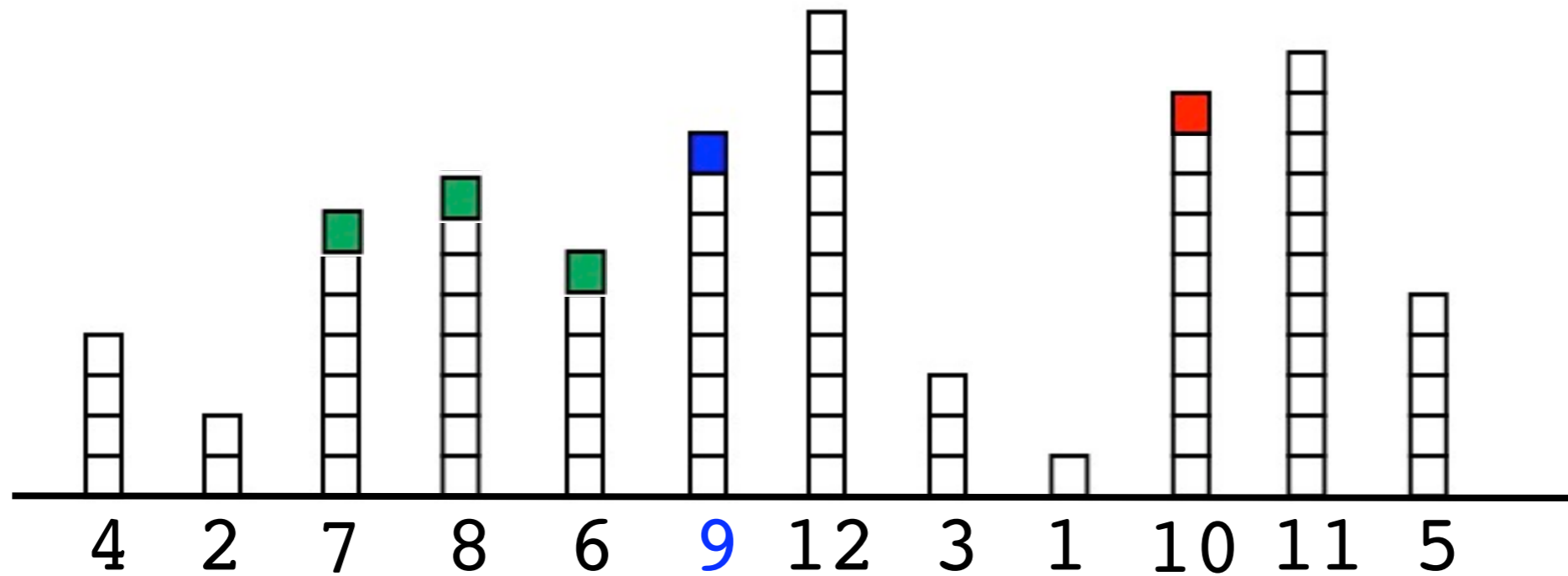
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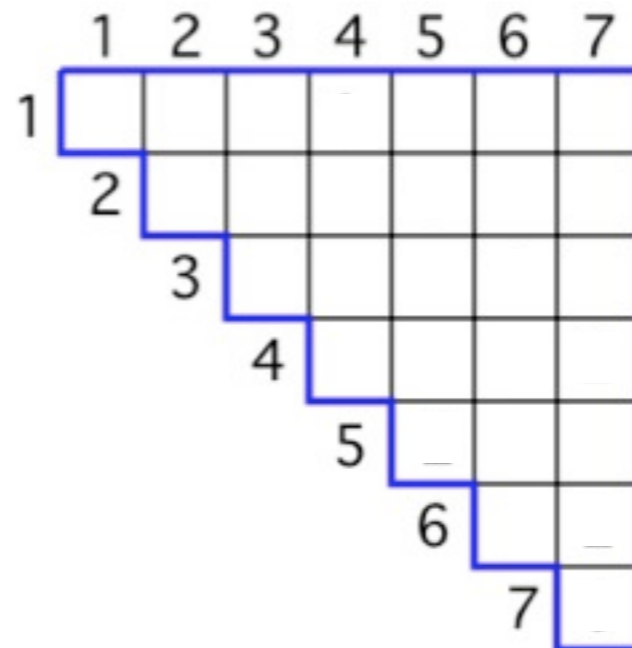
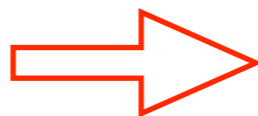
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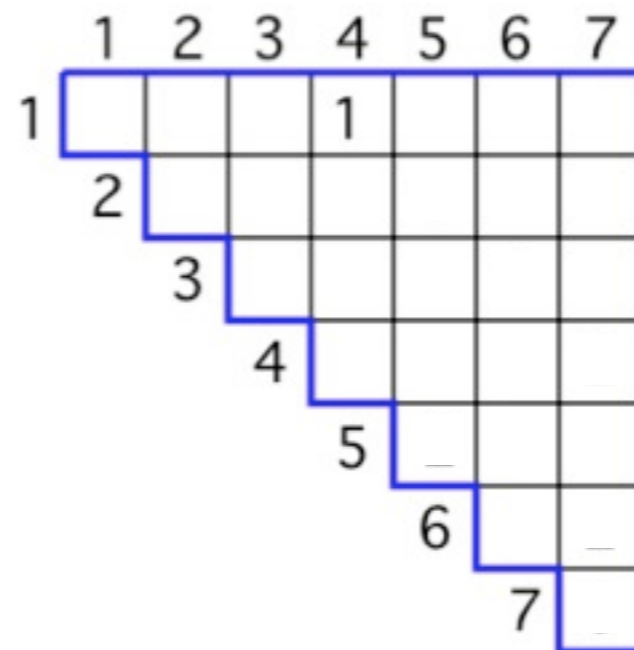
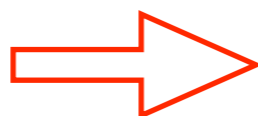
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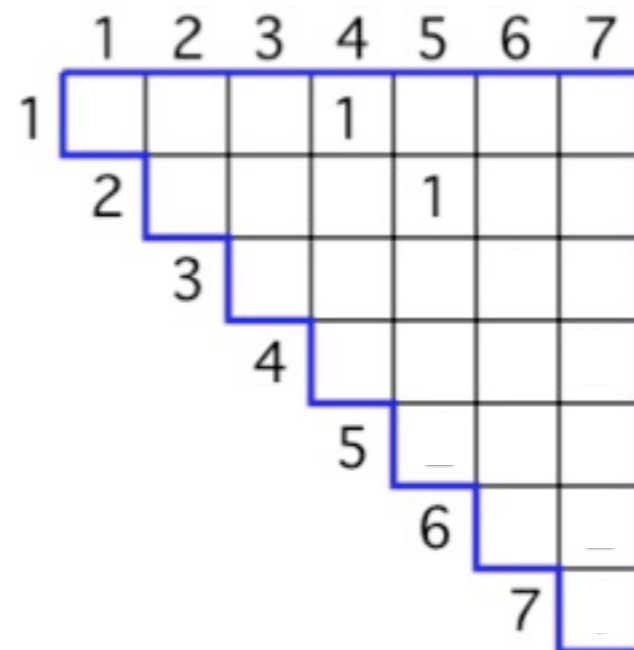
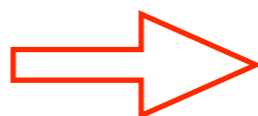
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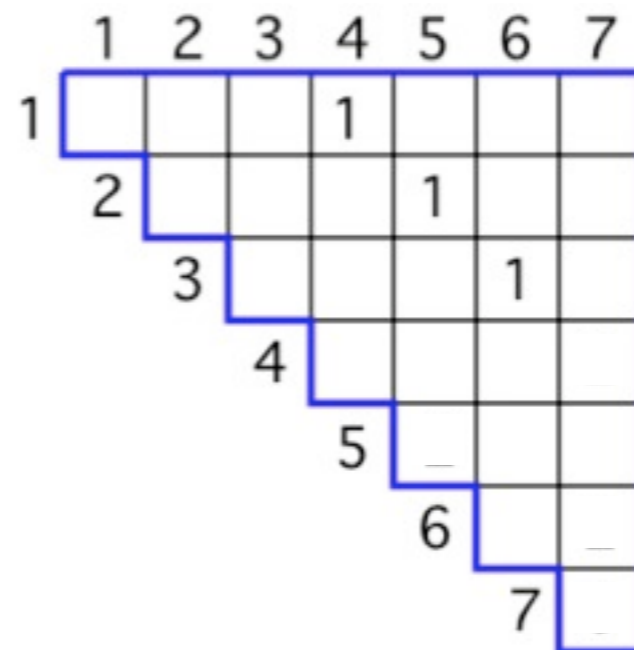
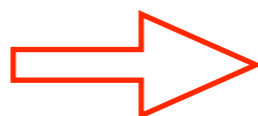
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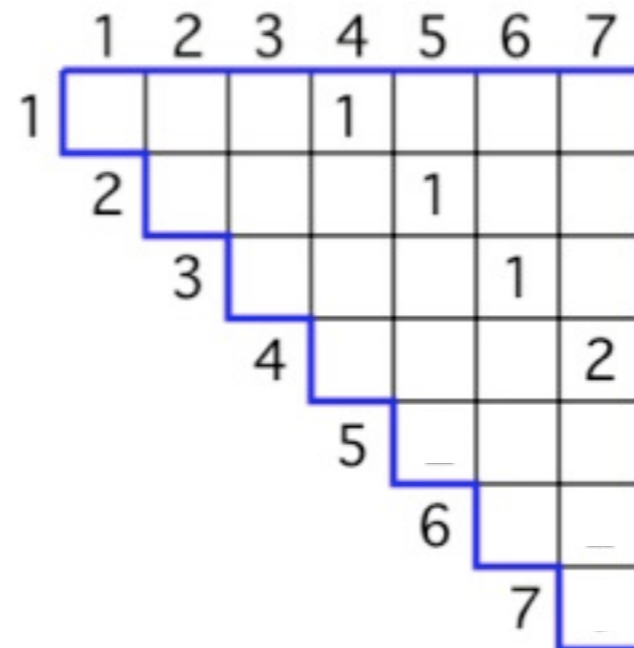
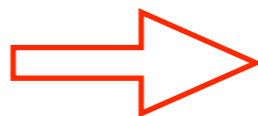
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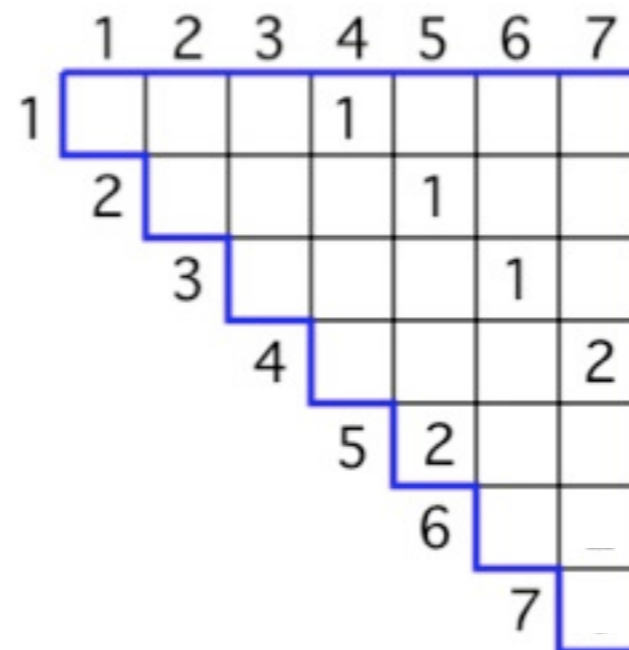
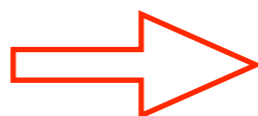
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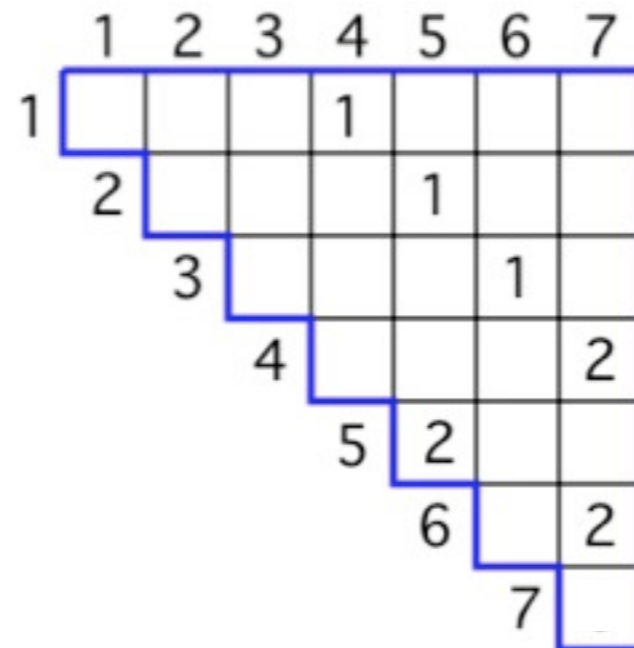
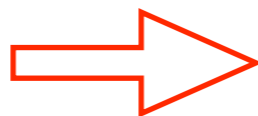
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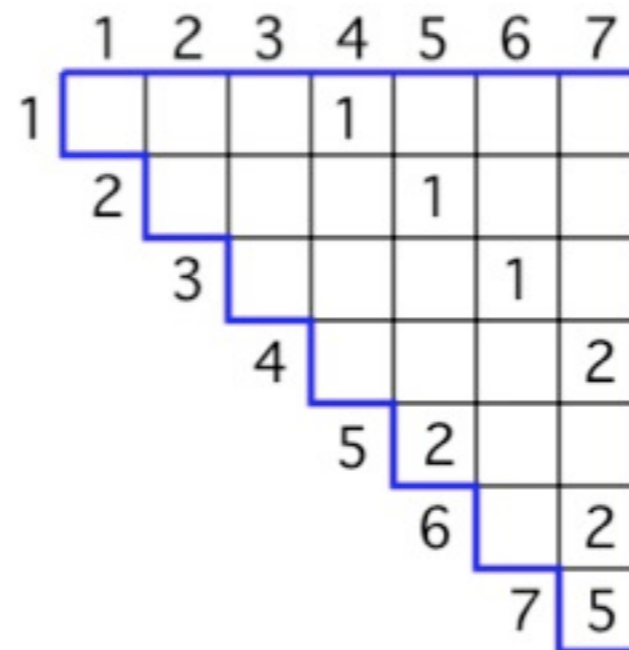
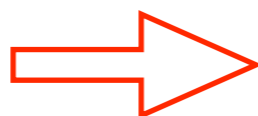
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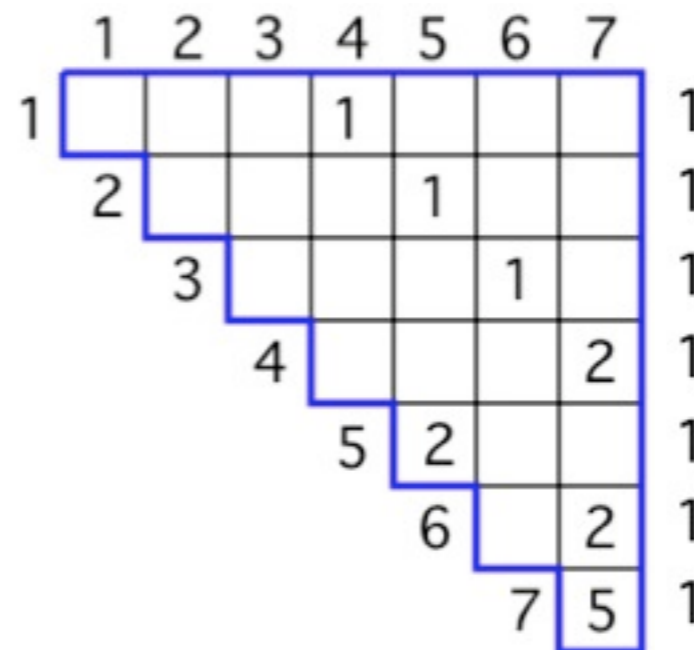
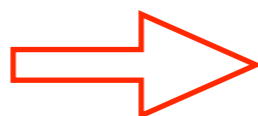
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Let  $\alpha$  be a filling permutation, to construct  $A(\alpha) \in \Pi[1, 1, \dots, 1]$  we simply set

