

```
> restart;
```

```
>
```

```
>
```

First functions

```
> with(LinearAlgebra):with(Groebner):
```

```
# transform x into a number in base m, as an array of size N  
# for example tm(3,4,2) = [ 0, 0, 1,1]
```

```
tm:=proc(x,N,m)  
  option remember;  
  local y,i,XX;  
  y:=x;  
  for i from 1 to N do  
    XX[i]:= y mod m;  
    y:=(y-(y mod m))/m;  
  od;  
  return([seq(XX[N-i+1], i=1..N)]);  
end:
```

```
# special function tm in the case m=2  
t2:=proc(x,N) option remember: return(tm(x,N,2)):end:
```

```
# to display long list of elements; only those with less than l  
characters are displayed
```

```
Affiche:=proc(x,l) local n,i,ens:ens:={};  
  n:=nops(x): for i from 1 to n do if length(x[i])<l then print  
(i);print(factor(x[i])) else ens:=ens union {i}:fi:od:  
  print("=====");  
  print("trop long",ens);  
end:
```

```
# reverse operation of tm -> Transform an array into a number  
itm :=proc(u,m) option remember: local l; l:=nops(u):  
return(add(u[i]*m^(l-i),i=1..l)):end:
```

```
# binary case  
it:=proc(u) option remember: return(itm(u,2)): end:
```

```
# set of words on {0,1} with N letters  
ense:= proc(N) option remember; return(seq(t2(i,N),i=0..2^N-1)  
);end:
```

```
# set of words on {0,...,m-1} with N letters  
ensem:= proc(N,m) option remember; return(seq(tm(i,N,m),i=0..  
m^N-1));end:
```

```
#Indicator function
```

```
ind := proc(a,b)  
  if a = b then return(1) else return(0):fi:  
end:
```

```
# special modulo
```

```

modu := proc(x, m)
  if x<1 then return(modu(x+m, m )):fi:
  if x>m then return(modu(x-m, m)):fi:
  return(x):
end:

```

in maple, the arrays start at 1: we then use this form of modulo, modu(7,6)=1, modu (6,6)=6

Illustration of the functions

```

> # Illustration
  tm(3,4,2);
  tm(324,4,5);
  itm([2, 2, 4, 4],5);

```

```
[0, 0, 1, 1]
```

```
[2, 2, 4, 4]
```

```
324
```

(1.1.1)

```

> ense(4);
  ensem(2,4);
  Affiche({ensem(2,3)},100);

```

```

[0, 0, 0, 0], [0, 0, 0, 1], [0, 0, 1, 0], [0, 0, 1, 1], [0, 1, 0, 0], [0, 1, 0, 1], [0, 1, 1, 0], [0,
  1, 1, 1], [1, 0, 0, 0], [1, 0, 0, 1], [1, 0, 1, 0], [1, 0, 1, 1], [1, 1, 0, 0], [1, 1, 0, 1], [1,
  1, 1, 0], [1, 1, 1, 1]

```

```

[0, 0], [0, 1], [0, 2], [0, 3], [1, 0], [1, 1], [1, 2], [1, 3], [2, 0], [2, 1], [2, 2], [2, 3],
[3, 0], [3, 1], [3, 2], [3, 3]

```

```
1
```

```
[0, 0]
```

```
2
```

```
[0, 1]
```

```
3
```

```
[0, 2]
```

```
4
```

```
[1, 0]
```

```
5
```

```
[1, 1]
```

```
6
```

```
[1, 2]
```

```
7
```

```
[2, 0]
```

```
8
```

```
[2, 1]
```

```
9
```

```
[2, 2]
```

```
"=====
```

(1.1.2)

"trop long", ∅

(1.1.2)

```
> # indicator -> Works also to test equality of anything
ind(1,2); ind(3,3);
ind([1,2],[2,3]); ind([1,2],[1,2]);
```

```
# modu
modu(7,6); modu(6,6); modu(0,6);
```

0
1
0
1
1
6
6

(1.1.3)

Example of Section 3.1

```
> P1 := x^2+y^2-z^2-3:
P2 := x^2+2*y^2-4:
P3 := y^2+3*z^2-x-2:
P4 := x*z-y^2+2:
```

```
> Basis ([P1,P2,P3], tdeg(x,y,z));
```

$[2z^2 - x - 1, 2y^2 + x - 1, x^2 - x - 3]$

(1.2.1)

```
> Basis ([P1,P2,P3], plex(x,y,z));
```

$[4z^4 - 6z^2 - 1, y^2 + z^2 - 1, -2z^2 + x + 1]$

(1.2.2)

```
> Basis ([P1,P2,P3,P4], tdeg(x,y,z));
Basis ([P1,P2,P3,P4], plex(x,y,z));
```

[1]

[1]

(1.2.3)

Section 3.1.1 - Ising

```
> unassign('M','T'); # for further computations
```

```
# Transition definition
```

```
# -> Initialisation
```

```
for u in ense(3) do
```

```
  for v in ense(3) do
```

```
    T[u,v]:=0:
```

```
  od:
```

```
od:
```

```
# -> Non zero transitions
```

```
for u in ense(3) do
```

```
  T[ u, [u[1],1-u[2],u[3]]]:= x^(2+(2*u[2]-1)*(2*u[1]+2*u
```

```
[3]-2));
```

```
od;
```

```
## product of M over 5 letters
```

```
MM:=proc(a,b,c,d,e) return( M[a,b]*M[b,c]*M[c,d]*M[d,e]):
```

```
end:
```

```
Z5:=proc(u) ## for a word u of size 5 and L = 3 we need the
```

```

following Z
  return( add( MM(u[1],v[1],v[2],v[3],u[5]) *T [v,[u
[2],u[3],u[4]]]/MM( op(u) )
-T[[u[2],u[3],u[4]],v], v in ense(3))) :
end:

```

```

Cycle9MT:=proc(a,b,c,d,e) option remember: local Vec:
  Vec :=[ a, b, c, d, e, 0, 0, 0, 0]:
  return( simplify( add(Z5( [seq( Vec[modu(i +j, 9)],j=1.
.5) ] ),i=1..9))) :
end:

```

$$\begin{aligned}
 T_{[0,0,0],[0,1,0]} &:= x^4 \\
 T_{[0,0,1],[0,1,1]} &:= x^2 \\
 T_{[0,1,0],[0,0,0]} &:= 1 \\
 T_{[0,1,1],[0,0,1]} &:= x^2 \\
 T_{[1,0,0],[1,1,0]} &:= x^2 \\
 T_{[1,0,1],[1,1,1]} &:= 1 \\
 T_{[1,1,0],[1,0,0]} &:= x^2 \\
 T_{[1,1,1],[1,0,1]} &:= x^4
 \end{aligned}
 \tag{1.3.1}$$

```

> for i in {0,1} do M[i,0]:=1-M[i,1]: od:

```

```

SystemEquation := { seq( Cycle9MT(op(u)), u in ense(5)),
  seq( M[op(u)]*g[op(u)]-1,u in ense(2))}:

```

```

#####

```

```

PolynomialSystem := map(x-> numer(normal(x)), SystemEquation)

```

```

: # keep the numerators only

```

```

Variables:= M[0,1],M[1,1] , seq(g[op(u)],u in ense(2)), x ;

```

```

#####

```

```

# Grobner

```

```

G:= Basis(PolynomialSystem , tdeg(Variables )):

```

```

Affiche(G,200);

```

$$\text{Variables} := M_{0,1}, M_{1,1}, g_{0,0}, g_{0,1}, g_{1,0}, g_{1,1}, x$$

1

$$g_{0,1} - g_{1,0}$$

2

$$g_{0,0} - g_{1,1}$$

3

$$M_{0,1} + M_{1,1} - 1$$

4

$$\frac{g_{1,0}g_{1,1} - g_{1,0} - g_{1,1}}{5} M_{1,1}g_{1,1} - 1$$

$$\frac{g_{1,0}M_{1,1} - g_{1,0} + 1}{6}$$

$$\frac{(x^2 + g_{1,1} - 1)(x^2 - g_{1,1} + 1)}{7}$$

=====
"trop long", \emptyset

(1.3.2)

Section 3.1.2 - Voter

```

>
# -> Initialisation
unassign('T','M');
> # Transition definition
NonZeroT:= NULL: # will be the set of nonzerotransitions
for u in ense(3) do
  for v in ense(3) do
    if ( (v=u) or (v[1]<>u[1]) or (v[3]<>u[3]) or (not v
[2] in {u[1],u[3]}) )
      then
        T[u,v]:=0:
      else
        NonZeroT:= NonZeroT ,T[u,v];
      fi:
    od:
  od:
od:
"Parameters = NonZeroTransition", NonZeroT;
>

## product of M over 5 letters
MM:=proc(a,b,c,d,e) return( M[a,b]*M[b,c]*M[c,d]*M[d,e]):
end:

Z5:=proc(u) ## for a word u of size 5 and L = 3 we need the
following Z
  return( add( MM(u[1],v[1],v[2],v[3],u[5]) *T[v,[u
[2],u[3],u[4]]]/MM( op(u) ) -T[[u[2],u[3],u[4]],v], v in ense
(3))):
end:

Cycle9MT:=proc(a,b,c,d,e) option remember: local Vec:
  Vec :=[ a, b, c, d, e, 0, 0, 0, 0]:
  return( simplify( add(Z5( [seq( Vec[modu(i +j, 9)],j=1.
.5) ] ),i=1..9))):
end:
>

for i in {0,1} do M[i,0]:=1-M[i,1]: od:

```

```

>
SystemEquation := { seq( Cycle9MT(op(u)), u in ense(5)), seq
( M[op(u)]*g[op(u)]-1,u in ense(2))}:
#####
PolynomialSystem:= map(x-> numer(normal(x)), SystemEquation):
# numerators only
Variables:= NonZeroT, M[0,1],M[1,1] , seq(g[op(u)],u in ense
(2)) :
#####
# grobner
G:= Basis(PolynomialSystem, plex( Variables)):
Affiche(G,2000);
unassign('M'); # for further computations

```

"Parameters = NonZeroTransition", $T_{[0,0,1],[0,1,1]}$, $T_{[0,1,0],[0,0,0]}$, $T_{[0,1,1],[0,0,1]}$,

$T_{[1,0,0],[1,1,0]}$, $T_{[1,0,1],[1,1,1]}$, $T_{[1,1,0],[1,0,0]}$

1

$$g_{1,0}g_{1,1} - g_{1,0} - g_{1,1}$$

2

$$g_{0,0}g_{0,1} - g_{0,0} - g_{0,1}$$

3

$$M_{1,1}g_{1,1} - 1$$

4

$$g_{1,0}M_{1,1} - g_{1,0} + 1$$

5

$$M_{0,1}g_{0,1} - 1$$

6

$$M_{0,1}g_{0,0} - g_{0,0} + 1$$

7

$$T_{[1,0,1],[1,1,1]}$$

8

$$T_{[0,1,0],[0,0,0]}$$

9

$$-M_{1,1}T_{[0,1,1],[0,0,1]}g_{0,0} - M_{1,1}T_{[1,1,0],[1,0,0]}g_{0,0} + T_{[0,0,1],[0,1,1]} + T_{[1,0,0],[1,1,0]}$$

"=====

"trop long", \emptyset

(1.4.1)

```

> latex(G);
[g_{1,0}g_{1,1}-g_{1,0}-g_{1,1},g_{0,0}g_{0,1}-
g_{0,0}-g_{0,1},M_{1,1}g_{1,1}-1,g_{1,0}M_{1,1}-g_{1,0}
+1,M_{0,1}g_{0,1}-1,
M_{0,1}g_{0,0}-g_{0,0}+1,T_{[1,0,1],[1,1,1]},
T_{[0,1,0],[0,0,0]},-M_{1,1}T_{[0,1,1],[0,0,1]}g_{0,0}-M_{1,1}T_{[1,0,0],[1,1,0]}]

```

```
[1,1,0],[1,0,0]]}g_{{0,0}}+T_{{[0,0,1],[0,1,1]}}+T_{{[1,0,0],[1,1,0]}}
```

```
> Remains:=map(normal,subs({seq(g[op(u)]=1/M[op(u)],u in ense  
(2))},G));
```

$$\text{Remains} := \left[-\frac{-1 + M_{1,1} + M_{1,0}}{M_{1,0}M_{1,1}}, -\frac{-1 + M_{0,1} + M_{0,0}}{M_{0,0}M_{0,1}}, 0, \frac{-1 + M_{1,1} + M_{1,0}}{M_{1,0}}, 0, \right. \\ \left. \frac{-1 + M_{0,1} + M_{0,0}}{M_{0,0}}, T_{[1,0,1],[1,1,1]}, T_{[0,1,0],[0,0,0]}, \right. \\ \left. \frac{1}{M_{0,0}} (T_{[0,0,1],[0,1,1]} M_{0,0} + T_{[1,0,0],[1,1,0]} M_{0,0} - M_{1,1} T_{[0,1,1],[0,0,1]} \right. \\ \left. - M_{1,1} T_{[1,1,0],[1,0,0]}) \right] \quad (1.4.2)$$

```
> latex(numer(Remains[9]));
```

```
M_{{0,0}}T_{{[0,0,1],[0,1,1]}}+M_{{0,0}}T_{{[1,0,0],[1,1,0]}}  
}-M_{{1,1}}  
}T_{{[0,1,1],[0,0,1]}}-M_{{1,1}}T_{{[1,1,0],[1,0,0]}}
```

Section 3.1.4 : all variants ...

3 color tasep

```
> # -> Initialisation  
unassign('T','M');  
K:=3;
```

```
> # Transition definition  
NonZeroT:= NULL: # will be the set of nonzerotransitions  
for u in ensem(2,K) do  
  for v in ensem(2,K) do
```

```
    if( not [u,v] in {[[2,1],[1,2]], [[2,0],[0,2]] ,  
[[1,0],[0,1]] } )
```

```
      then
```

```
        T[u,v]:=0:
```

```
      else
```

```
        NonZeroT := NonZeroT, T[u,v]:
```

```
      fi:
```

```
    od:
```

```
  od:
```

```
"Parameters = NonZeroTransition", NonZeroT;
```

```
>
```

```
## product of M over 4 letters
```

```
MM:=proc(a,b,c,d) return( M[a,b]*M[b,c]*M[c,d]): end:
```

```
Z4:=proc(u) ## for a word u of size 4 and L = 2 we need the  
following Z
```

```
  return( add( MM(u[1],v[1],v[2],u[4]) *T[v,[u[2],  
u[3]]]/MM( op(u) ) -T[[u[2],u[3]],v], v in ensem(2,K))):
```

```

end:
>
Cycle7MT:=proc(a,b,c,d) option remember: local Vec:
  Vec :=[ a, b, c, d, 0, 0, 0]:
  return( simplify( add(Z4( [seq( Vec[modu(i +j, 7)],j=1..4) ] ),i=1..7))):
end:
>
for i in {0,1,2} do M[i,0]:=1-M[i,1]-M[i,2]: od:
>
SystemEquation := { seq( Cycle7MT(op(u)), u in ensem(4,K)
), seq( M[op(u)]*g[op(u)]-1,u in ensem(2,K))}:
#####
PolynomialSystem:= map(x-> numer(normal(x)),
SystemEquation): # numerators only
Variables:= seq(seq(M[i,j],i=0..K-1),j=1..K-1) , seq(g[op
(u)],u in ensem(2,K)), NonZeroT :
#####
# grobner
G:= Basis(PolynomialSystem,plex( Variables)):
Affiche(G,2000);
unassign('M'); # for further computations
      K := 3
"Parameters = NonZeroTransition",  $T_{[1,0],[0,1]}$ ,  $T_{[2,0],[0,2]}$ ,  $T_{[2,1],[1,2]}$ 
      1
       $T_{[1,0],[0,1]} - T_{[2,0],[0,2]} + T_{[2,1],[1,2]}$ 
      2
       $g_{2,0}g_{2,1}g_{2,2} - g_{2,0}g_{2,1} - g_{2,0}g_{2,2} - g_{2,1}g_{2,2}$ 
      3
       $T_{[2,1],[1,2]} (g_{1,2} - g_{2,2})$ 
      4
       $T_{[2,0],[0,2]} (g_{1,2} - g_{2,2})$ 
      5
       $T_{[2,1],[1,2]} (g_{1,1} - g_{2,1})$ 
      6
       $T_{[2,0],[0,2]} (g_{1,1} - g_{2,1})$ 
      7
       $T_{[2,1],[1,2]} (g_{1,0} - g_{2,0})$ 
      8
       $T_{[2,0],[0,2]} (g_{1,0} - g_{2,0})$ 
      9
       $g_{1,0}g_{1,1}g_{1,2} - g_{1,0}g_{1,1} - g_{1,0}g_{1,2} - g_{1,1}g_{1,2}$ 

```


$$\begin{aligned}
& 10 \\
& T_{[2,0],[0,2]} (g_{0,2} - g_{2,2}) \\
& 11 \\
& T_{[2,0],[0,2]} (g_{0,1} - g_{2,1}) \\
& 12 \\
& T_{[2,1],[1,2]} (g_{0,1} g_{0,2} g_{2,2} - g_{0,2} g_{2,2} g_{2,1} - g_{0,1} g_{2,2} + g_{0,2} g_{2,1}) \\
& 13 \\
& T_{[2,1],[1,2]} (g_{0,1} g_{0,2} g_{2,0} - g_{0,1} g_{2,0} - g_{0,2} g_{2,0} - g_{0,2} g_{2,1}) \\
& 14 \\
& T_{[2,0],[0,2]} (g_{0,0} - g_{2,0}) \\
& 15 \\
& T_{[2,1],[1,2]} (-g_{0,1} g_{2,0} g_{2,2} + g_{0,0} g_{2,2} + g_{0,1} g_{2,0} + g_{0,1} g_{2,2}) \\
& 16 \\
& T_{[2,1],[1,2]} (g_{0,0} g_{2,1} - g_{0,1} g_{2,0}) \\
& 17 \\
& g_{0,0} g_{0,1} g_{0,2} - g_{0,0} g_{0,1} - g_{0,0} g_{0,2} - g_{0,1} g_{0,2} \\
& 18 \\
& M_{2,2} g_{2,2} - 1 \\
& 19 \\
& M_{2,2} g_{2,0} g_{2,1} - g_{2,0} g_{2,1} + g_{2,0} + g_{2,1} \\
& 20 \\
& T_{[2,1],[1,2]} (M_{2,2} g_{0,2} g_{2,1} + g_{0,1} g_{0,2} - g_{0,2} g_{2,1} - g_{0,1}) \\
& 21 \\
& T_{[2,1],[1,2]} (M_{2,2} g_{0,2} g_{2,0} + g_{0,0} g_{0,2} - g_{0,2} g_{2,0} - g_{0,0}) \\
& 22 \\
& T_{[2,1],[1,2]} (M_{2,2} g_{0,1} g_{2,0} - g_{0,1} g_{2,0} + g_{0,0} + g_{0,1}) \\
& 23 \\
& T_{[2,1],[1,2]} (M_{1,2} - M_{2,2}) \\
& 24 \\
& T_{[2,0],[0,2]} (M_{1,2} - M_{2,2}) \\
& 25 \\
& M_{1,2} g_{1,2} - 1 \\
& 26 \\
& M_{1,2} g_{1,0} g_{1,1} - g_{1,0} g_{1,1} + g_{1,0} + g_{1,1} \\
& 27 \\
& T_{[2,0],[0,2]} (M_{0,2} - M_{2,2}) \\
& 28 \\
& M_{0,2} g_{0,2} - 1
\end{aligned}$$

29

$$T_{[2, 1], [1, 2]} (M_{0, 2} g_{0, 1} - M_{2, 2} g_{2, 1} - g_{0, 1} + g_{2, 1})$$

30

$$T_{[2, 1], [1, 2]} (g_{0, 0} M_{0, 2} - M_{2, 2} g_{2, 0} - g_{0, 0} + g_{2, 0})$$

31

$$M_{0, 2} g_{0, 0} g_{0, 1} - g_{0, 0} g_{0, 1} + g_{0, 0} + g_{0, 1}$$

32

$$M_{2, 1} g_{2, 1} - 1$$

33

$$g_{2, 0} M_{2, 1} + M_{2, 2} g_{2, 0} - g_{2, 0} + 1$$

34

$$T_{[2, 1], [1, 2]} (M_{2, 1} g_{0, 1} g_{0, 2} - M_{2, 1} g_{0, 1} + M_{2, 2} g_{0, 2} - g_{0, 2})$$

35

$$T_{[2, 1], [1, 2]} (M_{2, 1} g_{0, 0} + M_{2, 1} g_{0, 1} + M_{2, 2} g_{0, 0} - g_{0, 0})$$

36

$$T_{[2, 1], [1, 2]} (M_{1, 1} - M_{2, 1})$$

37

$$T_{[2, 0], [0, 2]} (M_{1, 1} - M_{2, 1})$$

38

$$M_{1, 1} g_{1, 1} - 1$$

39

$$g_{1, 0} M_{1, 1} + M_{1, 2} g_{1, 0} - g_{1, 0} + 1$$

40

$$T_{[2, 0], [0, 2]} (M_{0, 1} - M_{2, 1})$$

41

$$T_{[2, 1], [1, 2]} (M_{0, 2} M_{2, 1} g_{2, 2} + M_{0, 1} g_{2, 2} - M_{2, 1} g_{2, 2} - M_{0, 1})$$

42

$$T_{[2, 1], [1, 2]} (M_{0, 1} g_{2, 0} + M_{0, 1} g_{2, 1} + M_{0, 2} g_{2, 0} - g_{2, 0})$$

43

$$M_{0, 1} g_{0, 1} - 1$$

44

$$M_{0, 1} g_{0, 0} + g_{0, 0} M_{0, 2} - g_{0, 0} + 1$$

45

$$T_{[2, 1], [1, 2]} (M_{0, 1} M_{2, 2} - M_{0, 2} M_{2, 1} - M_{0, 1} + M_{2, 1})$$

"=====

"trop long", \emptyset

(1.5.1.1)

▼ **particle i can overtake $i-1 \pmod 3$**

```

> # -> Initialisation
unassign('T','M');
K:=3;

> # Transition definition
NonZeroT:= NULL: # will be the set of nonzerotransitions
for u in ensem(2,K) do
  for v in ensem(2,K) do
    if( not [u,v] in {[[2,1],[1,2]], [[0,2],[2,0]] ,
[[1,0],[0,1]] } )
      then
        T[u,v]:=0:
      else
        NonZeroT := NonZeroT, T[u,v]:
      fi:
    od:
  od:

"Parameters = NonZeroTransition", NonZeroT;

>
## product of M over 4 letters
MM:=proc(a,b,c,d) return( M[a,b]*M[b,c]*M[c,d]): end:

Z4:=proc(u) ## for a word u of size 4 and L = 2 we need the
following Z
  return( add( MM(u[1],v[1],v[2],u[4]) *T[v,[u[2],
u[3]]]/MM( op(u) ) -T[[u[2],u[3]],v], v in ensem(2,K))) :
end:

>
Cycle7MT:=proc(a,b,c,d) option remember: local Vec:
  Vec :=[ a, b, c, d, 0, 0, 0]:
  return( simplify( add(Z4( [seq( Vec[modu(i +j, 7)],j=1..4) ] ),i=1..7))) :
end:

>
for i in {0,1,2} do M[i,0]:=1-M[i,1]-M[i,2]: od:

>
SystemEquation := { seq( Cycle7MT(op(u)), u in ensem(4,K)
), seq( M[op(u)]*g[op(u)]-1,u in ensem(2,K))}:
#####
PolynomialSystem:= map(x-> numer(normal(x)),
SystemEquation): # numerators only
Variables:= seq(seq(M[i,j],i=0..K-1),j=1..K-1) , seq(g[op
(u)],u in ensem(2,K)), NonZeroT :
#####
# grobner
G:= Basis(PolynomialSystem,plex( Variables)):
Affiche(G,2000);
unassign('M'); # for further computations

```

$K := 3$

"Parameters = NonZeroTransition", $T_{[0,2],[2,0]}$, $T_{[1,0],[0,1]}$, $T_{[2,1],[1,2]}$

$$\begin{aligned}
& 1 \\
& T_{[0, 2], [2, 0]} + T_{[1, 0], [0, 1]} + T_{[2, 1], [1, 2]} \\
& 2 \\
& g_{2, 0} g_{2, 1} g_{2, 2} - g_{2, 0} g_{2, 1} - g_{2, 0} g_{2, 2} - g_{2, 1} g_{2, 2} \\
& 3 \\
& T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2}) \\
& 4 \\
& T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1}) \\
& 5 \\
& T_{[2, 1], [1, 2]} (g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 1}) \\
& 6 \\
& T_{[2, 1], [1, 2]} (-g_{1, 2} g_{2, 0} g_{2, 1} + g_{1, 1} g_{2, 0} + g_{1, 2} g_{2, 0} + g_{1, 2} g_{2, 1}) \\
& 7 \\
& T_{[1, 0], [0, 1]} (g_{1, 1} g_{1, 2} g_{2, 1} - g_{1, 1} g_{2, 1} g_{2, 2} + g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 1}) \\
& 8 \\
& T_{[1, 0], [0, 1]} (g_{1, 1} g_{1, 2} g_{2, 0} - g_{1, 1} g_{2, 0} - g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 0}) \\
& 9 \\
& T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0}) \\
& 10 \\
& T_{[1, 0], [0, 1]} (g_{1, 0} g_{2, 2} - g_{1, 2} g_{2, 0}) \\
& 11 \\
& T_{[1, 0], [0, 1]} (-g_{1, 2} g_{2, 0} g_{2, 1} + g_{1, 0} g_{2, 1} + g_{1, 2} g_{2, 0} + g_{1, 2} g_{2, 1}) \\
& 12 \\
& T_{[2, 1], [1, 2]} (g_{1, 0} g_{1, 2} g_{2, 1} - g_{1, 0} g_{2, 1} - g_{1, 0} g_{2, 2} - g_{1, 2} g_{2, 1}) \\
& 13 \\
& T_{[2, 1], [1, 2]} (g_{1, 0} g_{1, 2} g_{2, 0} - g_{1, 0} g_{2, 0} g_{2, 2} + g_{1, 0} g_{2, 2} - g_{1, 2} g_{2, 0}) \\
& 14 \\
& g_{1, 0} g_{1, 1} g_{1, 2} - g_{1, 0} g_{1, 1} - g_{1, 0} g_{1, 2} - g_{1, 1} g_{1, 2} \\
& 15 \\
& T_{[1, 0], [0, 1]} g_{0, 2} - T_{[1, 0], [0, 1]} g_{1, 2} + T_{[2, 1], [1, 2]} g_{0, 2} - T_{[2, 1], [1, 2]} g_{2, 2} \\
& 16 \\
& T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2}) (g_{0, 2} - g_{2, 2}) \\
& 17 \\
& T_{[2, 1], [1, 2]} (g_{0, 2} g_{1, 1} - g_{0, 2} g_{2, 1} - g_{1, 2} g_{2, 1} + g_{2, 1} g_{2, 2}) \\
& 18 \\
& T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0}) (g_{0, 2} - g_{2, 2}) \\
& 19 \\
& g_{0, 1} T_{[1, 0], [0, 1]} - g_{1, 1} T_{[1, 0], [0, 1]} + T_{[2, 1], [1, 2]} g_{0, 1} - T_{[2, 1], [1, 2]} g_{2, 1}
\end{aligned}$$

