A rectangular area with a light-colored, marbled or stone-like texture. The text is centered within this area.

# Tri par pile combinant des symétries : quelques cas d'énumération

*Journées de combinatoire de Bordeaux 2013*

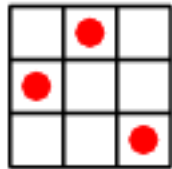
Olivier Guibert (LaBRI)

Travail en collaboration avec Mathilde Bouvel

14 février 2013

12345

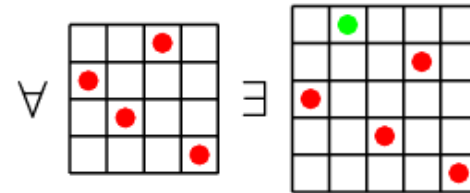
15243



12345

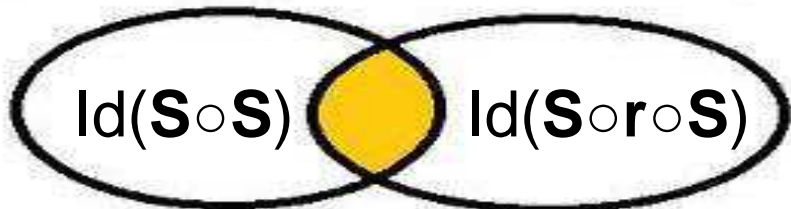
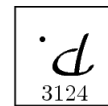
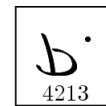
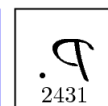
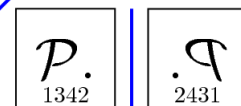
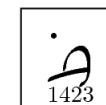
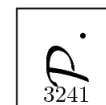
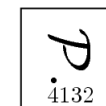
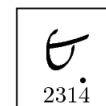
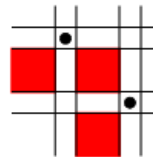
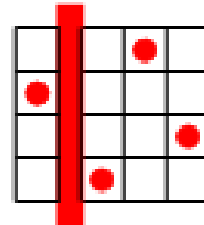
13245

31425



symétrie

12...n



- 
- 
- 
- 
- 

12...n

rc

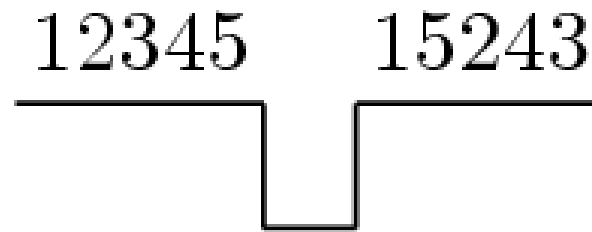
c



Tri par pile

# Tri par pile

D. Knuth  
1973



illustration

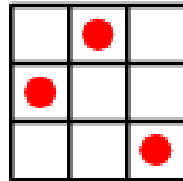


Id(**S**)

- Exemples :  $\varepsilon$ , 1, 12, 21, 123, 132, 213, 312, 321, 15243
- Contre-exemple : 231

# Tri par pile

- $\text{Id}(\mathbf{S}) = \text{Av}(2-3-1)$



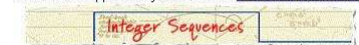
- Exemples : 1, 12, 21, 123, 132, 213, 312, 321, 15243
- Contre-exemples : 231 (**231**), 21637458 (**674**, **675**)

- Énumération

- [A000108](#) : 1, 2, 5, 14, 42, 132, ...

- $$\frac{(2n)!}{(n+1)!n!}$$

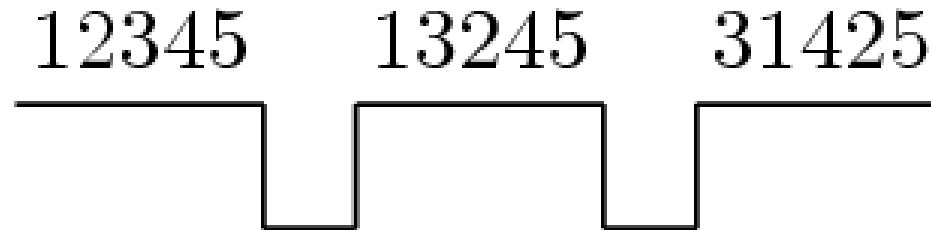
This site is supported by donations to The OEIS Foundation.



Permutations triables par deux  
passages dans la pile  
(c.-à-d. permutations 2-triables)

Deux décennies plus tard...

# Permutations 2-triangles



- 1992 : D. Zeilberger

1996 :

I. Goulden, J. West

S. Dulucq, S. Gire, O. Guibert, J. West

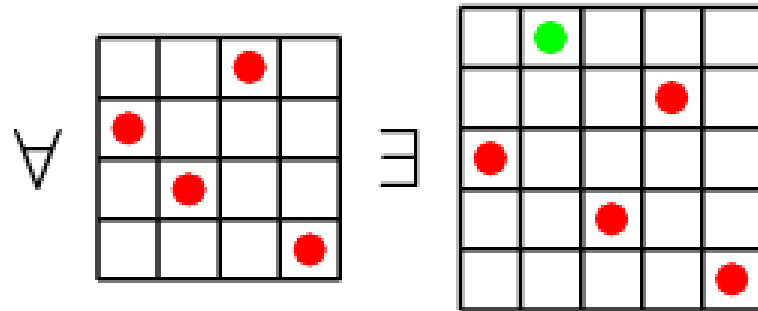
- $\text{Id}(\mathbf{S} \circ \mathbf{S})$

– Exemples : 15243, 231, 31425, 35241

– Contre-exemples : 2341, 3241

# Permutations 2-triangles

- $\text{Id}(\mathbf{S} \circ \mathbf{S}) = \text{Av}(2-3-4-1, 3-5-2-4-1)$  [J. West]



- Énumération

– [A000139](#) : 1, 2, 6, 22, 91, 408, ...