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# q-Counting Permutational Tesler matrices

**Theorem** (P.Levande)

For any  $\alpha \in S_n$  set

$$P_\alpha(q) = \sum_{\text{filling}(pf)=\alpha} q^{\text{area}(pf)}$$

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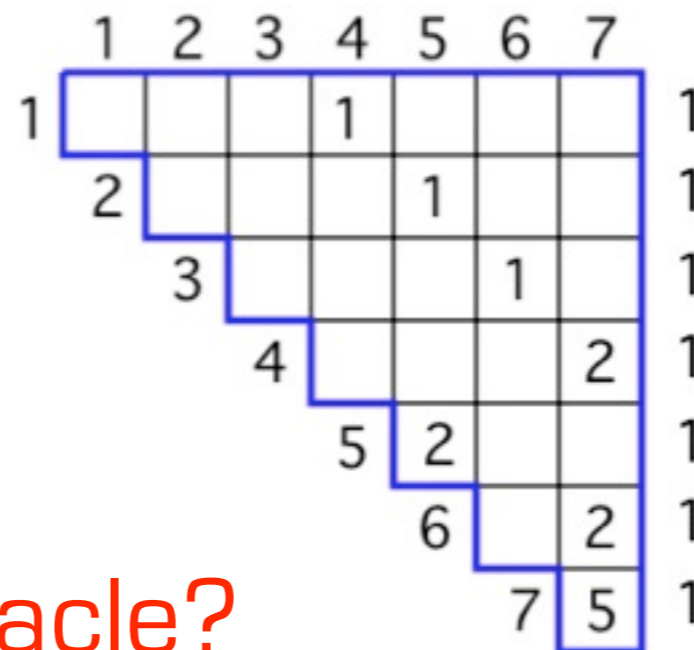
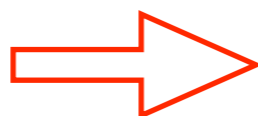
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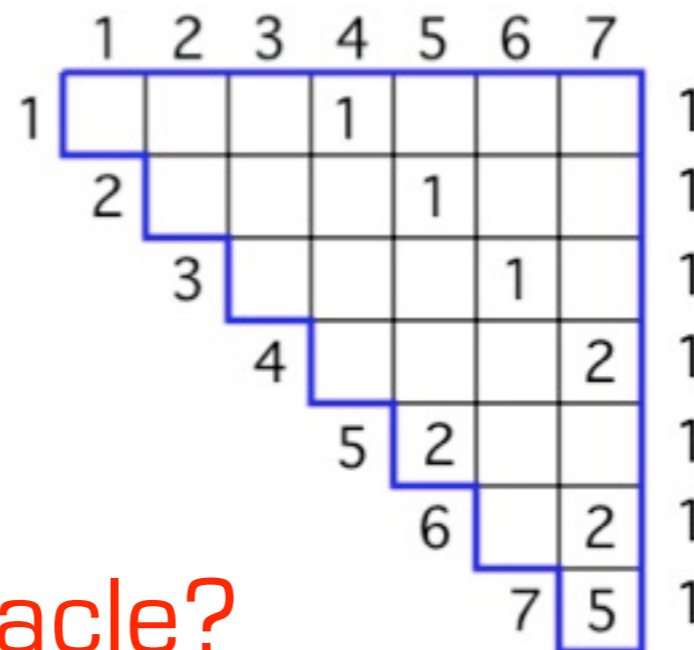
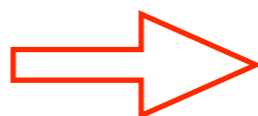
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next



# Constructing all preference functions with the same filling permutation

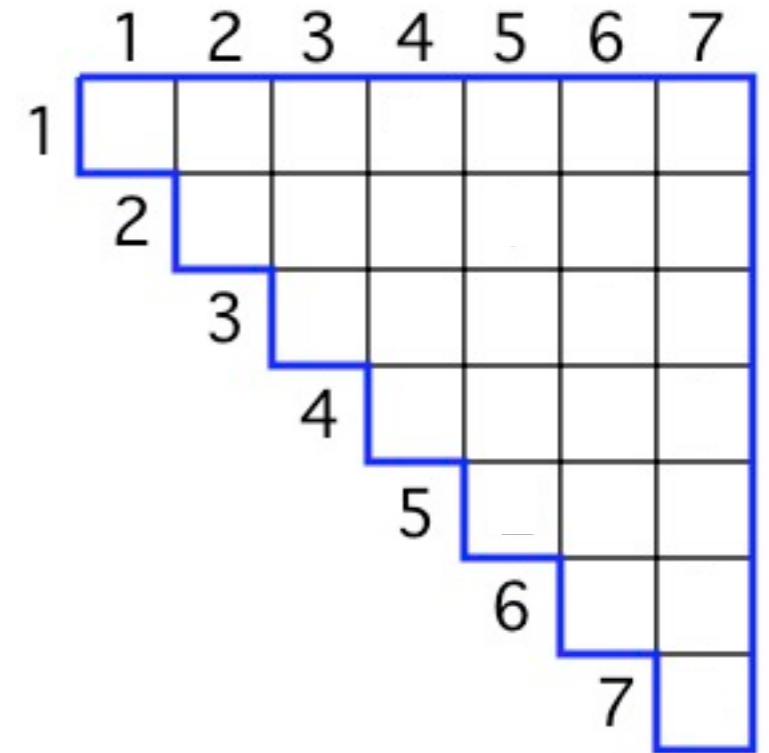


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3 6 1 4 7 2 5

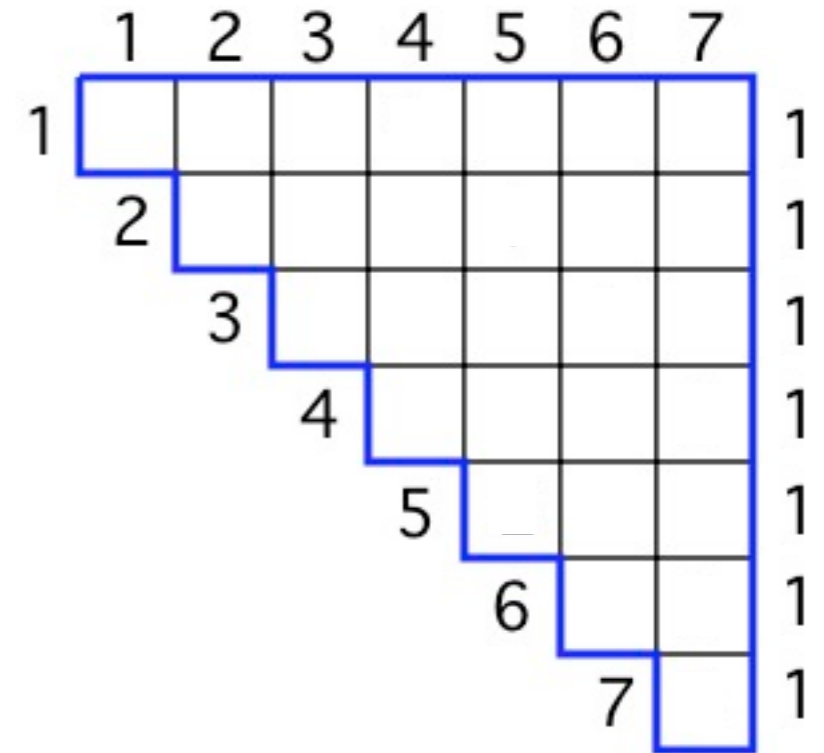
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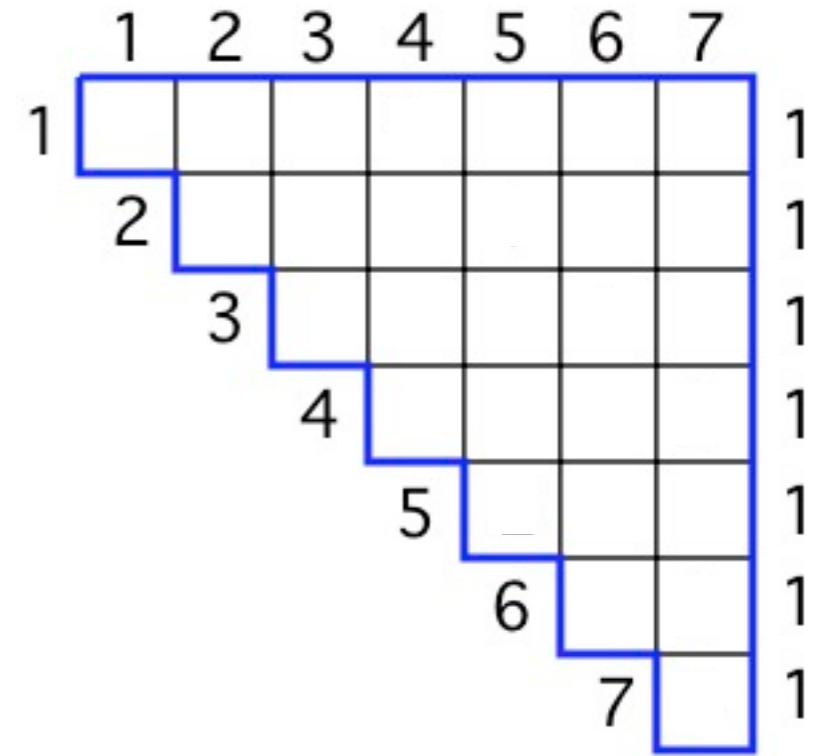
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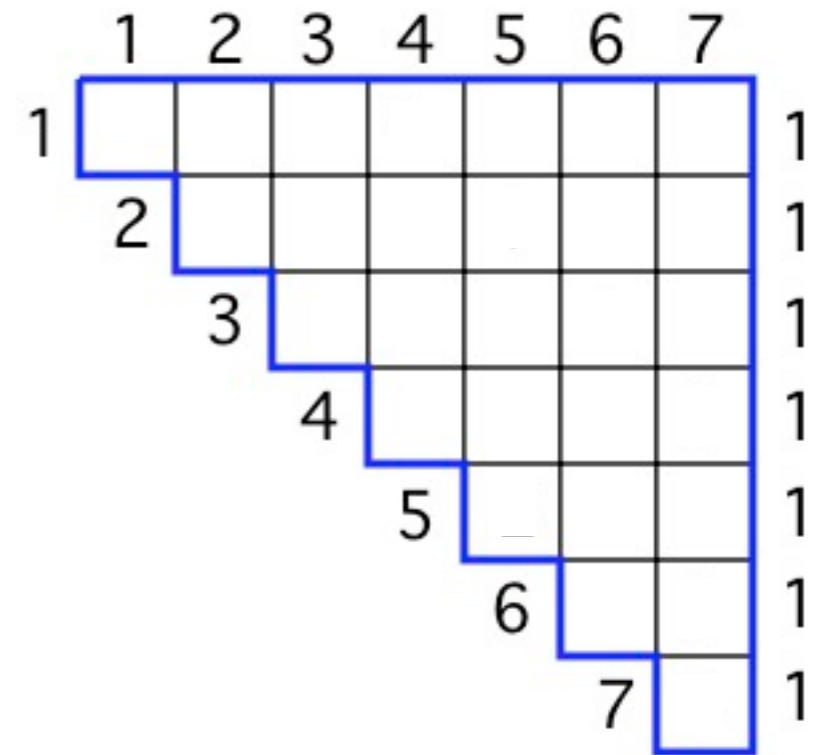
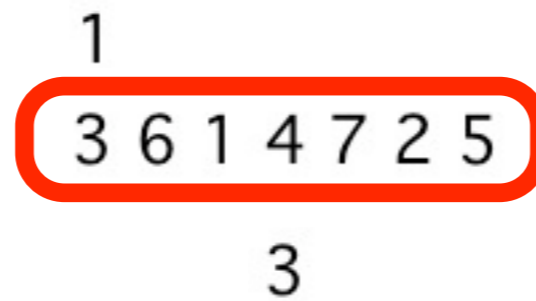