20

$$T_{[2,1],[1,2]}(g_{1,2} - g_{2,2})(g_{0,1} - g_{2,1})$$

21

$$T_{[2,1],[1,2]}(g_{1,1} - g_{2,1})(g_{0,1} - g_{2,1})$$

22

$$T_{[2,1],[1,2]}(g_{1,0} - g_{2,0})(g_{0,1} - g_{2,1})$$

23

$$T_{[2,1],[1,2]}(g_{0,1}g_{0,2}g_{2,2} - g_{0,2}g_{2,1}g_{2,2} - g_{0,1}g_{2,2} + g_{0,2}g_{2,1})$$

24

$$T_{[2,1],[1,2]}(g_{0,1}g_{0,2}g_{2,0} - g_{0,1}g_{2,0} - g_{0,2}g_{2,0} - g_{0,2}g_{2,1})$$

25

$$g_{0,0}T_{[1,0],[0,1]} - g_{1,0}T_{[1,0],[0,1]} + T_{[2,1],[1,2]}g_{0,0} - T_{[2,1],[1,2]}g_{2,0}$$

26

$$T_{[2,1],[1,2]}(-g_{0,1}g_{2,0}g_{2,2} + g_{0,0}g_{2,2} + g_{0,1}g_{2,0} + g_{0,1}g_{2,2})$$

27

$$T_{[2,1],[1,2]}(g_{0,0}g_{2,1} - g_{0,1}g_{2,0})$$

28

$$T_{[2,1],[1,2]}(-g_{0,1}g_{2,0}g_{2,2} + g_{0,0}g_{1,2} + g_{0,1}g_{2,0} + g_{0,1}g_{2,2} - g_{1,2}g_{2,0} + g_{2,0}g_{2,2})$$

29

$$T_{[2,1],[1,2]}(-g_{1,2}g_{2,0}g_{2,1} + g_{0,0}g_{1,1} - g_{0,1}g_{2,0} + g_{1,2}g_{2,0} + g_{1,2}g_{2,1} + g_{2,0}g_{2,1})$$

30

$$T_{[2,1],[1,2]}(g_{1,0} - g_{2,0})(g_{0,0} - g_{2,0})$$

31

$$g_{0,0}g_{0,1}g_{0,2} - g_{0,0}g_{0,1} - g_{0,0}g_{0,2} - g_{0,1}g_{0,2}$$

32

$$M_{2,2}g_{2,2} - 1$$

33

$$M_{2,2}g_{2,0}g_{2,1} - g_{2,0}g_{2,1} + g_{2,0} + g_{2,1}$$

34

$$T_{[2,1],[1,2]}(M_{2,2}g_{1,2}g_{2,1} - g_{1,1})$$

35

$$T_{[1,0],[0,1]}(M_{2,2}g_{1,2}g_{2,0} - g_{1,0})$$

36

$$T_{[1,0],[0,1]}(M_{2,2}g_{1,1}g_{2,0} - g_{1,0}g_{1,1} + g_{1,0} + g_{1,1})$$

37

$$T_{[2, 1], [1, 2]} (M_{2, 2} g_{1, 0} g_{2, 1} - g_{1, 0} g_{1, 1} + g_{1, 0} + g_{1, 1})$$

38

$$T_{[2, 1], [1, 2]} (M_{2, 2} g_{0, 2} g_{2, 1} + g_{0, 1} g_{0, 2} - g_{0, 2} g_{2, 1} - g_{0, 1})$$

39

$$T_{[2, 1], [1, 2]} (M_{2, 2} g_{0, 2} g_{2, 0} + g_{0, 0} g_{0, 2} - g_{0, 2} g_{2, 0} - g_{0, 0})$$

40

$$T_{[2, 1], [1, 2]} (M_{2, 2} g_{0, 1} g_{2, 0} - g_{0, 1} g_{2, 0} + g_{0, 0} + g_{0, 1})$$

41

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (M_{1, 2} - M_{2, 2})$$

42

$$M_{1, 2} g_{1, 2} - 1$$

43

$$T_{[2, 1], [1, 2]} (M_{1, 2} g_{1, 1} - M_{2, 2} g_{2, 1})$$

44

$$T_{[1, 0], [0, 1]} (M_{1, 2} g_{1, 1} g_{2, 1} - M_{2, 2} g_{1, 1} g_{2, 1} - M_{1, 2} g_{1, 1} + M_{2, 2} g_{2, 1})$$

45

$$T_{[1, 0], [0, 1]} (M_{1, 2} g_{1, 1} g_{2, 0} + M_{1, 2} g_{1, 1} g_{2, 2} - g_{1, 1} g_{2, 0} + g_{2, 0})$$

46

$$T_{[1, 0], [0, 1]} (M_{1, 2} g_{1, 0} - M_{2, 2} g_{2, 0})$$

47

$$T_{[2, 1], [1, 2]} (M_{1, 2} g_{1, 0} g_{2, 1} + M_{1, 2} g_{1, 0} g_{2, 2} - g_{1, 0} g_{2, 1} + g_{2, 1})$$

48

$$T_{[2, 1], [1, 2]} (M_{1, 2} g_{1, 0} g_{2, 0} - M_{2, 2} g_{1, 0} g_{2, 0} - M_{1, 2} g_{1, 0} + M_{2, 2} g_{2, 0})$$

49

$$M_{1, 2} g_{1, 0} g_{1, 1} - g_{1, 0} g_{1, 1} + g_{1, 0} + g_{1, 1}$$

50

$$T_{[2, 1], [1, 2]} (M_{1, 2} g_{0, 2} - M_{1, 2} g_{2, 2} - M_{2, 2} g_{0, 2} + 1)$$

51

$$T_{[2, 1], [1, 2]} (g_{0, 1} - g_{2, 1}) (M_{1, 2} - M_{2, 2})$$

52

$$T_{[2, 1], [1, 2]} (g_{0, 0} - g_{2, 0}) (M_{1, 2} - M_{2, 2})$$

53

$$M_{0, 2} T_{[1, 0], [0, 1]} + T_{[2, 1], [1, 2]} M_{0, 2} - M_{1, 2} T_{[1, 0], [0, 1]} - M_{2, 2} T_{[2, 1], [1, 2]}$$

54

$$T_{[2, 1], [1, 2]} (M_{0, 2} g_{1, 2} - M_{0, 2} g_{2, 2} - M_{2, 2} g_{1, 2} + 1)$$

55

$$T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1}) (M_{0, 2} - M_{2, 2})$$

56

$$T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0}) (M_{0, 2} - M_{2, 2})$$

57

$$M_{0, 2} g_{0, 2} - 1$$

58

$$T_{[2, 1], [1, 2]} (M_{0, 2} g_{0, 1} - M_{2, 2} g_{2, 1} - g_{0, 1} + g_{2, 1})$$

59

$$T_{[2, 1], [1, 2]} (g_{0, 0} M_{0, 2} - M_{2, 2} g_{2, 0} - g_{0, 0} + g_{2, 0})$$

60

$$M_{0, 2} g_{0, 0} g_{0, 1} - g_{0, 0} g_{0, 1} + g_{0, 0} + g_{0, 1}$$

61

$$T_{[2, 1], [1, 2]} (M_{1, 2} - M_{2, 2}) (M_{0, 2} - M_{2, 2})$$

62

$$M_{2, 1} g_{2, 1} - 1$$

63

$$g_{2, 0} M_{2, 1} + M_{2, 2} g_{2, 0} - g_{2, 0} + 1$$

64

$$T_{[2, 1], [1, 2]} (M_{2, 1} g_{1, 1} - M_{2, 2} g_{1, 2})$$

65

$$T_{[1, 0], [0, 1]} (M_{2, 2} g_{1, 1} g_{1, 2} + M_{2, 1} g_{1, 1} - M_{2, 2} g_{1, 2} - g_{1, 1})$$

66

$$T_{[2, 1], [1, 2]} (-M_{2, 2} g_{1, 0} g_{1, 2} + M_{2, 1} g_{1, 0} + M_{2, 2} g_{1, 0} + M_{2, 2} g_{1, 2})$$

67

$$T_{[1, 0], [0, 1]} (M_{2, 1} g_{1, 0} + M_{2, 2} g_{1, 0} + M_{2, 2} g_{1, 2} - g_{1, 0})$$

68

$$T_{[2, 1], [1, 2]} (M_{2, 1} g_{0, 1} g_{0, 2} - M_{2, 1} g_{0, 1} + M_{2, 2} g_{0, 2} - g_{0, 2})$$

69

$$T_{[2, 1], [1, 2]} (M_{2, 1} g_{0, 0} + M_{2, 1} g_{0, 1} + M_{2, 2} g_{0, 0} - g_{0, 0})$$

70

$$T_{[2, 1], [1, 2]} (-M_{1, 2} M_{2, 1} g_{2, 2} + M_{1, 1})$$

71

$$T_{[1, 0], [0, 1]} (-M_{1, 2} M_{2, 1} g_{2, 2} + M_{1, 2} g_{2, 2} + M_{1, 1} - 1)$$

72

$$M_{1, 1} g_{1, 1} - 1$$

73

$$g_{1, 0} M_{1, 1} + M_{1, 2} g_{1, 0} - g_{1, 0} + 1$$

74

$$-M_{1, 2} M_{2, 1} T_{[1, 0], [0, 1]} g_{2, 2} + M_{1, 2} T_{[1, 0], [0, 1]} g_{2, 2} + M_{0, 1} T_{[1, 0], [0, 1]}$$

$$\begin{aligned}
& + M_{0,1} T_{[2,1],[1,2]} - M_{2,1} T_{[2,1],[1,2]} - T_{[1,0],[0,1]} \\
& \qquad \qquad \qquad 75 \\
& T_{[2,1],[1,2]} (M_{0,2} M_{2,1} g_{2,2} + M_{0,1} g_{2,2} - M_{2,1} g_{2,2} - M_{0,1}) \\
& \qquad \qquad \qquad 76 \\
& T_{[2,1],[1,2]} (M_{0,1} g_{2,0} + M_{0,1} g_{2,1} + M_{0,2} g_{2,0} - g_{2,0}) \\
& \qquad \qquad \qquad 77 \\
& T_{[2,1],[1,2]} (M_{0,2} M_{2,1} g_{2,2} + M_{0,1} g_{1,2} - M_{2,1} g_{1,2} - M_{0,1}) \\
& \qquad \qquad \qquad 78 \\
& T_{[2,1],[1,2]} (M_{0,1} g_{1,1} - M_{0,1} g_{2,1} - M_{2,2} g_{1,2} + 1) \\
& \qquad \qquad \qquad 79 \\
& T_{[2,1],[1,2]} (-M_{2,2} g_{1,0} g_{1,2} + M_{0,1} g_{1,0} + M_{0,1} g_{2,1} + M_{0,2} g_{2,0} + M_{2,2} g_{1,0} \\
& \quad + M_{2,2} g_{1,2} - M_{2,2} g_{2,0} - 1) \\
& \qquad \qquad \qquad 80 \\
& \qquad \qquad \qquad M_{0,1} g_{0,1} - 1 \\
& \qquad \qquad \qquad 81 \\
& \qquad \qquad \qquad M_{0,1} g_{0,0} + g_{0,0} M_{0,2} - g_{0,0} + 1 \\
& \qquad \qquad \qquad 82 \\
& T_{[2,1],[1,2]} (M_{0,1} M_{2,2} - M_{0,2} M_{2,1} - M_{0,1} + M_{2,1}) \\
& \qquad \qquad \qquad 83 \\
& T_{[2,1],[1,2]} (M_{0,1} M_{1,2} - M_{0,2} M_{2,1} - M_{1,2} M_{2,1} + M_{2,1} M_{2,2} - M_{0,1} + M_{2,1}) \\
& \qquad \qquad \qquad \text{"====="} \\
& \qquad \qquad \qquad \text{"trop long", } \emptyset
\end{aligned}$$

(1.5.2.1)

▼ **Variant with parameter $T[[a,b],[b,a]]$**

```

> # -> Initialisation
unassign('T','M');
K:=3;

> # Transition definition
NonZeroT:= NULL: # will be the set of nonzerotransitions
for u in ensem(2,K) do
  for v in ensem(2,K) do
    if( u=v or (u[1]<>v[2]) or(v[1]<>u[2]) )
      then
        T[u,v]:=0:
      else
        NonZeroT := NonZeroT, T[u,v]:
      fi:
    od:
  od:
od:

"Parameters = NonZeroTransition", NonZeroT;

```



```
## product of M over 4 letters
MM:=proc(a,b,c,d) return( M[a,b]*M[b,c]*M[c,d]): end:
```

```
Z4:=proc(u) ## for a word u of size 4 and L = 2 we need the
following Z
return( add( MM(u[1],v[1],v[2],u[4]) *T[v,[u[2],
u[3]]]/MM( op(u) ) -T[[u[2],u[3]],v], v in ensem(2,K))):
end:
```

```
> Cycle7MT:=proc(a,b,c,d) option remember: local Vec:
Vec :=[ a, b, c, d, 0, 0, 0]:
return( simplify( add(Z4( [seq( Vec[modu(i +j, 7)],j=1..
.4] ),i=1..7))):
end:
```

```
> for i in {0,1,2} do M[i,0]:=1-M[i,1]-M[i,2]: od:
```

```
> SystemEquation := { seq( Cycle7MT(op(u)), u in ensem(4,K)),
seq( M[op(u)]*g[op(u)]-1,u in ensem(2,K))}:
#####
```

```
PolynomialSystem:= map(x-> numer(normal(x)),
```

```
SystemEquation): # numerators only
```

```
Variables:= seq(seq(M[i,j],i=0..K-1),j=1..K-1) , seq(g[op
(u)],u in ensem(2,K)), NonZeroT :
```

```
#####
```

```
# grobner
```

```
G:= Basis(PolynomialSystem,tdeg( Variables)):
```

```
Affiche(G,500);
```

```
unassign('M'); # for further computations
```

```
K := 3
```

"Parameters = NonZeroTransition", $T_{[0,1],[1,0]}$, $T_{[0,2],[2,0]}$, $T_{[1,0],[0,1]}$, $T_{[1,2],[2,1]}$,

$T_{[2,0],[0,2]}$, $T_{[2,1],[1,2]}$

1

$$T_{[1,0],[0,1]} g_{0,2} - g_{1,2} T_{[1,0],[0,1]} - g_{0,2} T_{[1,2],[2,1]} + g_{1,2} T_{[1,2],[2,1]} \\ - T_{[2,0],[0,2]} g_{0,2} + T_{[2,0],[0,2]} g_{2,2} + g_{0,2} T_{[2,1],[1,2]} - g_{2,2} T_{[2,1],[1,2]}$$

2

$$g_{0,1} T_{[1,0],[0,1]} - g_{1,1} T_{[1,0],[0,1]} - g_{0,1} T_{[1,2],[2,1]} + g_{1,1} T_{[1,2],[2,1]} \\ - T_{[2,0],[0,2]} g_{0,1} + T_{[2,0],[0,2]} g_{2,1} + g_{0,1} T_{[2,1],[1,2]} - g_{2,1} T_{[2,1],[1,2]}$$

3

$$g_{0,0} T_{[1,0],[0,1]} - g_{1,0} T_{[1,0],[0,1]} - g_{0,0} T_{[1,2],[2,1]} + g_{1,0} T_{[1,2],[2,1]} \\ - T_{[2,0],[0,2]} g_{0,0} + T_{[2,0],[0,2]} g_{2,0} + g_{0,0} T_{[2,1],[1,2]} - g_{2,0} T_{[2,1],[1,2]}$$

4

$$M_{0,2} T_{[1,0],[0,1]} - M_{0,2} T_{[1,2],[2,1]} - T_{[2,0],[0,2]} M_{0,2} + M_{0,2} T_{[2,1],[1,2]} \\ - M_{1,2} T_{[1,0],[0,1]} + M_{1,2} T_{[1,2],[2,1]} + T_{[2,0],[0,2]} M_{2,2} - M_{2,2} T_{[2,1],[1,2]}$$

5

$$M_{0,1} T_{[1,0],[0,1]} - M_{0,1} T_{[1,2],[2,1]} - M_{0,1} T_{[2,0],[0,2]} + M_{0,1} T_{[2,1],[1,2]}$$

$$-M_{1,1} T_{[1,0],[0,1]} + M_{1,1} T_{[1,2],[2,1]} + M_{2,1} T_{[2,0],[0,2]} - M_{2,1} T_{[2,1],[1,2]}$$

7

$$(g_{1,2} - g_{2,2}) (T_{[2,0],[0,2]} + T_{[1,2],[2,1]} - T_{[2,1],[1,2]} + T_{[0,1],[1,0]} - T_{[0,2],[2,0]} - T_{[1,0],[0,1]})$$

8

$$(g_{1,1} - g_{2,1}) (T_{[2,0],[0,2]} + T_{[1,2],[2,1]} - T_{[2,1],[1,2]} + T_{[0,1],[1,0]} - T_{[0,2],[2,0]} - T_{[1,0],[0,1]})$$

9

$$(g_{1,0} - g_{2,0}) (T_{[2,0],[0,2]} + T_{[1,2],[2,1]} - T_{[2,1],[1,2]} + T_{[0,1],[1,0]} - T_{[0,2],[2,0]} - T_{[1,0],[0,1]})$$

10

$$T_{[0,1],[1,0]} g_{0,2} - g_{2,2} T_{[0,1],[1,0]} - g_{0,2} T_{[0,2],[2,0]} + g_{2,2} T_{[0,2],[2,0]} - g_{1,2} T_{[1,0],[0,1]} + g_{2,2} T_{[1,0],[0,1]} + g_{1,2} T_{[1,2],[2,1]} - g_{2,2} T_{[1,2],[2,1]}$$

11

$$T_{[0,1],[1,0]} g_{0,1} - g_{2,1} T_{[0,1],[1,0]} - g_{0,1} T_{[0,2],[2,0]} + g_{2,1} T_{[0,2],[2,0]} - g_{1,1} T_{[1,0],[0,1]} + g_{2,1} T_{[1,0],[0,1]} + g_{1,1} T_{[1,2],[2,1]} - g_{2,1} T_{[1,2],[2,1]}$$

12

$$T_{[0,1],[1,0]} g_{0,0} - g_{2,0} T_{[0,1],[1,0]} - g_{0,0} T_{[0,2],[2,0]} + g_{2,0} T_{[0,2],[2,0]} - g_{1,0} T_{[1,0],[0,1]} + g_{2,0} T_{[1,0],[0,1]} + g_{1,0} T_{[1,2],[2,1]} - g_{2,0} T_{[1,2],[2,1]}$$

13

$$(T_{[2,0],[0,2]} + T_{[1,2],[2,1]} - T_{[2,1],[1,2]} + T_{[0,1],[1,0]} - T_{[0,2],[2,0]} - T_{[1,0],[0,1]}) (M_{1,2} - M_{2,2})$$

14

$$T_{[0,1],[1,0]} M_{0,2} - M_{0,2} T_{[0,2],[2,0]} - M_{1,2} T_{[1,0],[0,1]} + M_{1,2} T_{[1,2],[2,1]} - M_{2,2} T_{[0,1],[1,0]} + T_{[0,2],[2,0]} M_{2,2} + M_{2,2} T_{[1,0],[0,1]} - M_{2,2} T_{[1,2],[2,1]}$$

15

$$(T_{[2,0],[0,2]} + T_{[1,2],[2,1]} - T_{[2,1],[1,2]} + T_{[0,1],[1,0]} - T_{[0,2],[2,0]} - T_{[1,0],[0,1]}) (M_{1,1} - M_{2,1})$$

16

$$M_{0,1} T_{[0,1],[1,0]} - M_{0,1} T_{[0,2],[2,0]} - M_{1,1} T_{[1,0],[0,1]} + M_{1,1} T_{[1,2],[2,1]} - M_{2,1} T_{[0,1],[1,0]} + M_{2,1} T_{[0,2],[2,0]} + M_{2,1} T_{[1,0],[0,1]} - M_{2,1} T_{[1,2],[2,1]}$$

17

$$M_{2,2} g_{2,2} - 1$$

18

$$M_{2,1} g_{2,1} - 1$$

19

$$g_{2,0} M_{2,1} + M_{2,2} g_{2,0} - g_{2,0} + 1$$

20

$$M_{1,2} g_{1,2} - 1$$

21

$$M_{1,1} g_{1,1} - 1$$

22

$$g_{1,0} M_{1,1} + M_{1,2} g_{1,0} - g_{1,0} + 1$$

23

$$M_{0,2} g_{0,2} - 1$$

24

$$M_{0,1} g_{0,1} - 1$$

25

$$M_{0,1} g_{0,0} + M_{0,2} g_{0,0} - g_{0,0} + 1$$

26

$$T_{[2,1],[1,2]}(g_{1,2} - g_{2,2})(g_{0,2} - g_{2,2})$$

27

$$T_{[2,1],[1,2]}(g_{1,2} - g_{2,2})(g_{0,1} - g_{2,1})$$

28

$$T_{[2,1],[1,2]}(g_{1,2} - g_{2,2})(g_{0,0} - g_{2,0})$$

29

$$T_{[2,1],[1,2]}(M_{0,2} g_{1,2} - M_{0,2} g_{2,2} - M_{2,2} g_{1,2} + 1)$$

30

$$T_{[2,1],[1,2]}(g_{1,2} - g_{2,2})(M_{0,1} - M_{2,1})$$

31

$$T_{[2,1],[1,2]}(g_{1,1} - g_{2,1})(g_{0,2} - g_{2,2})$$

32

$$T_{[2,1],[1,2]}(g_{1,1} - g_{2,1})(g_{0,1} - g_{2,1})$$

33

$$T_{[2,1],[1,2]}(g_{1,1} - g_{2,1})(g_{0,0} - g_{2,0})$$

34

$$T_{[2,1],[1,2]}(g_{1,1} - g_{2,1})(M_{0,2} - M_{2,2})$$

35

$$T_{[2,1],[1,2]}(M_{1,1} g_{2,1} - M_{1,2} g_{2,2} + M_{2,1} g_{1,1} - M_{2,2} g_{1,2})$$

36

$$T_{[2,1],[1,2]}(M_{0,1} g_{1,1} - M_{0,1} g_{2,1} + M_{1,1} g_{2,1} - M_{1,2} g_{2,2} - M_{2,2} g_{1,2} + 1)$$

37

$$T_{[2,1],[1,2]}(g_{1,0} - g_{2,0})(g_{0,2} - g_{2,2})$$

38

$$T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0}) (g_{0, 1} - g_{2, 1})$$

39

$$T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0}) (g_{0, 0} - g_{2, 0})$$

40

$$T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0}) (M_{0, 2} - M_{2, 2})$$

41

$$T_{[2, 1], [1, 2]} (M_{0, 1} g_{1, 0} - M_{0, 1} g_{2, 0} - M_{2, 1} g_{1, 0} - M_{2, 2} g_{2, 0} + g_{2, 0} - 1)$$

42

$$T_{[2, 1], [1, 2]} (M_{1, 2} g_{0, 2} - M_{1, 2} g_{2, 2} - M_{2, 2} g_{0, 2} + 1)$$

43

$$T_{[2, 1], [1, 2]} (g_{0, 2} - g_{2, 2}) (M_{1, 1} - M_{2, 1})$$

44

$$T_{[2, 1], [1, 2]} (g_{0, 1} - g_{2, 1}) (M_{1, 2} - M_{2, 2})$$

45

$$T_{[2, 1], [1, 2]} (M_{1, 1} g_{0, 1} - M_{1, 1} g_{2, 1} - M_{2, 1} g_{0, 1} + 1)$$

46

$$T_{[2, 1], [1, 2]} (g_{0, 0} - g_{2, 0}) (M_{1, 2} - M_{2, 2})$$

47

$$T_{[2, 1], [1, 2]} (M_{1, 1} g_{0, 0} - M_{1, 1} g_{2, 0} - M_{2, 1} g_{0, 0} - M_{2, 2} g_{2, 0} + g_{2, 0} - 1)$$

48

$$T_{[2, 1], [1, 2]} (M_{1, 2} - M_{2, 2}) (M_{0, 2} - M_{2, 2})$$

49

$$T_{[2, 1], [1, 2]} (M_{1, 2} - M_{2, 2}) (M_{0, 1} - M_{2, 1})$$

50

$$T_{[2, 1], [1, 2]} (M_{1, 1} - M_{2, 1}) (M_{0, 2} - M_{2, 2})$$

51

$$T_{[2, 1], [1, 2]} (M_{1, 1} - M_{2, 1}) (M_{0, 1} - M_{2, 1})$$

52

$$(g_{1, 2} - g_{2, 2}) (T_{[2, 0], [0, 2]} - T_{[2, 1], [1, 2]}) (T_{[1, 0], [0, 1]} - T_{[1, 2], [2, 1]})$$

53

$$(g_{1, 1} - g_{2, 1}) (T_{[2, 0], [0, 2]} - T_{[2, 1], [1, 2]}) (T_{[1, 0], [0, 1]} - T_{[1, 2], [2, 1]})$$

54

$$(g_{1, 0} - g_{2, 0}) (T_{[2, 0], [0, 2]} - T_{[2, 1], [1, 2]}) (T_{[1, 0], [0, 1]} - T_{[1, 2], [2, 1]})$$

55

$$(T_{[2, 0], [0, 2]} - T_{[2, 1], [1, 2]}) (T_{[1, 0], [0, 1]} - T_{[1, 2], [2, 1]}) (M_{1, 2} - M_{2, 2})$$

56

$$(T_{[2, 0], [0, 2]} - T_{[2, 1], [1, 2]}) (T_{[1, 0], [0, 1]} - T_{[1, 2], [2, 1]}) (M_{1, 1} - M_{2, 1})$$

57

$$T_{[2, 0], [0, 2]} (g_{1, 2} - g_{2, 2}) (g_{0, 2} - g_{2, 2})$$

58

$$T_{[2, 0], [0, 2]} (g_{1, 2} - g_{2, 2}) (g_{0, 1} - g_{2, 1})$$

59

$$T_{[2, 0], [0, 2]} (g_{1, 2} - g_{2, 2}) (g_{0, 0} - g_{2, 0})$$

60

$$T_{[2, 0], [0, 2]} (M_{0, 2} g_{1, 2} - M_{0, 2} g_{2, 2} - M_{2, 2} g_{1, 2} + 1)$$