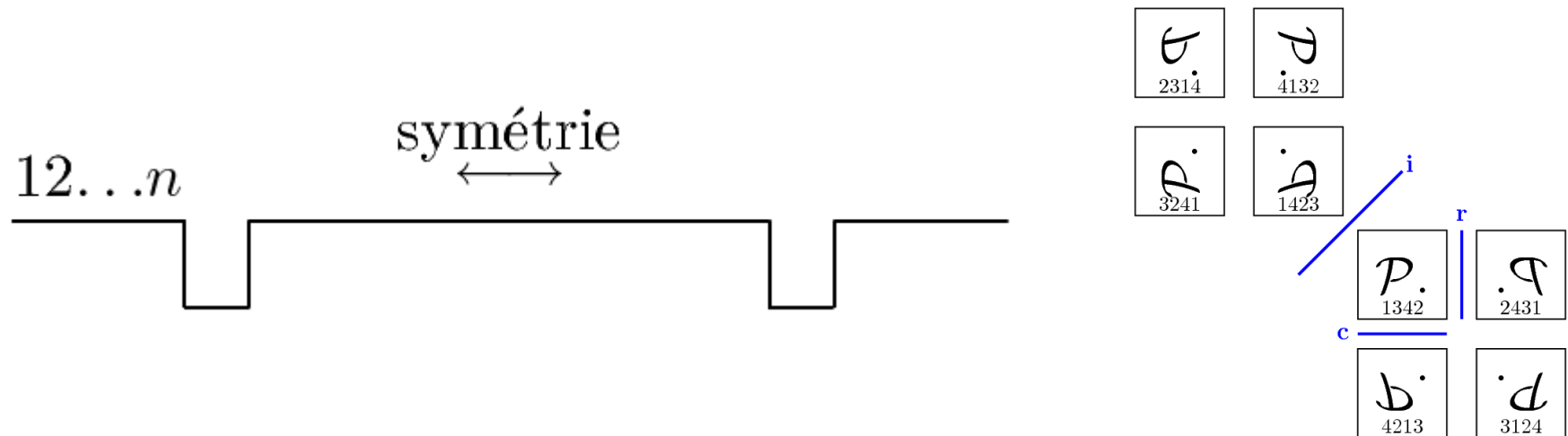
– 
$$\frac{2(3n)!}{(2n+1)!(n+1)!}$$



# Permutations 2-triangles combinant une symétrie

Deux autres décennies après : aujourd'hui !

# Permutations 2-triangles combinant une symétrie

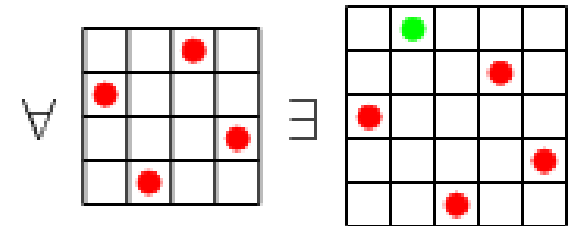
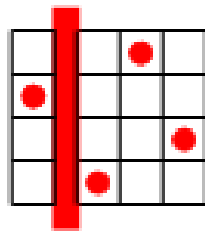


- $\text{Id}(\mathbf{S} \circ \{id, r, c, i, r \circ c, i \circ r, i \circ c, i \circ c \circ r\} \circ \mathbf{S})$
- Conjectures 2010 : M. Albert, M. Atkinson, M. Bouvel, A. Claesson, M. Dukes

# 2-triangles combinant une symétrie

- $\text{Id}(\mathbf{S} \circ \mathbf{c} \circ \mathbf{S}) = \text{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{r} \circ \mathbf{S}) = \text{Av}(2-3-1) \quad \frac{(2n)!}{(n+1)!n!}$

- $\text{Id}(\mathbf{S} \circ \mathbf{S}) = \text{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{c} \circ \mathbf{r} \circ \mathbf{S}) = \text{Av}(2-3-4-1, 3-5-2-4-1)$   
 $\text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}) = \text{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{c} \circ \mathbf{S})$   
 $= \text{Av}(1-3-4-2, \underline{31}-4-2) = \text{Av}(1-3-4-2, 3-5-1-4-2)$



A000139 : 1, 2, 6, 22, 91, 408, ...

$$\frac{2(3n)!}{(2n+1)!(n+1)!}$$

- $\text{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{S}) = \text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{c} \circ \mathbf{S}) = \text{Av}(3-4-1-2, 3-4-\underline{21})$

A001181 : 1, 2, 6, 22, 92, 422, ...

$$\frac{\sum_{m=1}^n \binom{n+1}{m-1} \binom{n+1}{m} \binom{n+1}{m+1}}{\binom{n+1}{1} \binom{n+1}{2}}$$

$$\text{Id}(\mathbf{S} \circ \mathbf{S}) \leftrightarrow \text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$$

$$\text{Av}(3-2-1-4, \cancel{2}-4-1-3-5) = \mathbf{r} \circ \mathbf{i} \circ \text{Id}(\mathbf{S} \circ \mathbf{S})$$