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1  
3 6 1 4 7 2 5



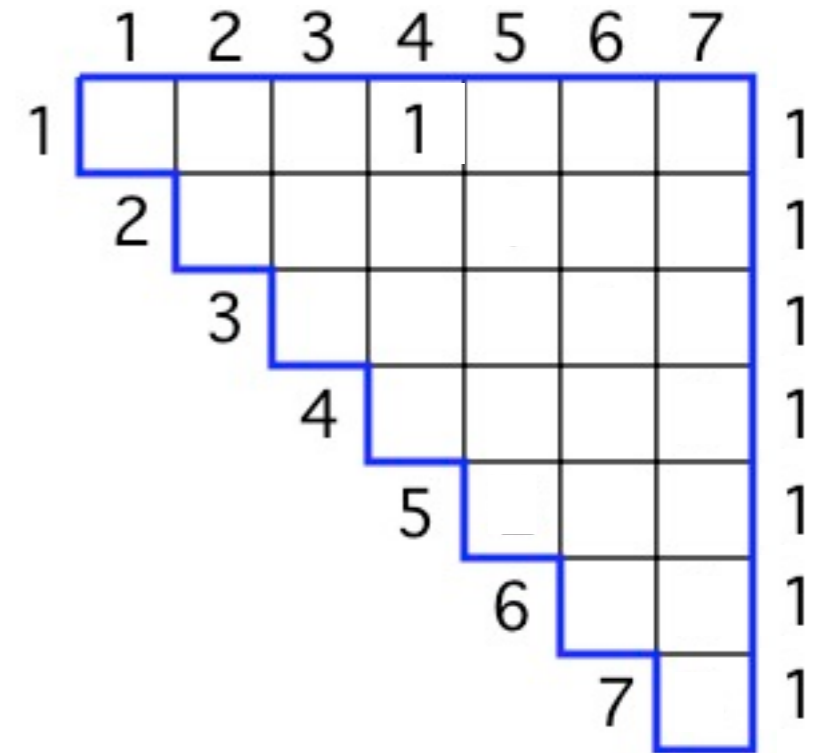
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3 6 1 4 7 2 5