61

$$T_{[2, 0], [0, 2]} (g_{1, 2} - g_{2, 2}) (M_{0, 1} - M_{2, 1})$$

62

$$T_{[2, 0], [0, 2]} (g_{1, 1} - g_{2, 1}) (g_{0, 2} - g_{2, 2})$$

63

$$T_{[2, 0], [0, 2]} (g_{1, 1} - g_{2, 1}) (g_{0, 1} - g_{2, 1})$$

64

$$T_{[2, 0], [0, 2]} (g_{1, 1} - g_{2, 1}) (g_{0, 0} - g_{2, 0})$$

65

$$T_{[2, 0], [0, 2]} (g_{1, 1} - g_{2, 1}) (M_{0, 2} - M_{2, 2})$$

66

$$T_{[2, 0], [0, 2]} (M_{0, 1} g_{1, 1} - M_{0, 1} g_{2, 1} - M_{2, 1} g_{1, 1} + 1)$$

67

$$T_{[2, 0], [0, 2]} (g_{1, 0} - g_{2, 0}) (g_{0, 2} - g_{2, 2})$$

68

$$T_{[2, 0], [0, 2]} (g_{1, 0} - g_{2, 0}) (g_{0, 1} - g_{2, 1})$$

69

$$T_{[2, 0], [0, 2]} (g_{1, 0} - g_{2, 0}) (g_{0, 0} - g_{2, 0})$$

70

$$T_{[2, 0], [0, 2]} (g_{1, 0} - g_{2, 0}) (M_{0, 2} - M_{2, 2})$$

71

$$T_{[2, 0], [0, 2]} (M_{0, 1} g_{1, 0} - M_{0, 1} g_{2, 0} - M_{2, 1} g_{1, 0} - M_{2, 2} g_{2, 0} + g_{2, 0} - 1)$$

72

$$T_{[2, 0], [0, 2]} (M_{1, 2} g_{0, 2} - M_{1, 2} g_{2, 2} - M_{2, 2} g_{0, 2} + 1)$$

73

$$T_{[2, 0], [0, 2]} (g_{0, 2} - g_{2, 2}) (M_{1, 1} - M_{2, 1})$$

74

$$T_{[2, 0], [0, 2]} (g_{0, 1} - g_{2, 1}) (M_{1, 2} - M_{2, 2})$$

75

$$T_{[2, 0], [0, 2]} (M_{1, 1} g_{0, 1} - M_{1, 1} g_{2, 1} - M_{2, 1} g_{0, 1} + 1)$$

76

$$T_{[2, 0], [0, 2]} (g_{0, 0} - g_{2, 0}) (M_{1, 2} - M_{2, 2})$$

77

$$T_{[2, 0], [0, 2]} (M_{0, 1} g_{2, 0} + M_{0, 2} g_{2, 0} + M_{0, 2} g_{2, 2} + M_{2, 1} g_{0, 0} + M_{2, 2} g_{0, 0} + M_{2, 2} g_{0, 2} - g_{0, 0} - g_{2, 0})$$

78

$$T_{[2, 0], [0, 2]} (M_{0, 1} g_{2, 0} + M_{0, 2} g_{2, 0} + M_{0, 2} g_{2, 2} + M_{1, 1} g_{0, 0} - M_{1, 1} g_{2, 0} + M_{2, 2} g_{0, 0} + M_{2, 2} g_{0, 2} - M_{2, 2} g_{2, 0} - g_{0, 0} - 1)$$

79

$$T_{[2, 0], [0, 2]} (M_{1, 2} - M_{2, 2}) (M_{0, 2} - M_{2, 2})$$

80

$$T_{[2, 0], [0, 2]} (M_{1, 2} - M_{2, 2}) (M_{0, 1} - M_{2, 1})$$

83

$$T_{[2, 0], [0, 2]} (M_{1, 1} - M_{2, 1}) (M_{0, 1} - M_{2, 1})$$

89

$$(g_{1, 2} - g_{2, 2}) (T_{[1, 2], [2, 1]} + T_{[2, 1], [1, 2]}) (T_{[1, 0], [0, 1]} - T_{[2, 0], [0, 2]})$$

90

$$(g_{1, 1} - g_{2, 1}) (T_{[1, 2], [2, 1]} + T_{[2, 1], [1, 2]}) (T_{[1, 0], [0, 1]} - T_{[2, 0], [0, 2]})$$

91

$$(g_{1, 0} - g_{2, 0}) (T_{[1, 2], [2, 1]} + T_{[2, 1], [1, 2]}) (T_{[1, 0], [0, 1]} - T_{[2, 0], [0, 2]})$$

92

$$(T_{[1, 2], [2, 1]} + T_{[2, 1], [1, 2]}) (T_{[1, 0], [0, 1]} - T_{[2, 0], [0, 2]}) (M_{1, 2} - M_{2, 2})$$

93

$$(T_{[1, 2], [2, 1]} + T_{[2, 1], [1, 2]}) (T_{[1, 0], [0, 1]} - T_{[2, 0], [0, 2]}) (M_{1, 1} - M_{2, 1})$$

95

$$(g_{1, 2} - g_{2, 2}) (T_{[1, 2], [2, 1]} + T_{[2, 1], [1, 2]}) (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]})$$

96

$$(g_{1, 1} - g_{2, 1}) (T_{[1, 2], [2, 1]} + T_{[2, 1], [1, 2]}) (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]})$$

97

$$(g_{1, 0} - g_{2, 0}) (T_{[1, 2], [2, 1]} + T_{[2, 1], [1, 2]}) (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]})$$

98

$$(g_{0, 2} - g_{2, 2}) (T_{[2, 0], [0, 2]} + T_{[1, 2], [2, 1]}) (T_{[0, 2], [2, 0]} + T_{[2, 0], [0, 2]})$$

99

$$(g_{0, 1} - g_{2, 1}) (T_{[2, 0], [0, 2]} + T_{[1, 2], [2, 1]}) (T_{[0, 2], [2, 0]} + T_{[2, 0], [0, 2]})$$

100

$$(g_{0, 0} - g_{2, 0}) (T_{[2, 0], [0, 2]} + T_{[1, 2], [2, 1]}) (T_{[0, 2], [2, 0]} + T_{[2, 0], [0, 2]})$$

101

$$(T_{[1, 2], [2, 1]} + T_{[2, 1], [1, 2]}) (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]}) (M_{1, 2} - M_{2, 2})$$

102

$$(T_{[2,0],[0,2]} + T_{[1,2],[2,1]}) (T_{[0,2],[2,0]} + T_{[2,0],[0,2]}) (M_{0,2} - M_{2,2})$$

103

$$(T_{[1,2],[2,1]} + T_{[2,1],[1,2]}) (T_{[0,2],[2,0]} + T_{[2,1],[1,2]}) (M_{1,1} - M_{2,1})$$

104

$$(T_{[2,0],[0,2]} + T_{[1,2],[2,1]}) (T_{[0,2],[2,0]} + T_{[2,0],[0,2]}) (M_{0,1} - M_{2,1})$$

105

$$-(g_{1,1}g_{2,2} - g_{1,2}g_{2,1}) (T_{[1,2],[2,1]} + T_{[2,1],[1,2]})$$

106

$$-(g_{0,1}g_{2,2} - g_{0,2}g_{2,1}) (T_{[2,0],[0,2]} + T_{[1,2],[2,1]})$$

107

$$(T_{[1,2],[2,1]} + T_{[2,1],[1,2]}) (M_{1,1}g_{2,1} - M_{1,2}g_{2,2})$$

108

$$(T_{[2,0],[0,2]} + T_{[1,2],[2,1]}) (M_{0,1}g_{2,1} - M_{0,2}g_{2,2})$$

110

$$T_{[1,2],[2,1]} (g_{1,2} - g_{2,2}) (-g_{1,2} + g_{0,2})$$

111

$$T_{[1,2],[2,1]} (g_{1,2} - g_{2,2}) (g_{0,1} - g_{1,1})$$

112

$$T_{[1,2],[2,1]} (g_{1,2} - g_{2,2}) (g_{0,0} - g_{1,0})$$

113

$$T_{[1,2],[2,1]} (M_{0,2}g_{1,2} - M_{0,2}g_{2,2} + M_{1,2}g_{2,2} - 1)$$

114

$$(T_{[1,2],[2,1]} + T_{[2,1],[1,2]}) (M_{1,1}g_{1,2} - M_{2,1}g_{2,2})$$

115

$$M_{0,1}T_{[1,2],[2,1]}g_{1,2} - M_{0,1}T_{[1,2],[2,1]}g_{2,2} + M_{1,1}T_{[1,2],[2,1]}g_{2,2} \\ + M_{1,1}T_{[2,1],[1,2]}g_{1,2} - M_{2,1}T_{[1,2],[2,1]}g_{2,2} - M_{2,1}T_{[2,1],[1,2]}g_{2,2}$$

116

$$-T_{[1,2],[2,1]}g_{0,1}g_{2,2} + T_{[1,2],[2,1]}g_{0,2}g_{1,1} - T_{[1,2],[2,1]}g_{1,1}g_{1,2} \\ + T_{[1,2],[2,1]}g_{1,1}g_{2,2} - T_{[2,0],[0,2]}g_{0,1}g_{2,2} + T_{[2,0],[0,2]}g_{0,2}g_{2,1} \\ + T_{[2,1],[1,2]}g_{1,1}g_{2,2} - T_{[2,1],[1,2]}g_{1,2}g_{2,1}$$

117

$$T_{[1,2],[2,1]} (g_{1,1} - g_{2,1}) (g_{0,1} - g_{1,1})$$

118

$$T_{[1,2],[2,1]} (g_{1,1} - g_{2,1}) (g_{0,0} - g_{1,0})$$

119

$$(T_{[1,2],[2,1]} + T_{[2,1],[1,2]}) (M_{1,2}g_{1,1} - M_{2,2}g_{2,1})$$

120

$$M_{0,2} T_{[1,2],[2,1]} g_{1,1} - M_{0,2} T_{[1,2],[2,1]} g_{2,1} + M_{1,2} T_{[1,2],[2,1]} g_{2,1} \\ + M_{1,2} T_{[2,1],[1,2]} g_{1,1} - M_{2,2} T_{[1,2],[2,1]} g_{2,1} - M_{2,2} T_{[2,1],[1,2]} g_{2,1}$$

121

$$-M_{1,1} T_{[2,1],[1,2]} g_{2,1} + M_{1,2} T_{[2,1],[1,2]} g_{2,2} + M_{2,1} T_{[1,2],[2,1]} g_{1,1} \\ - M_{2,2} T_{[1,2],[2,1]} g_{1,2}$$

122

$$M_{0,1} T_{[1,2],[2,1]} g_{1,1} + M_{0,1} T_{[2,0],[0,2]} g_{2,1} - M_{0,2} T_{[1,2],[2,1]} g_{2,2} \\ - M_{0,2} T_{[2,0],[0,2]} g_{2,2} - M_{1,1} T_{[2,1],[1,2]} g_{2,1} + M_{1,2} T_{[1,2],[2,1]} g_{2,2} \\ + M_{1,2} T_{[2,1],[1,2]} g_{2,2} - T_{[1,2],[2,1]}$$

123

$$T_{[1,2],[2,1]} (g_{1,0} - g_{2,0}) (-g_{1,2} + g_{0,2})$$

124

$$T_{[1,2],[2,1]} (g_{1,0} - g_{2,0}) (g_{0,1} - g_{1,1})$$

125

$$T_{[1,2],[2,1]} (g_{1,0} - g_{2,0}) (g_{0,0} - g_{1,0})$$

126

$$T_{[1,2],[2,1]} (g_{1,0} - g_{2,0}) (-M_{1,2} + M_{0,2})$$

128

$$T_{[1,2],[2,1]} (M_{1,2} g_{0,2} - M_{2,2} g_{0,2} + M_{2,2} g_{1,2} - 1)$$

129

$$M_{1,1} T_{[1,2],[2,1]} g_{0,2} + M_{1,1} T_{[2,1],[1,2]} g_{1,2} - M_{2,1} T_{[1,2],[2,1]} g_{0,2} \\ + M_{2,1} T_{[1,2],[2,1]} g_{1,2} - M_{2,1} T_{[1,2],[2,1]} g_{2,2} - M_{2,1} T_{[2,1],[1,2]} g_{2,2}$$

130

$$(T_{[2,0],[0,2]} + T_{[1,2],[2,1]}) (M_{0,1} g_{0,2} - M_{2,1} g_{2,2})$$

131

$$M_{1,2} T_{[1,2],[2,1]} g_{0,1} + M_{1,2} T_{[2,1],[1,2]} g_{1,1} - M_{2,2} T_{[1,2],[2,1]} g_{0,1} \\ + M_{2,2} T_{[1,2],[2,1]} g_{1,1} - M_{2,2} T_{[1,2],[2,1]} g_{2,1} - M_{2,2} T_{[2,1],[1,2]} g_{2,1}$$

132

$$(T_{[2,0],[0,2]} + T_{[1,2],[2,1]}) (M_{0,2} g_{0,1} - M_{2,2} g_{2,1})$$

133

$$(T_{[2,0],[0,2]} + T_{[1,2],[2,1]}) (M_{2,1} g_{0,1} - M_{2,2} g_{0,2})$$

134

$$M_{1,1} T_{[1,2],[2,1]} g_{0,1} + M_{1,1} T_{[2,1],[1,2]} g_{2,1} - M_{1,2} T_{[2,1],[1,2]} g_{2,2} \\ + M_{2,1} T_{[2,0],[0,2]} g_{0,1} - M_{2,2} T_{[1,2],[2,1]} g_{0,2} + M_{2,2} T_{[1,2],[2,1]} g_{1,2} \\ - M_{2,2} T_{[2,0],[0,2]} g_{0,2} - T_{[1,2],[2,1]}$$

135

$$T_{[1, 2], [2, 1]} (g_{0, 0} - g_{1, 0}) (M_{1, 2} - M_{2, 2})$$

139

$$T_{[1, 2], [2, 1]} (M_{1, 2} - M_{2, 2}) (-M_{1, 2} + M_{0, 2})$$

140

$$-(T_{[1, 2], [2, 1]} + T_{[2, 1], [1, 2]}) (M_{1, 1} M_{2, 2} - M_{1, 2} M_{2, 1})$$

141

$$T_{[1, 2], [2, 1]} (M_{1, 2} - M_{2, 2}) (M_{0, 1} - M_{1, 1})$$

144

$$T_{[1, 2], [2, 1]} (M_{1, 1} - M_{2, 1}) (M_{0, 1} - M_{1, 1})$$

146

$$\begin{aligned} & M_{1, 2} T_{[1, 0], [0, 1]} g_{1, 0} - M_{1, 2} T_{[2, 0], [0, 2]} g_{1, 0} - M_{2, 2} T_{[1, 0], [0, 1]} g_{2, 0} \\ & + M_{2, 2} T_{[2, 0], [0, 2]} g_{2, 0} - T_{[0, 2], [2, 0]} g_{1, 0} + T_{[0, 2], [2, 0]} g_{2, 0} - T_{[1, 0], [0, 1]} g_{1, 0} \\ & + T_{[1, 0], [0, 1]} g_{2, 0} + T_{[2, 0], [0, 2]} g_{1, 0} - T_{[2, 0], [0, 2]} g_{2, 0} - T_{[2, 1], [1, 2]} g_{1, 0} \\ & + T_{[2, 1], [1, 2]} g_{2, 0} \end{aligned}$$

161

$$-(g_{1, 0} g_{2, 2} - g_{1, 2} g_{2, 0}) (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]})$$

162

$$-(g_{1, 0} g_{2, 1} - g_{1, 1} g_{2, 0}) (-T_{[2, 0], [0, 2]} + T_{[2, 1], [1, 2]} + T_{[0, 2], [2, 0]} + T_{[1, 0], [0, 1]})$$

163

$$-(g_{0, 0} g_{2, 2} - g_{0, 2} g_{2, 0}) (T_{[0, 2], [2, 0]} + T_{[2, 0], [0, 2]})$$

165

$$(T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]}) (M_{1, 1} g_{2, 0} + M_{1, 2} g_{2, 0} + M_{1, 2} g_{2, 2} - g_{2, 0})$$

166

$$(T_{[0, 2], [2, 0]} + T_{[2, 0], [0, 2]}) (M_{0, 1} g_{2, 0} + M_{0, 2} g_{2, 0} + M_{0, 2} g_{2, 2} - g_{2, 0})$$

167

$$(T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]}) (M_{1, 1} g_{1, 2} - M_{2, 1} g_{2, 2} - g_{1, 2} + g_{2, 2})$$

168

$$-T_{[0, 2], [2, 0]} (g_{0, 1} g_{1, 2} - g_{0, 1} g_{2, 2} - g_{0, 2} g_{1, 1} + g_{0, 2} g_{2, 1} + g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 1})$$

171

$$\begin{aligned} & M_{0, 1} T_{[0, 2], [2, 0]} g_{1, 1} - M_{0, 2} T_{[0, 2], [2, 0]} g_{1, 2} - M_{2, 1} T_{[2, 0], [0, 2]} g_{1, 0} \\ & - M_{2, 1} T_{[2, 0], [0, 2]} g_{1, 1} - M_{2, 2} T_{[2, 0], [0, 2]} g_{1, 0} + T_{[2, 0], [0, 2]} g_{1, 0} + T_{[0, 1], [1, 0]} \\ & - T_{[0, 2], [2, 0]} - T_{[1, 0], [0, 1]} + T_{[1, 2], [2, 1]} + T_{[2, 0], [0, 2]} - T_{[2, 1], [1, 2]} \end{aligned}$$

172

$$\begin{aligned} & -T_{[0, 2], [2, 0]} g_{0, 0} g_{1, 2} + T_{[0, 2], [2, 0]} g_{0, 2} g_{1, 0} - T_{[2, 0], [0, 2]} g_{0, 0} g_{2, 2} \\ & + T_{[2, 0], [0, 2]} g_{0, 2} g_{2, 0} + T_{[2, 1], [1, 2]} g_{1, 0} g_{2, 2} - T_{[2, 1], [1, 2]} g_{1, 2} g_{2, 0} \end{aligned}$$

174

$$(T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]}) (M_{1, 2} g_{1, 0} - M_{2, 2} g_{2, 0})$$

175

$$(T_{[0,2],[2,0]} + T_{[2,1],[1,2]}) (M_{2,1}g_{1,0} + M_{2,2}g_{1,0} + M_{2,2}g_{1,2} - g_{1,0})$$

177

$$T_{[0,2],[2,0]} (g_{0,2} - g_{2,2}) (-g_{1,2} + g_{0,2})$$

178

$$T_{[0,2],[2,0]} (-g_{1,2} + g_{0,2}) (g_{0,1} - g_{2,1})$$

179

$$\begin{aligned} & T_{[0,2],[2,0]}g_{0,0}g_{0,2} - T_{[0,2],[2,0]}g_{0,0}g_{1,2} - T_{[0,2],[2,0]}g_{0,0}g_{2,2} \\ & + T_{[0,2],[2,0]}g_{1,0}g_{2,2} - T_{[2,0],[0,2]}g_{0,0}g_{2,2} + T_{[2,0],[0,2]}g_{0,2}g_{2,0} \\ & + T_{[2,1],[1,2]}g_{1,0}g_{2,2} - T_{[2,1],[1,2]}g_{1,2}g_{2,0} \end{aligned}$$

180

$$T_{[0,2],[2,0]} (M_{0,2}g_{1,2} + M_{2,2}g_{0,2} - M_{2,2}g_{1,2} - 1)$$

181

$$T_{[0,2],[2,0]} (M_{0,2}g_{2,2} + M_{1,2}g_{0,2} - M_{1,2}g_{2,2} - 1)$$

182

$$\begin{aligned} & M_{0,1}T_{[0,2],[2,0]}g_{1,2} + M_{0,1}T_{[2,0],[0,2]}g_{0,2} + M_{2,1}T_{[0,2],[2,0]}g_{0,2} \\ & - M_{2,1}T_{[0,2],[2,0]}g_{1,2} - M_{2,1}T_{[0,2],[2,0]}g_{2,2} - M_{2,1}T_{[2,0],[0,2]}g_{2,2} \\ & - T_{[0,2],[2,0]}g_{0,2} + T_{[0,2],[2,0]}g_{2,2} - T_{[2,0],[0,2]}g_{0,2} + T_{[2,0],[0,2]}g_{2,2} \end{aligned}$$

183

$$\begin{aligned} & M_{0,1}T_{[0,2],[2,0]}g_{2,2} + M_{0,1}T_{[2,0],[0,2]}g_{0,2} + M_{1,1}T_{[0,2],[2,0]}g_{0,2} \\ & - M_{1,1}T_{[0,2],[2,0]}g_{2,2} - M_{2,1}T_{[0,2],[2,0]}g_{2,2} - M_{2,1}T_{[2,0],[0,2]}g_{2,2} \\ & - T_{[0,2],[2,0]}g_{0,2} + T_{[0,2],[2,0]}g_{2,2} - T_{[2,0],[0,2]}g_{0,2} + T_{[2,0],[0,2]}g_{2,2} \end{aligned}$$

184

$$(T_{[0,2],[2,0]} + T_{[2,0],[0,2]}) (M_{0,1}g_{0,2} - M_{2,1}g_{2,2} - g_{0,2} + g_{2,2})$$

185

$$T_{[0,2],[2,0]} (g_{0,1} - g_{2,1}) (g_{0,1} - g_{1,1})$$

191

$$\begin{aligned} & M_{0,2}T_{[0,2],[2,0]}g_{2,2} + M_{1,1}T_{[0,2],[2,0]}g_{0,1} - M_{1,1}T_{[2,0],[0,2]}g_{2,0} \\ & - M_{1,1}T_{[2,0],[0,2]}g_{2,1} - M_{1,2}T_{[0,2],[2,0]}g_{2,2} - M_{1,2}T_{[2,0],[0,2]}g_{2,0} \\ & + T_{[2,0],[0,2]}g_{2,0} - T_{[0,1],[1,0]} + T_{[1,0],[0,1]} - T_{[1,2],[2,1]} - T_{[2,0],[0,2]} \\ & + T_{[2,1],[1,2]} \end{aligned}$$

192

$$T_{[0,2],[2,0]} (g_{0,0} - g_{2,0}) (g_{0,0} - g_{1,0})$$

193

$$\begin{aligned} & M_{0,2}T_{[0,2],[2,0]}g_{1,0} + M_{0,2}T_{[2,0],[0,2]}g_{0,0} + M_{2,2}T_{[0,2],[2,0]}g_{0,0} \\ & - M_{2,2}T_{[0,2],[2,0]}g_{1,0} - M_{2,2}T_{[0,2],[2,0]}g_{2,0} - M_{2,2}T_{[2,0],[0,2]}g_{2,0} \end{aligned}$$

194

$$M_{0,2} T_{[0,2],[2,0]} g_{2,0} + M_{0,2} T_{[2,0],[0,2]} g_{0,0} + M_{1,2} T_{[0,2],[2,0]} g_{0,0} \\ - M_{1,2} T_{[0,2],[2,0]} g_{2,0} - M_{2,2} T_{[0,2],[2,0]} g_{2,0} - M_{2,2} T_{[2,0],[0,2]} g_{2,0}$$

195

$$(T_{[0,2],[2,0]} + T_{[2,0],[0,2]}) (M_{0,2} g_{0,0} - M_{2,2} g_{2,0})$$

198

$$-(T_{[0,2],[2,0]} + T_{[2,1],[1,2]}) (M_{1,1} M_{2,2} - M_{1,2} M_{2,1} + M_{1,2} - M_{2,2})$$

199

$$T_{[0,2],[2,0]} (M_{0,2} - M_{2,2}) (-M_{1,2} + M_{0,2})$$

203

$$T_{[0,2],[2,0]} (M_{0,1} - M_{2,1}) (M_{0,1} - M_{1,1})$$

204

$$g_{2,0} g_{2,1} g_{2,2} - g_{2,0} g_{2,1} - g_{2,0} g_{2,2} - g_{2,1} g_{2,2}$$

205

$$M_{2,2} g_{2,0} g_{2,1} - g_{2,0} g_{2,1} + g_{2,0} + g_{2,1}$$

206

$$g_{1,0} g_{1,1} g_{1,2} - g_{1,0} g_{1,1} - g_{1,0} g_{1,2} - g_{1,1} g_{1,2}$$

207

$$M_{1,2} g_{1,0} g_{1,1} - g_{1,0} g_{1,1} + g_{1,0} + g_{1,1}$$

208

$$g_{0,0} g_{0,1} g_{0,2} - g_{0,0} g_{0,1} - g_{0,0} g_{0,2} - g_{0,1} g_{0,2}$$

209

$$M_{0,2} g_{0,0} g_{0,1} - g_{0,0} g_{0,1} + g_{0,0} + g_{0,1}$$

210

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{1,2} - g_{2,2})^2$$

211

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{1,2} - g_{2,2}) (g_{1,1} - g_{2,1})$$

212

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{1,2} - g_{2,2}) (g_{1,0} - g_{2,0})$$

213

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (M_{1,2} g_{2,2} + M_{2,2} g_{1,2} - 2)$$

214

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{1,2} - g_{2,2}) (M_{1,1} - M_{2,1})$$

215

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{1,1} - g_{2,1})^2$$

216

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{1,1} - g_{2,1}) (g_{1,0} - g_{2,0})$$

217

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{1,1} - g_{2,1}) (M_{1,2} - M_{2,2})$$

218

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{1,0} - g_{2,0})^2$$

219

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{1,0} - g_{2,0}) (M_{1,2} - M_{2,2})$$

220

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (M_{1,1} g_{2,0} + M_{1,2} g_{2,0} + M_{2,1} g_{1,0} + M_{2,2} g_{1,0} - g_{1,0} - g_{2,0} + 2)$$

221

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{0,2} - g_{2,2})^2$$

222

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{0,2} - g_{2,2}) (g_{0,1} - g_{2,1})$$

223

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{0,2} - g_{2,2}) (g_{0,0} - g_{2,0})$$

224

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (M_{0,2} g_{2,2} + M_{2,2} g_{0,2} - 2)$$

225

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{0,2} - g_{2,2}) (M_{0,1} - M_{2,1})$$

226

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{0,1} - g_{2,1})^2$$

227

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{0,1} - g_{2,1}) (g_{0,0} - g_{2,0})$$

228

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{0,1} - g_{2,1}) (M_{0,2} - M_{2,2})$$

229

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (M_{0,1} g_{2,1} + M_{2,1} g_{0,1} - 2)$$