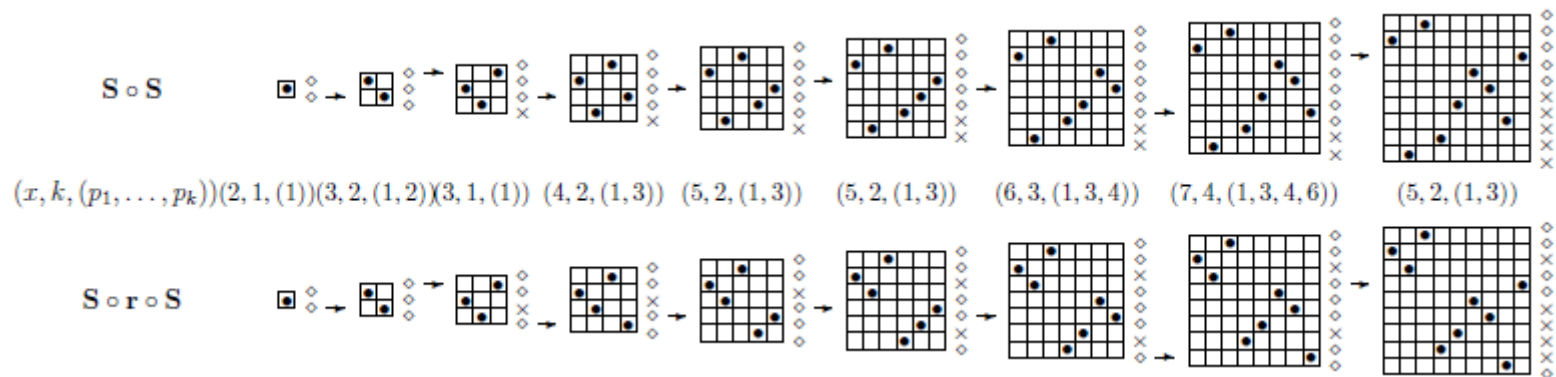
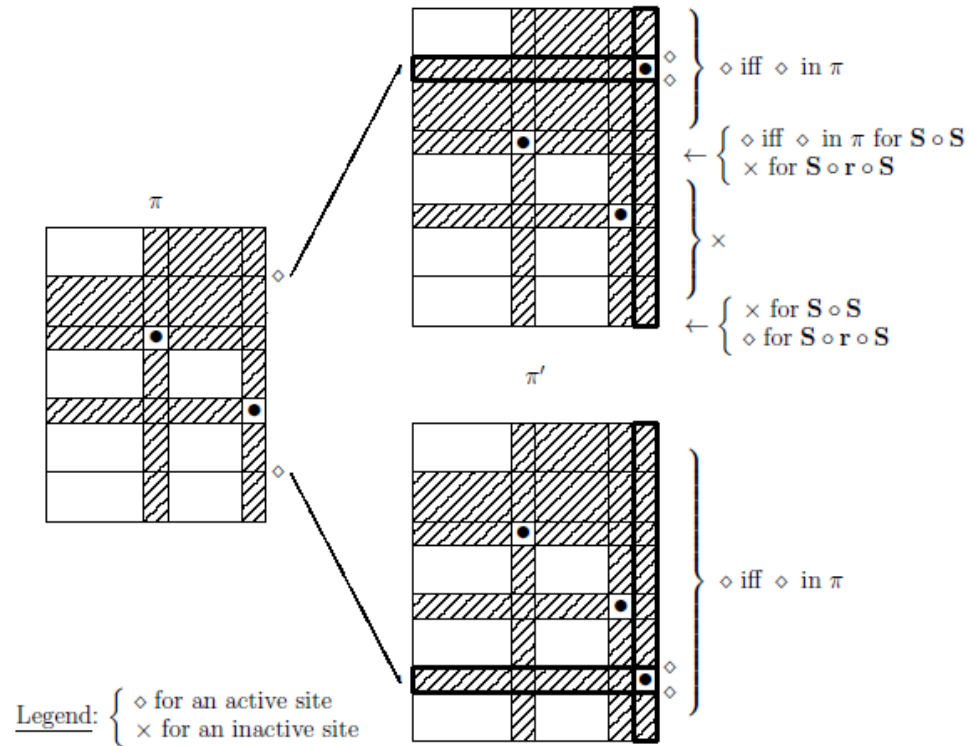
(1996)

$$\left\{ \begin{array}{l} (2, 1, (1)) \\ (x, k, (p_1, \dots, p_k)) \end{array} \right. \rightsquigarrow \begin{array}{l} (2 + p_j, j, (p_1, \dots, p_{j-1}, i)) \\ \text{for } 1 \leq j \leq k \text{ and } p_{j-1} < i \leq p_j \\ (x + 1, k + 1, (p_1, \dots, p_k, i)) \\ \text{for } p_k < i \leq x \end{array}$$

$$\text{Av}(3-2-4-1, \cancel{2}-4-1-5-3) = \mathbf{r} \circ \mathbf{i} \circ \text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$$

Équidistribution : (taille,) ensemble des descentes, nombre de saillants supérieurs gauche et droit, statistique de Zeilberger, position du plus grand élément, position du deuxième saillant supérieur gauche dans la permutation et dans son miroir

$$\text{Id}(\mathbf{S} \circ \mathbf{S}) \leftrightarrow \text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$$



# Id( $\mathbf{S} \circ \mathbf{i} \circ \mathbf{S}$ ) $\leftrightarrow$ Permutations de Baxter

Id( $\mathbf{S} \circ \mathbf{i} \circ \mathbf{S}$ )

$n, lmax, des, comp$

$$\begin{cases} (2, 0) \\ (q, r) \end{cases} \rightsquigarrow \begin{cases} (i+1, q+r-i) \text{ for } 1 \leq i \leq q \\ (q, r-j) \text{ for } 1 \leq j \leq r \end{cases}$$

Baxter tordues = Av(2-41-3, 3-41-2)