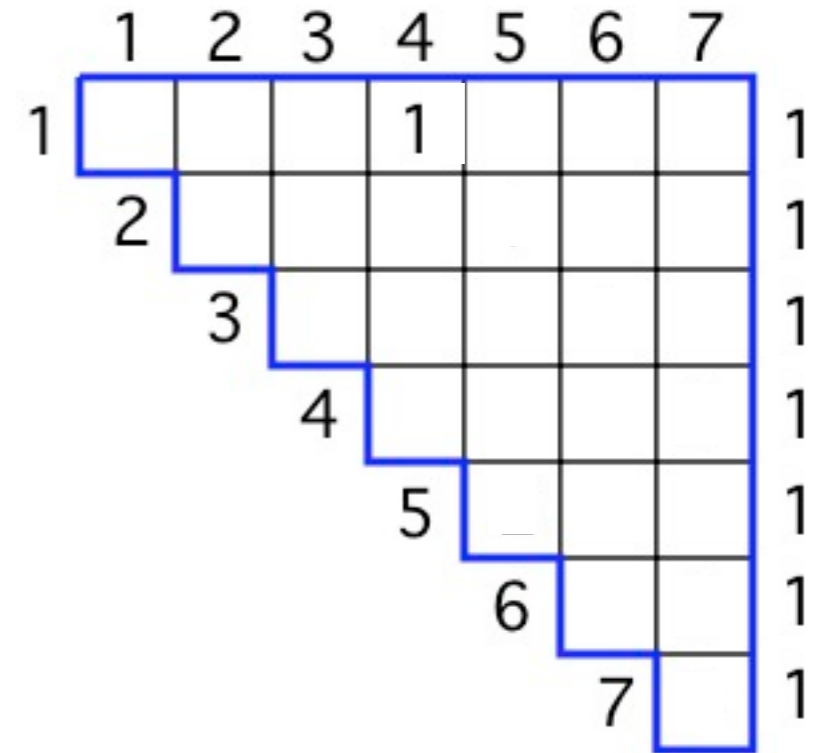
3



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1 2  
3 6 1 4 7 2 5

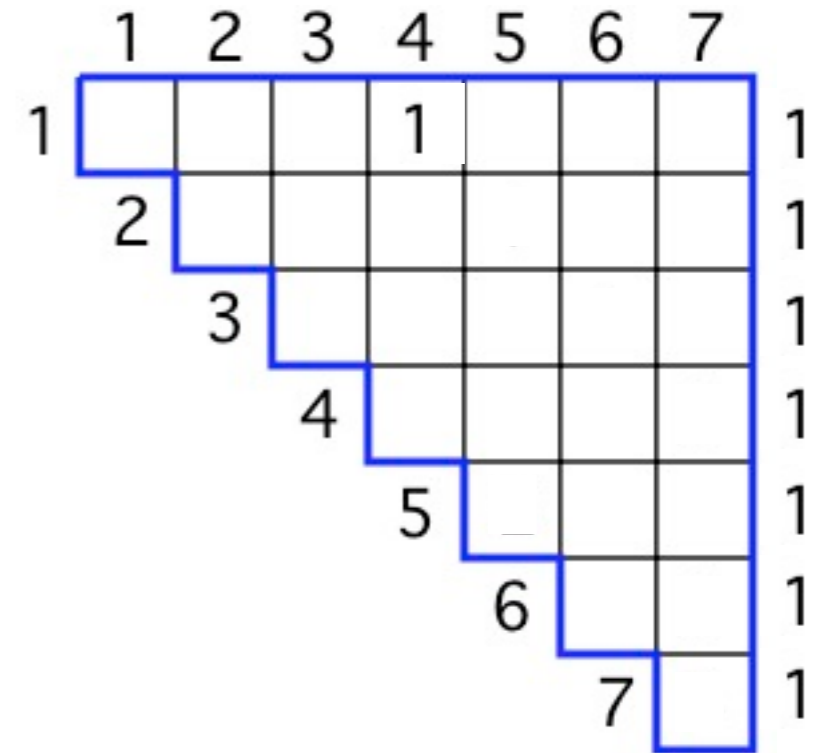
3



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1 2  
3 6 1 4 7 2 5

3  
↓  
6

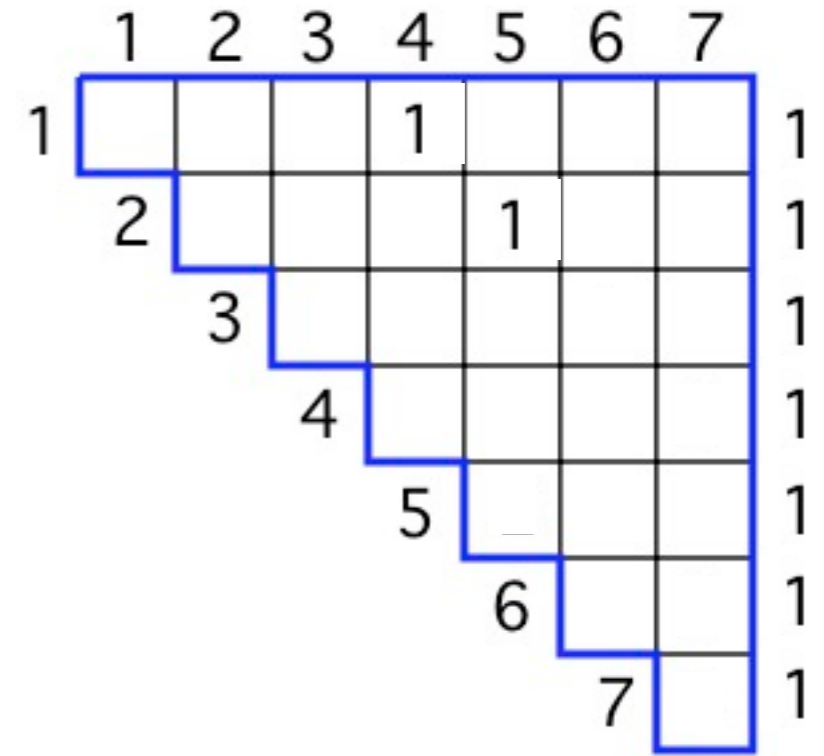




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1 2  
3 6 1 4 7 2 5

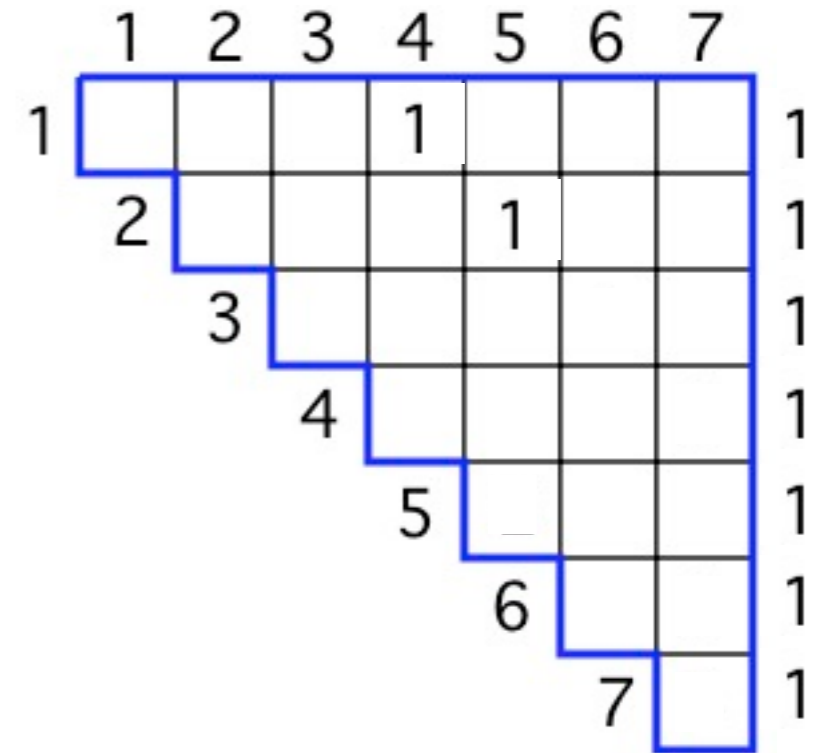
3  
↓  
6



# Constructing all preference functions with the same filling permutation

1 2 3  
3 6 1 4 7 2 5

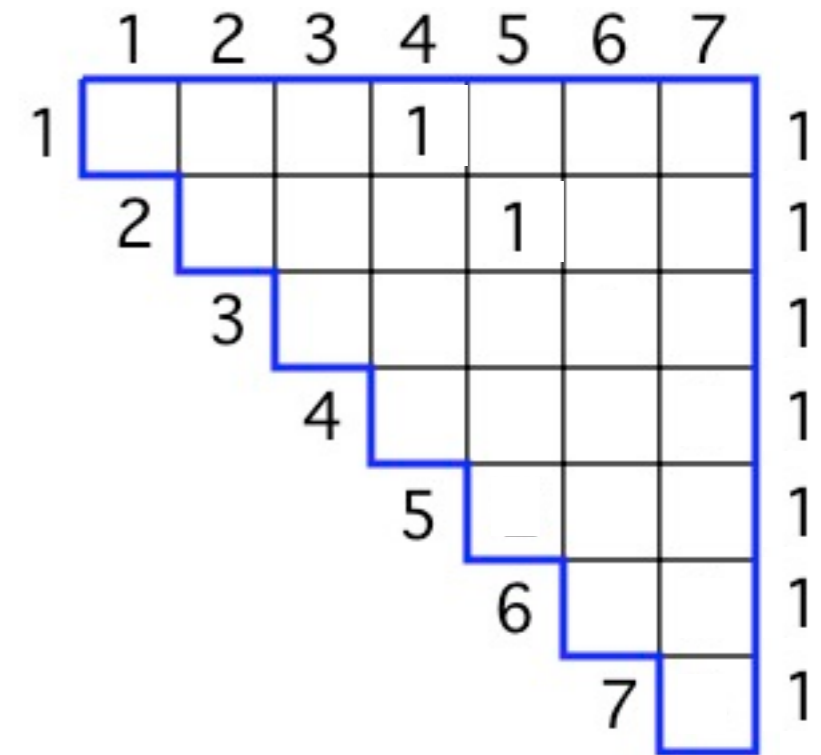
3  
↓  
6



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1 2 3  
3 6 1 4 7 2 5

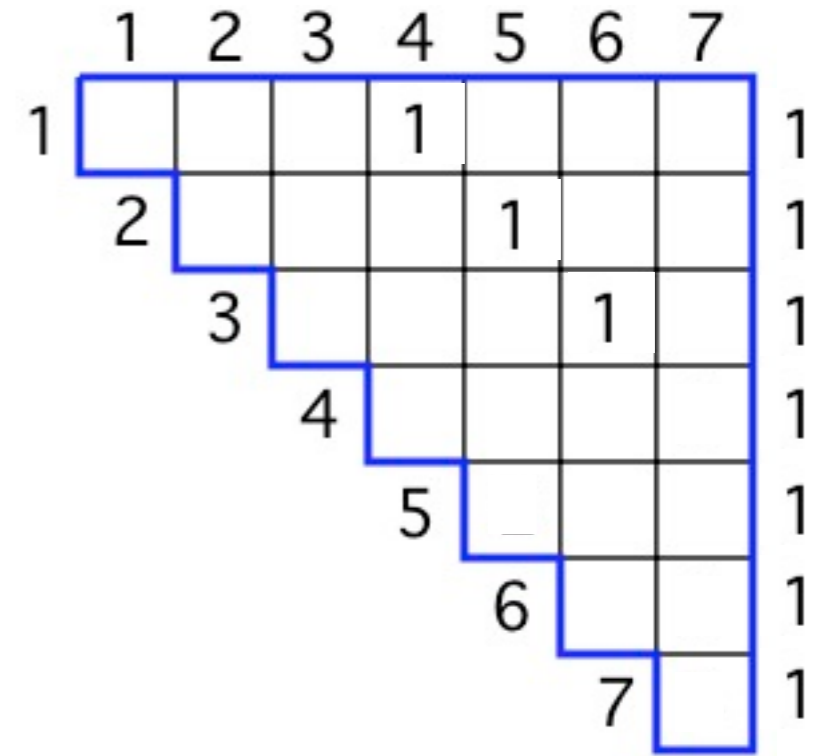
3  
↓  
6  
↓  
1



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1 2 3  
3 6 1 4 7 2 5

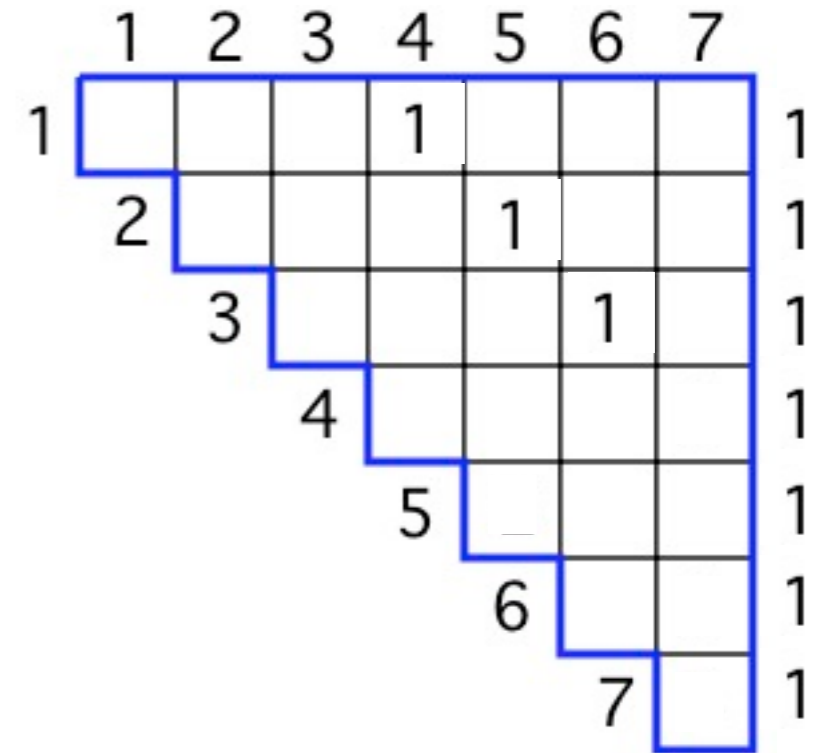
3  
↓  
6  
↓  
1



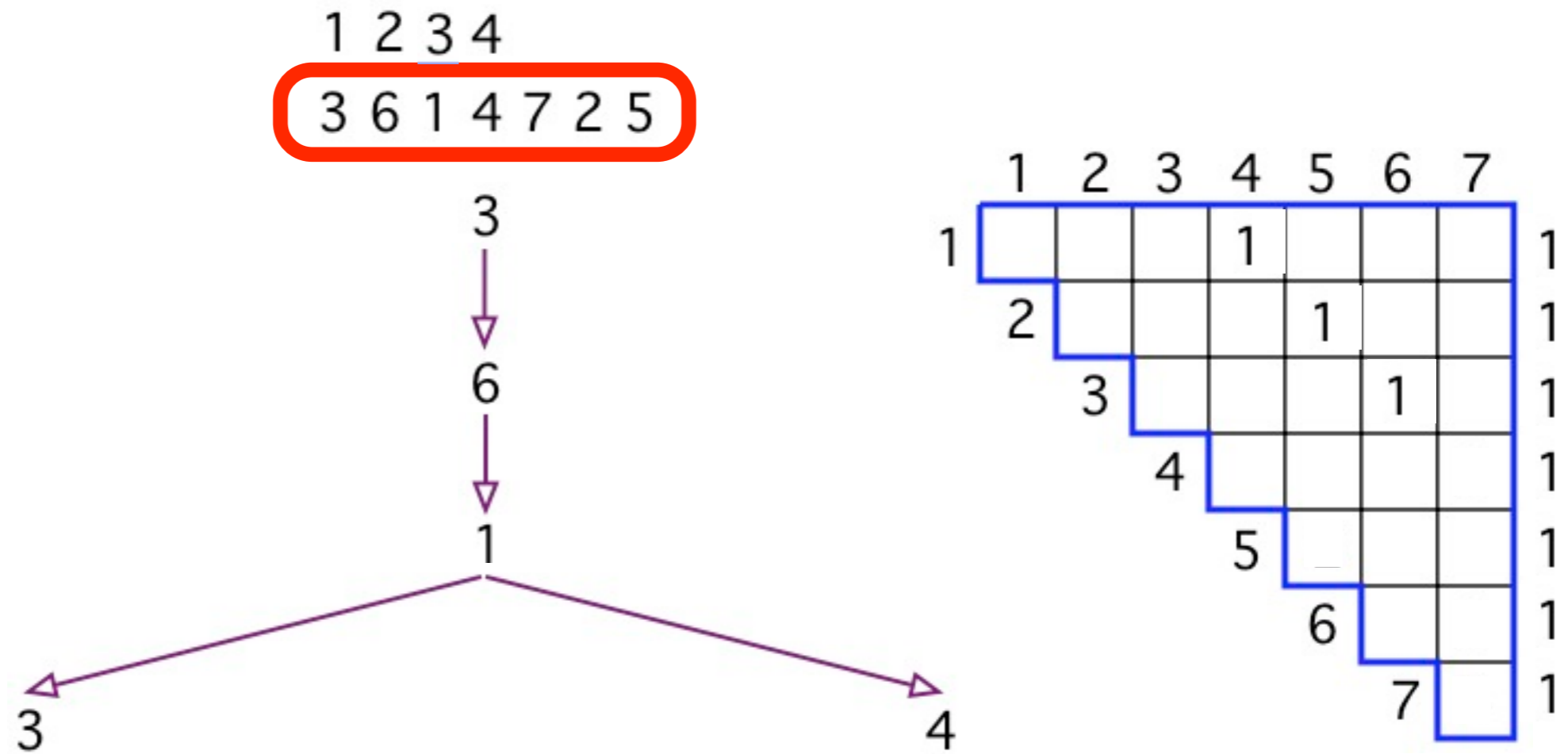
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1 2 3 4  
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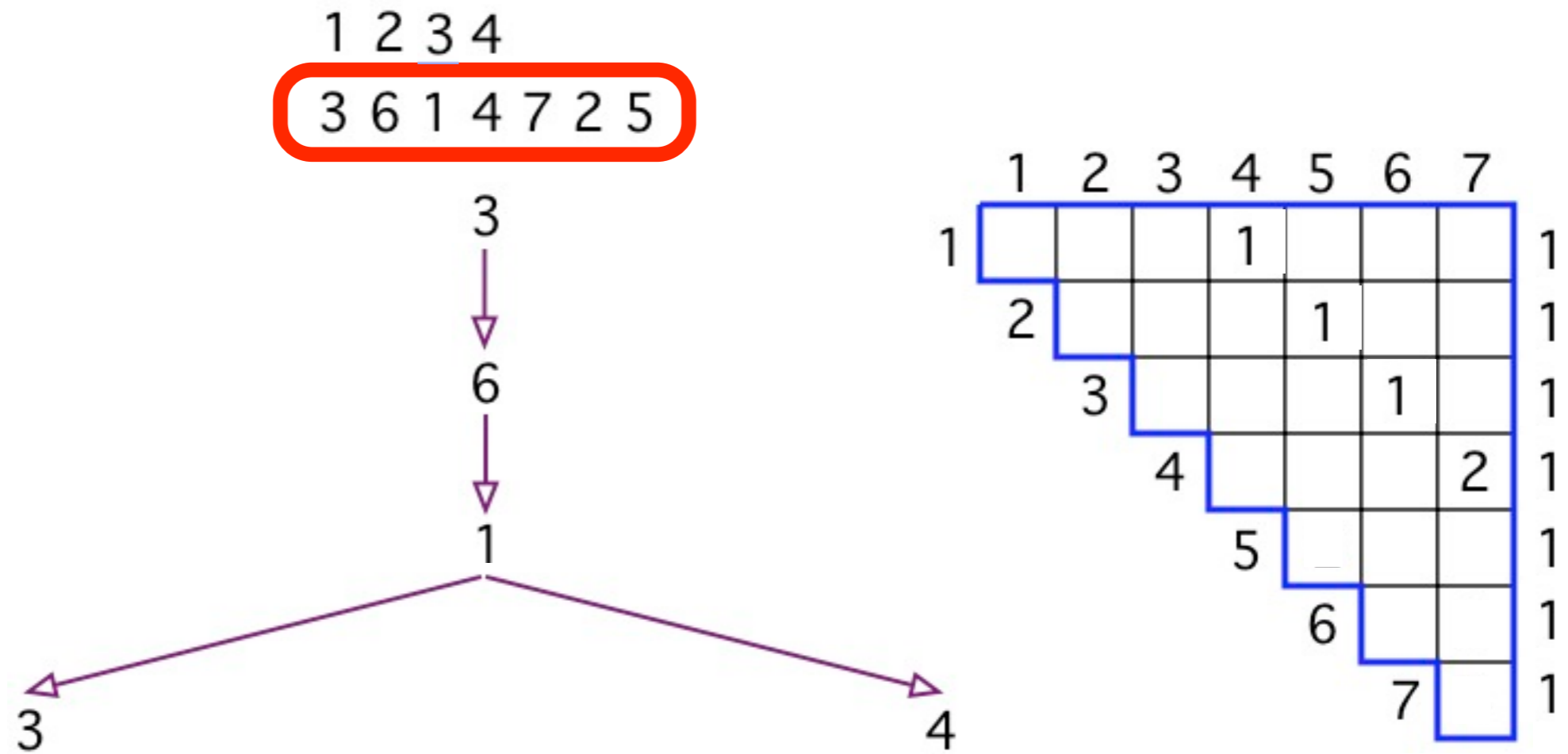
3  
↓  
6  
↓  
1