230

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{0,0} - g_{2,0})^2$$

231

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (g_{0,0} - g_{2,0}) (M_{0,2} - M_{2,2})$$

232

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (M_{1,2} - M_{2,2})^2$$

233

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (M_{1,2} - M_{2,2}) (M_{1,1} - M_{2,1})$$

234

$$T_{[2,0],[0,2]} T_{[2,1],[1,2]} (M_{0,2} - M_{2,2})^2$$

236

$$T_{[2, 0], [0, 2]} T_{[2, 1], [1, 2]} (M_{1, 1} - M_{2, 1})^2$$

237

$$T_{[2, 0], [0, 2]} T_{[2, 1], [1, 2]} (M_{0, 1} - M_{2, 1})^2$$

239

$$T_{[1, 2], [2, 1]} T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2})^2$$

240

$$T_{[2, 1], [1, 2]} (T_{[1, 2], [2, 1]} g_{1, 1} g_{1, 2} - 2 T_{[1, 2], [2, 1]} g_{1, 1} g_{2, 2} + T_{[1, 2], [2, 1]} g_{2, 1} g_{2, 2} - T_{[2, 1], [1, 2]} g_{1, 1} g_{2, 2} + T_{[2, 1], [1, 2]} g_{1, 2} g_{2, 1})$$

241

$$T_{[1, 2], [2, 1]} T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2}) (g_{1, 0} - g_{2, 0})$$

242

$$T_{[1, 2], [2, 1]} T_{[2, 1], [1, 2]} (M_{1, 2} g_{2, 2} + M_{2, 2} g_{1, 2} - 2)$$

243

$$T_{[2, 1], [1, 2]} (M_{1, 1} T_{[1, 2], [2, 1]} g_{2, 2} + M_{1, 1} T_{[2, 1], [1, 2]} g_{1, 2} + M_{2, 1} T_{[1, 2], [2, 1]} g_{1, 2} - 2 M_{2, 1} T_{[1, 2], [2, 1]} g_{2, 2} - M_{2, 1} T_{[2, 1], [1, 2]} g_{2, 2})$$

244

$$T_{[1, 2], [2, 1]} T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1})^2$$

245

$$T_{[1, 2], [2, 1]} T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1}) (g_{1, 0} - g_{2, 0})$$

246

$$T_{[2, 1], [1, 2]} (M_{1, 2} T_{[1, 2], [2, 1]} g_{2, 1} + M_{1, 2} T_{[2, 1], [1, 2]} g_{1, 1} + M_{2, 2} T_{[1, 2], [2, 1]} g_{1, 1} - 2 M_{2, 2} T_{[1, 2], [2, 1]} g_{2, 1} - M_{2, 2} T_{[2, 1], [1, 2]} g_{2, 1})$$

247

$$T_{[1, 2], [2, 1]} T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0})^2$$

248

$$T_{[1, 2], [2, 1]} T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0}) (M_{1, 2} - M_{2, 2})$$

250

$$T_{[1, 2], [2, 1]} T_{[2, 1], [1, 2]} (M_{1, 2} - M_{2, 2})^2$$

251

$$T_{[2, 1], [1, 2]} (M_{1, 1} M_{1, 2} T_{[1, 2], [2, 1]} - 2 M_{1, 1} M_{2, 2} T_{[1, 2], [2, 1]} - M_{1, 1} M_{2, 2} T_{[2, 1], [1, 2]} + M_{2, 1} T_{[2, 1], [1, 2]} M_{1, 2} + M_{2, 1} M_{2, 2} T_{[1, 2], [2, 1]})$$

252

$$T_{[1, 2], [2, 1]} T_{[2, 1], [1, 2]} (M_{1, 1} - M_{2, 1})^2$$

254

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2})^2$$

255

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2}) (g_{1, 1} - g_{2, 1})$$

256

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2}) (g_{1, 0} - g_{2, 0})$$

257

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (M_{1, 2} g_{2, 2} + M_{2, 2} g_{1, 2} - 2)$$

258

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2}) (M_{1, 1} - M_{2, 1})$$

259

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1})^2$$

260

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1}) (g_{1, 0} - g_{2, 0})$$

261

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1}) (M_{1, 2} - M_{2, 2})$$

262

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0})^2$$

265

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (M_{1, 2} - M_{2, 2})^2$$

267

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (M_{1, 1} - M_{2, 1})^2$$

270

$$T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2})^2 (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]})$$

272

$$T_{[2, 1], [1, 2]} (g_{1, 0} g_{1, 2} - 2 g_{1, 0} g_{2, 2} + g_{2, 0} g_{2, 2}) (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]})$$

273

$$T_{[2, 1], [1, 2]} (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]}) (M_{1, 2} g_{2, 2} + M_{2, 2} g_{1, 2} - 2)$$

275

$$T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1})^2 (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]})$$

278

$$T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0})^2 (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]})$$

279

$$T_{[2, 1], [1, 2]} (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]}) (M_{1, 2} g_{2, 0} + M_{2, 2} g_{1, 0} - 2 M_{2, 2} g_{2, 0})$$

280

$$T_{[2, 1], [1, 2]} (M_{1, 2} - M_{2, 2})^2 (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]})$$

282

$$T_{[2, 1], [1, 2]} (M_{1, 1} - M_{2, 1})^2 (T_{[0, 2], [2, 0]} + T_{[2, 1], [1, 2]})$$

284

$$T_{[2, 1], [1, 2]} (M_{1, 1} g_{2, 1}^2 + M_{1, 2} g_{1, 1} g_{2, 2} - M_{1, 2} g_{2, 1} g_{2, 2} - g_{2, 1})$$

285

$$T_{[2, 1], [1, 2]} (g_{1, 0} g_{1, 1} g_{2, 2} + g_{1, 2} g_{2, 0} g_{2, 1} - g_{1, 0} g_{2, 1} - g_{1, 0} g_{2, 2} - g_{1, 1} g_{2, 0} - g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 0} - g_{1, 2} g_{2, 1})$$

287

$$-T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2}) (g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 1})$$

288

$$-T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1}) (g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 1})$$

289

$$T_{[2, 1], [1, 2]} (g_{1, 0} g_{1, 2} g_{2, 1} + g_{1, 1} g_{2, 0} g_{2, 2} - g_{1, 0} g_{2, 1} - g_{1, 0} g_{2, 2} - g_{1, 1} g_{2, 0} - g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 0} - g_{1, 2} g_{2, 1})$$

290

$$T_{[2, 1], [1, 2]} (M_{1, 2} g_{1, 1} g_{2, 2} + M_{2, 2} g_{1, 2} g_{2, 1} - g_{1, 1} - g_{2, 1})$$

291

$$T_{[2, 1], [1, 2]} (M_{1, 1} g_{1, 2} g_{2, 1} - M_{1, 1} g_{2, 1} g_{2, 2} + M_{1, 2} g_{2, 2}^2 - g_{2, 2})$$

297

$$T_{[2, 1], [1, 2]} (M_{1, 1} M_{1, 2} g_{2, 1} - M_{1, 1} M_{2, 2} g_{2, 1} - M_{1, 2}^2 g_{2, 2} + M_{1, 2})$$

298

$$T_{[2, 1], [1, 2]} (M_{1, 1}^2 g_{2, 1} - M_{1, 1} M_{1, 2} g_{2, 2} + M_{1, 2} M_{2, 1} g_{2, 2} - M_{1, 1})$$

299

$$T_{[2, 1], [1, 2]} (M_{1, 1} g_{1, 2} g_{2, 0} + M_{2, 1} g_{1, 0} g_{2, 2} - g_{1, 0} g_{1, 2} - g_{2, 0} g_{2, 2} + g_{1, 0} + g_{1, 2} + g_{2, 0} + g_{2, 2})$$

305

$$T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2}) (M_{1, 1} g_{1, 2} - M_{2, 1} g_{2, 2})$$

306

$$-T_{[2, 1], [1, 2]} (-M_{1, 2} g_{2, 0} g_{2, 2} - M_{2, 2} g_{1, 0} g_{1, 2} + M_{1, 1} g_{2, 0} + M_{1, 2} g_{2, 0} + M_{1, 2} g_{2, 2} + M_{2, 1} g_{1, 0} + M_{2, 2} g_{1, 0} + M_{2, 2} g_{1, 2})$$

307

$$T_{[2, 1], [1, 2]} (M_{1, 1} M_{2, 2} g_{1, 2} + M_{1, 2} M_{2, 1} g_{2, 2} - M_{1, 1} - M_{2, 1})$$

308

$$T_{[2, 1], [1, 2]} (M_{1, 1} - M_{2, 1}) (M_{1, 1} g_{1, 2} - M_{2, 1} g_{2, 2})$$

309

$$T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1}) (M_{1, 2} g_{1, 1} - M_{2, 2} g_{2, 1})$$

310

$$T_{[2, 1], [1, 2]} (M_{1, 2} - M_{2, 2}) (M_{1, 2} g_{1, 1} - M_{2, 2} g_{2, 1})$$

311

$$T_{[2, 1], [1, 2]} (M_{1, 1} M_{2, 2} g_{2, 0} + M_{1, 2} M_{2, 1} g_{1, 0} + M_{1, 2} M_{2, 2} g_{1, 0} + M_{1, 2} M_{2, 2} g_{2, 0} - M_{1, 2} g_{2, 0} - M_{2, 2} g_{1, 0} + M_{1, 2} + M_{2, 2})$$

317

$$-T_{[2, 1], [1, 2]} (M_{1, 2} - M_{2, 2}) (M_{1, 1} M_{2, 2} - M_{1, 2} M_{2, 1})$$

318

$$-T_{[2, 1], [1, 2]} (M_{1, 1} - M_{2, 1}) (M_{1, 1} M_{2, 2} - M_{1, 2} M_{2, 1})$$

334

$$T_{[1, 2], [2, 1]} T_{[2, 0], [0, 2]} (g_{1, 2} - g_{2, 2})^2$$

335

$$T_{[2, 0], [0, 2]} (T_{[1, 2], [2, 1]} g_{1, 1} g_{1, 2} - 2 T_{[1, 2], [2, 1]} g_{1, 1} g_{2, 2} + T_{[1, 2], [2, 1]} g_{2, 1} g_{2, 2} - T_{[2, 1], [1, 2]} g_{1, 1} g_{2, 2} + T_{[2, 1], [1, 2]} g_{1, 2} g_{2, 1})$$

338

$$T_{[2, 0], [0, 2]} (T_{[1, 2], [2, 1]} + T_{[2, 1], [1, 2]}) (M_{1, 1} g_{2, 2} + M_{2, 1} g_{1, 2} - 2 M_{2, 1} g_{2, 2})$$

339

$$T_{[1, 2], [2, 1]} T_{[2, 0], [0, 2]} (g_{1, 1} - g_{2, 1})^2$$

341

$$T_{[2, 0], [0, 2]} (T_{[1, 2], [2, 1]} + T_{[2, 1], [1, 2]}) (M_{1, 2} g_{2, 1} + M_{2, 2} g_{1, 1} - 2 M_{2, 2} g_{2, 1})$$

342

$$T_{[1, 2], [2, 1]} T_{[2, 0], [0, 2]} (g_{1, 0} - g_{2, 0})^2$$

346

$$T_{[2, 0], [0, 2]} (g_{0, 2} - g_{2, 2})^2 (T_{[2, 0], [0, 2]} + T_{[1, 2], [2, 1]})$$

347

$$T_{[2, 0], [0, 2]} (g_{0, 1} g_{0, 2} - 2 g_{0, 1} g_{2, 2} + g_{2, 1} g_{2, 2}) (T_{[2, 0], [0, 2]} + T_{[1, 2], [2, 1]})$$

348

$$T_{[2, 0], [0, 2]} (g_{0, 2} - g_{2, 2}) (g_{0, 0} - g_{2, 0}) (T_{[2, 0], [0, 2]} + T_{[1, 2], [2, 1]})$$

349

$$T_{[2, 0], [0, 2]} (T_{[2, 0], [0, 2]} + T_{[1, 2], [2, 1]}) (M_{0, 2} g_{2, 2} + M_{2, 2} g_{0, 2} - 2)$$

350

$$T_{[2, 0], [0, 2]} (T_{[2, 0], [0, 2]} + T_{[1, 2], [2, 1]}) (M_{0, 1} g_{2, 2} + M_{2, 1} g_{0, 2} - 2 M_{2, 1} g_{2, 2})$$

351

$$T_{[2, 0], [0, 2]} (g_{0, 1} - g_{2, 1})^2 (T_{[2, 0], [0, 2]} + T_{[1, 2], [2, 1]})$$

352

$$T_{[2, 0], [0, 2]} (g_{0, 1} - g_{2, 1}) (g_{0, 0} - g_{2, 0}) (T_{[2, 0], [0, 2]} + T_{[1, 2], [2, 1]})$$

353

$$T_{[2, 0], [0, 2]} (T_{[2, 0], [0, 2]} + T_{[1, 2], [2, 1]}) (M_{0, 2} g_{2, 1} + M_{2, 2} g_{0, 1} - 2 M_{2, 2} g_{2, 1})$$

354

$$T_{[2,0],[0,2]} (g_{0,0} - g_{2,0})^2 (T_{[2,0],[0,2]} + T_{[1,2],[2,1]})$$

356

$$T_{[1,2],[2,1]} T_{[2,0],[0,2]} (M_{1,2} - M_{2,2})^2$$

357

$$T_{[2,0],[0,2]} (M_{1,1} M_{1,2} T_{[1,2],[2,1]} - 2 M_{1,1} M_{2,2} T_{[1,2],[2,1]} - M_{1,1} M_{2,2} T_{[2,1],[1,2]} + M_{2,1} T_{[2,1],[1,2]} M_{1,2} + M_{2,1} M_{2,2} T_{[1,2],[2,1]})$$

358

$$T_{[2,0],[0,2]} (M_{0,2} - M_{2,2})^2 (T_{[2,0],[0,2]} + T_{[1,2],[2,1]})$$

359

$$T_{[2,0],[0,2]} (T_{[2,0],[0,2]} + T_{[1,2],[2,1]}) (M_{0,1} M_{0,2} - 2 M_{0,1} M_{2,2} + M_{2,1} M_{2,2})$$

360

$$T_{[1,2],[2,1]} T_{[2,0],[0,2]} (M_{1,1} - M_{2,1})^2$$

361

$$T_{[2,0],[0,2]} (M_{0,1} - M_{2,1})^2 (T_{[2,0],[0,2]} + T_{[1,2],[2,1]})$$

399

$$-T_{[2,0],[0,2]} (g_{0,2} - g_{2,2}) (g_{0,0} g_{2,2} - g_{0,2} g_{2,0})$$

401

$$-T_{[2,0],[0,2]} (g_{0,0} - g_{2,0}) (g_{0,0} g_{2,2} - g_{0,2} g_{2,0})$$

415

$$T_{[2,0],[0,2]} (g_{0,2} - g_{2,2}) (M_{0,1} g_{0,2} - M_{2,1} g_{2,2} - g_{0,2} + g_{2,2})$$

418

$$T_{[2,0],[0,2]} (M_{0,1} - M_{2,1}) (M_{0,1} g_{0,2} - M_{2,1} g_{2,2} - g_{0,2} + g_{2,2})$$

419

$$T_{[2,0],[0,2]} (g_{0,0} - g_{2,0}) (M_{0,2} g_{0,0} - M_{2,2} g_{2,0})$$

420

$$T_{[2,0],[0,2]} (M_{0,2} - M_{2,2}) (M_{0,2} g_{0,0} - M_{2,2} g_{2,0})$$

437

$$(g_{1,1} g_{2,0} g_{2,2} - g_{1,1} g_{2,0} - g_{1,1} g_{2,2} - g_{1,2} g_{2,0}) (T_{[1,2],[2,1]} + T_{[2,1],[1,2]})$$

438

$$(g_{0,1} g_{2,0} g_{2,2} - g_{0,1} g_{2,0} - g_{0,1} g_{2,2} - g_{0,2} g_{2,0}) (T_{[2,0],[0,2]} + T_{[1,2],[2,1]})$$

439

$$-(T_{[1,2],[2,1]} + T_{[2,1],[1,2]}) (-M_{1,2} g_{2,0} g_{2,2} + M_{1,1} g_{2,0} + M_{1,2} g_{2,0} + M_{1,2} g_{2,2})$$

441

$$(g_{1,0} g_{1,1} g_{2,2} - g_{1,0} g_{2,1} - g_{1,0} g_{2,2} - g_{1,1} g_{2,2}) (T_{[1,2],[2,1]} + T_{[2,1],[1,2]})$$

442

$$(T_{[1,2],[2,1]} + T_{[2,1],[1,2]}) (M_{2,1} g_{1,0} g_{2,2} - g_{1,0} g_{1,2} + g_{1,0} + g_{1,2})$$

449

$$(g_{1,0}g_{1,2}g_{2,0} - g_{1,0}g_{2,0}g_{2,2} + g_{1,0}g_{2,2} - g_{1,2}g_{2,0}) (T_{[1,2],[2,1]} + T_{[2,1],[1,2]})$$

450

$$(g_{1,0}g_{1,1}g_{2,0} - g_{1,0}g_{2,0}g_{2,1} + g_{1,0}g_{2,1} - g_{1,1}g_{2,0}) (T_{[1,2],[2,1]} + T_{[2,1],[1,2]})$$

452

$$(T_{[1,2],[2,1]} + T_{[2,1],[1,2]}) (M_{1,2}g_{1,0}g_{2,0} - M_{2,2}g_{1,0}g_{2,0} - M_{1,2}g_{1,0} + M_{2,2}g_{2,0})$$

454

$$g_{0,0}g_{0,2}g_{2,0}T_{[1,2],[2,1]} - T_{[1,2],[2,1]}g_{0,0}g_{2,0}g_{2,2} + T_{[2,0],[0,2]}g_{0,0}^2g_{2,2} - 2T_{[2,0],[0,2]}g_{0,0}g_{2,0}g_{2,2} + T_{[2,0],[0,2]}g_{0,2}g_{2,0}^2 + T_{[1,2],[2,1]}g_{0,0}g_{2,2} - T_{[1,2],[2,1]}g_{0,2}g_{2,0} + T_{[2,0],[0,2]}g_{0,0}g_{2,2} - T_{[2,0],[0,2]}g_{0,2}g_{2,0}$$

457

$$(g_{0,0}g_{0,1}g_{2,0} - g_{0,0}g_{2,0}g_{2,1} + g_{0,0}g_{2,1} - g_{0,1}g_{2,0}) (T_{[2,0],[0,2]} + T_{[1,2],[2,1]})$$

460

$$(T_{[1,2],[2,1]} + T_{[2,1],[1,2]}) (M_{1,1}M_{2,2}g_{2,0} + M_{1,2}M_{2,2}g_{2,0} - M_{1,2}g_{2,0} + M_{1,2})$$

461

$$-M_{1,2}T_{[2,1],[1,2]}g_{2,0}g_{2,2} + M_{2,2}g_{1,0}g_{1,2}T_{[1,2],[2,1]} + M_{1,1}T_{[2,1],[1,2]}g_{2,0} + M_{1,2}T_{[2,1],[1,2]}g_{2,0} + M_{1,2}T_{[2,1],[1,2]}g_{2,2} - M_{2,1}T_{[1,2],[2,1]}g_{1,0} - M_{2,2}T_{[1,2],[2,1]}g_{1,0} - M_{2,2}T_{[1,2],[2,1]}g_{1,2}$$

496

$$(g_{1,0}g_{2,1}g_{2,2} - g_{1,0}g_{2,1} - g_{1,0}g_{2,2} - g_{1,2}g_{2,1}) (T_{[0,2],[2,0]} + T_{[2,1],[1,2]})$$

504

$$T_{[0,2],[2,0]} (M_{0,2}g_{1,2}g_{2,2} + g_{0,2} - g_{1,2} - g_{2,2})$$

513

$$g_{1,1}g_{1,2}g_{2,1}T_{[0,2],[2,0]} - T_{[0,2],[2,0]}g_{1,1}g_{2,1}g_{2,2} + T_{[2,1],[1,2]}g_{1,1}^2g_{2,2} - 2T_{[2,1],[1,2]}g_{1,1}g_{2,1}g_{2,2} + T_{[2,1],[1,2]}g_{1,2}g_{2,1}^2 + T_{[0,2],[2,0]}g_{1,1}g_{2,2} - T_{[0,2],[2,0]}g_{1,2}g_{2,1} + T_{[2,1],[1,2]}g_{1,1}g_{2,2} - T_{[2,1],[1,2]}g_{1,2}g_{2,1}$$

559

$$T_{[2,0],[0,2]}T_{[2,1],[1,2]} (M_{1,2}g_{2,2}^2 + g_{1,2} - 2g_{2,2})$$

560

$$T_{[2,0],[0,2]}T_{[2,1],[1,2]} (M_{0,2}g_{2,2}^2 + g_{0,2} - 2g_{2,2})$$

568

$$T_{[2,0],[0,2]}T_{[2,1],[1,2]} (M_{0,1}g_{2,1}^2 + g_{0,1} - 2g_{2,1})$$

572

$$T_{[2,0],[0,2]}T_{[2,1],[1,2]} (M_{1,1}g_{2,0}^2 + M_{1,2}g_{2,0}^2 - g_{2,0}^2 - g_{1,0} + 2g_{2,0})$$

577

$$T_{[1, 2], [2, 1]} T_{[2, 1], [1, 2]} (M_{1, 2} g_{2, 2}^2 + g_{1, 2} - 2 g_{2, 2})$$

588

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (M_{1, 2} g_{2, 2}^2 + g_{1, 2} - 2 g_{2, 2})$$

607

$$-T_{[2, 1], [1, 2]} (-M_{1, 2} g_{2, 0} g_{2, 2}^2 + M_{1, 1} g_{2, 0} g_{2, 2} + M_{1, 2} g_{2, 0} g_{2, 2} + M_{1, 2} g_{2, 2}^2 + M_{2, 1} g_{1, 0} g_{2, 2} - g_{1, 0} g_{1, 2} + g_{1, 0} + g_{1, 2})$$

609

$$T_{[2, 1], [1, 2]} (M_{1, 2} g_{1, 1} g_{2, 2}^2 - g_{1, 1} g_{2, 2} + g_{1, 2} g_{2, 1} - g_{2, 1} g_{2, 2})$$

613

$$T_{[2, 1], [1, 2]} (M_{1, 2} M_{2, 1} g_{2, 2}^2 + M_{1, 1} g_{1, 2} - M_{1, 1} g_{2, 2} - M_{2, 1} g_{2, 2})$$

618

$$T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2}) (g_{1, 1} g_{2, 0} g_{2, 2} - g_{1, 1} g_{2, 0} - g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 0})$$

620

$$T_{[2, 1], [1, 2]} (g_{1, 1} g_{2, 0} g_{2, 2} - g_{1, 1} g_{2, 0} - g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 0}) (g_{1, 0} - g_{2, 0})$$

621

$$T_{[2, 1], [1, 2]} (M_{1, 2} g_{1, 0} g_{2, 0} g_{2, 2} - M_{1, 2} g_{2, 0}^2 g_{2, 2} + M_{1, 1} g_{2, 0}^2 - M_{1, 2} g_{1, 0} g_{2, 2} + M_{1, 2} g_{2, 0}^2 + M_{1, 2} g_{2, 0} g_{2, 2} - g_{1, 0} g_{2, 0} + g_{2, 0})$$

623

$$-T_{[2, 1], [1, 2]} (-M_{1, 2}^2 g_{2, 0} g_{2, 2} + M_{1, 1} M_{1, 2} g_{2, 0} - M_{1, 1} M_{2, 2} g_{2, 0} + M_{1, 2}^2 g_{2, 0} + M_{1, 2}^2 g_{2, 2} - M_{1, 2} M_{2, 2} g_{2, 0} + M_{1, 2} g_{2, 0} - M_{1, 2})$$

627

$$T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1}) (g_{1, 0} g_{1, 1} g_{2, 2} - g_{1, 0} g_{2, 1} - g_{1, 0} g_{2, 2} - g_{1, 1} g_{2, 2})$$

629

$$T_{[2, 1], [1, 2]} (M_{2, 1} g_{1, 0}^2 g_{2, 2} - g_{1, 0}^2 g_{1, 2} + g_{1, 0} g_{1, 2} g_{2, 0} - g_{1, 0} g_{2, 0} g_{2, 2} + g_{1, 0}^2 + g_{1, 0} g_{1, 2} + g_{1, 0} g_{2, 2} - g_{1, 2} g_{2, 0})$$

646

$$T_{[2, 1], [1, 2]} (M_{1, 1} M_{2, 2} g_{2, 0}^2 + M_{1, 2} M_{2, 2} g_{2, 0}^2 + M_{1, 2} g_{1, 0} g_{2, 0} - M_{1, 2} g_{2, 0}^2 - M_{2, 2} g_{1, 0} g_{2, 0} - M_{1, 2} g_{1, 0} + M_{1, 2} g_{2, 0} + M_{2, 2} g_{2, 0})$$

648

$$T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2}) (g_{1, 0} g_{1, 2} g_{2, 0} - g_{1, 0} g_{2, 0} g_{2, 2} + g_{1, 0} g_{2, 2} - g_{1, 2} g_{2, 0})$$

649

$$T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0}) (g_{1, 0} g_{1, 2} g_{2, 0} - g_{1, 0} g_{2, 0} g_{2, 2} + g_{1, 0} g_{2, 2} - g_{1, 2} g_{2, 0})$$

650

$$T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1}) (g_{1, 0} g_{1, 1} g_{2, 0} - g_{1, 0} g_{2, 0} g_{2, 1} + g_{1, 0} g_{2, 1} - g_{1, 1} g_{2, 0})$$

651

$$T_{[2,1],[1,2]}(g_{1,0} - g_{2,0})(g_{1,0}g_{1,1}g_{2,0} - g_{1,0}g_{2,0}g_{2,1} + g_{1,0}g_{2,1} - g_{1,1}g_{2,0})$$

652

$$T_{[2,1],[1,2]}(g_{1,0} - g_{2,0})(M_{1,2}g_{1,0}g_{2,0} - M_{2,2}g_{1,0}g_{2,0} - M_{1,2}g_{1,0} + M_{2,2}g_{2,0})$$

653

$$T_{[2,1],[1,2]}(M_{1,2} - M_{2,2})(M_{1,2}g_{1,0}g_{2,0} - M_{2,2}g_{1,0}g_{2,0} - M_{1,2}g_{1,0} + M_{2,2}g_{2,0})$$

655

$$T_{[2,1],[1,2]}(M_{1,2} - M_{2,2})(M_{1,1}M_{2,2}g_{2,0} + M_{1,2}M_{2,2}g_{2,0} - M_{1,2}g_{2,0} + M_{1,2})$$