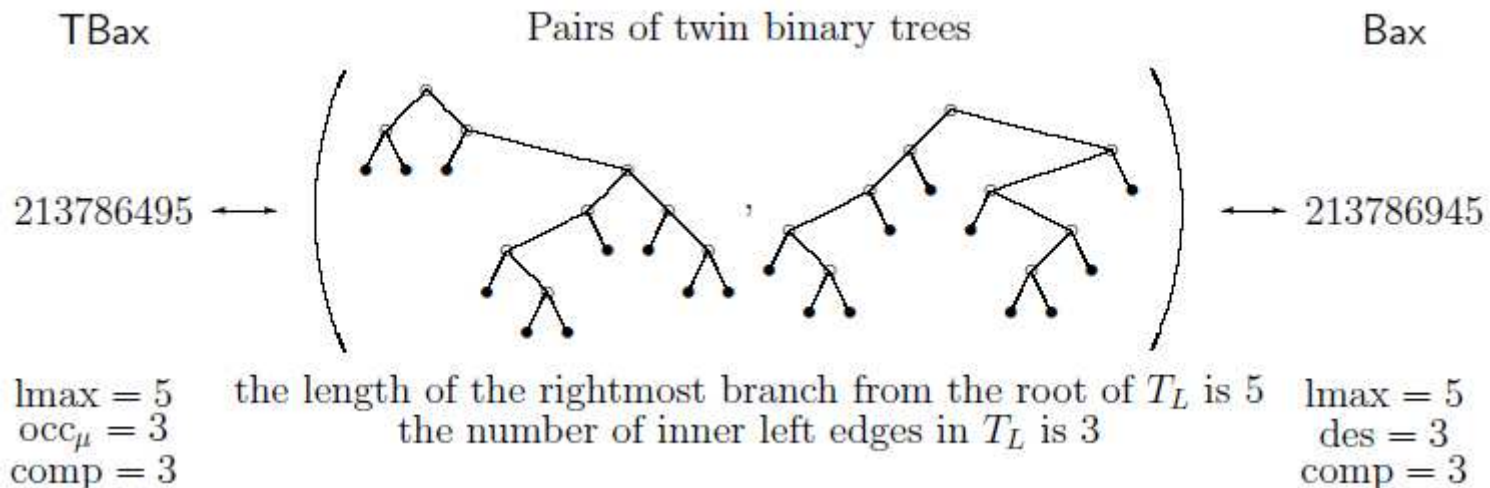
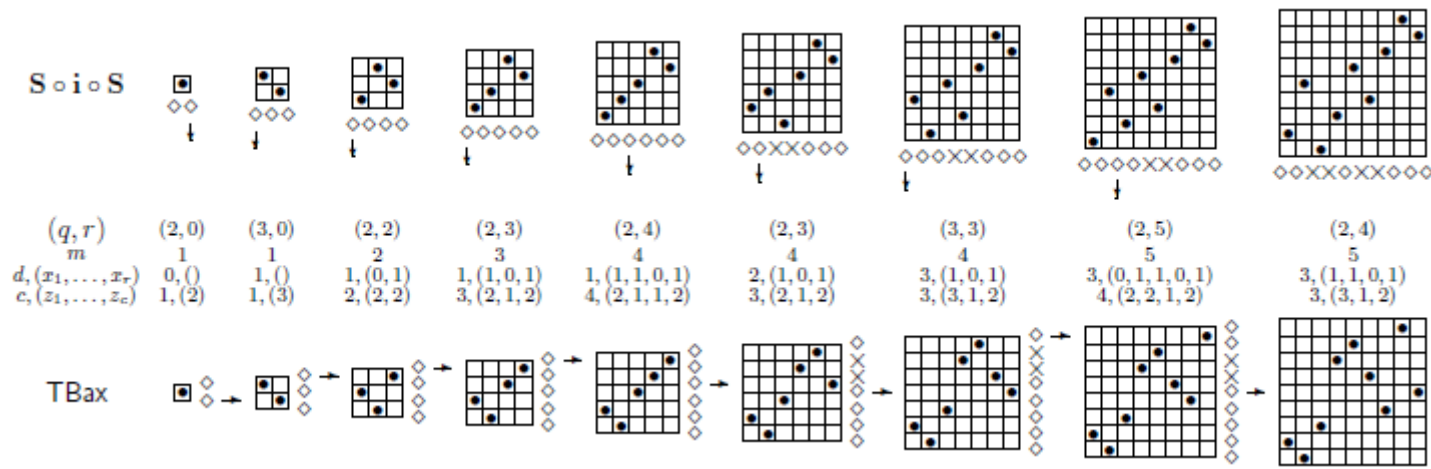
$n, max, \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \blacksquare & \blacksquare & \\ \hline & & \bullet \\ \hline \end{array}, comp$

[S. Giraudo (2012) via arbres jumeaux]

Baxter = Av(2-41-3, 3-14-2)  $n, lmax, des, comp$

Légende :  $n$  = taille,  $lmax$  = nombre de saillants supérieurs gauche,  $des$  = nombre de descentes,  $comp$  = nombre de composantes

# Id( $\mathbf{S} \circ \mathbf{i} \circ \mathbf{S}$ ) $\leftrightarrow$ Permutations de Baxter



$$\text{Id}(\mathbf{S} \circ \mathbf{S}) \cap \text{Id}(\mathbf{S} \circ r \circ \mathbf{S})$$

(intersection des deux ensembles en bijection)



# $\text{Id}(\mathbf{S} \circ \mathbf{S}) \cap \text{Id}(\mathbf{S} \circ r \circ \mathbf{S})$

## $\text{Id}(\mathbf{S} \circ \mathbf{S}) \cap \text{Id}(\mathbf{S} \circ r \circ \mathbf{S})$

↕ taille  $n$ , nombre de vallées à **droite** et **gauche** de 1,  
nombre de doubles **montées** et **descentes**

$$\mathcal{L} = \epsilon + \begin{array}{c} \mathcal{L} \\ / \cdot \cdot \backslash \\ \cdot \end{array} \mathcal{L} + \begin{array}{c} \mathcal{L} \\ \cdot \cdot / \backslash \\ \cdot \end{array} \mathcal{L} + \text{---} \mathcal{L} + \text{---} \mathcal{L}$$

longueur  $n-1$ , nombre de  $\begin{array}{c} / \\ \cdot \end{array}$ ,  $\begin{array}{c} \backslash \\ \cdot \end{array}$ ,  $\text{---}$  et  $\text{---}$

## Énumération

– 1, 2, 6, 20, 72, 272, 1064, ... cf. [A071356](#)

$$- \sum_{b,r,m,v} \binom{n-1}{2(b+r), m+v} \text{Catalan}_{b+r} \binom{b+r}{b, r} \binom{m+v}{m, v}$$

# $\text{Id}(\mathbf{S} \circ \mathbf{S}) \cap \text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$

$\text{Id}(\mathbf{S} \circ \mathbf{S}) \cap \text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$

taille  $n$ , nombre de vallées à droite et gauche de 1, nombre de doubles montées et descentes