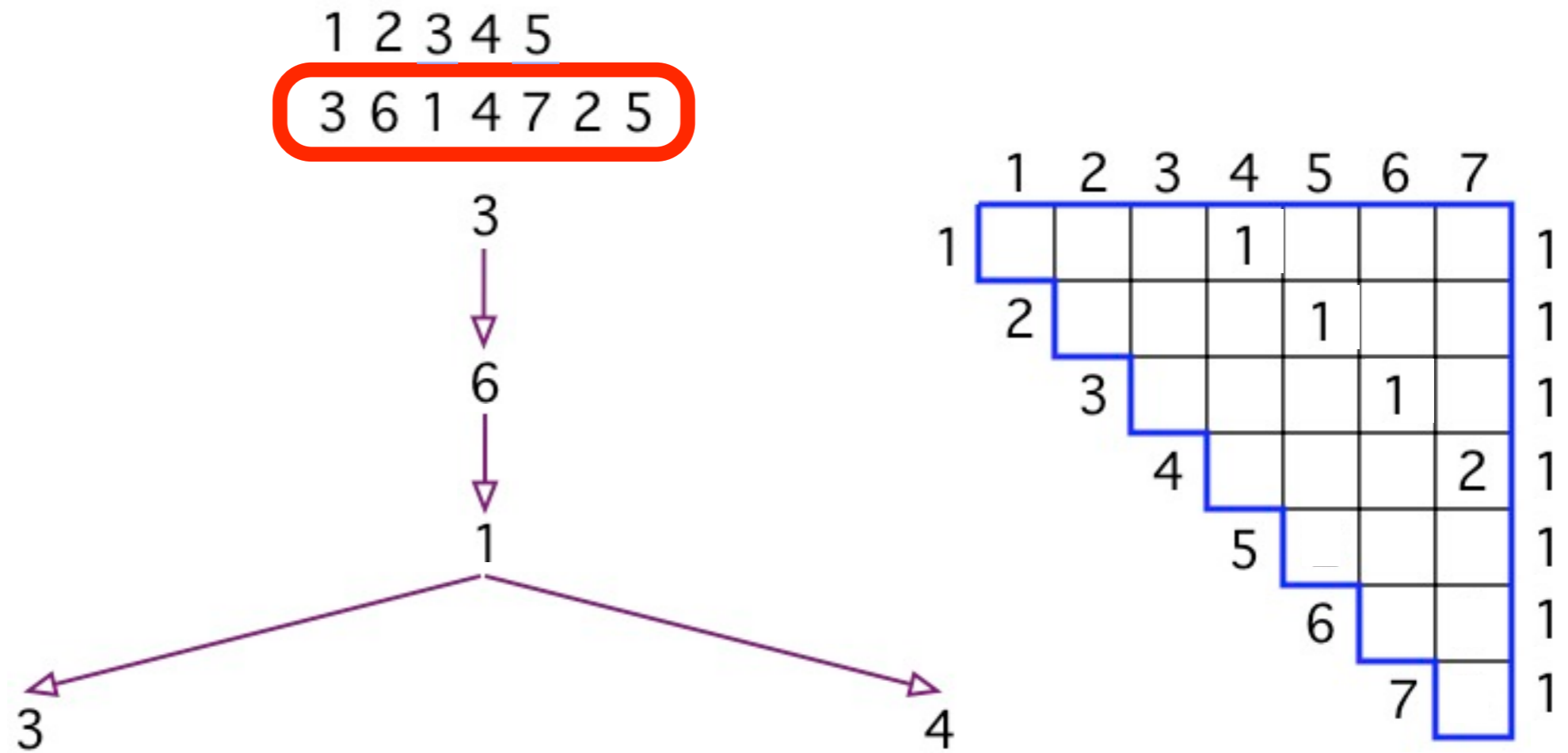
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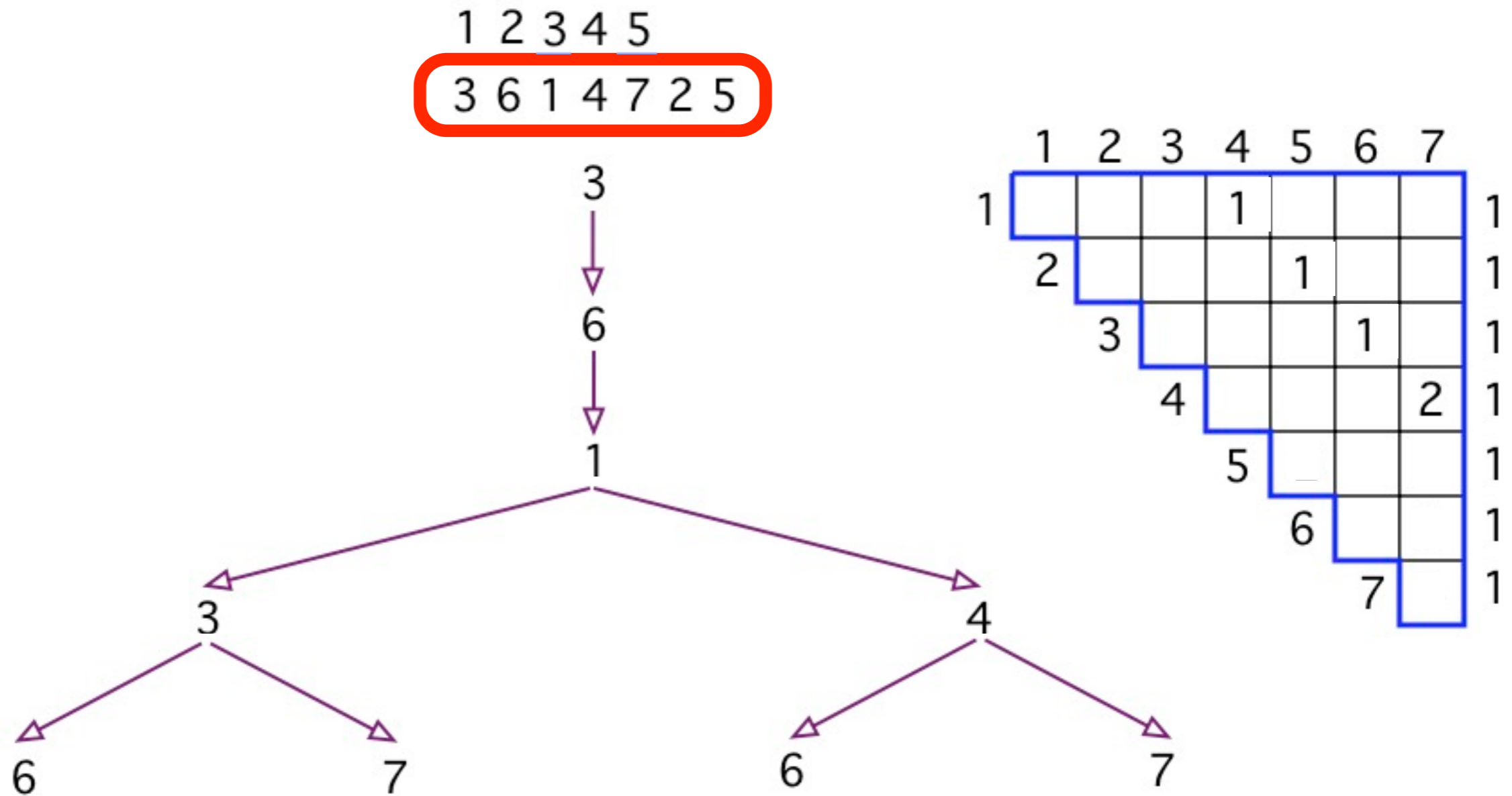


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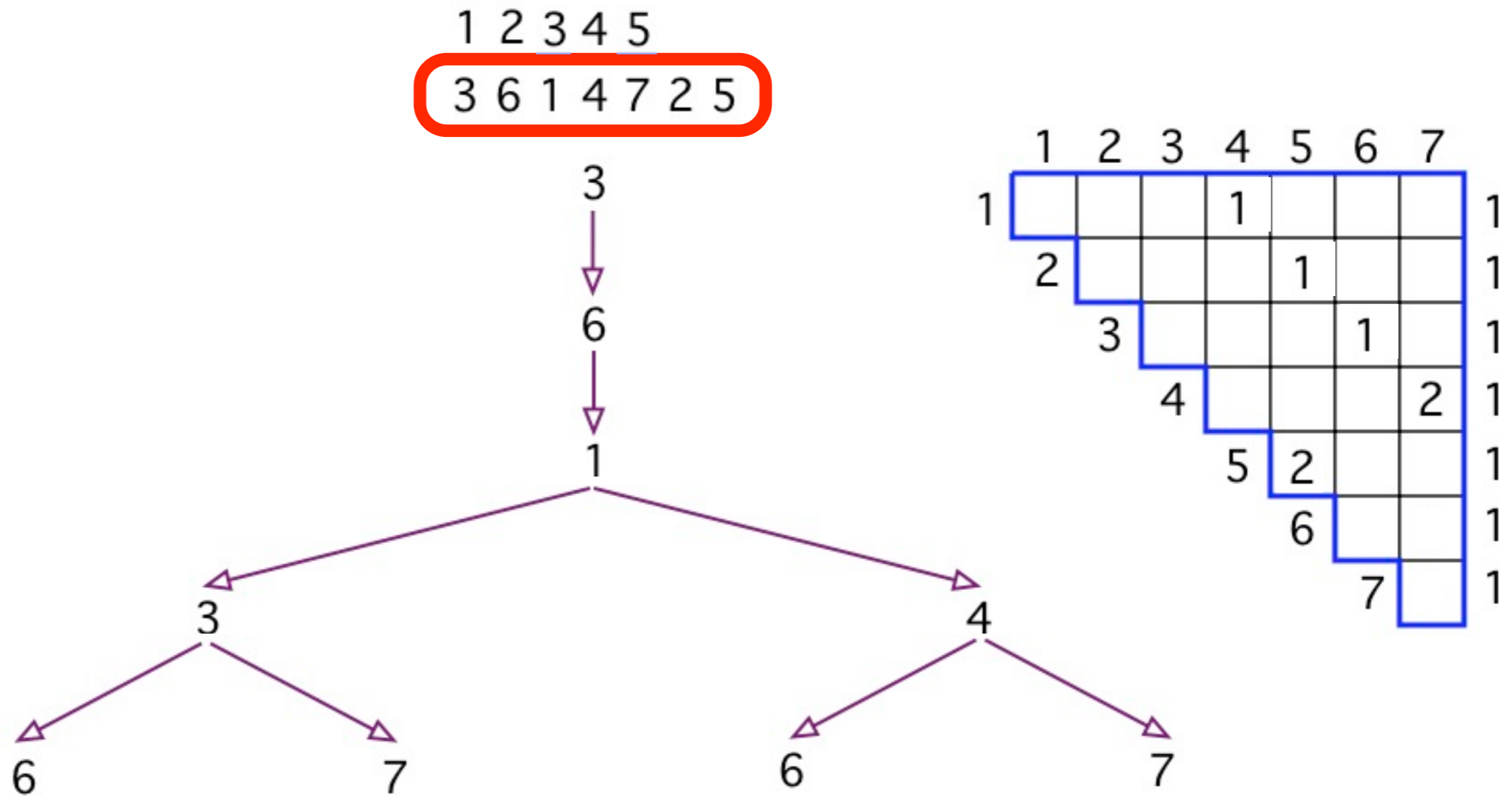