665

$$T_{[2,0],[0,2]}(g_{0,2}g_{2,0}^2 - g_{2,0}^2g_{2,2} + g_{0,0}g_{2,2} - g_{0,2}g_{2,0})(T_{[2,0],[0,2]} + T_{[1,2],[2,1]})$$

683

$$T_{[2,0],[0,2]}(g_{0,2} - g_{2,2})(g_{0,1}g_{0,2}g_{2,1} - g_{0,1}g_{2,1}g_{2,2} + g_{0,1}g_{2,2} - g_{0,2}g_{2,1})$$

684

$$T_{[2,0],[0,2]}(g_{0,1} - g_{2,1})(g_{0,1}g_{0,2}g_{2,1} - g_{0,1}g_{2,1}g_{2,2} + g_{0,1}g_{2,2} - g_{0,2}g_{2,1})$$

685

$$T_{[2,0],[0,2]}(g_{0,1} - g_{2,1})(g_{0,0}g_{0,1}g_{2,1} - g_{0,1}g_{2,0}g_{2,1} - g_{0,0}g_{2,1} + g_{0,1}g_{2,0})$$

686

$$T_{[2,0],[0,2]}(g_{0,1} - g_{2,1})(M_{0,2}g_{0,1}g_{2,1} - M_{2,2}g_{0,1}g_{2,1} - M_{0,2}g_{0,1} + M_{2,2}g_{2,1})$$

687

$$T_{[2,0],[0,2]}(g_{0,0} - g_{2,0})(g_{0,0}g_{0,1}g_{2,1} - g_{0,1}g_{2,0}g_{2,1} - g_{0,0}g_{2,1} + g_{0,1}g_{2,0})$$

688

$$T_{[2,0],[0,2]}(M_{0,2} - M_{2,2})(M_{0,2}g_{0,1}g_{2,1} - M_{2,2}g_{0,1}g_{2,1} - M_{0,2}g_{0,1} + M_{2,2}g_{2,1})$$

731

$$T_{[2,1],[1,2]}(M_{0,1}g_{2,1} - M_{0,2}g_{2,2})(M_{0,1}g_{2,1}g_{2,2} - M_{0,1}g_{2,1} + M_{0,2}g_{2,2} - g_{2,2})$$

732

$$-T_{[2,1],[1,2]}(g_{0,1}g_{0,2}g_{2,2} - g_{0,2}g_{2,1}g_{2,2} - g_{0,1}g_{2,2} + g_{0,2}g_{2,1})(g_{0,1}g_{2,2} - g_{0,2}g_{2,1})$$

733

$$T_{[2,1],[1,2]}(M_{0,1}g_{0,2} - M_{2,1}g_{2,2})(M_{0,1}g_{0,2}g_{2,2} - M_{2,1}g_{0,2}g_{2,2} - M_{0,1}g_{0,2} + M_{2,1}g_{2,2})$$

734

$$-T_{[2,1],[1,2]}(g_{0,0}g_{2,1} - g_{0,1}g_{2,0})(g_{0,0}g_{0,1}g_{2,0} - g_{0,0}g_{2,0}g_{2,1} + g_{0,0}g_{2,1} - g_{0,1}g_{2,0})$$

736

$$T_{[2,1],[1,2]}(M_{2,1}g_{0,1} - M_{2,2}g_{0,2})(M_{2,1}g_{0,1}g_{0,2} - M_{2,1}g_{0,1} + M_{2,2}g_{0,2} - g_{0,2})$$

737

$$-T_{[2,0],[0,2]}(g_{1,0}g_{1,2}g_{2,2} - g_{1,2}g_{2,0}g_{2,2} - g_{1,0}g_{2,2} + g_{1,2}g_{2,0})(g_{1,0}g_{2,2} - g_{1,2}g_{2,0})$$

$$-g_{1,2}g_{2,0})$$

739

$$-T_{[2,0],[0,2]}(g_{1,0}g_{1,1}g_{2,1} - g_{1,1}g_{2,0}g_{2,1} - g_{1,0}g_{2,1} + g_{1,1}g_{2,0})(g_{1,0}g_{2,1} - g_{1,1}g_{2,0})$$

745

$$-T_{[1,0],[0,1]}(g_{1,0}g_{1,2}g_{2,2} - g_{1,2}g_{2,0}g_{2,2} - g_{1,0}g_{2,2} + g_{1,2}g_{2,0})(g_{1,0}g_{2,2} - g_{1,2}g_{2,0})$$

747

$$-T_{[1,0],[0,1]}(g_{1,0}g_{1,1}g_{2,1} - g_{1,1}g_{2,0}g_{2,1} - g_{1,0}g_{2,1} + g_{1,1}g_{2,0})(g_{1,0}g_{2,1} - g_{1,1}g_{2,0})$$

"=====

"trop long", {6, 81, 82, 84, 85, 86, 87, 88, 94, 109, 127, 136, 137, 138, 142, 143, 145, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 164, 169, 170, 173, 176, 186, 187, 188, 189, 190, 196, 197, 200, 201, 202, 235, 238, 249, 253, 263, 264, 266, 268, 269, 271, 274, 276, 277, 281, 283, 286, 292, 293, 294, 295, 296, 300, 301, 302, 303, 304, 312, 313, 314, 315, 316, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 336, 337, 340, 343, 344, 345, 355, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 400, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 416, 417, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 440, 443, 444, 445, 446, 447, 448, 451, 453, 455, 456, 458, 459, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 497, 498, 499, 500, 501, 502, 503, 505, 506, 507, 508, 509, 510, 511, 512, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 561, 562, 563, 564, 565, 566, 567, 569, 570, 571, 573, 574, 575, 576, 578, 579, 580, 581, 582, 583, 584, 585, 586, 587, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 600, 601, 602, 603, 604, 605, 606, 608, 610, 611, 612, 614, 615, 616, 617, 619, 622, 624, 625, 626, 628, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 640, 641, 642, 643, 644, 645, 647, 654, 656, 657, 658, 659, 660, 661, 662, 663, 664, 666, 667, 668, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 680, 681, 682, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728, 729, 730, 735, 738, 740, 741, 742, 743, 744, 746, 748, 749, 750, 751, 752, 753, 754}

(1.5.3.1)

"Parameters = NonZeroTransition", $T_{[0, 2], [2, 0]}$, $T_{[1, 0], [0, 1]}$, $T_{[2, 1], [1, 2]}$

- 1
 $T_{[0, 2], [2, 0]} + T_{[1, 0], [0, 1]} + T_{[2, 1], [1, 2]}$
- 2
 $g_{2, 0} g_{2, 1} g_{2, 2} - g_{2, 0} g_{2, 1} - g_{2, 0} g_{2, 2} - g_{2, 1} g_{2, 2}$
- 3
 $T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2})$
- 4
 $T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1})$
- 5
 $T_{[2, 1], [1, 2]} (g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 1})$
- 6
 $T_{[2, 1], [1, 2]} (-g_{1, 2} g_{2, 0} g_{2, 1} + g_{1, 1} g_{2, 0} + g_{1, 2} g_{2, 0} + g_{1, 2} g_{2, 1})$
- 7
 $T_{[1, 0], [0, 1]} (g_{1, 2} g_{2, 1} g_{1, 1} - g_{2, 1} g_{1, 1} g_{2, 2} + g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 1})$
- 8
 $T_{[1, 0], [0, 1]} (g_{1, 2} g_{2, 0} g_{1, 1} - g_{1, 1} g_{2, 0} - g_{1, 1} g_{2, 2} - g_{1, 2} g_{2, 0})$
- 9
 $T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0})$
- 10
 $T_{[1, 0], [0, 1]} (g_{1, 0} g_{2, 2} - g_{1, 2} g_{2, 0})$
- 11
 $T_{[1, 0], [0, 1]} (-g_{1, 2} g_{2, 0} g_{2, 1} + g_{1, 0} g_{2, 1} + g_{1, 2} g_{2, 0} + g_{1, 2} g_{2, 1})$
- 12
 $T_{[2, 1], [1, 2]} (g_{1, 0} g_{1, 2} g_{2, 1} - g_{1, 0} g_{2, 1} - g_{1, 0} g_{2, 2} - g_{1, 2} g_{2, 1})$
- 13
 $T_{[2, 1], [1, 2]} (g_{1, 0} g_{1, 2} g_{2, 0} - g_{1, 0} g_{2, 0} g_{2, 2} + g_{1, 0} g_{2, 2} - g_{1, 2} g_{2, 0})$
- 14
 $g_{1, 0} g_{1, 1} g_{1, 2} - g_{1, 0} g_{1, 1} - g_{1, 0} g_{1, 2} - g_{1, 1} g_{1, 2}$
- 15
 $T_{[1, 0], [0, 1]} g_{0, 2} - T_{[1, 0], [0, 1]} g_{1, 2} + T_{[2, 1], [1, 2]} g_{0, 2} - T_{[2, 1], [1, 2]} g_{2, 2}$
- 16
 $T_{[2, 1], [1, 2]} (g_{1, 2} - g_{2, 2}) (g_{0, 2} - g_{2, 2})$
- 17
 $T_{[2, 1], [1, 2]} (g_{0, 2} g_{1, 1} - g_{0, 2} g_{2, 1} - g_{1, 2} g_{2, 1} + g_{2, 1} g_{2, 2})$
- 18
 $T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0}) (g_{0, 2} - g_{2, 2})$
- 19

$$g_{0,1} T_{[1,0],[0,1]} - g_{1,1} T_{[1,0],[0,1]} + T_{[2,1],[1,2]} g_{0,1} - T_{[2,1],[1,2]} g_{2,1}$$

20

$$T_{[2,1],[1,2]} (g_{1,2} - g_{2,2}) (g_{0,1} - g_{2,1})$$

21

$$T_{[2,1],[1,2]} (g_{1,1} - g_{2,1}) (g_{0,1} - g_{2,1})$$

22

$$T_{[2,1],[1,2]} (g_{1,0} - g_{2,0}) (g_{0,1} - g_{2,1})$$

23

$$T_{[2,1],[1,2]} (g_{0,1} g_{0,2} g_{2,2} - g_{0,2} g_{2,1} g_{2,2} - g_{0,1} g_{2,2} + g_{0,2} g_{2,1})$$

24

$$T_{[2,1],[1,2]} (g_{0,1} g_{0,2} g_{2,0} - g_{0,1} g_{2,0} - g_{0,2} g_{2,0} - g_{0,2} g_{2,1})$$

25

$$g_{0,0} T_{[1,0],[0,1]} - g_{1,0} T_{[1,0],[0,1]} + T_{[2,1],[1,2]} g_{0,0} - T_{[2,1],[1,2]} g_{2,0}$$

26

$$T_{[2,1],[1,2]} (-g_{0,1} g_{2,0} g_{2,2} + g_{0,0} g_{2,2} + g_{0,1} g_{2,0} + g_{0,1} g_{2,2})$$

27

$$T_{[2,1],[1,2]} (g_{0,0} g_{2,1} - g_{0,1} g_{2,0})$$

28

$$T_{[2,1],[1,2]} (-g_{0,1} g_{2,0} g_{2,2} + g_{0,0} g_{1,2} + g_{0,1} g_{2,0} + g_{0,1} g_{2,2} - g_{1,2} g_{2,0} + g_{2,0} g_{2,2})$$

29

$$T_{[2,1],[1,2]} (-g_{1,2} g_{2,0} g_{2,1} + g_{0,0} g_{1,1} - g_{0,1} g_{2,0} + g_{1,2} g_{2,0} + g_{1,2} g_{2,1} + g_{2,0} g_{2,1})$$

30

$$T_{[2,1],[1,2]} (g_{1,0} - g_{2,0}) (g_{0,0} - g_{2,0})$$

31

$$g_{0,0} g_{0,1} g_{0,2} - g_{0,0} g_{0,1} - g_{0,0} g_{0,2} - g_{0,1} g_{0,2}$$

32

$$M_{2,2} g_{2,2} - 1$$

33

$$M_{2,2} g_{2,0} g_{2,1} - g_{2,0} g_{2,1} + g_{2,0} + g_{2,1}$$

34

$$T_{[2,1],[1,2]} (M_{2,2} g_{1,2} g_{2,1} - g_{1,1})$$

35

$$T_{[1,0],[0,1]} (M_{2,2} g_{1,2} g_{2,0} - g_{1,0})$$

36

$$T_{[1,0],[0,1]} (M_{2,2} g_{2,0} g_{1,1} - g_{1,0} g_{1,1} + g_{1,0} + g_{1,1})$$

37

$$T_{[2,1],[1,2]} (M_{2,2} g_{1,0} g_{2,1} - g_{1,0} g_{1,1} + g_{1,0} + g_{1,1})$$

38

$$T_{[2, 1], [1, 2]} (M_{2, 2} g_{0, 2} g_{2, 1} + g_{0, 1} g_{0, 2} - g_{0, 2} g_{2, 1} - g_{0, 1})$$

39

$$T_{[2, 1], [1, 2]} (M_{2, 2} g_{0, 2} g_{2, 0} + g_{0, 0} g_{0, 2} - g_{0, 2} g_{2, 0} - g_{0, 0})$$

40

$$T_{[2, 1], [1, 2]} (M_{2, 2} g_{0, 1} g_{2, 0} - g_{0, 1} g_{2, 0} + g_{0, 0} + g_{0, 1})$$

41

$$T_{[1, 0], [0, 1]} T_{[2, 1], [1, 2]} (M_{1, 2} - M_{2, 2})$$

42

$$M_{1, 2} g_{1, 2} - 1$$

43

$$T_{[2, 1], [1, 2]} (M_{1, 2} g_{1, 1} - M_{2, 2} g_{2, 1})$$

44

$$T_{[1, 0], [0, 1]} (M_{1, 2} g_{2, 1} g_{1, 1} - M_{2, 2} g_{2, 1} g_{1, 1} - M_{1, 2} g_{1, 1} + M_{2, 2} g_{2, 1})$$

45

$$T_{[1, 0], [0, 1]} (M_{1, 2} g_{2, 0} g_{1, 1} + M_{1, 2} g_{1, 1} g_{2, 2} - g_{1, 1} g_{2, 0} + g_{2, 0})$$

46

$$T_{[1, 0], [0, 1]} (M_{1, 2} g_{1, 0} - M_{2, 2} g_{2, 0})$$

47

$$T_{[2, 1], [1, 2]} (M_{1, 2} g_{1, 0} g_{2, 1} + M_{1, 2} g_{1, 0} g_{2, 2} - g_{1, 0} g_{2, 1} + g_{2, 1})$$

48

$$T_{[2, 1], [1, 2]} (M_{1, 2} g_{1, 0} g_{2, 0} - M_{2, 2} g_{1, 0} g_{2, 0} - M_{1, 2} g_{1, 0} + M_{2, 2} g_{2, 0})$$

49

$$M_{1, 2} g_{1, 0} g_{1, 1} - g_{1, 0} g_{1, 1} + g_{1, 0} + g_{1, 1}$$

50

$$T_{[2, 1], [1, 2]} (M_{1, 2} g_{0, 2} - M_{1, 2} g_{2, 2} - M_{2, 2} g_{0, 2} + 1)$$

51

$$T_{[2, 1], [1, 2]} (g_{0, 1} - g_{2, 1}) (M_{1, 2} - M_{2, 2})$$

52

$$T_{[2, 1], [1, 2]} (g_{0, 0} - g_{2, 0}) (M_{1, 2} - M_{2, 2})$$

53

$$M_{0, 2} T_{[1, 0], [0, 1]} + T_{[2, 1], [1, 2]} M_{0, 2} - M_{1, 2} T_{[1, 0], [0, 1]} - M_{2, 2} T_{[2, 1], [1, 2]}$$

54

$$T_{[2, 1], [1, 2]} (M_{0, 2} g_{1, 2} - M_{0, 2} g_{2, 2} - M_{2, 2} g_{1, 2} + 1)$$

55

$$T_{[2, 1], [1, 2]} (g_{1, 1} - g_{2, 1}) (M_{0, 2} - M_{2, 2})$$

56

$$T_{[2, 1], [1, 2]} (g_{1, 0} - g_{2, 0}) (M_{0, 2} - M_{2, 2})$$

57

$$M_{0,2}g_{0,2} - 1$$

58

$$T_{[2,1],[1,2]}(M_{0,2}g_{0,1} - M_{2,2}g_{2,1} - g_{0,1} + g_{2,1})$$

59

$$T_{[2,1],[1,2]}(g_{0,0}M_{0,2} - M_{2,2}g_{2,0} - g_{0,0} + g_{2,0})$$

60

$$M_{0,2}g_{0,0}g_{0,1} - g_{0,0}g_{0,1} + g_{0,0} + g_{0,1}$$

61

$$T_{[2,1],[1,2]}(M_{1,2} - M_{2,2})(M_{0,2} - M_{2,2})$$

62

$$M_{2,1}g_{2,1} - 1$$

63

$$g_{2,0}M_{2,1} + M_{2,2}g_{2,0} - g_{2,0} + 1$$

64

$$T_{[2,1],[1,2]}(M_{2,1}g_{1,1} - M_{2,2}g_{1,2})$$

65

$$T_{[1,0],[0,1]}(M_{2,2}g_{1,1}g_{1,2} + M_{2,1}g_{1,1} - M_{2,2}g_{1,2} - g_{1,1})$$

66

$$T_{[2,1],[1,2]}(-M_{2,2}g_{1,0}g_{1,2} + M_{2,1}g_{1,0} + M_{2,2}g_{1,0} + M_{2,2}g_{1,2})$$

67

$$T_{[1,0],[0,1]}(M_{2,1}g_{1,0} + M_{2,2}g_{1,0} + M_{2,2}g_{1,2} - g_{1,0})$$

68

$$T_{[2,1],[1,2]}(M_{2,1}g_{0,1}g_{0,2} - M_{2,1}g_{0,1} + M_{2,2}g_{0,2} - g_{0,2})$$

69

$$T_{[2,1],[1,2]}(M_{2,1}g_{0,0} + M_{2,1}g_{0,1} + M_{2,2}g_{0,0} - g_{0,0})$$

70

$$T_{[2,1],[1,2]}(-M_{1,2}M_{2,1}g_{2,2} + M_{1,1})$$

71

$$T_{[1,0],[0,1]}(-M_{1,2}M_{2,1}g_{2,2} + M_{1,2}g_{2,2} + M_{1,1} - 1)$$

72

$$M_{1,1}g_{1,1} - 1$$

73

$$g_{1,0}M_{1,1} + M_{1,2}g_{1,0} - g_{1,0} + 1$$

74

$$\begin{aligned} & -M_{1,2}M_{2,1}T_{[1,0],[0,1]}g_{2,2} + M_{1,2}T_{[1,0],[0,1]}g_{2,2} + M_{0,1}T_{[1,0],[0,1]} + M_{0,1}T_{[2,1],[1,2]} \\ & - M_{2,1}T_{[2,1],[1,2]} - T_{[1,0],[0,1]} \end{aligned}$$

75

$$T_{[2,1],[1,2]} (M_{0,2} M_{2,1} g_{2,2} + M_{0,1} g_{2,2} - M_{2,1} g_{2,2} - M_{0,1})$$

76

$$T_{[2,1],[1,2]} (M_{0,1} g_{2,0} + M_{0,1} g_{2,1} + M_{0,2} g_{2,0} - g_{2,0})$$

77

$$T_{[2,1],[1,2]} (M_{0,2} M_{2,1} g_{2,2} + M_{0,1} g_{1,2} - M_{2,1} g_{1,2} - M_{0,1})$$

78

$$T_{[2,1],[1,2]} (M_{0,1} g_{1,1} - M_{0,1} g_{2,1} - M_{2,2} g_{1,2} + 1)$$

79

$$T_{[2,1],[1,2]} (-M_{2,2} g_{1,0} g_{1,2} + M_{0,1} g_{1,0} + M_{0,1} g_{2,1} + M_{0,2} g_{2,0} + M_{2,2} g_{1,0} + M_{2,2} g_{1,2} - M_{2,2} g_{2,0} - 1)$$

80

$$M_{0,1} g_{0,1} - 1$$

81

$$M_{0,1} g_{0,0} + g_{0,0} M_{0,2} - g_{0,0} + 1$$

82

$$T_{[2,1],[1,2]} (M_{0,1} M_{2,2} - M_{0,2} M_{2,1} - M_{0,1} + M_{2,1})$$

83

$$T_{[2,1],[1,2]} (M_{0,1} M_{1,2} - M_{0,2} M_{2,1} - M_{1,2} M_{2,1} + M_{2,1} M_{2,2} - M_{0,1} + M_{2,1})$$

"=====

"trop long", ∅

(1.5.1)

> **PolynomialSystem;**

$$\{0, -T_{[1,0],[0,1]}, -T_{[1,0],[0,1]} (M_{0,1} M_{1,2} - M_{1,1} M_{0,2} - M_{0,1} + M_{1,1}),$$

$$-T_{[1,0],[0,1]} (M_{0,1} M_{2,2} - M_{2,1} M_{0,2} - M_{0,1} + M_{2,1}), M_{0,1} g_{0,1} - 1, M_{0,2} g_{0,2} - 1,$$

$$M_{1,1} g_{1,1} - 1, M_{1,2} g_{1,2} - 1, M_{2,1} g_{2,1} - 1, M_{2,2} g_{2,2} - 1, -M_{0,1} g_{0,0} - g_{0,0} M_{0,2}$$

$$+ g_{0,0} - 1, -g_{1,0} M_{1,1} - g_{1,0} M_{1,2} + g_{1,0} - 1, -g_{2,0} M_{2,1} - g_{2,0} M_{2,2} + g_{2,0} - 1\}$$

(1.5.2)

> **subs({seq([Variables][i]=(i+3)/2,i=1..nops([Variables]))}, SystemEquation);**

$$\left\{ -53, 0, 10, 20, 29, -\frac{147}{4}, -\frac{47}{2}, -\frac{19}{2}, \frac{19}{52}, \frac{19}{88}, \frac{33}{2}, \frac{49}{2}, \frac{79}{2} \right\}$$

(1.5.3)

> **ensem(2,3);**

$$[0, 0], [0, 1], [0, 2], [1, 0], [1, 1], [1, 2], [2, 0], [2, 1], [2, 2]$$

(1.5.4)

>

Section 3.2 - Hidden Markov chain

> **K:=3;**
unassign('T','M');

Indices:= { seq(seq([u,v], u in ensem(3,K),v in ensem(3,K))) }:

for u in ensem(3,K) do for v in ensem(3,K) do
T[u,v]:=0:Indices:=Indices minus {[u,u]}:

```

od:od:

#definition of T
T[[0,0,0],[0,1,0]]:=255:
T[[0,0,0],[0,2,0]]:=15:
T[[0,1,0],[0,0,0]]:=294:
T[[0,2,0],[0,0,0]]:=294:
T[[0,1,0],[0,2,0]]:=49:
T[[0,2,0],[0,1,0]]:=49:
Indices:=Indices minus {[[0,0,0],[0,1,0]],[[0,0,0],[0,2,0]],[[0,1,0],[0,0,0]],[[0,2,0],[0,0,0]],[[0,1,0],[0,2,0]],[[0,2,0],[0,1,0]]}:
# projection function:

NonZeroT:= { seq(T[op(A)], A in Indices)}:

proj:=proc(x) if x>0 then return(1) else return(0):fi:end:

# Condition projection
EQQ:=NULL:
for u in ensem(3,K) do
  for v in ensem(3,K) do
    if map(proj,u)=map(proj,v) then
      EQQ:=EQQ, seq( add((T[u,w]-T[v,w])*ind
(map(proj,w),b)*(1-ind(map(proj,u),b)), w in ensem(3,K)),b in
ensem(3)):
      fi:
    od:
  od:
od:

# T' as given in section 3.2 is clearly correct.

M:= Matrix([[7/15, 1/3, 1/5],[1/2,1/6,1/3],[1/6,1/2,1/3]]):
Rho:=Matrix([35/89,29/89,25/89]):
# checking that Rho is a left main eigenvalue of M
Rho.M -Rho ;

# Projected measure
MesMarkov:=proc(v) local n: n:=nops(v): # the
colour are labeled 0..2
return(Rho[1,v[1]+1]*mul(M[v[i]+1,v[i+1]+1],i=1..n-1)):# but
the matrices entries are labeled 1-3
end: # the
shift +1 is here to compensate that.

Mes:=proc(u) local n:
n:=nops(u):
return( add(MesMarkov(v)*mul(ind(proj(v[i]),u[i]),i=1..n),
v in ensem(n,3)));
end:
Mes([1,1,1])/Mes([1,1]);
Mes([1,1])/Mes([1]);

## adaptation of MM to this matricial expression

```



```

"Parameters = NonZeroTransition", NonZeroT;
>
## product of M over 4 letters
MM:=proc(a,b,c,d) return( M[a,b]*M[b,c]*M[c,d]): end:

Z4:=proc(u) ## for a word u of size 4 and L = 2 we need the
following Z
return( add( MM(u[1],v[1],v[2],u[4]) *T[v,[u[2],u
[3]]]/MM( op(u) ) -T[[u[2],u[3]],v], v in ensem(2,K)))
end:

>
Cycle7MT:=proc(a,b,c,d) option remember: local Vec:
Vec :=[ a, b, c, d, 0, 0, 0]:
return( simplify( add(Z4( [seq( Vec[modu(i +j, 7)],j=1.
.4) ] ),i=1..7)))
end:

>
for i in {0,1} do M[i,0]:=1-M[i,1]: od:

>
SystemEquation := { seq( Cycle7MT(op(u)), u in ensem(4,K)),
seq( M[op(u)]*g[op(u)]-1,u in ensem(2,K)), (M[0,1]-M[1,1])*
x-1}:
#####
PolynomialSystem:= map(x-> numer(normal(x)), SystemEquation):
# numerators only
Variables:= seq(g[op(u)],u in ensem(2,K)),x, seq(seq(M[i,j],
i=0..K-1),j=1..K-1) , NonZeroT :
#####
# grobner
G:= Basis(PolynomialSystem,plex( Variables)):
Affiche(G,2000);
unassign('M'); # for further computations

```

$K := 2$

"Parameters = NonZeroTransition", $t_{0,1}, t_{0,2}, t_{0,3}, t_{1,0}, t_{1,2}, t_{1,3}, t_{2,0}, t_{2,1}, t_{2,3}, t_{3,0}, t_{3,1}, t_{3,2}$

1

$t_{3,0}$

2

$t_{2,1}$

3

$t_{1,2}$

4

$t_{0,3}$

5

$t_{0,1}t_{1,0} + t_{0,1}t_{2,0} + t_{0,2}t_{1,0} + t_{0,2}t_{2,0} - t_{1,3}t_{3,1} - t_{1,3}t_{3,2} - t_{2,3}t_{3,1} - t_{2,3}t_{3,2}$