$\mathbf{c} \circ \mathbf{r} \circ (\text{Id}(\mathbf{S} \circ \mathbf{S}) \cap \text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}))$

taille  $n$ , nombre de pics à gauche et droite de  $n$ , nombre de doubles montées et descentes

Histoire de Laguerre [J. Françon, X. G. Viennot]

longueur  $n-1$ , nombre de  $\begin{array}{c} \diagdown \\ [1] \end{array}$ ,  $\begin{array}{c} \diagdown \\ h\max \end{array}$ ,  $\begin{array}{c} \text{---} \\ [1] \end{array}$  et  $\begin{array}{c} \text{---} \\ [1] \end{array}$

$$\mathcal{L} = \epsilon + \begin{array}{c} \mathcal{L} \\ / \cdot \end{array} \begin{array}{c} \cdot \\ \backslash \end{array} \mathcal{L} + \begin{array}{c} \mathcal{L} \\ / \cdot \end{array} \begin{array}{c} \cdot \\ \backslash \end{array} \mathcal{L} + \begin{array}{c} \text{---} \\ \cdot \end{array} \mathcal{L} + \begin{array}{c} \text{---} \\ \cdot \end{array} \mathcal{L}$$

longueur  $n-1$ , nombre de  $\begin{array}{c} / \cdot \\ \cdot \end{array}$ ,  $\begin{array}{c} / \cdot \\ \cdot \end{array}$ ,  $\begin{array}{c} \text{---} \\ \cdot \end{array}$  et  $\begin{array}{c} \text{---} \\ \cdot \end{array}$

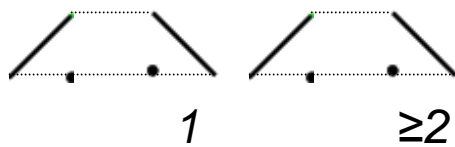
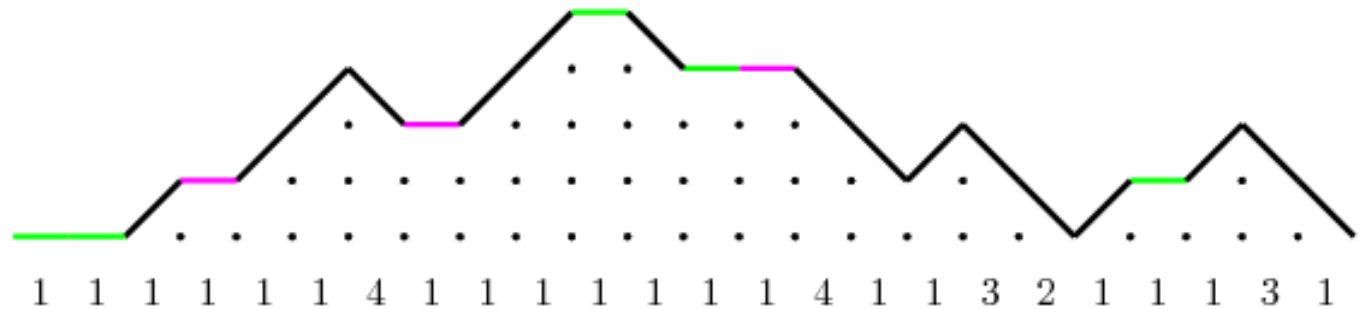
$$\text{Id}(\mathbf{S} \circ \mathbf{S}) \cap \text{Id}(\mathbf{S} \circ r \circ \mathbf{S})$$

25 24 19 23 11 21 8 20 7 9 3 6 5 1 4 2 17 13 10 12 16 15 14 18 22

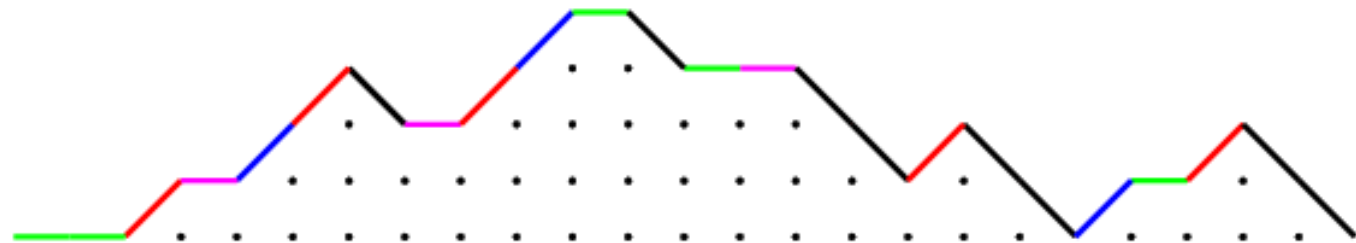
↕ rc

4 8 12 11 10 14 16 13 9 24 22 25 21 20 23 17 19 6 18 5 15 3 7 2 1

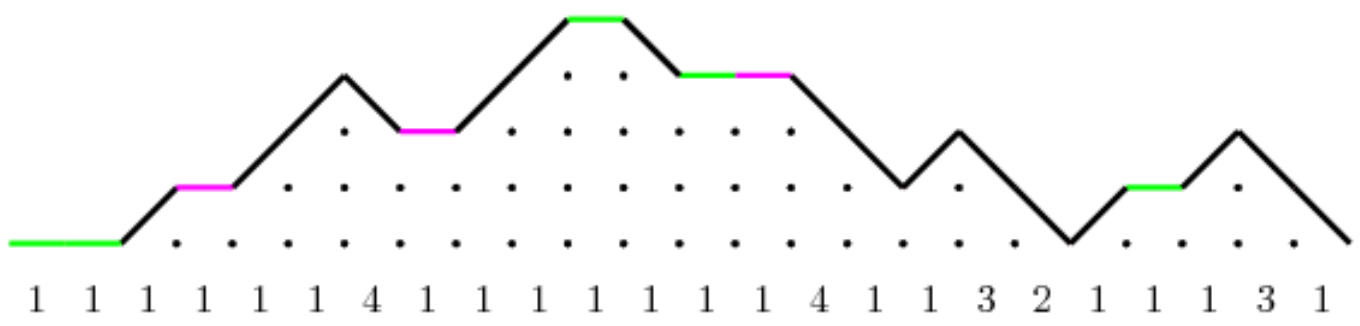
↕



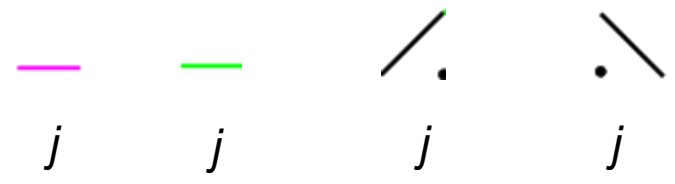
↕



*i*-décomposition



*i*<sup>e</sup> pas :