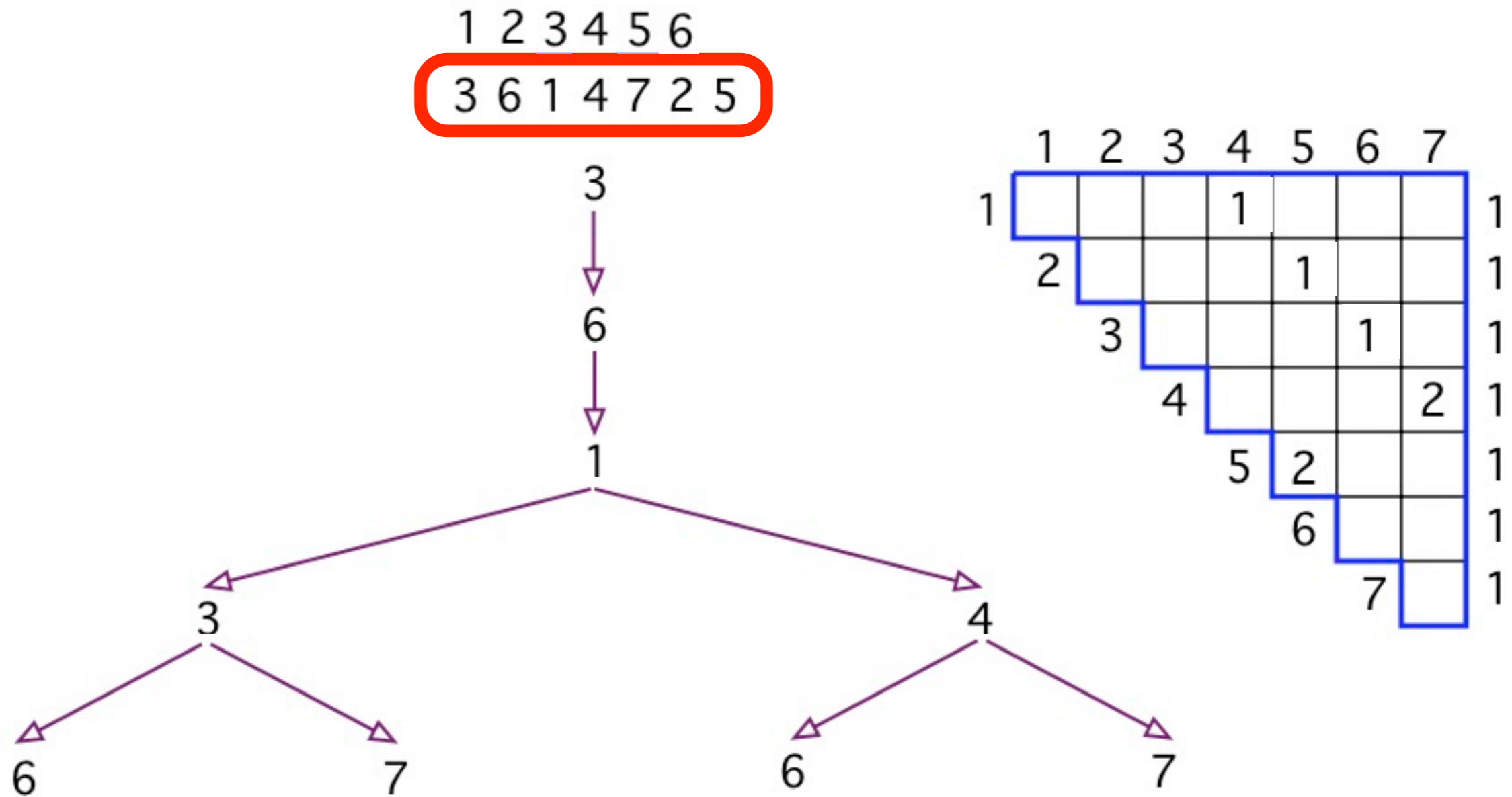
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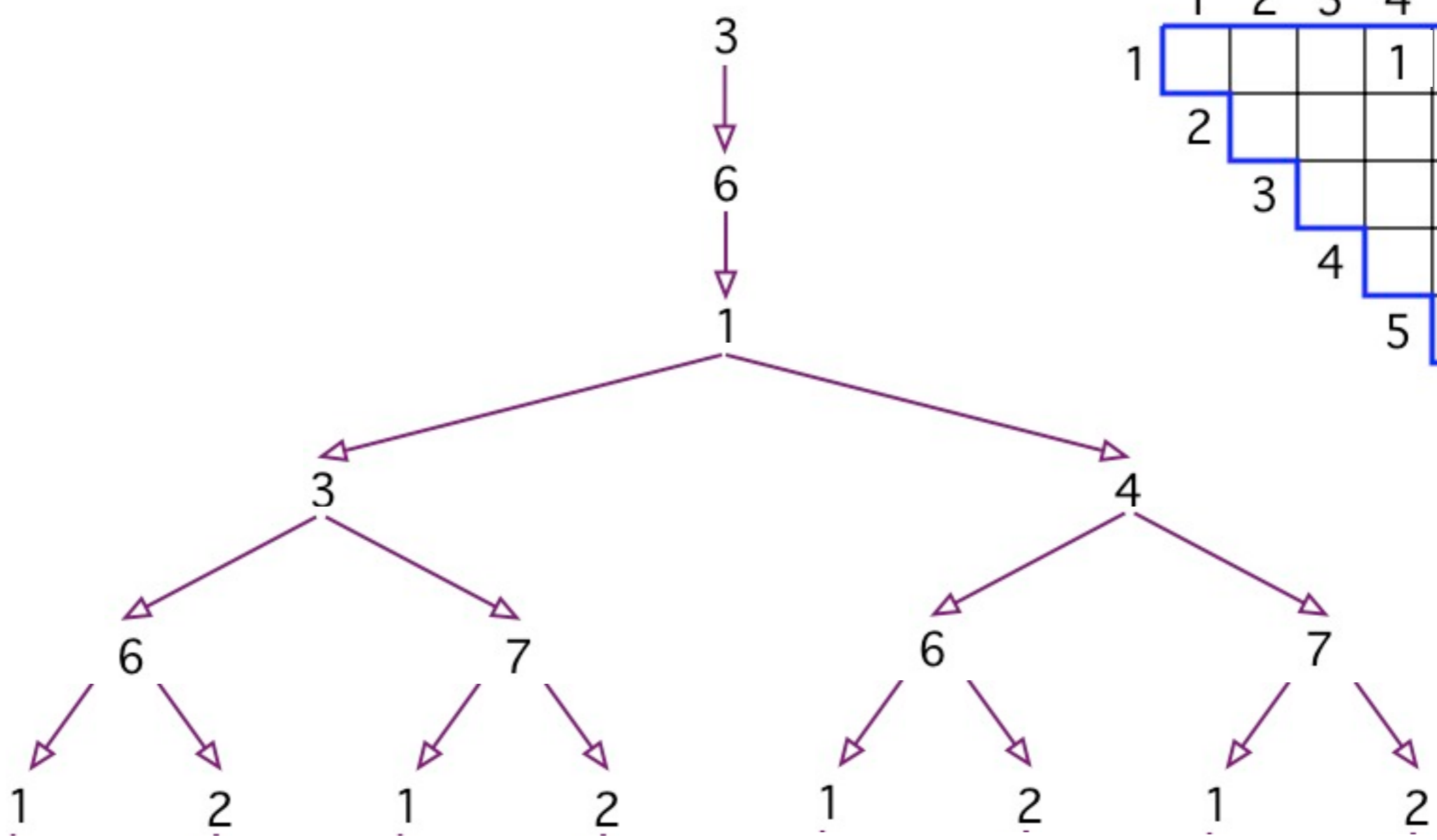
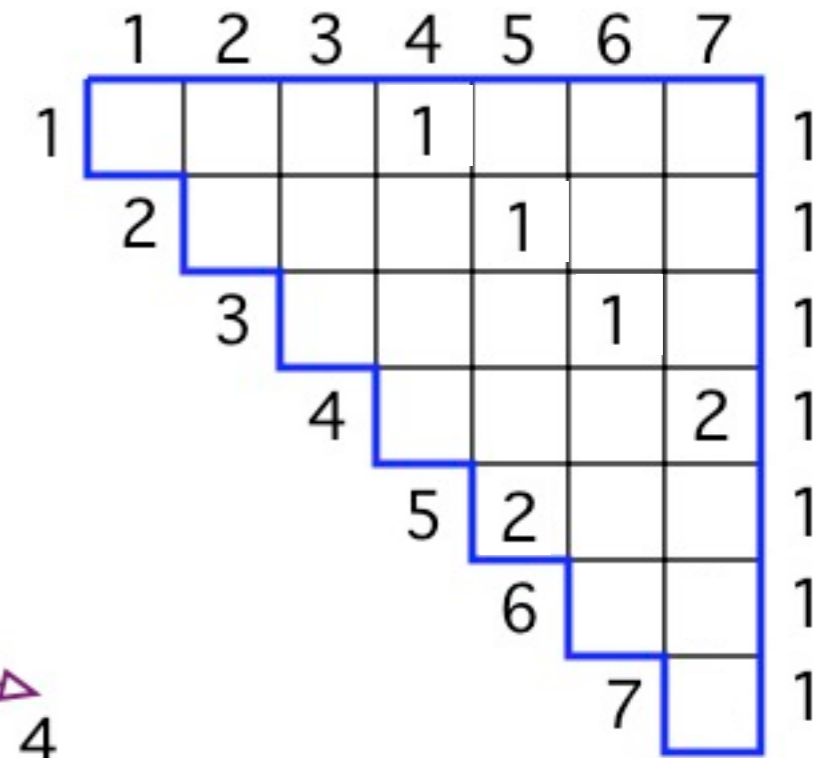
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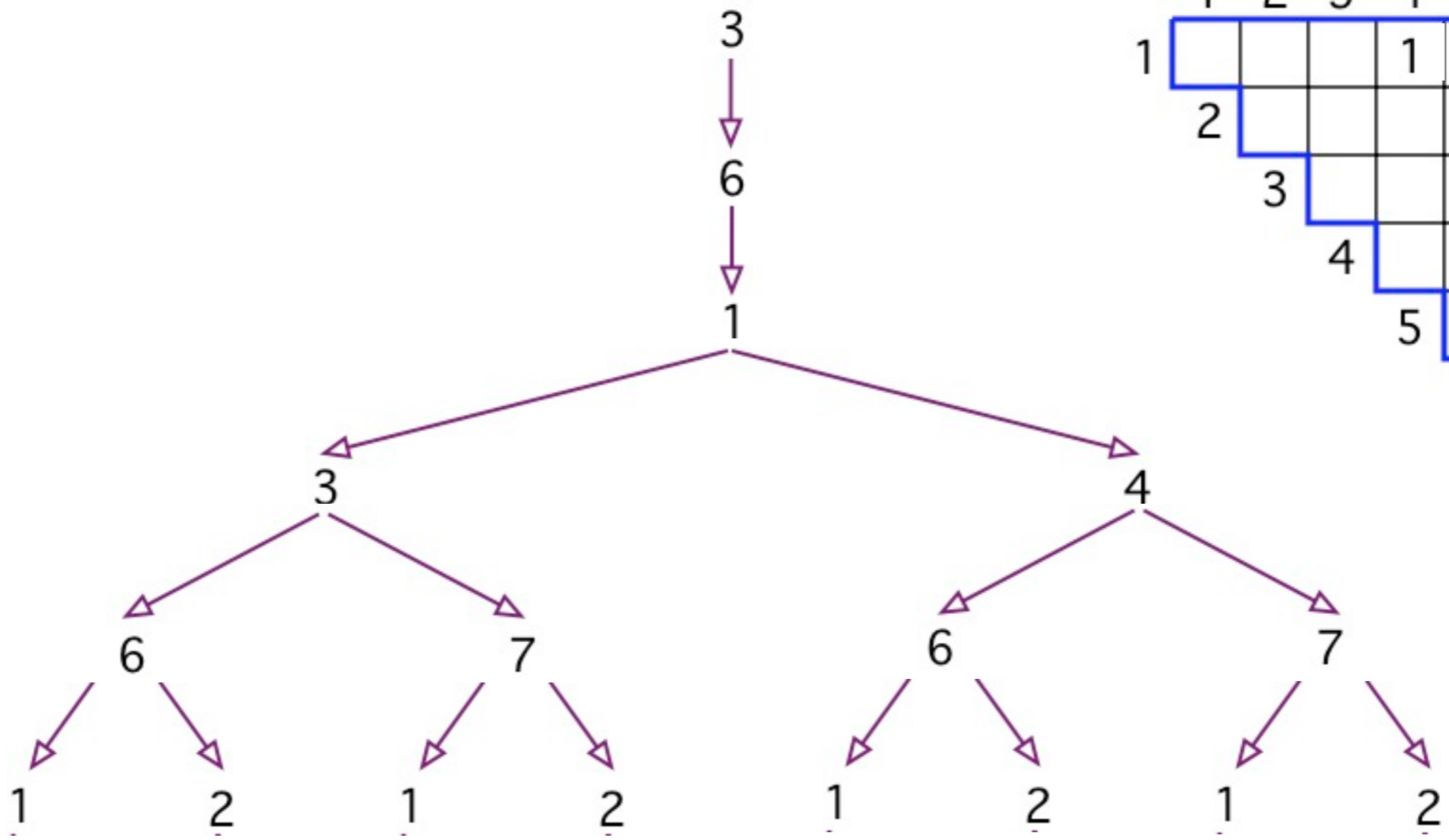
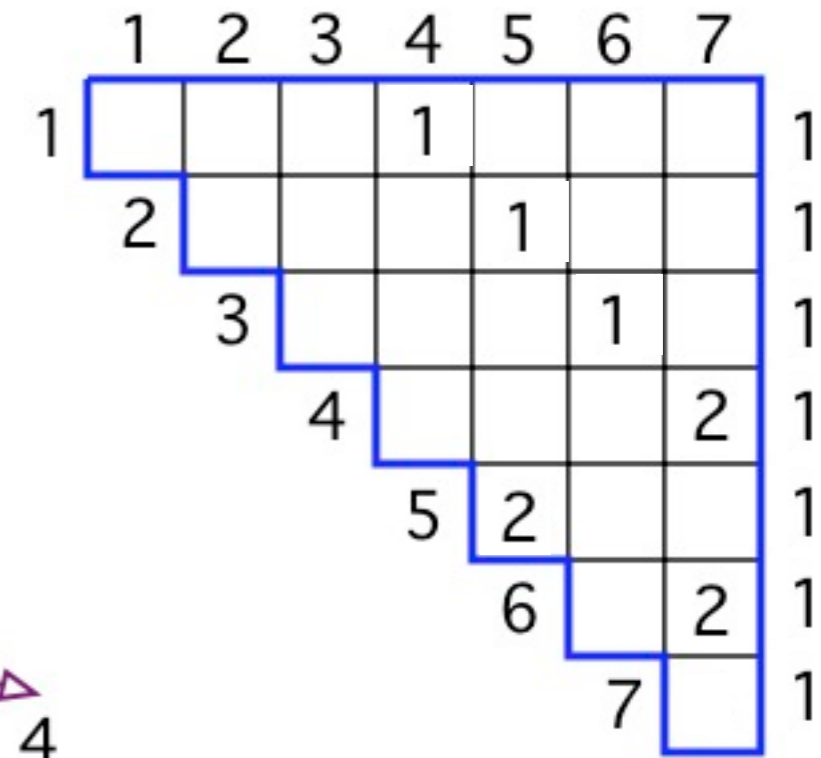
1 2 3 4 5 6