6

$$M_{1,1}^2 t_{1,0} + M_{1,1}^2 t_{2,0} - M_{1,1}^2 t_{3,1} - M_{1,1}^2 t_{3,2} - M_{1,1} t_{1,0} - M_{1,1} t_{1,3} - M_{1,1} t_{2,0} - M_{1,1} t_{2,3} + t_{1,3} + t_{2,3}$$

7

$$M_{1,1}^2 t_{0,1} t_{3,1} + M_{1,1}^2 t_{0,1} t_{3,2} + M_{1,1}^2 t_{0,2} t_{3,1} + M_{1,1}^2 t_{0,2} t_{3,2} - M_{1,1}^2 t_{1,3} t_{3,1} - M_{1,1}^2 t_{1,3} t_{3,2} - M_{1,1}^2 t_{2,3} t_{3,1} - M_{1,1}^2 t_{2,3} t_{3,2} + M_{1,1} t_{0,1} t_{1,3} + M_{1,1} t_{0,1} t_{2,3} + M_{1,1} t_{0,2} t_{1,3} + M_{1,1} t_{0,2} t_{2,3} + M_{1,1} t_{1,3} t_{3,1} + M_{1,1} t_{1,3} t_{3,2} + M_{1,1} t_{2,3} t_{3,1} + M_{1,1} t_{2,3} t_{3,2} - t_{0,1} t_{1,3} - t_{0,1} t_{2,3} - t_{0,2} t_{1,3} - t_{0,2} t_{2,3}$$

8

$$t_{1,3} M_{0,1} + M_{0,1} t_{2,3} + M_{1,1} t_{1,0} + M_{1,1} t_{2,0} - t_{1,3} - t_{2,3}$$

9

$$M_{0,1} t_{0,1} + M_{0,1} t_{0,2} + M_{1,1} t_{3,1} + M_{1,1} t_{3,2} - t_{0,1} - t_{0,2}$$

10

$$M_{0,1} M_{1,1} t_{1,0} + M_{0,1} M_{1,1} t_{2,0} - M_{0,1} M_{1,1} t_{3,1} - M_{0,1} M_{1,1} t_{3,2} - M_{0,1} t_{1,0} - M_{0,1} t_{2,0} + M_{1,1} t_{3,1} + M_{1,1} t_{3,2}$$

11

$$x t_{1,0} t_{1,3} + x t_{1,0} t_{2,3} + x t_{1,3} t_{2,0} - x t_{1,3} t_{3,1} - x t_{1,3} t_{3,2} + x t_{2,0} t_{2,3} - x t_{2,3} t_{3,1} - x t_{2,3} t_{3,2} - M_{1,1} t_{1,0}^2 - M_{1,1} t_{1,0} t_{1,3} - 2 M_{1,1} t_{1,0} t_{2,0} - M_{1,1} t_{1,0} t_{2,3} + M_{1,1} t_{1,0} t_{3,1} + M_{1,1} t_{1,0} t_{3,2} - M_{1,1} t_{1,3} t_{2,0} + M_{1,1} t_{1,3} t_{3,1} + M_{1,1} t_{1,3} t_{3,2} - M_{1,1} t_{2,0}^2 - M_{1,1} t_{2,0} t_{2,3} + M_{1,1} t_{2,0} t_{3,1} + M_{1,1} t_{2,0} t_{3,2} + M_{1,1} t_{2,3} t_{3,1} + M_{1,1} t_{2,3} t_{3,2} + t_{1,0}^2 + t_{1,0} t_{1,3} + 2 t_{1,0} t_{2,0} + t_{1,0} t_{2,3} + t_{1,3}^2 + t_{1,3} t_{2,0} + 2 t_{1,3} t_{2,3} + t_{1,3} t_{3,1} + t_{1,3} t_{3,2} + t_{2,0}^2 + t_{2,0} t_{2,3} + t_{2,3}^2 + t_{2,3} t_{3,1} + t_{2,3} t_{3,2}$$

12

$$x t_{0,1} t_{3,1} + x t_{0,1} t_{3,2} + x t_{0,2} t_{3,1} + x t_{0,2} t_{3,2} - x t_{1,3} t_{3,1} - x t_{1,3} t_{3,2} - x t_{2,3} t_{3,1} - x t_{2,3} t_{3,2} - M_{1,1} t_{0,1} t_{3,1} - M_{1,1} t_{0,1} t_{3,2} - M_{1,1} t_{0,2} t_{3,1} - M_{1,1} t_{0,2} t_{3,2} + M_{1,1} t_{1,0} t_{3,1} + M_{1,1} t_{1,0} t_{3,2} + M_{1,1} t_{1,3} t_{3,1} + M_{1,1} t_{1,3} t_{3,2} + M_{1,1} t_{2,0} t_{3,1} + M_{1,1} t_{2,0} t_{3,2} + M_{1,1} t_{2,3} t_{3,1} + M_{1,1} t_{2,3} t_{3,2} - M_{1,1} t_{3,1}^2 - 2 M_{1,1} t_{3,1} t_{3,2} - M_{1,1} t_{3,2}^2 - t_{0,1} t_{1,3} - t_{0,1} t_{2,3} - t_{0,1} t_{3,1} - t_{0,1} t_{3,2} - t_{0,2} t_{1,3} - t_{0,2} t_{2,3} - t_{0,2} t_{3,1} - t_{0,2} t_{3,2} - t_{1,0} t_{3,1} - t_{1,0} t_{3,2} - t_{1,3} t_{3,1} - t_{1,3} t_{3,2} - t_{2,0} t_{3,1} - t_{2,0} t_{3,2} - t_{2,3} t_{3,1} - t_{2,3} t_{3,2}$$

13

$$x M_{1,1} t_{1,3} + x M_{1,1} t_{2,3} + x M_{1,1} t_{3,1} + x M_{1,1} t_{3,2} - x t_{1,3} - x t_{2,3} + M_{1,1} t_{1,0} + M_{1,1} t_{2,0} - M_{1,1} t_{3,1} - M_{1,1} t_{3,2} - t_{1,0} - t_{2,0}$$

14

$$x M_{1,1} t_{1,0} + x M_{1,1} t_{2,0} - x M_{1,1} t_{3,1} - x M_{1,1} t_{3,2} - M_{1,1} t_{1,0} - M_{1,1} t_{2,0} + M_{1,1} t_{3,1} + M_{1,1} t_{3,2} + t_{1,0} + t_{1,3} + t_{2,0} + t_{2,3}$$

15

$$x M_{1,1} t_{0,1} + x M_{1,1} t_{0,2} + x M_{1,1} t_{3,1} + x M_{1,1} t_{3,2} - x t_{0,1} - x t_{0,2} + t_{0,1} + t_{0,2}$$

16

$$x M_{0,1} - x M_{1,1} - 1$$

17

$$M_{1,1} t_{1,0} + M_{1,1} t_{2,0} - M_{1,1} t_{3,1} - M_{1,1} t_{3,2} + g_{1,1} t_{1,3} + g_{1,1} t_{2,3} - t_{1,0} - t_{1,3} - t_{2,0} - t_{2,3}$$

18

$$x t_{1,0} + x t_{2,0} - x t_{3,1} - x t_{3,2} - M_{1,1} t_{1,0} - M_{1,1} t_{2,0} + M_{1,1} t_{3,1} + M_{1,1} t_{3,2} + g_{1,1} t_{1,0} + g_{1,1} t_{2,0} + t_{1,3} + t_{2,3} + t_{3,1} + t_{3,2}$$

19

$$M_{1,1} g_{1,1} - 1$$

20

$$x g_{1,1} t_{0,1} + x g_{1,1} t_{0,2} - x t_{0,1} - x t_{0,2} - x t_{3,1} - x t_{3,2} - g_{1,1} t_{0,1} - g_{1,1} t_{0,2}$$

21

$$M_{1,1} t_{1,0} + M_{1,1} t_{2,0} - M_{1,1} t_{3,1} - M_{1,1} t_{3,2} + g_{1,0} t_{3,1} + g_{1,0} t_{3,2} - t_{1,3} - t_{2,3} - t_{3,1} - t_{3,2}$$

22

$$-x t_{0,1} - x t_{0,2} + x t_{1,3} + x t_{2,3} - M_{1,1} t_{1,0} - M_{1,1} t_{2,0} + M_{1,1} t_{3,1} + M_{1,1} t_{3,2} + g_{1,0} t_{0,1} + g_{1,0} t_{0,2} + t_{1,0} + t_{1,3} + t_{2,0} + t_{2,3}$$

23

$$g_{1,0} M_{1,1} - g_{1,0} + 1$$

24

$$x g_{1,0} t_{1,0} + x g_{1,0} t_{2,0} - x t_{1,0} - x t_{1,3} - x t_{2,0} - x t_{2,3} + g_{1,0} t_{1,3} + g_{1,0} t_{2,3}$$

25

$$g_{1,0} g_{1,1} - g_{1,0} - g_{1,1}$$

26

$$x t_{1,0} + x t_{2,0} - x t_{3,1} - x t_{3,2} - M_{1,1} t_{1,0} - M_{1,1} t_{2,0} + M_{1,1} t_{3,1} + M_{1,1} t_{3,2} + g_{0,1} t_{3,1} + g_{0,1} t_{3,2} + t_{1,0} + t_{1,3} + t_{2,0} + t_{2,3}$$

27

$$M_{1,1} t_{1,0} + M_{1,1} t_{2,0} - M_{1,1} t_{3,1} - M_{1,1} t_{3,2} + g_{0,1} t_{0,1} + g_{0,1} t_{0,2} - t_{0,1} - t_{0,2} - t_{1,0} - t_{2,0}$$

28

$$M_{1,1} g_{0,1} t_{1,0} + M_{1,1} g_{0,1} t_{2,0} - g_{0,1} t_{1,3} - g_{0,1} t_{2,3} + t_{1,3} + t_{2,3}$$

29

$$M_{0,1} g_{0,1} - 1$$

30

$$x g_{0,1} t_{1,3} + x g_{0,1} t_{2,3} - x t_{1,0} - x t_{1,3} - x t_{2,0} - x t_{2,3} + g_{0,1} t_{1,0} + g_{0,1} t_{2,0}$$

31

$$x M_{1,1} g_{0,1} - x + g_{0,1}$$

32

$$x g_{0,1} - x g_{1,1} + g_{0,1} g_{1,1}$$

33

$$g_{0,1} g_{1,0} t_{1,0} - g_{0,1} g_{1,0} t_{1,3} + g_{0,1} g_{1,0} t_{2,0} - g_{0,1} g_{1,0} t_{2,3} - g_{0,1} t_{1,0} - g_{0,1} t_{2,0} + g_{1,0} t_{1,3} + g_{1,0} t_{2,3}$$

34

$$x g_{0,1} g_{1,0} - x g_{0,1} - x g_{1,0} + g_{0,1} g_{1,0}$$

35

$$g_{0,0} t_{3,1} + g_{0,0} t_{3,2} - g_{1,1} t_{0,1} - g_{1,1} t_{0,2}$$

36

$$-x t_{0,1} - x t_{0,2} + x t_{1,3} + x t_{2,3} - M_{1,1} t_{1,0} - M_{1,1} t_{2,0} + M_{1,1} t_{3,1} + M_{1,1} t_{3,2} + g_{0,0} t_{1,3} + g_{0,0} t_{2,3} + t_{0,1} + t_{0,2} + t_{1,0} + t_{2,0}$$

37

$$M_{1,1} t_{1,0} + M_{1,1} t_{2,0} - M_{1,1} t_{3,1} - M_{1,1} t_{3,2} + g_{0,0} t_{1,0} + g_{0,0} t_{2,0} - t_{1,0} - t_{1,3} - t_{2,0} - t_{2,3}$$

38

$$g_{0,0} M_{0,1} - g_{0,0} + 1$$

39

$$x M_{1,1} g_{0,0} - x g_{0,0} + x + g_{0,0}$$

40

$$x g_{0,0} g_{1,1} - x g_{0,0} - x g_{1,1} - g_{0,0} g_{1,1}$$

41

$$-x g_{0,0} + x g_{1,0} + g_{0,0} g_{1,0}$$

42

$$g_{0,0} g_{0,1} - g_{0,0} - g_{0,1}$$

"=====

"trop long", ∅

(1.7.1)

> collect(G[6],M[1,1]);latex(%);

```
(t_{1,0} + t_{2,0} - t_{3,1} - t_{3,2}) M_{1,1}^2 + (-t_{1,0} - t_{1,3} - t_{2,0} - t_{2,3}) M_{1,1} + t_{1,3} + t_{2,3}
\left( t_{\{1,0\}} + t_{\{2,0\}} - t_{\{3,1\}} - t_{\{3,2\}} \right) M_{\{1,1\}}^2
+ \left( -t_{\{1,0\}} - t_{\{1,3\}} - t_{\{2,0\}} - t_{\{2,3\}} \right) M_{\{1,1\}} + t_{\{1,3\}} + t_{\{2,3\}}
```

▼ Section 3.3.2 - Two colour general product case


```

> # -> Initialisation
unassign('T','M','u','v','rho');
K:=2;

> # Transition definition
NonZeroT:= NULL: # will be the set of nonzerotransitions
for u in ensem(2,K) do
  for v in ensem(2,K) do
    if u=v then T[u,u]:=0:
                  t[it(u),it(v)]:=0:
    else
      T[u,v]:=t[it(u),it(v)]:
      NonZeroT := NonZeroT, T[u,v]:
    fi:
  od:
od:

"Parameters = NonZeroTransition", NonZeroT;

>
Z2:=proc(u) ## for a word u of size 4 and L = 2 we need the
following Z
  return( add( rho[v[1]]*rho[v[2]] *T[v,[u[1],u[2]]]
/(rho[u[1]]*rho[u[2]]) -T[[u[1],u[2]],v], v in ensem(2,K))) :
end:

>
Cycle2MT:=proc(a,b) option remember:
  return( simplify(Z2([a,b])+Z2([b,a]))):
end:

>
rho[0]:=1-rho[1]:

>
SystemEquation := { seq( Cycle2MT(op(u)), u in ensem(2,K))}:

#####
PolynomialSystem:= map(x-> numer(normal(x)), SystemEquation):
# numerators only
Variables:= g[0],g[1], rho[1] , NonZeroT :
#####
# grobner
G:= Basis(PolynomialSystem,tdeg( Variables)):
Affiche(G,20000);
unassign('M'); # for further computations
                K := 2

"Parameters = NonZeroTransition",  $t_{0,1}, t_{0,2}, t_{0,3}, t_{1,0}, t_{1,2}, t_{1,3}, t_{2,0}, t_{2,1}, t_{2,3}, t_{3,0}, t_{3,1}, t_{3,2}$ 
1
 $\rho_1 t_{0,1} t_{3,0} + \rho_1 t_{0,1} t_{3,1} + \rho_1 t_{0,1} t_{3,2} + \rho_1 t_{0,2} t_{3,0} + \rho_1 t_{0,2} t_{3,1} + \rho_1 t_{0,2} t_{3,2} + \rho_1 t_{0,3} t_{3,1}$ 
 $+ \rho_1 t_{0,3} t_{3,2} + \rho_1 t_{1,0} t_{3,0} + \rho_1 t_{1,0} t_{3,1} + \rho_1 t_{1,0} t_{3,2} + \rho_1 t_{1,3} t_{3,0} + \rho_1 t_{2,0} t_{3,0}$ 
 $+ \rho_1 t_{2,0} t_{3,1} + \rho_1 t_{2,0} t_{3,2} + \rho_1 t_{2,3} t_{3,0} - t_{0,1} t_{3,0} - t_{0,1} t_{3,1} - t_{0,1} t_{3,2} - t_{0,2} t_{3,0}$ 
 $- t_{0,2} t_{3,1} - t_{0,2} t_{3,2} - t_{0,3} t_{3,1} - t_{0,3} t_{3,2}$ 

```

2

$$\begin{aligned} &\rho_1 t_{0,1} t_{1,3} + \rho_1 t_{0,1} t_{2,3} + \rho_1 t_{0,2} t_{1,3} + \rho_1 t_{0,2} t_{2,3} + \rho_1 t_{0,3} t_{1,0} + \rho_1 t_{0,3} t_{1,3} + \rho_1 t_{0,3} t_{2,0} \\ &\quad + \rho_1 t_{0,3} t_{2,3} - \rho_1 t_{1,0} t_{3,0} - \rho_1 t_{1,0} t_{3,1} - \rho_1 t_{1,0} t_{3,2} - \rho_1 t_{1,3} t_{3,0} - \rho_1 t_{2,0} t_{3,0} \\ &\quad - \rho_1 t_{2,0} t_{3,1} - \rho_1 t_{2,0} t_{3,2} - \rho_1 t_{2,3} t_{3,0} - t_{0,1} t_{1,3} - t_{0,1} t_{2,3} + t_{0,1} t_{3,0} + t_{0,1} t_{3,1} \\ &\quad + t_{0,1} t_{3,2} - t_{0,2} t_{1,3} - t_{0,2} t_{2,3} + t_{0,2} t_{3,0} + t_{0,2} t_{3,1} + t_{0,2} t_{3,2} - t_{0,3} t_{1,0} - t_{0,3} t_{1,3} \\ &\quad - t_{0,3} t_{2,0} - t_{0,3} t_{2,3} + t_{0,3} t_{3,1} + t_{0,3} t_{3,2} \end{aligned}$$

3

$$\rho_1^2 t_{0,3} - t_{1,3} \rho_1^2 - t_{2,3} \rho_1^2 - \rho_1^2 t_{3,0} - \rho_1^2 t_{3,1} - \rho_1^2 t_{3,2} - 2 \rho_1 t_{0,3} + t_{1,3} \rho_1 + t_{2,3} \rho_1 + t_{0,3}$$

4

$$\begin{aligned} &\rho_1^2 t_{0,1} + \rho_1^2 t_{0,2} + \rho_1^2 t_{1,0} + t_{1,3} \rho_1^2 + \rho_1^2 t_{2,0} + t_{2,3} \rho_1^2 + \rho_1^2 t_{3,1} + \rho_1^2 t_{3,2} - 2 \rho_1 t_{0,1} \\ &\quad - 2 \rho_1 t_{0,2} - \rho_1 t_{1,0} - t_{1,3} \rho_1 - \rho_1 t_{2,0} - t_{2,3} \rho_1 + t_{0,1} + t_{0,2} \end{aligned}$$

5

$$\begin{aligned} &-t_{0,1}^2 t_{3,0}^2 - 2 t_{0,1}^2 t_{3,0} t_{3,1} - 2 t_{0,1}^2 t_{3,0} t_{3,2} - t_{0,1}^2 t_{3,1}^2 - 2 t_{0,1}^2 t_{3,1} t_{3,2} - t_{0,1}^2 t_{3,2}^2 \\ &\quad - 2 t_{0,1} t_{0,2} t_{3,0}^2 - 4 t_{0,1} t_{0,2} t_{3,0} t_{3,1} - 4 t_{0,1} t_{0,2} t_{3,0} t_{3,2} - 2 t_{0,1} t_{0,2} t_{3,1}^2 \\ &\quad - 4 t_{0,1} t_{0,2} t_{3,1} t_{3,2} - 2 t_{0,1} t_{0,2} t_{3,2}^2 - 2 t_{0,1} t_{0,3} t_{3,0} t_{3,1} - 2 t_{0,1} t_{0,3} t_{3,0} t_{3,2} \\ &\quad - 2 t_{0,1} t_{0,3} t_{3,1}^2 - 4 t_{0,1} t_{0,3} t_{3,1} t_{3,2} - 2 t_{0,1} t_{0,3} t_{3,2}^2 + t_{0,1} t_{1,0} t_{1,3} t_{3,0} \\ &\quad + t_{0,1} t_{1,0} t_{1,3} t_{3,1} + t_{0,1} t_{1,0} t_{1,3} t_{3,2} + t_{0,1} t_{1,0} t_{2,3} t_{3,0} + t_{0,1} t_{1,0} t_{2,3} t_{3,1} \\ &\quad + t_{0,1} t_{1,0} t_{2,3} t_{3,2} + t_{0,1} t_{1,3}^2 t_{3,0} + t_{0,1} t_{1,3} t_{2,0} t_{3,0} + t_{0,1} t_{1,3} t_{2,0} t_{3,1} \\ &\quad + t_{0,1} t_{1,3} t_{2,0} t_{3,2} + 2 t_{0,1} t_{1,3} t_{2,3} t_{3,0} + t_{0,1} t_{2,0} t_{2,3} t_{3,0} + t_{0,1} t_{2,0} t_{2,3} t_{3,1} \\ &\quad + t_{0,1} t_{2,0} t_{2,3} t_{3,2} + t_{0,1} t_{2,3}^2 t_{3,0} - t_{0,2}^2 t_{3,0}^2 - 2 t_{0,2}^2 t_{3,0} t_{3,1} - 2 t_{0,2}^2 t_{3,0} t_{3,2} - t_{0,2}^2 t_{3,1}^2 \\ &\quad - 2 t_{0,2}^2 t_{3,1} t_{3,2} - t_{0,2}^2 t_{3,2}^2 - 2 t_{0,2} t_{0,3} t_{3,0} t_{3,1} - 2 t_{0,2} t_{0,3} t_{3,0} t_{3,2} - 2 t_{0,2} t_{0,3} t_{3,1}^2 \\ &\quad - 4 t_{0,2} t_{0,3} t_{3,1} t_{3,2} - 2 t_{0,2} t_{0,3} t_{3,2}^2 + t_{0,2} t_{1,0} t_{1,3} t_{3,0} + t_{0,2} t_{1,0} t_{1,3} t_{3,1} \\ &\quad + t_{0,2} t_{1,0} t_{1,3} t_{3,2} + t_{0,2} t_{1,0} t_{2,3} t_{3,0} + t_{0,2} t_{1,0} t_{2,3} t_{3,1} + t_{0,2} t_{1,0} t_{2,3} t_{3,2} + t_{0,2} \\ &\quad t_{1,3} t_{3,0} + t_{0,2} t_{1,3} t_{2,0} t_{3,0} + t_{0,2} t_{1,3} t_{2,0} t_{3,1} + t_{0,2} t_{1,3} t_{2,0} t_{3,2} + 2 t_{0,2} t_{1,3} t_{2,3} t_{3,0} \\ &\quad + t_{0,2} t_{2,0} t_{2,3} t_{3,0} + t_{0,2} t_{2,0} t_{2,3} t_{3,1} + t_{0,2} t_{2,0} t_{2,3} t_{3,2} + t_{0,2} t_{2,3}^2 t_{3,0} - t_{0,3}^2 t_{3,1}^2 - 2 \\ &\quad t_{0,3}^2 t_{3,1} t_{3,2} - t_{0,3}^2 t_{3,2}^2 + t_{0,3} t_{1,0}^2 t_{3,0} + t_{0,3} t_{1,0}^2 t_{3,1} + t_{0,3} t_{1,0}^2 t_{3,2} + 2 t_{0,3} t_{1,0} t_{1,3} t_{3,0} \\ &\quad + t_{0,3} t_{1,0} t_{1,3} t_{3,1} + t_{0,3} t_{1,0} t_{1,3} t_{3,2} + 2 t_{0,3} t_{1,0} t_{2,0} t_{3,0} + 2 t_{0,3} t_{1,0} t_{2,0} t_{3,1} \\ &\quad + 2 t_{0,3} t_{1,0} t_{2,0} t_{3,2} + 2 t_{0,3} t_{1,0} t_{2,3} t_{3,0} + t_{0,3} t_{1,0} t_{2,3} t_{3,1} + t_{0,3} t_{1,0} t_{2,3} t_{3,2} + t_{0,3} \\ &\quad t_{1,3} t_{3,0} + 2 t_{0,3} t_{1,3} t_{2,0} t_{3,0} + t_{0,3} t_{1,3} t_{2,0} t_{3,1} + t_{0,3} t_{1,3} t_{2,0} t_{3,2} + 2 t_{0,3} t_{1,3} t_{2,3} t_{3,0} \\ &\quad + t_{0,3} t_{2,0}^2 t_{3,0} + t_{0,3} t_{2,0}^2 t_{3,1} + t_{0,3} t_{2,0}^2 t_{3,2} + 2 t_{0,3} t_{2,0} t_{2,3} t_{3,0} + t_{0,3} t_{2,0} t_{2,3} t_{3,1} \\ &\quad + t_{0,3} t_{2,0} t_{2,3} t_{3,2} + t_{0,3} t_{2,3}^2 t_{3,0} \end{aligned}$$

"=====

"trop long", \emptyset

(1.8.1)

> ## simplification of the formulas

for i from 1 to 4 do collect(G[i],rho[1]);od

$$\begin{aligned} & (t_{0,1}t_{3,0} + t_{0,1}t_{3,1} + t_{0,1}t_{3,2} + t_{0,2}t_{3,0} + t_{0,2}t_{3,1} + t_{0,2}t_{3,2} + t_{0,3}t_{3,1} + t_{0,3}t_{3,2} \\ & \quad + t_{1,0}t_{3,0} + t_{1,0}t_{3,1} + t_{1,0}t_{3,2} + t_{1,3}t_{3,0} + t_{2,0}t_{3,0} + t_{2,0}t_{3,1} + t_{2,0}t_{3,2} + t_{2,3}t_{3,0}) \\ & \quad \rho_1 - t_{0,1}t_{3,0} - t_{0,1}t_{3,1} - t_{0,1}t_{3,2} - t_{0,2}t_{3,0} - t_{0,2}t_{3,1} - t_{0,2}t_{3,2} - t_{0,3}t_{3,1} - t_{0,3}t_{3,2} \\ & (t_{0,1}t_{1,3} + t_{0,1}t_{2,3} + t_{0,2}t_{1,3} + t_{0,2}t_{2,3} + t_{0,3}t_{1,0} + t_{0,3}t_{1,3} + t_{0,3}t_{2,0} + t_{0,3}t_{2,3} \\ & \quad - t_{1,0}t_{3,0} - t_{1,0}t_{3,1} - t_{1,0}t_{3,2} - t_{1,3}t_{3,0} - t_{2,0}t_{3,0} - t_{2,0}t_{3,1} - t_{2,0}t_{3,2} - t_{2,3}t_{3,0}) \\ & \quad \rho_1 - t_{0,1}t_{1,3} - t_{0,1}t_{2,3} + t_{0,1}t_{3,0} + t_{0,1}t_{3,1} + t_{0,1}t_{3,2} - t_{0,2}t_{1,3} - t_{0,2}t_{2,3} + t_{0,2}t_{3,0} \\ & \quad + t_{0,2}t_{3,1} + t_{0,2}t_{3,2} - t_{0,3}t_{1,0} - t_{0,3}t_{1,3} - t_{0,3}t_{2,0} - t_{0,3}t_{2,3} + t_{0,3}t_{3,1} + t_{0,3}t_{3,2} \\ & \quad (t_{0,3} - t_{1,3} - t_{2,3} - t_{3,0} - t_{3,1} - t_{3,2}) \rho_1^2 + (-2t_{0,3} + t_{1,3} + t_{2,3}) \rho_1 + t_{0,3} \\ & (t_{0,1} + t_{0,2} + t_{1,0} + t_{1,3} + t_{2,0} + t_{2,3} + t_{3,1} + t_{3,2}) \rho_1^2 + (-2t_{0,1} - 2t_{0,2} - t_{1,0} - t_{1,3} \\ & \quad - t_{2,0} - t_{2,3}) \rho_1 + t_{0,1} + t_{0,2} \end{aligned} \tag{1.8.2}$$