$\sqcup$  : *i*  $\sqcup$      $\sqcup$  *i*     $\sqcup$  *i*  $\sqcup$     *i* si *j*=1  
*j*<sup>e</sup>    (*j*=1)    (*j*=1)    (*j*=1)    *i* si *j*>1

- $\sqcup$  1
- $\sqcup$  2 1
- $\sqcup$  3  $\sqcup$  2 1
- 4  $\sqcup$  3  $\sqcup$  2 1
- 4  $\sqcup$  5  $\sqcup$  3  $\sqcup$  2 1
- 4  $\sqcup$  6  $\sqcup$  5  $\sqcup$  3  $\sqcup$  2 1
- 4 8  $\sqcup$  6  $\sqcup$  5  $\sqcup$  3 7 2 1
- 4 8  $\sqcup$  9  $\sqcup$  6  $\sqcup$  5  $\sqcup$  3 7 2 1
- 4 8  $\sqcup$  10  $\sqcup$  9  $\sqcup$  6  $\sqcup$  5  $\sqcup$  3 7 2 1
- 4 8  $\sqcup$  11 10  $\sqcup$  9  $\sqcup$  6  $\sqcup$  5  $\sqcup$  3 7 2 1
- 4 8 12 11 10  $\sqcup$  13 9  $\sqcup$  6  $\sqcup$  5  $\sqcup$  3 7 2 1
- 4 8 12 11 10 14  $\sqcup$  13 9  $\sqcup$  6  $\sqcup$  5  $\sqcup$  3 7 2 1
- 4 8 12 11 10 14  $\sqcup$  13 9  $\sqcup$  6  $\sqcup$  5 15 3 7 2 1
- 4 8 12 11 10 14 16 13 9  $\sqcup$  6  $\sqcup$  5 15 3 7 2 1
- 4 8 12 11 10 14 16 13 9  $\sqcup$  17  $\sqcup$  6  $\sqcup$  5 15 3 7 2 1
- 4 8 12 11 10 14 16 13 9  $\sqcup$  17  $\sqcup$  6 18 5 15 3 7 2 1
- 4 8 12 11 10 14 16 13 9  $\sqcup$  17 19 6 18 5 15 3 7 2 1
- 4 8 12 11 10 14 16 13 9  $\sqcup$  20  $\sqcup$  17 19 6 18 5 15 3 7 2 1
- 4 8 12 11 10 14 16 13 9  $\sqcup$  21 20  $\sqcup$  17 19 6 18 5 15 3 7 2 1
- 4 8 12 11 10 14 16 13 9  $\sqcup$  22  $\sqcup$  21 20  $\sqcup$  17 19 6 18 5 15 3 7 2 1
- 4 8 12 11 10 14 16 13 9  $\sqcup$  22  $\sqcup$  21 20 23 17 19 6 18 5 15 3 7 2 1
- 4 8 12 11 10 14 16 13 9 24 22  $\sqcup$  21 20 23 17 19 6 18 5 15 3 7 2 1

4 8 12 11 10 14 16 13 9 24 22 25 21 20 23 17 19 6 18 5 15 3 7 2 1

$\text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{c} \circ \mathbf{S} \circ \mathbf{c} \circ \mathbf{S})$

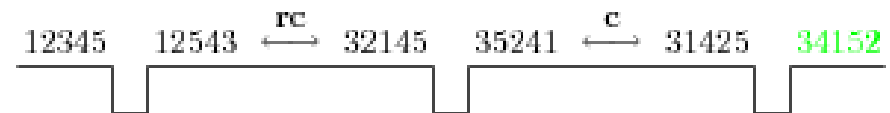
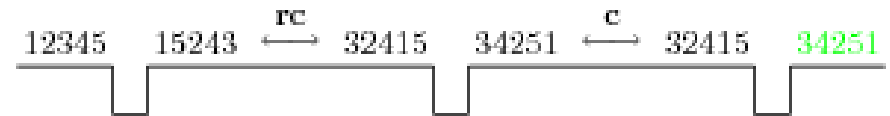
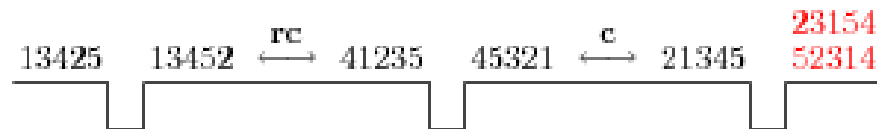
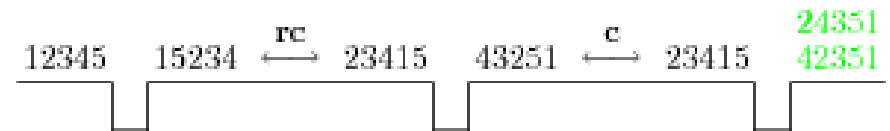
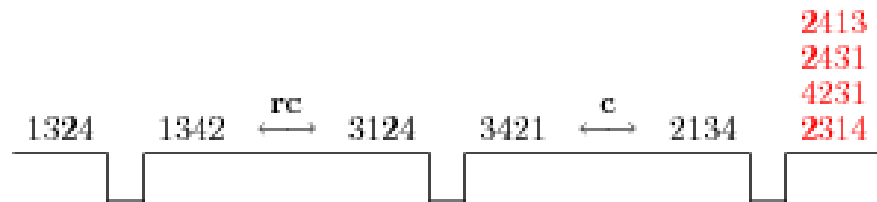
$$\text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{c} \circ \mathbf{S} \circ \mathbf{c} \circ \mathbf{S})$$

$$= \text{Av}(2-4-1-3,$$

$$2-4-\underline{31} = 2-4-3-\cancel{5}-1,$$

$$4-2-\underline{31} = 4-2-3-\cancel{5}-1,$$

$$2-3-\underline{141} \text{ c.-à-d. } \forall 2-3-\underline{14} \exists e / 3-4-\underline{251} \vee 3-4-\underline{152})$$