3 6 1 4 7 2 5



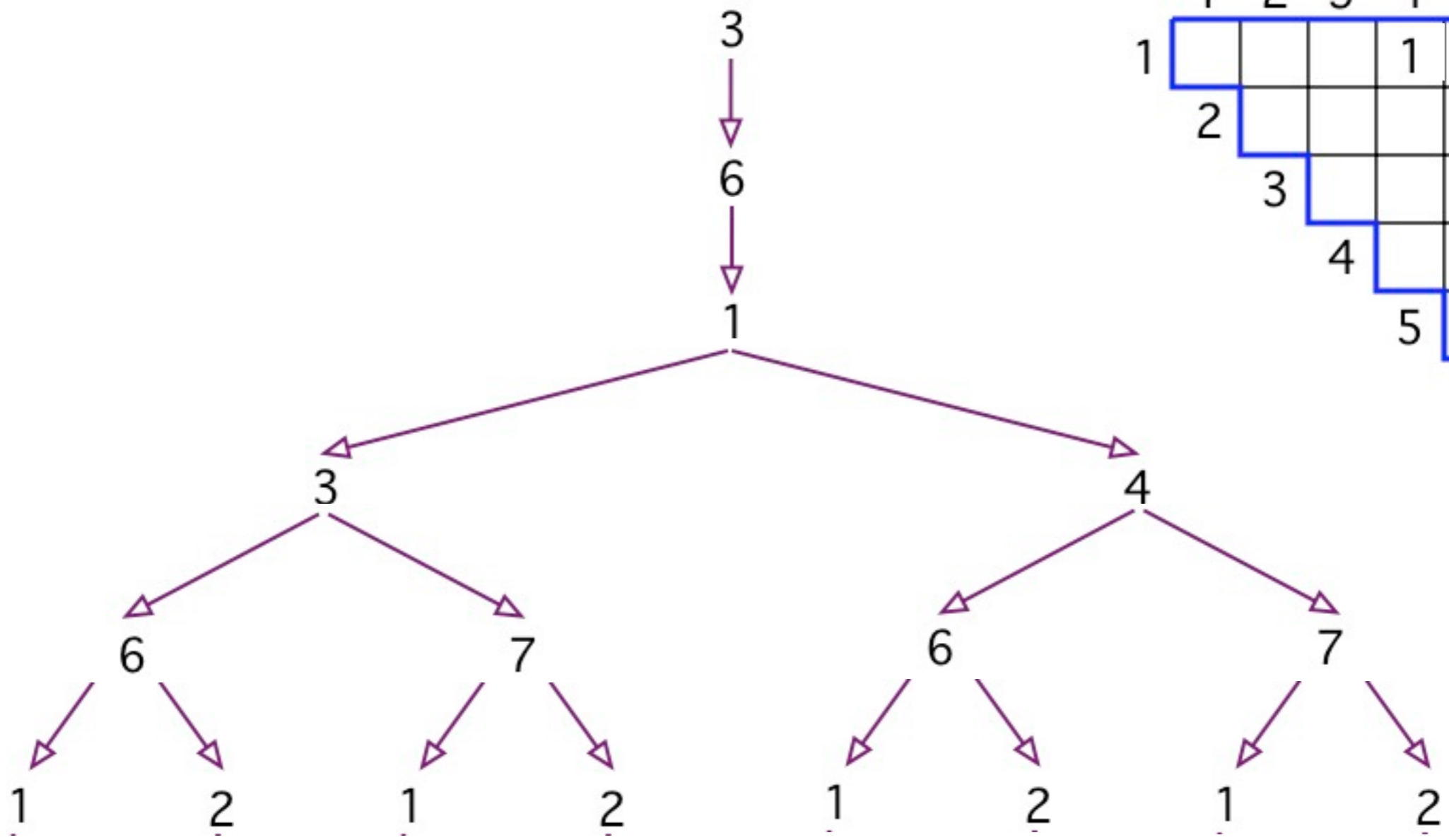
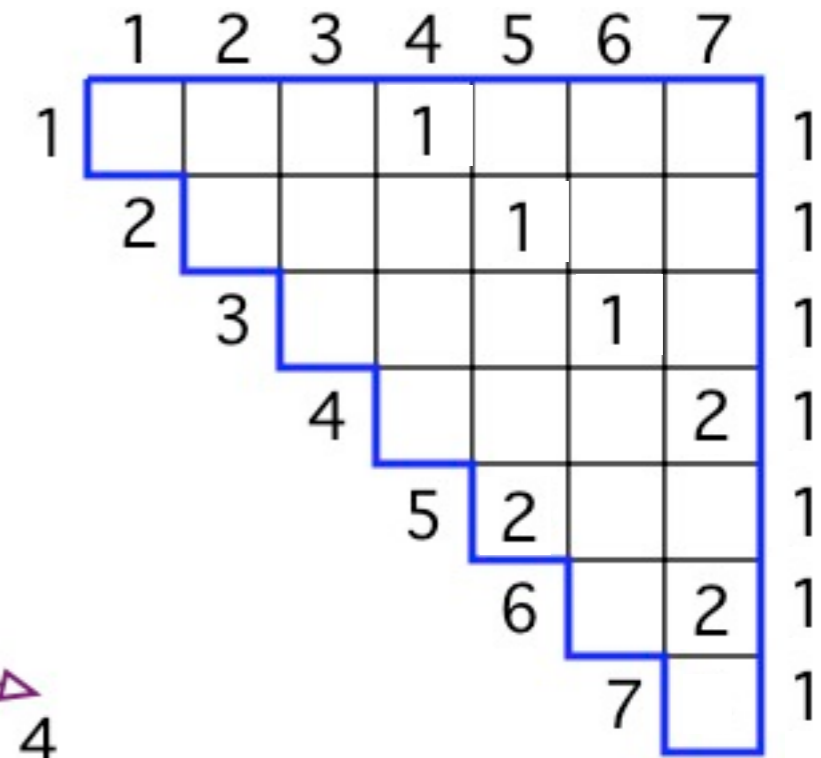
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1 2 3 4 5 6  
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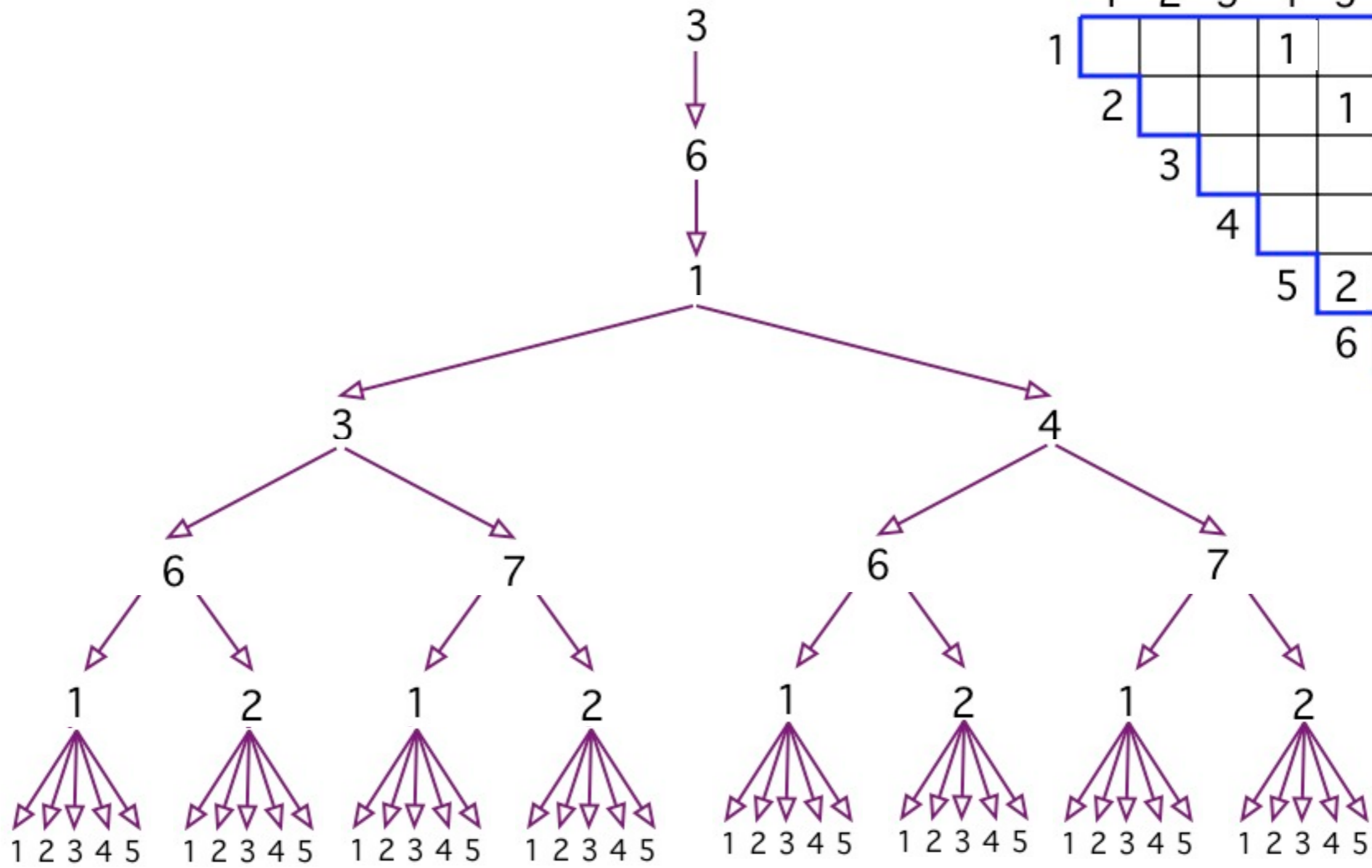
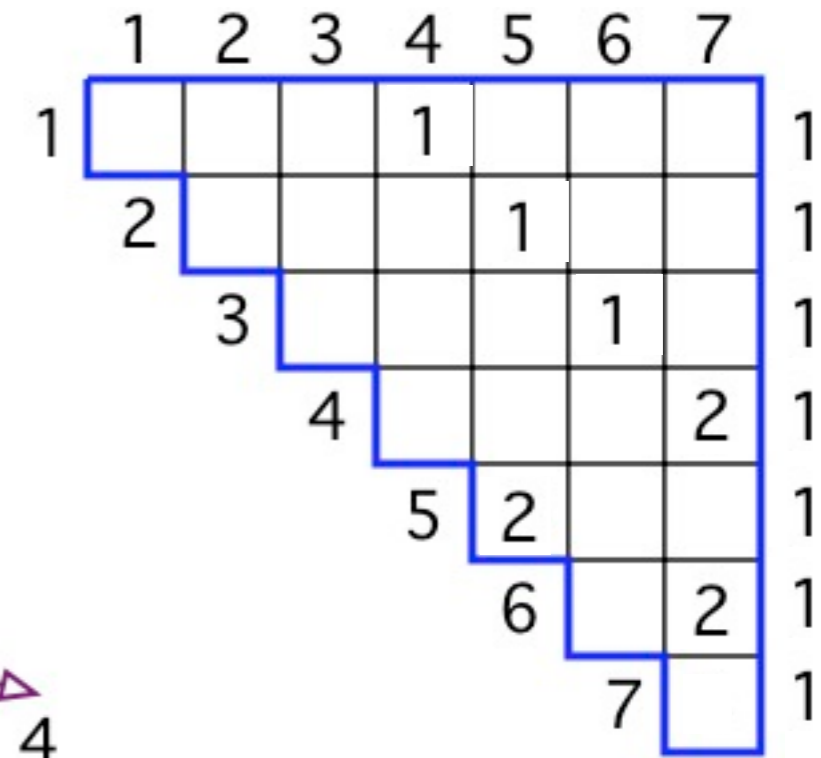
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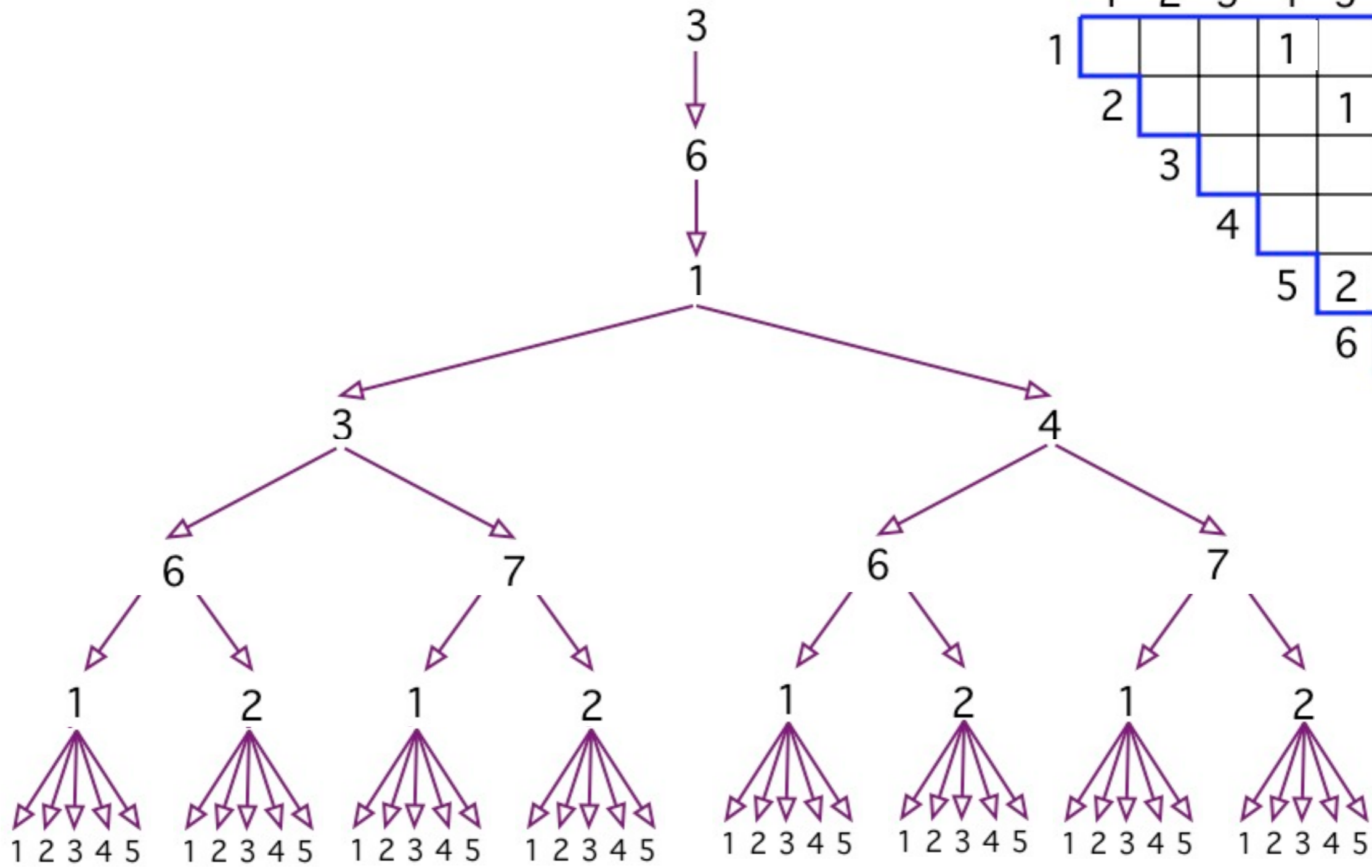
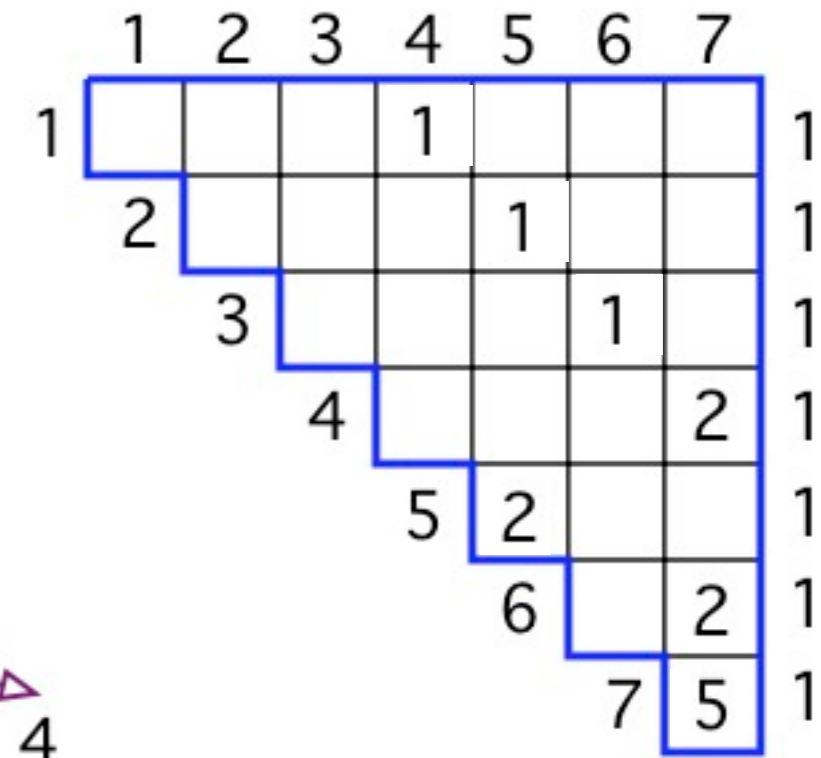