> lex := proc(x,y) if x[1]<y[1] or (x[1]=y[1] and x[2]<y[2]) then return(1) else return(0):fi:end:

Smaller:=proc(x,y) if x<y then return(1) else return(0)

:fi:end:

add(add(t[op(u)]*t[op(v)]*lex(u,v), u in {ensem(2,4)}),v in {ensem(2,4)})

$$\begin{aligned} & t_{0,1}t_{0,2} + t_{0,1}t_{0,3} + t_{0,1}t_{1,0} + t_{0,1}t_{1,2} + t_{0,1}t_{1,3} + t_{0,1}t_{2,0} + t_{0,1}t_{2,1} + t_{0,1}t_{2,3} + t_{0,1}t_{3,0} \\ & \quad + t_{0,1}t_{3,1} + t_{0,1}t_{3,2} + t_{0,2}t_{0,3} + t_{0,2}t_{1,0} + t_{0,2}t_{1,2} + t_{0,2}t_{1,3} + t_{0,2}t_{2,0} + t_{0,2}t_{2,1} \\ & \quad + t_{0,2}t_{2,3} + t_{0,2}t_{3,0} + t_{0,2}t_{3,1} + t_{0,2}t_{3,2} + t_{0,3}t_{1,0} + t_{0,3}t_{1,2} + t_{0,3}t_{1,3} + t_{0,3}t_{2,0} \\ & \quad + t_{0,3}t_{2,1} + t_{0,3}t_{2,3} + t_{0,3}t_{3,0} + t_{0,3}t_{3,1} + t_{0,3}t_{3,2} + t_{1,0}t_{1,2} + t_{1,0}t_{1,3} + t_{1,0}t_{2,0} \\ & \quad + t_{1,0}t_{2,1} + t_{1,0}t_{2,3} + t_{1,0}t_{3,0} + t_{1,0}t_{3,1} + t_{1,0}t_{3,2} + t_{1,2}t_{1,3} + t_{1,2}t_{2,0} + t_{1,2}t_{2,1} \\ & \quad + t_{1,2}t_{2,3} + t_{1,2}t_{3,0} + t_{1,2}t_{3,1} + t_{1,2}t_{3,2} + t_{1,3}t_{2,0} + t_{1,3}t_{2,1} + t_{1,3}t_{2,3} + t_{1,3}t_{3,0} \\ & \quad + t_{1,3}t_{3,1} + t_{1,3}t_{3,2} + t_{2,0}t_{2,1} + t_{2,0}t_{2,3} + t_{2,0}t_{3,0} + t_{2,0}t_{3,1} + t_{2,0}t_{3,2} + t_{2,1}t_{2,3} \\ & \quad + t_{2,1}t_{3,0} + t_{2,1}t_{3,1} + t_{2,1}t_{3,2} + t_{2,3}t_{3,0} + t_{2,3}t_{3,1} + t_{2,3}t_{3,2} + t_{3,0}t_{3,1} + t_{3,0}t_{3,2} \\ & \quad + t_{3,1}t_{3,2} \end{aligned} \tag{1.8.3}$$

> # factorisation of G[5]

S:={seq(t[op(u)]=1,u in ensem(2,4))};

$$S := \{0=1, t_{0,1}=1, t_{0,2}=1, t_{0,3}=1, t_{1,0}=1, t_{1,2}=1, t_{1,3}=1, t_{2,0}=1, t_{2,1}=1, t_{2,3}=1, \\ t_{3,0}=1, t_{3,1}=1, t_{3,2}=1\} \tag{1.8.4}$$

> m:=5;

pos:=factor(add(op(i,G[m])*ind(sign(subs(S,op(i,G[m])))),1),i=1..nops(G[m]));

neg:=factor(add(op(i,G[m])*ind(sign(subs(S,op(i,G[m])))), -1),i=1..nops(G[m]));

$m := 5$

```
pos := (t1,0t3,0 + t1,0t3,1 + t1,0t3,2 + t1,3t3,0 + t2,0t3,0 + t2,0t3,1 + t2,0t3,2
      + t2,3t3,0) (t0,1t1,3 + t0,1t2,3 + t0,2t1,3 + t0,2t2,3 + t0,3t1,0 + t0,3t1,3 + t0,3t2,0
      + t0,3t2,3)
```

```
neg := -(t0,1t3,0 + t0,1t3,1 + t0,1t3,2 + t0,2t3,0 + t0,2t3,1 + t0,2t3,2 + t0,3t3,1
      + t0,3t3,2)2 (1.8.5)
```

```
> normal(G[5]-pos-neg);
```

0

(1.8.6)

```
> latex(pos+neg);
```

```
\left( t_{\{1,0\}}t_{\{3,0\}}+t_{\{1,0\}}t_{\{3,1\}}+t_{\{1,0\}}t_{\{3,2\}}+t_{\{1,3\}}t_{\{3,0\}}+t_{\{2,0\}}t_{\{3,0\}}+t_{\{2,0\}}t_{\{3,1\}}+t_{\{2,0\}}t_{\{3,2\}}+t_{\{2,3\}}t_{\{3,0\}} \right) \left( t_{\{0,1\}}t_{\{1,3\}}+t_{\{0,1\}}t_{\{2,3\}}+t_{\{0,2\}}t_{\{1,3\}}+t_{\{0,2\}}t_{\{2,3\}}+t_{\{0,3\}}t_{\{1,0\}}+t_{\{0,3\}}t_{\{1,3\}}+t_{\{0,3\}}t_{\{2,0\}}+t_{\{0,3\}}t_{\{2,3\}} \right) - \left( t_{\{0,1\}}t_{\{3,0\}}+t_{\{0,1\}}t_{\{3,1\}}+t_{\{0,1\}}t_{\{3,2\}}+t_{\{0,2\}}t_{\{3,0\}}+t_{\{0,2\}}t_{\{3,1\}}+t_{\{0,2\}}t_{\{3,2\}}+t_{\{0,3\}}t_{\{3,1\}}+t_{\{0,3\}}t_{\{3,2\}} \right) ^{2}
```

Section 3.4 2D Applications

Functions to run (once) for the applications that follows

```
> # -> Initialisation
```

```
unassign('T','M','u','v','rho','K');
```

```
Gamma0:= [[0,0],[0,1],[1,0]]:
```

```
Gamma1:= [[0,0],[0,1],[1,0],[2,0]]:
```

```
Gamma2:= [[0,0],[0,1],[1,0],[2,0],[1,1]]:
```

```
## squares encoded following this order:
```

```
## d c
```

```
## a b
```

```
##
```

```
>
```

```
Prod :=proc(u) mul(rho[u[i]],i=1..4):end:
```

```
Zsquare:=proc(u) ## for a square-word u of size 4 we need the following Z
```

```
return( add( Prod(v) * T[v,u] / Prod(u) - T[u,v], v in ensem(4,K))):
```

```
end:
```

```
CartesianProduct:=proc(E1,E2,E3,E4) option remember:
```

```

return({seq(seq(seq(seq( [e1, e2, e3, e4], e1 in E1), e2
in E2), e3 in E3), e4 in E4)}):
end:

```

```

## partial Z... Sommatation on lacking variables

```

```

ZP:=proc(u) local E, i, Externe:

```

```

# we will put -2 in the unit square when they are outside
the domain

```

```

Externe:={}:

```

```

for i from 1 to 4 do

```

```

if u[i] = -2

```

```

then

```

```

E[i] := {seq(j, j= 0 .. K-1)} ;

```

```

Externe:=Externe union { i };

```

```

else

```

```

E[i]:= {u[i]}: fi:

```

```

od:

```

```

return( add(Zsquare(w)*mul(rho[w[i]], i in
Externe), w in CartesianProduct( E[1],E[2],E[3],E[4]))):
end:

```

```

rank:=proc(element,Seq) # provide the rank of an element in
a list (where it appears once), and return -2 if the
element is absent

```

```

if not element in Seq then return(-2) else return(max
({seq(ind(element,Seq[j])*j,j=1..nops(Seq)})):

```

```

fi:

```

```

end:

```

```

SetOfCornersOfSquaresThatIntersects:=proc(S) # position of
the corners bottom, left that intersects the domain S

```

```

return({seq( u+[-1,0], u in S),
seq( u+[-1,-1], u in S),
seq( u+[0,-1], u in S),
seq( u, u in S) }):

```

```

end:

```

```

# the square whose bottom-left vertex is s

```

```

SquareWithCorners:=proc(s) return( [ s +[0,0], s+[1,0], s+
[1,1], s+[0,1]]):end:

```

```

## Equation of stability of the mass of the config u
indexed by domain, ou Domain

```

```

EquationDomain:=proc(Domain, u) local equation,s,S,W,i,C:
equation:=0:

```

```

# addition on all squares that intersect the set of
interest

```

```

for s in SetOfCornersOfSquaresThatIntersects(Domain) do

```

```

S:= SquareWithCorners(s):

```

```

W:= [seq(rank(S[i],Domain),i=1..4)]:

```

```

for i from 1 to 4 do

```

```

if W[i]=-2 then C[i]:=-2 else C[i]:= u [W

```

```

[i]]:fi:

```

```

od:

```

```

        equation := equation + ZP([seq(C[i],i=1..4)]):
    od:
return(equation):
end:

```

▼ All T zeroes except $T[0111 | 1000]=a$, $T[1000|0111]=1$

```

> unassign('T','rho');
rho[0] := 1-rho[1]:
K:=2;
# Transition definition
NonZeroT:= NULL: # will be the set of nonzerotransitions
for u in ensem(4,K) do # u=(u1,u2,u3,u4)
  for v in ensem(4,K) do
    if([u,v]<> [[0,1,1,1],[1,0,0,0]] and [u,v] <> [
[1,0,0,0],[0,1,1,1]]) then
      T[u,v] := 0:
    else
      NonZeroT:=NonZeroT, T[u,v]:
    fi:
  od:
od:
"Parameters", NonZeroT;
"Parameters",  $T_{[0,1,1,1],[1,0,0,0]}$ ,  $T_{[1,0,0,0],[0,1,1,1]}$  (1.9.1.1)

```

```

> EquationParameter := T[[0, 1, 1, 1], [1, 0, 0, 0]]-a, T[
[1, 0, 0, 0], [0, 1, 1, 1]]-1;
EquationGamma0:= seq(EquationDomain (Gamma0, u), u in
ensem(3,K)):
EquationGamma1and2 := seq( EquationDomain (Gamma2, u)
- EquationDomain (Gamma1, [seq(u
[i],i=1..4)]) , u in ensem(5,K)):
SystemEquation := [EquationParameter ,EquationGamma0,
EquationGamma1and2 ,seq(g[i]*rho[i]-1,i=0..1)]:
#####
PolynomialSystem:= map(x-> numer(normal(x)),
SystemEquation): # numerators only
Variables:= rho[1] , g[1] ,g[0], a, NonZeroT :
#####
# grobner
G := Basis(PolynomialSystem, tdeg( Variables )):
Affiche(G,20000);

```

$$EquationParameter := T_{[0,1,1,1],[1,0,0,0]} - a, T_{[1,0,0,0],[0,1,1,1]} - 1$$

1

$$T_{[1,0,0,0],[0,1,1,1]} - 1$$

2

$$-T_{[0,1,1,1],[1,0,0,0]} + a$$

3

$$T_{[0,1,1,1],[1,0,0,0]} g_0 - T_{[0,1,1,1],[1,0,0,0]} - g_1 + 1$$

4

$$\rho_1 T_{[0,1,1,1],[1,0,0,0]} - g_1 - \rho_1 + 2$$

5

$$g_0 g_1 - g_0 - g_1$$

6

$$g_0 \rho_1 - g_0 + 1$$

7

$$g_1^2 - T_{[0,1,1,1],[1,0,0,0]} - 2 g_1 + 1$$

8

$$g_1 \rho_1 - 1$$

"=====

"trop long", \emptyset

(1.9.1.1.2)

> normal(subs(g[1]=1/rho[1],g[0]=1/rho[0],G));solve(%,
{NonZeroT,rho[1]});

$$\left[T_{[1,0,0,0],[0,1,1,1]} - 1, -T_{[0,1,1,1],[1,0,0,0]} + a, \right.$$

$$\left. - \frac{\rho_1^2 T_{[0,1,1,1],[1,0,0,0]} - \rho_1^2 + 2 \rho_1 - 1}{(-1 + \rho_1) \rho_1}, \right.$$

$$\left. \frac{\rho_1^2 T_{[0,1,1,1],[1,0,0,0]} - \rho_1^2 + 2 \rho_1 - 1}{\rho_1}, 0, 0, \right.$$

$$\left. - \frac{\rho_1^2 T_{[0,1,1,1],[1,0,0,0]} - \rho_1^2 + 2 \rho_1 - 1}{\rho_1^2}, 0 \right]$$

{ $T_{[0,1,1,1],[1,0,0,0]} = a, T_{[1,0,0,0],[0,1,1,1]} = 1, \rho_1 = \text{RootOf}(-1 + (a - 1) _Z^2 + 2 _Z)$ } (1.9.1.1.3)

$$\frac{-1 + \sqrt{a}}{a - 1}, -\frac{1 + \sqrt{a}}{a - 1}$$

(1.9.1.1.4)

▼ All T zeroes except $T[0101 | 1010] = a, T[1010|0101] = 1$

> unassign('T','rho');

rho[0] := 1-rho[1];

K:=2;

Transition definition

NonZeroT:= NULL: # will be the set of nonzerotransitions

for u in ensem(4,K) do # u=(u1,u2,u3,u4)

for v in ensem(4,K) do

if([u,v]<> [[0,1,0,1],[1,0,1,0]] and [u,v] <> [1,0,1,0],[0,1,0,1]]) then

T[u,v] := 0:

```

else
NonZeroT:=NonZeroT, T[u,v]:
fi:
od:
od:
"Parameters", NonZeroT;
"Parameters", T[0, 1, 0, 1], [1, 0, 1, 0], T[1, 0, 1, 0], [0, 1, 0, 1]
(1.9.1.2.1)

```

```

>
EquationParameter := T[[0,1,0,1],[1,0,1,0]] - a , T[[1,0,
1,0],[0,1,0,1]] - b;
EquationGamma0:= seq(EquationDomain (Gamma0, u), u in
ensem(3,K)):
EquationGamma1and2 := seq( EquationDomain (Gamma2, u)
- EquationDomain (Gamma1, [seq(u
[i],i=1..4)]), u in ensem(5,K)):
SystemEquation := [EquationParameter ,EquationGamma0,
EquationGamma1and2 ,seq(g[i]*rho[i]-1,i=0..1)]:
#####
PolynomialSystem:= map(x-> numer(normal(x)),
SystemEquation): # numerators only
Variables:= rho[1] , g[1] ,g[0], a, b, NonZeroT :
#####
# grobner
G := Basis(PolynomialSystem, tdeg( Variables )):
Affiche(G,20000);

```

$$\begin{aligned}
\text{EquationParameter} &:= T_{[0, 1, 0, 1], [1, 0, 1, 0]} - a, T_{[1, 0, 1, 0], [0, 1, 0, 1]} - b \\
&1 \\
&T_{[0, 1, 0, 1], [1, 0, 1, 0]} - T_{[1, 0, 1, 0], [0, 1, 0, 1]} \\
&2 \\
&-T_{[1, 0, 1, 0], [0, 1, 0, 1]} + b \\
&3 \\
&a - T_{[1, 0, 1, 0], [0, 1, 0, 1]} \\
&4 \\
&g_0 g_1 - g_0 - g_1 \\
&5 \\
&g_0 \rho_1 - g_0 + 1 \\
&6 \\
&g_1 \rho_1 - 1
\end{aligned}$$

"=====

"trop long", ∅ (1.9.1.2.2)

```

> normal(subs(g[0]=1/rho[0],g[1]=1/rho[1],G));
[T[0, 1, 0, 1], [1, 0, 1, 0] - T[1, 0, 1, 0], [0, 1, 0, 1] - T[1, 0, 1, 0], [0, 1, 0, 1] + b, a
- T[1, 0, 1, 0], [0, 1, 0, 1], 0, 0, 0]
(1.9.1.2.3)

```

▼ All T zeroes except T[0011 | 0110]
> unassign('T','rho');


```

rho[0] := 1-rho[1]:
K:=2;
# Transition definition
NonZeroT:= NULL: # will be the set of nonzerotransitions
for u in ensem(4,K) do # u=(u1,u2,u3,u4)
  for v in ensem(4,K) do
    if([u,v]<> [[0,0,1,1],[0,1,1,0]]) then
      T[u,v] := 0:
    else
      NonZeroT:=NonZeroT, T[u,v]:
    fi:
  od:
od:
"Parameters", NonZeroT;
"Parameters", T[0, 0, 1, 1], [0, 1, 1, 0]

```

(1.9.1.3.1)

```

>
EquationGamma0:= seq(EquationDomain (Gamma0, u), u in
ensem(3,K)):
EquationGamma1and2 := seq( EquationDomain (Gamma2, u)
- EquationDomain (Gamma1, [seq(u
[i],i=1..4)]) , u in ensem(5,K)):
SystemEquation := [EquationGamma0, EquationGamma1and2 ,
seq(g[i]*rho[i]-1,i=0..1)]:
#####
PolynomialSystem:= map(x-> numer(normal(x)),
SystemEquation): # numerators only
Variables:= rho[1] , g[1] ,g[0], NonZeroT :
#####
# grobner
G := Basis(PolynomialSystem, tdeg( Variables )):
Affiche(G,20000);

```

```

1
T[0, 0, 1, 1], [0, 1, 1, 0]
2
g0g1 - g0 - g1
3
g0ρ1 - g0 + 1
4
g1ρ1 - 1
"=====
"trop long", ∅

```

(1.9.1.3.2)

▼ All T zeroes except T[0011 | 0110], T[0110 | 1100], T[1100 | 1001], T[1001 | 0011]

```

> unassign('T','rho');
rho[0] := 1-rho[1]:
K:=2;
# Transition definition
NonZeroT:= NULL: # will be the set of nonzerotransitions
for u in ensem(4,K) do # u=(u1,u2,u3,u4)

```

```

    for v in ensem(4,K) do
      if( not [u,v] in { [[0,0,1,1],[0,1,1,0]],[[0,1,1,0],[1,1,0,0]],[[1,1,0,0],[1,0,0,1]],[[1,0,0,1],[0,0,1,1]]} ) then
        T[u,v] := 0:
      else
        NonZeroT:=NonZeroT, T[u,v]:
      fi:
    od:
  od:
  "Parameters", NonZeroT;
  "Parameters", T[0,0,1,1],[0,1,1,0], T[0,1,1,0],[1,1,0,0], T[1,0,0,1],[0,0,1,1],
  T[1,1,0,0],[1,0,0,1]

```

(1.9.1.4.1)

```

> EquationGamma0:= seq(EquationDomain (Gamma0, u), u in
ensem(3,K)):
EquationGamma1and2 := seq( EquationDomain (Gamma2, u)
- EquationDomain (Gamma1, [seq(u
[i],i=1..4)]) , u in ensem(5,K)):
SystemEquation := [EquationGamma0, EquationGamma1and2 ,
seq(g[i]*rho[i]-1,i=0..1)]:
#####
PolynomialSystem:= map(x-> numer(normal(x)),
SystemEquation): # numerators only
Variables:= rho[1] , g[1] ,g[0], NonZeroT :
#####
# grobner
G := Basis(PolynomialSystem, tdeg( Variables )):
Affiche(G,20000);

```

```

1
T[1,0,0,1],[0,0,1,1] - T[1,1,0,0],[1,0,0,1]
2
T[0,1,1,0],[1,1,0,0] - T[1,1,0,0],[1,0,0,1]
3
T[0,0,1,1],[0,1,1,0] - T[1,1,0,0],[1,0,0,1]
4
g0g1 - g0 - g1
5
g0ρ1 - g0 + 1
6
g1ρ1 - 1
"=====
"trop long", ∅

```

(1.9.1.4.2)

>

▼ All T zeroes except T[abcd|a(1-b)(1-c)d]
> unassign('T','rho');

```

rho[0] := 1-rho[1]:
K:=2;
# Transition definition
NonZeroT:= NULL: # will be the set of nonzerotransitions
for u in ensem(4,K) do # u=(u1,u2,u3,u4)
  for v in ensem(4,K) do
    if( u <> [v[1],1-v[2],1-v[3],v[4]] ) then
      T[u,v] := 0:
    else
      NonZeroT:=NonZeroT, T[u,v]:
    fi:
  od:
od:
"Parameters", NonZeroT;

```

"Parameters", $T_{[0,0,0,0],[0,1,1,0]}$, $T_{[0,0,0,1],[0,1,1,1]}$, $T_{[0,0,1,0],[0,1,0,0]}$, (1.9.1.5.1)

$T_{[0,0,1,1],[0,1,0,1]}$, $T_{[0,1,0,0],[0,0,1,0]}$, $T_{[0,1,0,1],[0,0,1,1]}$,
 $T_{[0,1,1,0],[0,0,0,0]}$, $T_{[0,1,1,1],[0,0,0,1]}$, $T_{[1,0,0,0],[1,1,1,0]}$,
 $T_{[1,0,0,1],[1,1,1,1]}$, $T_{[1,0,1,0],[1,1,0,0]}$, $T_{[1,0,1,1],[1,1,0,1]}$,
 $T_{[1,1,0,0],[1,0,1,0]}$, $T_{[1,1,0,1],[1,0,1,1]}$, $T_{[1,1,1,0],[1,0,0,0]}$,
 $T_{[1,1,1,1],[1,0,0,1]}$

```

>
EquationGamma0:= seq(EquationDomain (Gamma0, u), u in
ensem(3,K)):
EquationGamma1and2 := seq( EquationDomain (Gamma2, u)
- EquationDomain (Gamma1, [seq(u
[i],i=1..4)]), u in ensem(5,K)):
SystemEquation := [EquationGamma0, EquationGamma1and2 ,
seq(g[i]*rho[i]-1,i=0..1)]:
#####
PolynomialSystem:= map(x-> numer(normal(x)),
SystemEquation): # numerators only
Variables:= rho[1] , g[1] ,g[0], NonZeroT :
#####
# grobner
G := Basis(PolynomialSystem, plex( Variables )):
subs(
Affiche(G,20000);

```

1

$(T_{[1,0,1,0],[1,1,0,0]} - T_{[1,0,1,1],[1,1,0,1]} - T_{[1,1,0,0],[1,0,1,0]}$
 $+ T_{[1,1,0,1],[1,0,1,1]}) (T_{[1,0,0,1],[1,1,1,1]} - T_{[1,1,1,1],[1,0,0,1]})$

2

$T_{[1,0,0,0],[1,1,1,0]} T_{[1,1,1,1],[1,0,0,1]} - T_{[1,0,0,1],[1,1,1,1]} T_{[1,1,1,0],[1,0,0,0]}$
 $+ T_{[1,0,1,0],[1,1,0,0]} T_{[1,1,1,1],[1,0,0,1]}$
 $- T_{[1,0,1,1],[1,1,0,1]} T_{[1,1,1,1],[1,0,0,1]}$
 $- T_{[1,1,0,0],[1,0,1,0]} T_{[1,1,1,1],[1,0,0,1]}$
 $+ T_{[1,1,0,1],[1,0,1,1]} T_{[1,1,1,1],[1,0,0,1]}$

3

$$\begin{aligned}
& (T_{[1, 0, 1, 0], [1, 1, 0, 0]} - T_{[1, 0, 1, 1], [1, 1, 0, 1]} - T_{[1, 1, 0, 0], [1, 0, 1, 0]} \\
& \quad + T_{[1, 1, 0, 1], [1, 0, 1, 1]}) (T_{[1, 0, 1, 0], [1, 1, 0, 0]} + T_{[1, 0, 0, 0], [1, 1, 1, 0]} \\
& \quad - T_{[1, 0, 1, 1], [1, 1, 0, 1]} - T_{[1, 1, 0, 0], [1, 0, 1, 0]} + T_{[1, 1, 0, 1], [1, 0, 1, 1]} \\
& \quad - T_{[1, 1, 1, 0], [1, 0, 0, 0]})
\end{aligned}$$

4

$$\begin{aligned}
& T_{[0, 0, 1, 1], [0, 1, 0, 1]} - T_{[0, 1, 0, 1], [0, 0, 1, 1]} - T_{[1, 0, 1, 0], [1, 1, 0, 0]} \\
& \quad + T_{[1, 1, 0, 0], [1, 0, 1, 0]}
\end{aligned}$$

5

$$\begin{aligned}
& T_{[0, 0, 1, 0], [0, 1, 0, 0]} - T_{[0, 1, 0, 0], [0, 0, 1, 0]} - 2 T_{[1, 0, 1, 0], [1, 1, 0, 0]} \\
& \quad + T_{[1, 0, 1, 1], [1, 1, 0, 1]} + 2 T_{[1, 1, 0, 0], [1, 0, 1, 0]} - T_{[1, 1, 0, 1], [1, 0, 1, 1]}
\end{aligned}$$

6

$$\begin{aligned}
& T_{[0, 0, 0, 1], [0, 1, 1, 1]} T_{[1, 1, 1, 1], [1, 0, 0, 1]} - T_{[0, 1, 1, 1], [0, 0, 0, 1]} T_{[1, 0, 0, 1], [1, 1, 1, 1]} \\
& \quad - T_{[1, 0, 1, 0], [1, 1, 0, 0]} T_{[1, 1, 1, 1], [1, 0, 0, 1]} \\
& \quad + T_{[1, 0, 1, 1], [1, 1, 0, 1]} T_{[1, 1, 1, 1], [1, 0, 0, 1]} \\
& \quad + T_{[1, 1, 0, 0], [1, 0, 1, 0]} T_{[1, 1, 1, 1], [1, 0, 0, 1]} \\
& \quad - T_{[1, 1, 0, 1], [1, 0, 1, 1]} T_{[1, 1, 1, 1], [1, 0, 0, 1]}
\end{aligned}$$