1, 2, 6, 20, 72, 272, 1064, ... cf. [A071356](#)

# Id(**S** ◦ **r** ◦ **c** ◦ **S** ◦ **c** ◦ **S**)

## Id(**S** ◦ **r** ◦ **c** ◦ **S** ◦ **c** ◦ **S**)

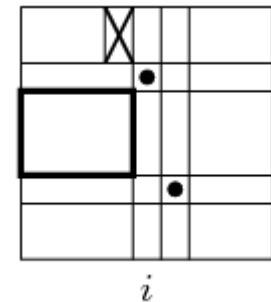
↑ taille  $n$ , nombre de pics  $\pi_i$  dont  $moyen_i$  est  $=0$  et  $>0$ ,  
 nombre de doubles **montées** et **descentes**

↕

$$\mathcal{L} = \epsilon + \begin{array}{c} \mathcal{L} \\ / \cdot \cdot \backslash \\ \cdot \end{array} \mathcal{L} + \begin{array}{c} \mathcal{L} \\ / \cdot \cdot \backslash \\ \cdot \end{array} \mathcal{L} + \text{---} \mathcal{L} + \text{---} \mathcal{L}$$

longueur  $n-1$ , nombre de  $\begin{array}{c} / \cdot \\ \cdot \end{array}$ ,  $\begin{array}{c} / \cdot \\ \cdot \end{array}$ ,  $\text{---}$  et  $\text{---}$

Déf. :  $moyen_i = \#\{ \pi_j / \pi_{i+1} < \pi_j < \pi_i, \forall 1 \leq j < i \}$   
 pour  $\pi \in \text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{c} \circ \mathbf{S} \circ \mathbf{c} \circ \mathbf{S})$   
 et  $\pi_{i-1} < \pi_i > \pi_{i+1}$



Cas particulier : tous  $moyen_i = 0$  correspond à Av(**2-31**)

# Id(**S** ◦ **r** ◦ **c** ◦ **S** ◦ **c** ◦ **S**)

## Id(**S** ◦ **r** ◦ **c** ◦ **S** ◦ **c** ◦ **S**)

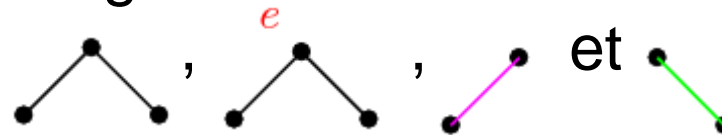


taille  $n$ , nombre de pics  $\pi_i$  dont  $moyen_i$  est  $=0$  et  $>0$ ,  
nombre de doubles **montées** et **descentes**

Arbre binaire dont branche gauche partiellement **étiquetée**



$n \bullet$ , nombre de



Chemin Motzkin bi-coloré tq montée init. partiellement **étiquetée**



longueur  $n-1$ , nombre de  $\diagup$ ,  $\overset{e}{\diagup}$ ,  $\text{—}$  et  $\text{—}$

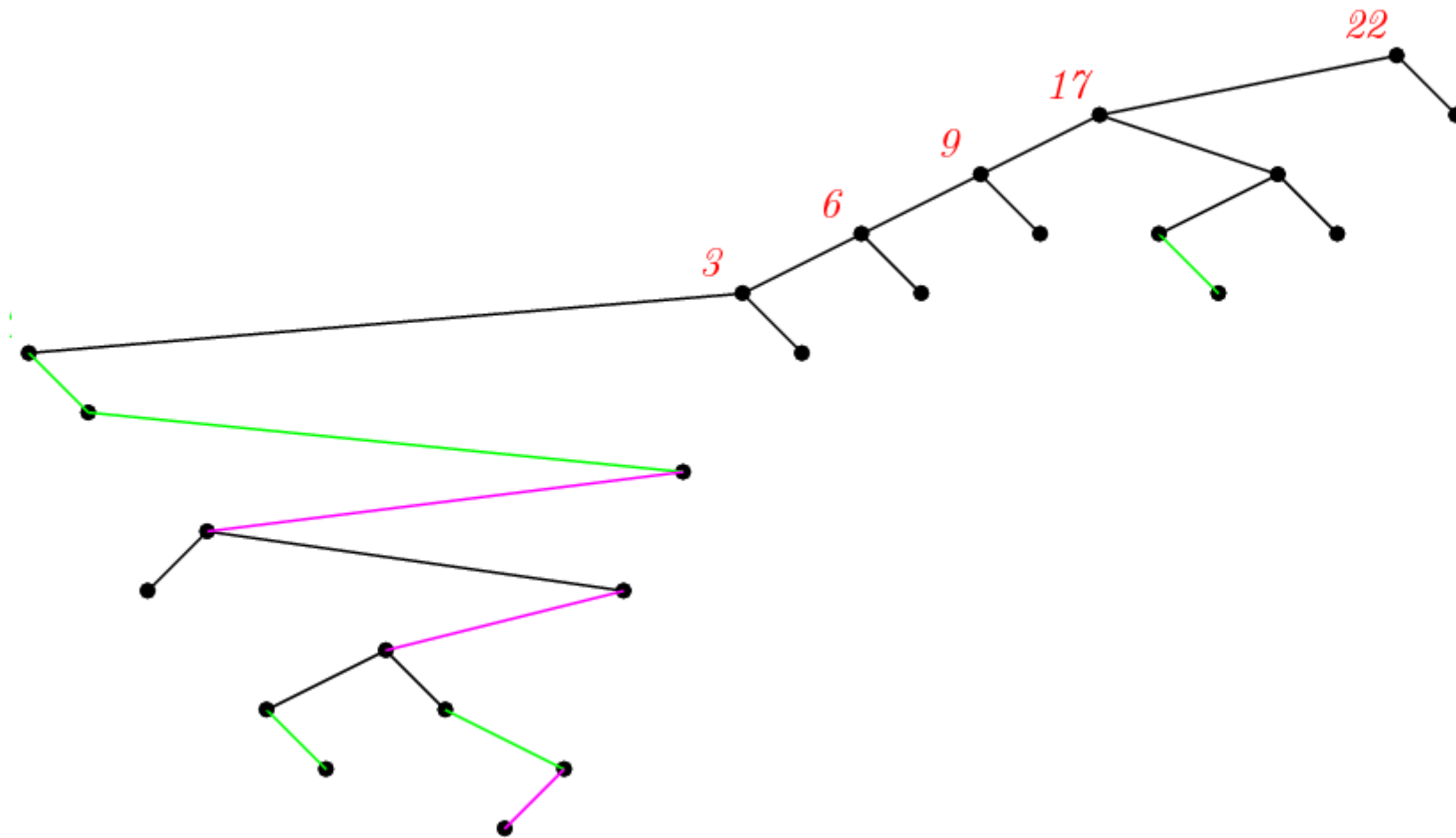
$$\mathcal{L} = \epsilon + \begin{array}{c} \mathcal{L} \\ \diagup \cdot \cdot \diagdown \end{array} \mathcal{L} + \begin{array}{c} \mathcal{L} \\ \overset{e}{\diagup} \cdot \cdot \diagdown \end{array} \mathcal{L} + \text{—} \mathcal{L} + \text{—} \mathcal{L}$$