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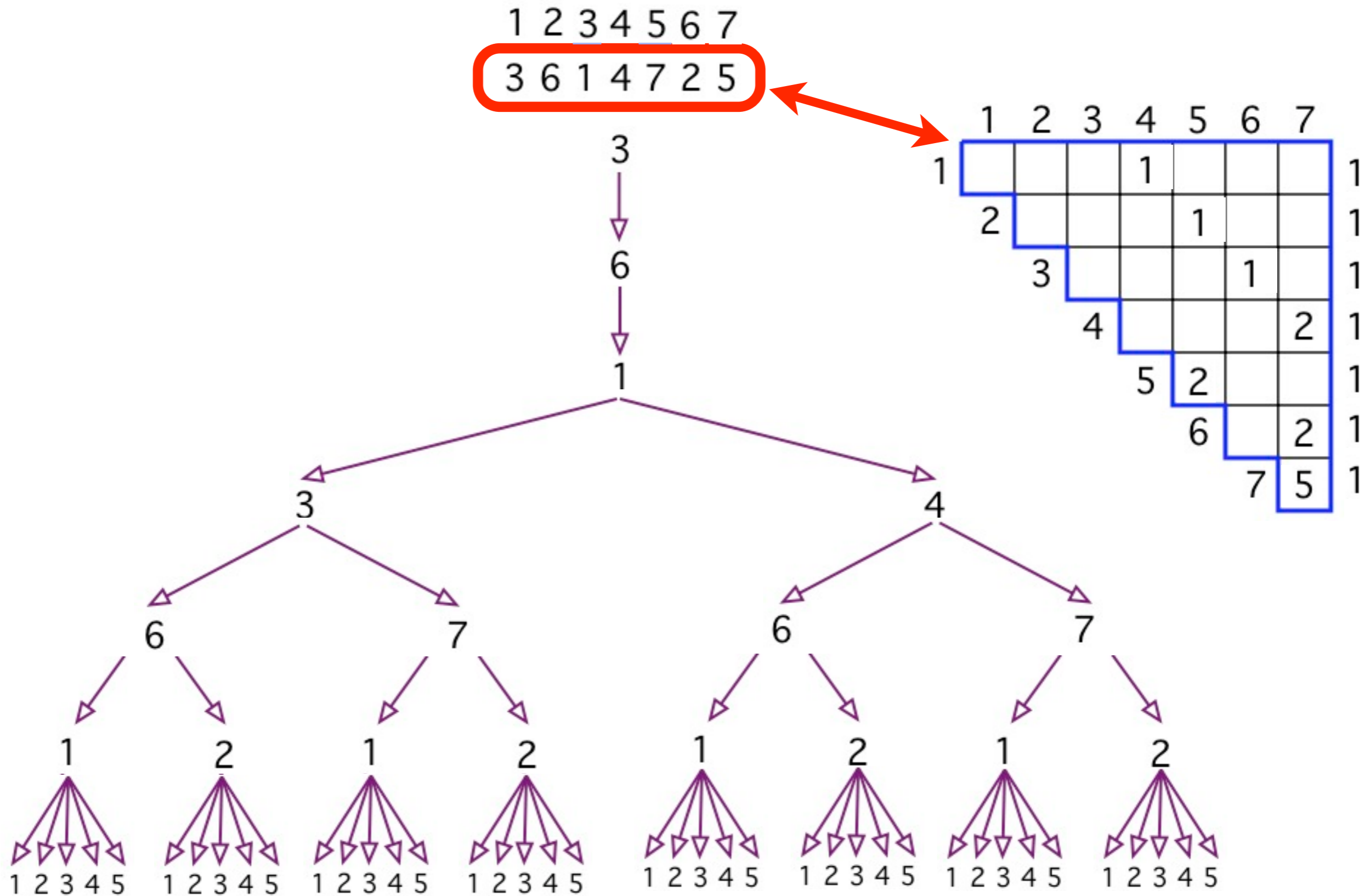
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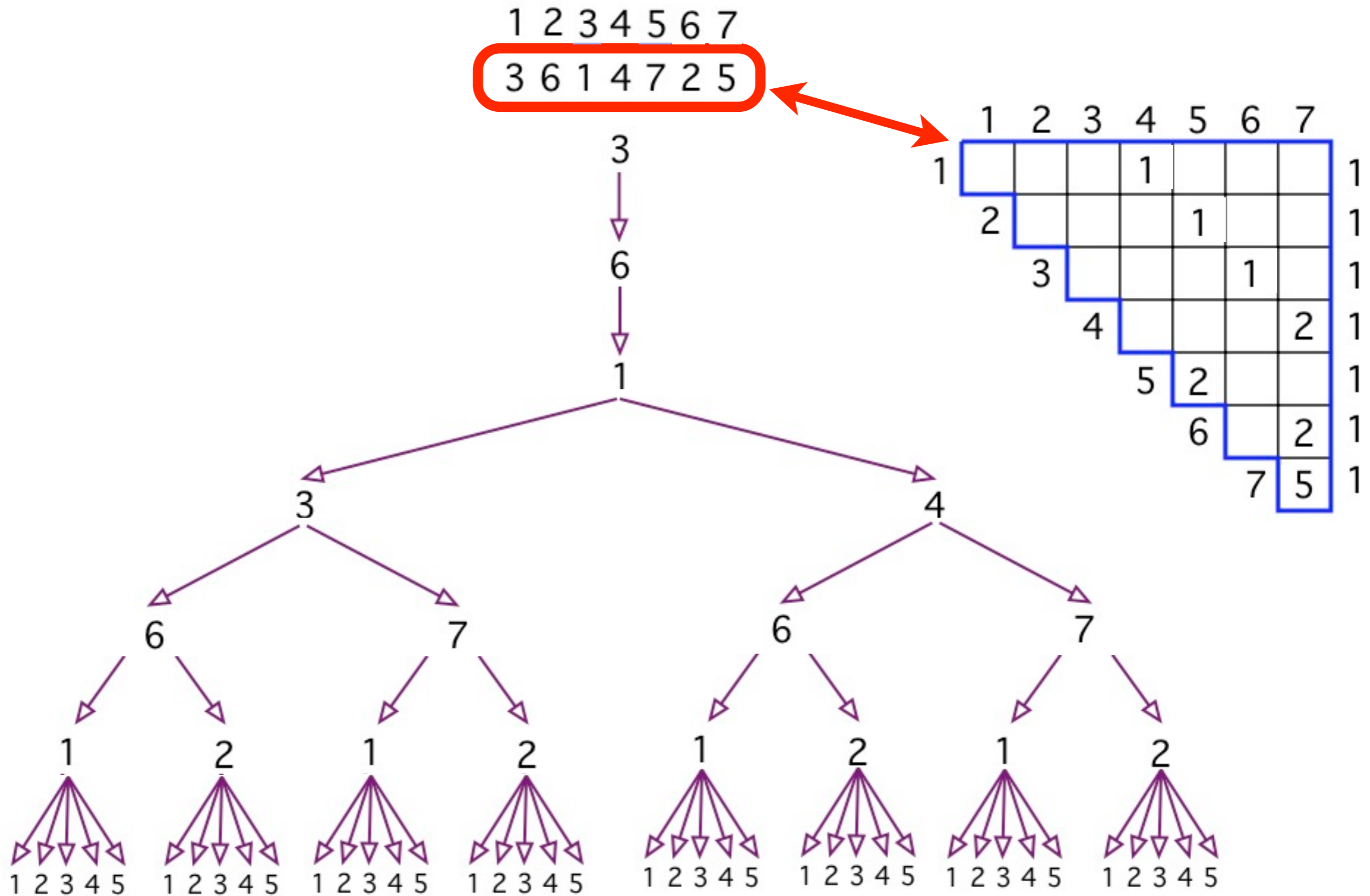




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next



The Miracle is due to  
a property of permutations

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**Lemma**

# The Miracle is due to a property of permutations

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