7

$$\begin{aligned}
& T_{[0, 0, 0, 1], [0, 1, 1, 1]} T_{[1, 1, 1, 0], [1, 0, 0, 0]} - T_{[0, 1, 1, 1], [0, 0, 0, 1]} T_{[1, 0, 0, 0], [1, 1, 1, 0]} \\
& \quad - T_{[0, 1, 1, 1], [0, 0, 0, 1]} T_{[1, 0, 1, 0], [1, 1, 0, 0]} \\
& \quad + T_{[0, 1, 1, 1], [0, 0, 0, 1]} T_{[1, 0, 1, 1], [1, 1, 0, 1]} \\
& \quad + T_{[0, 1, 1, 1], [0, 0, 0, 1]} T_{[1, 1, 0, 0], [1, 0, 1, 0]} \\
& \quad - T_{[0, 1, 1, 1], [0, 0, 0, 1]} T_{[1, 1, 0, 1], [1, 0, 1, 1]} \\
& \quad - T_{[1, 0, 1, 0], [1, 1, 0, 0]} T_{[1, 1, 1, 0], [1, 0, 0, 0]} \\
& \quad + T_{[1, 0, 1, 1], [1, 1, 0, 1]} T_{[1, 1, 1, 0], [1, 0, 0, 0]} \\
& \quad + T_{[1, 1, 0, 0], [1, 0, 1, 0]} T_{[1, 1, 1, 0], [1, 0, 0, 0]} \\
& \quad - T_{[1, 1, 0, 1], [1, 0, 1, 1]} T_{[1, 1, 1, 0], [1, 0, 0, 0]}
\end{aligned}$$

8

$$\begin{aligned}
& (T_{[1, 0, 1, 0], [1, 1, 0, 0]} - T_{[1, 0, 1, 1], [1, 1, 0, 1]} - T_{[1, 1, 0, 0], [1, 0, 1, 0]} \\
& \quad + T_{[1, 1, 0, 1], [1, 0, 1, 1]}) (-T_{[1, 0, 1, 0], [1, 1, 0, 0]} + T_{[0, 0, 0, 1], [0, 1, 1, 1]} \\
& \quad - T_{[0, 1, 1, 1], [0, 0, 0, 1]} + T_{[1, 0, 1, 1], [1, 1, 0, 1]} + T_{[1, 1, 0, 0], [1, 0, 1, 0]} \\
& \quad - T_{[1, 1, 0, 1], [1, 0, 1, 1]})
\end{aligned}$$

9

$$T_{[0, 0, 0, 0], [0, 1, 1, 0]} T_{[1, 1, 1, 1], [1, 0, 0, 1]} - T_{[0, 1, 1, 0], [0, 0, 0, 0]} T_{[1, 0, 0, 1], [1, 1, 1, 1]}$$

10

$$T_{[0, 0, 0, 0], [0, 1, 1, 0]} T_{[1, 1, 1, 0], [1, 0, 0, 0]} - T_{[0, 1, 1, 0], [0, 0, 0, 0]} T_{[1, 0, 0, 0], [1, 1, 1, 0]}$$

$$\begin{aligned}
& -T_{[0, 1, 1, 0], [0, 0, 0, 0]} T_{[1, 0, 1, 0], [1, 1, 0, 0]} \\
& + T_{[0, 1, 1, 0], [0, 0, 0, 0]} T_{[1, 0, 1, 1], [1, 1, 0, 1]} \\
& + T_{[0, 1, 1, 0], [0, 0, 0, 0]} T_{[1, 1, 0, 0], [1, 0, 1, 0]} \\
& - T_{[0, 1, 1, 0], [0, 0, 0, 0]} T_{[1, 1, 0, 1], [1, 0, 1, 1]}
\end{aligned}$$

11

$$\begin{aligned}
& (T_{[1, 0, 1, 0], [1, 1, 0, 0]} - T_{[1, 0, 1, 1], [1, 1, 0, 1]} - T_{[1, 1, 0, 0], [1, 0, 1, 0]} \\
& + T_{[1, 1, 0, 1], [1, 0, 1, 1]}) (T_{[0, 0, 0, 0], [0, 1, 1, 0]} - T_{[0, 1, 1, 0], [0, 0, 0, 0]})
\end{aligned}$$

12

$$\begin{aligned}
& T_{[0, 0, 0, 0], [0, 1, 1, 0]} T_{[0, 1, 1, 1], [0, 0, 0, 1]} - T_{[0, 0, 0, 1], [0, 1, 1, 1]} T_{[0, 1, 1, 0], [0, 0, 0, 0]} \\
& + T_{[0, 1, 1, 0], [0, 0, 0, 0]} T_{[1, 0, 1, 0], [1, 1, 0, 0]} \\
& - T_{[0, 1, 1, 0], [0, 0, 0, 0]} T_{[1, 0, 1, 1], [1, 1, 0, 1]} \\
& - T_{[0, 1, 1, 0], [0, 0, 0, 0]} T_{[1, 1, 0, 0], [1, 0, 1, 0]} \\
& + T_{[0, 1, 1, 0], [0, 0, 0, 0]} T_{[1, 1, 0, 1], [1, 0, 1, 1]}
\end{aligned}$$

13

$$\begin{aligned}
& (g_0 - 2) (T_{[1, 0, 1, 0], [1, 1, 0, 0]} - T_{[1, 0, 1, 1], [1, 1, 0, 1]} - T_{[1, 1, 0, 0], [1, 0, 1, 0]} \\
& + T_{[1, 1, 0, 1], [1, 0, 1, 1]})
\end{aligned}$$

14

$$\begin{aligned}
& T_{[1, 1, 1, 1], [1, 0, 0, 1]} g_0^2 - 2 T_{[1, 1, 1, 1], [1, 0, 0, 1]} g_0 - T_{[1, 0, 0, 1], [1, 1, 1, 1]} \\
& + T_{[1, 1, 1, 1], [1, 0, 0, 1]}
\end{aligned}$$

15

$$\begin{aligned}
& T_{[1, 1, 1, 0], [1, 0, 0, 0]} g_0^2 - 2 T_{[1, 1, 1, 0], [1, 0, 0, 0]} g_0 - T_{[1, 0, 0, 0], [1, 1, 1, 0]} \\
& - T_{[1, 0, 1, 0], [1, 1, 0, 0]} + T_{[1, 0, 1, 1], [1, 1, 0, 1]} + T_{[1, 1, 0, 0], [1, 0, 1, 0]} \\
& - T_{[1, 1, 0, 1], [1, 0, 1, 1]} + T_{[1, 1, 1, 0], [1, 0, 0, 0]}
\end{aligned}$$

16

$$\begin{aligned}
& T_{[0, 1, 1, 1], [0, 0, 0, 1]} g_0^2 - 2 T_{[0, 1, 1, 1], [0, 0, 0, 1]} g_0 - T_{[0, 0, 0, 1], [0, 1, 1, 1]} \\
& + T_{[0, 1, 1, 1], [0, 0, 0, 1]} + T_{[1, 0, 1, 0], [1, 1, 0, 0]} - T_{[1, 0, 1, 1], [1, 1, 0, 1]} \\
& - T_{[1, 1, 0, 0], [1, 0, 1, 0]} + T_{[1, 1, 0, 1], [1, 0, 1, 1]}
\end{aligned}$$

17

$$\begin{aligned}
& T_{[0, 1, 1, 0], [0, 0, 0, 0]} g_0^2 - 2 T_{[0, 1, 1, 0], [0, 0, 0, 0]} g_0 - T_{[0, 0, 0, 0], [0, 1, 1, 0]} \\
& + T_{[0, 1, 1, 0], [0, 0, 0, 0]}
\end{aligned}$$

18

$$\begin{aligned}
& (g_1 - 2) (T_{[1, 0, 1, 0], [1, 1, 0, 0]} - T_{[1, 0, 1, 1], [1, 1, 0, 1]} - T_{[1, 1, 0, 0], [1, 0, 1, 0]} \\
& + T_{[1, 1, 0, 1], [1, 0, 1, 1]})
\end{aligned}$$

19

$$T_{[1, 0, 0, 1], [1, 1, 1, 1]} g_1 - T_{[1, 1, 1, 1], [1, 0, 0, 1]} g_0 - T_{[1, 0, 0, 1], [1, 1, 1, 1]}$$

$$+ T_{[1, 1, 1, 1], [1, 0, 0, 1]}$$

20

$$\begin{aligned} & T_{[1, 0, 0, 0], [1, 1, 1, 0]} g_1 - T_{[1, 1, 1, 0], [1, 0, 0, 0]} g_0 - T_{[1, 0, 0, 0], [1, 1, 1, 0]} \\ & + T_{[1, 0, 1, 0], [1, 1, 0, 0]} - T_{[1, 0, 1, 1], [1, 1, 0, 1]} - T_{[1, 1, 0, 0], [1, 0, 1, 0]} \\ & + T_{[1, 1, 0, 1], [1, 0, 1, 1]} + T_{[1, 1, 1, 0], [1, 0, 0, 0]} \end{aligned}$$

21

$$\begin{aligned} & T_{[0, 0, 0, 1], [0, 1, 1, 1]} g_1 - T_{[0, 1, 1, 1], [0, 0, 0, 1]} g_0 - T_{[0, 0, 0, 1], [0, 1, 1, 1]} \\ & + T_{[0, 1, 1, 1], [0, 0, 0, 1]} - T_{[1, 0, 1, 0], [1, 1, 0, 0]} + T_{[1, 0, 1, 1], [1, 1, 0, 1]} \\ & + T_{[1, 1, 0, 0], [1, 0, 1, 0]} - T_{[1, 1, 0, 1], [1, 0, 1, 1]} \end{aligned}$$

22

$$\begin{aligned} & T_{[0, 0, 0, 0], [0, 1, 1, 0]} g_1 - T_{[0, 1, 1, 0], [0, 0, 0, 0]} g_0 - T_{[0, 0, 0, 0], [0, 1, 1, 0]} \\ & + T_{[0, 1, 1, 0], [0, 0, 0, 0]} \end{aligned}$$

23

$$g_0 g_1 - g_0 - g_1$$

24

$$\begin{aligned} & (2 \rho_1 - 1) (T_{[1, 0, 1, 0], [1, 1, 0, 0]} - T_{[1, 0, 1, 1], [1, 1, 0, 1]} - T_{[1, 1, 0, 0], [1, 0, 1, 0]} \\ & + T_{[1, 1, 0, 1], [1, 0, 1, 1]}) \end{aligned}$$

25

$$\begin{aligned} & T_{[1, 0, 0, 1], [1, 1, 1, 1]} \rho_1 + T_{[1, 1, 1, 1], [1, 0, 0, 1]} g_0 - T_{[1, 1, 1, 1], [1, 0, 0, 1]} \rho_1 \\ & - T_{[1, 0, 0, 1], [1, 1, 1, 1]} - T_{[1, 1, 1, 1], [1, 0, 0, 1]} \end{aligned}$$

26

$$\begin{aligned} & 2 \rho_1 T_{[1, 0, 0, 0], [1, 1, 1, 0]} + 2 T_{[1, 1, 1, 0], [1, 0, 0, 0]} g_0 - 2 T_{[1, 1, 1, 0], [1, 0, 0, 0]} \rho_1 \\ & - 2 T_{[1, 0, 0, 0], [1, 1, 1, 0]} - T_{[1, 0, 1, 0], [1, 1, 0, 0]} + T_{[1, 0, 1, 1], [1, 1, 0, 1]} \\ & + T_{[1, 1, 0, 0], [1, 0, 1, 0]} - T_{[1, 1, 0, 1], [1, 0, 1, 1]} - 2 T_{[1, 1, 1, 0], [1, 0, 0, 0]} \end{aligned}$$

27

$$\begin{aligned} & 2 T_{[0, 0, 0, 1], [0, 1, 1, 1]} \rho_1 + 2 T_{[0, 1, 1, 1], [0, 0, 0, 1]} g_0 - 2 T_{[0, 1, 1, 1], [0, 0, 0, 1]} \rho_1 \\ & - 2 T_{[0, 0, 0, 1], [0, 1, 1, 1]} - 2 T_{[0, 1, 1, 1], [0, 0, 0, 1]} + T_{[1, 0, 1, 0], [1, 1, 0, 0]} \\ & - T_{[1, 0, 1, 1], [1, 1, 0, 1]} - T_{[1, 1, 0, 0], [1, 0, 1, 0]} + T_{[1, 1, 0, 1], [1, 0, 1, 1]} \end{aligned}$$

28

$$\begin{aligned} & T_{[0, 0, 0, 0], [0, 1, 1, 0]} \rho_1 + T_{[0, 1, 1, 0], [0, 0, 0, 0]} g_0 - T_{[0, 1, 1, 0], [0, 0, 0, 0]} \rho_1 \\ & - T_{[0, 0, 0, 0], [0, 1, 1, 0]} - T_{[0, 1, 1, 0], [0, 0, 0, 0]} \end{aligned}$$

29

$$g_0 \rho_1 - g_0 + 1$$

30

$$g_1 \rho_1 - 1$$

"=====

"trop long", \emptyset

(1.9.1.5.2)

>

3 color case : All T zeroes except $T[i^A|(i+1 \bmod 3)^4]$

> unassign('T','rho');

```
K:=3;
# Transition definition
NonZeroT:= NULL: # will be the set of nonzerotransitions
for u in ensem(4,K) do # u=(u1,u2,u3,u4)
  for v in ensem(4,K) do
    if( not [u,v] in {seq([[i,i,i,i],[(i+1) mod 3,
(i+1) mod 3,(i+1) mod 3, (i+1) mod 3] ,i=0..3)} ) then
      T[u,v] := 0:
    else
      NonZeroT:=NonZeroT, T[u,v]:
    fi:
  od:
od:
```

```
T[[0, 0, 0, 0], [1, 1, 1, 1]]:=a[0]:
T[[1, 1, 1, 1], [2, 2, 2, 2]]:=a[1]:
T[[2, 2, 2, 2], [0, 0, 0, 0]]:=a[2]:
```

"Parameters", NonZeroT;

$K := 3$

"Parameters", a_0, a_1, a_2

(1.9.1.6.1)

```
> EquationGamma0:= seq(EquationDomain (Gamma0, u), u in
ensem(3,K)):
EquationGamma1and2 := seq( EquationDomain (Gamma2, u)
- EquationDomain (Gamma1, [seq(u
[i],i=1..4)]), u in ensem(5,K)):
SystemEquation := [ rho[0]+rho[1]+rho[2]-1,
EquationGamma0, EquationGamma1and2 ]:#,seq(g[i]*rho[i]-1,
i=0..2]):
#####
PolynomialSystem:= map(x-> numer(normal(x)),
SystemEquation): # numerators only
Variables:= seq( rho[i], i=0..2), seq( g[i], i=0..2),
NonZeroT :
#####
# grobner
G := Basis(PolynomialSystem, tdeg ( Variables )):

#### A BIT LONG
Affiche(G,2000);
```

1

$\rho_0 + \rho_1 + \rho_2 - 1$

2

$$a_1 \rho_1^4 - a_2 \rho_2^4$$

3

$$a_0 \rho_1^4 + 4 a_0 \rho_1^3 \rho_2 + 6 a_0 \rho_1^2 \rho_2^2 + 4 a_0 \rho_1 \rho_2^3 + a_0 \rho_2^4 - a_2 \rho_2^4 - 4 a_0 \rho_1^3 - 12 a_0 \rho_1^2 \rho_2 - 12 a_0 \rho_1 \rho_2^2 - 4 a_0 \rho_2^3 + 6 a_0 \rho_1^2 + 12 a_0 \rho_1 \rho_2 + 6 a_0 \rho_2^2 - 4 a_0 \rho_1 - 4 a_0 \rho_2 + a_0$$

4

$$4 a_0 a_1 \rho_1^3 \rho_2 + 6 a_0 a_1 \rho_1^2 \rho_2^2 + 4 a_0 a_1 \rho_1 \rho_2^3 + a_0 a_1 \rho_2^4 + a_0 a_2 \rho_2^4 - a_1 a_2 \rho_2^4 - 4 a_0 a_1 \rho_1^3 - 12 a_0 a_1 \rho_1^2 \rho_2 - 12 a_0 a_1 \rho_1 \rho_2^2 - 4 a_0 a_1 \rho_2^3 + 6 a_0 a_1 \rho_1^2 + 12 a_0 a_1 \rho_1 \rho_2 + 6 a_0 a_1 \rho_2^2 - 4 a_0 a_1 \rho_1 - 4 a_0 a_1 \rho_2 + a_0 a_1$$

5

$$10 a_0 a_1 \rho_1^2 \rho_2^3 + 10 a_0 a_1 \rho_1 \rho_2^4 + 3 a_0 a_1 \rho_2^5 - 2 a_0 a_2 \rho_1 \rho_2^4 - 5 a_0 a_2 \rho_2^5 + 2 a_1 a_2 \rho_1 \rho_2^4 - 3 a_1 a_2 \rho_2^5 - 30 a_0 a_1 \rho_1^2 \rho_2^2 - 40 a_0 a_1 \rho_1 \rho_2^3 - 15 a_0 a_1 \rho_2^4 + 5 a_0 a_2 \rho_2^4 + 3 a_1 a_2 \rho_2^4 + 30 a_0 a_1 \rho_1^2 \rho_2 + 60 a_0 a_1 \rho_1 \rho_2^2 + 30 a_0 a_1 \rho_2^3 - 10 a_0 a_1 \rho_1^2 - 40 a_0 a_1 \rho_1 \rho_2 - 30 a_0 a_1 \rho_2^2 + 10 a_0 a_1 \rho_1 + 15 a_0 a_1 \rho_2 - 3 a_0 a_1$$

6

$$2 a_0 a_1 \rho_1 \rho_2^5 + a_0 a_1 \rho_2^6 + 2 a_0 a_2 \rho_1^2 \rho_2^4 + 6 a_0 a_2 \rho_1 \rho_2^5 + 5 a_0 a_2 \rho_2^6 - 2 a_1 a_2 \rho_1^2 \rho_2^4 + 2 a_1 a_2 \rho_1 \rho_2^5 - a_1 a_2 \rho_2^6 - 10 a_0 a_1 \rho_1 \rho_2^4 - 6 a_0 a_1 \rho_2^5 - 6 a_0 a_2 \rho_1 \rho_2^4 - 10 a_0 a_2 \rho_2^5 - 2 a_1 a_2 \rho_1 \rho_2^4 + 2 a_1 a_2 \rho_2^5 + 20 a_0 a_1 \rho_1 \rho_2^3 + 15 a_0 a_1 \rho_2^4 + 5 a_0 a_2 \rho_2^4 - a_1 a_2 \rho_2^4 - 20 a_0 a_1 \rho_1 \rho_2^2 - 20 a_0 a_1 \rho_2^3 + 10 a_0 a_1 \rho_1 \rho_2 + 15 a_0 a_1 \rho_2^2 - 2 a_0 a_1 \rho_1 - 6 a_0 a_1 \rho_2 + a_0 a_1$$

7

$$a_0 a_1 \rho_2^7 - 20 a_0 a_2 \rho_1^3 \rho_2^4 - 70 a_0 a_2 \rho_1^2 \rho_2^5 - 84 a_0 a_2 \rho_1 \rho_2^6 - 35 a_0 a_2 \rho_2^7 + 20 a_1 a_2 \rho_1^3 \rho_2^4 - 10 a_1 a_2 \rho_1^2 \rho_2^5 + 4 a_1 a_2 \rho_1 \rho_2^6 - a_1 a_2 \rho_2^7 - 7 a_0 a_1 \rho_2^6 + 70 a_0 a_2 \rho_1^2 \rho_2^4 + 168 a_0 a_2 \rho_1 \rho_2^5 + 105 a_0 a_2 \rho_2^6 + 10 a_1 a_2 \rho_1^2 \rho_2^4 - 8 a_1 a_2 \rho_1 \rho_2^5 + 3 a_1 a_2 \rho_2^6 + 21 a_0 a_1 \rho_2^5 - 84 a_0 a_2 \rho_1 \rho_2^4 - 105 a_0 a_2 \rho_2^5 + 4 a_1 a_2 \rho_1 \rho_2^4 - 3 a_1 a_2 \rho_2^5 - 35 a_0 a_1 \rho_2^4 + 35 a_0 a_2 \rho_2^4 + a_1 a_2 \rho_2^4 + 35 a_0 a_1 \rho_2^3 - 21 a_0 a_1 \rho_2^2 + 7 a_0 a_1 \rho_2 - a_0 a_1$$

"=====

> ## Comparison with the announced result

```
Gp:=map(normal,
  Basis([add(rho[i],i=0..2)-1,
    seq(a[(i+1) mod 3]*rho[(i+1) mod 3]^4-a[i]*rho[i]
    ^4,i=0..2)],tdeg(Variables))):
Affiche(Gp,2000);
```

1

$$\rho_0 + \rho_1 + \rho_2 - 1$$

2

$$a_1 \rho_1^4 - a_2 \rho_2^4$$

3

$$\begin{aligned} & a_0 \rho_1^4 + 4 a_0 \rho_1^3 \rho_2 + 6 a_0 \rho_1^2 \rho_2^2 + 4 a_0 \rho_1 \rho_2^3 + a_0 \rho_2^4 - a_2 \rho_2^4 - 4 a_0 \rho_1^3 - 12 a_0 \\ & \rho_1^2 \rho_2 - 12 a_0 \rho_1 \rho_2^2 - 4 a_0 \rho_2^3 + 6 a_0 \rho_1^2 + 12 a_0 \rho_1 \rho_2 + 6 a_0 \rho_2^2 - 4 a_0 \rho_1 \\ & - 4 a_0 \rho_2 + a_0 \end{aligned}$$

4

$$\begin{aligned} & 4 a_0 a_1 \rho_1^3 \rho_2 + 6 a_0 a_1 \rho_1^2 \rho_2^2 + 4 a_0 a_1 \rho_1 \rho_2^3 + a_0 a_1 \rho_2^4 + a_0 a_2 \rho_2^4 - a_1 a_2 \rho_2^4 \\ & - 4 a_0 a_1 \rho_1^3 - 12 a_0 a_1 \rho_1^2 \rho_2 - 12 a_0 a_1 \rho_1 \rho_2^2 - 4 a_0 a_1 \rho_2^3 + 6 a_0 a_1 \rho_1^2 \\ & + 12 a_0 a_1 \rho_1 \rho_2 + 6 a_0 a_1 \rho_2^2 - 4 a_0 a_1 \rho_1 - 4 a_0 a_1 \rho_2 + a_0 a_1 \end{aligned}$$

5

$$\begin{aligned} & 10 a_0 a_1 \rho_1^2 \rho_2^3 + 10 a_0 a_1 \rho_1 \rho_2^4 + 3 a_0 a_1 \rho_2^5 - 2 a_0 a_2 \rho_1 \rho_2^4 - 5 a_0 a_2 \rho_2^5 \\ & + 2 a_1 a_2 \rho_1 \rho_2^4 - 3 a_1 a_2 \rho_2^5 - 30 a_0 a_1 \rho_1^2 \rho_2^2 - 40 a_0 a_1 \rho_1 \rho_2^3 - 15 a_0 a_1 \rho_2^4 \\ & + 5 a_0 a_2 \rho_2^4 + 3 a_1 a_2 \rho_2^4 + 30 a_0 a_1 \rho_1^2 \rho_2 + 60 a_0 a_1 \rho_1 \rho_2^2 + 30 a_0 a_1 \rho_2^3 \\ & - 10 a_0 a_1 \rho_1^2 - 40 a_0 a_1 \rho_1 \rho_2 - 30 a_0 a_1 \rho_2^2 + 10 a_0 a_1 \rho_1 + 15 a_0 a_1 \rho_2 \\ & - 3 a_0 a_1 \end{aligned}$$

6

$$\begin{aligned} & 2 a_0 a_1 \rho_1 \rho_2^5 + a_0 a_1 \rho_2^6 + 2 a_0 a_2 \rho_1^2 \rho_2^4 + 6 a_0 a_2 \rho_1 \rho_2^5 + 5 a_0 a_2 \rho_2^6 - 2 a_1 a_2 \rho_1^2 \rho_2^4 \\ & + 2 a_1 a_2 \rho_1 \rho_2^5 - a_1 a_2 \rho_2^6 - 10 a_0 a_1 \rho_1 \rho_2^4 - 6 a_0 a_1 \rho_2^5 - 6 a_0 a_2 \rho_1 \rho_2^4 \\ & - 10 a_0 a_2 \rho_2^5 - 2 a_1 a_2 \rho_1 \rho_2^4 + 2 a_1 a_2 \rho_2^5 + 20 a_0 a_1 \rho_1 \rho_2^3 + 15 a_0 a_1 \rho_2^4 \\ & + 5 a_0 a_2 \rho_2^4 - a_1 a_2 \rho_2^4 - 20 a_0 a_1 \rho_1 \rho_2^2 - 20 a_0 a_1 \rho_2^3 + 10 a_0 a_1 \rho_1 \rho_2 \\ & + 15 a_0 a_1 \rho_2^2 - 2 a_0 a_1 \rho_1 - 6 a_0 a_1 \rho_2 + a_0 a_1 \end{aligned}$$

7

$$a_0 a_1 \rho_2^7 - 20 a_0 a_2 \rho_1^3 \rho_2^4 - 70 a_0 a_2 \rho_1^2 \rho_2^5 - 84 a_0 a_2 \rho_1 \rho_2^6 - 35 a_0 a_2 \rho_2^7$$

$$\begin{aligned}
&+ 20 a_1 a_2 \rho_1^3 \rho_2^4 - 10 a_1 a_2 \rho_1^2 \rho_2^5 + 4 a_1 a_2 \rho_1 \rho_2^6 - a_1 a_2 \rho_2^7 - 7 a_0 a_1 \rho_2^6 \\
&+ 70 a_0 a_2 \rho_1^2 \rho_2^4 + 168 a_0 a_2 \rho_1 \rho_2^5 + 105 a_0 a_2 \rho_2^6 + 10 a_1 a_2 \rho_1^2 \rho_2^4 \\
&- 8 a_1 a_2 \rho_1 \rho_2^5 + 3 a_1 a_2 \rho_2^6 + 21 a_0 a_1 \rho_2^5 - 84 a_0 a_2 \rho_1 \rho_2^4 - 105 a_0 a_2 \rho_2^5 \\
&+ 4 a_1 a_2 \rho_1 \rho_2^4 - 3 a_1 a_2 \rho_2^5 - 35 a_0 a_1 \rho_2^4 + 35 a_0 a_2 \rho_2^4 + a_1 a_2 \rho_2^4 \\
&+ 35 a_0 a_1 \rho_2^3 - 21 a_0 a_1 \rho_2^2 + 7 a_0 a_1 \rho_2 - a_0 a_1
\end{aligned}$$

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(1.9.1.6.4)

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