longueur  $n-1$ , nombre de  $\diagup$ ,  $\overset{e}{\diagup}$ ,  $\text{—}$  et  $\text{—}$

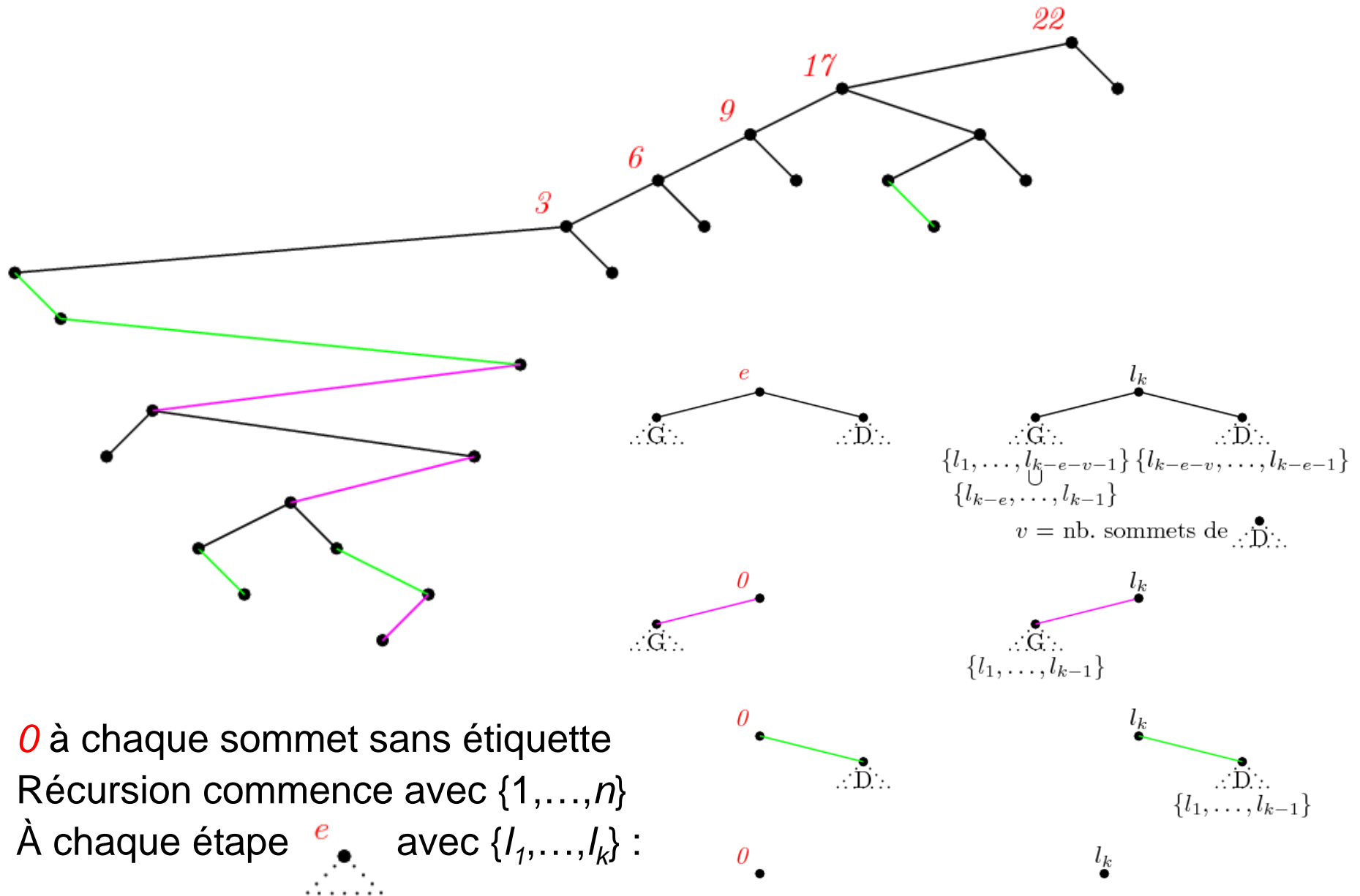


$\text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{c} \circ \mathbf{S} \circ \mathbf{c} \circ \mathbf{S})$

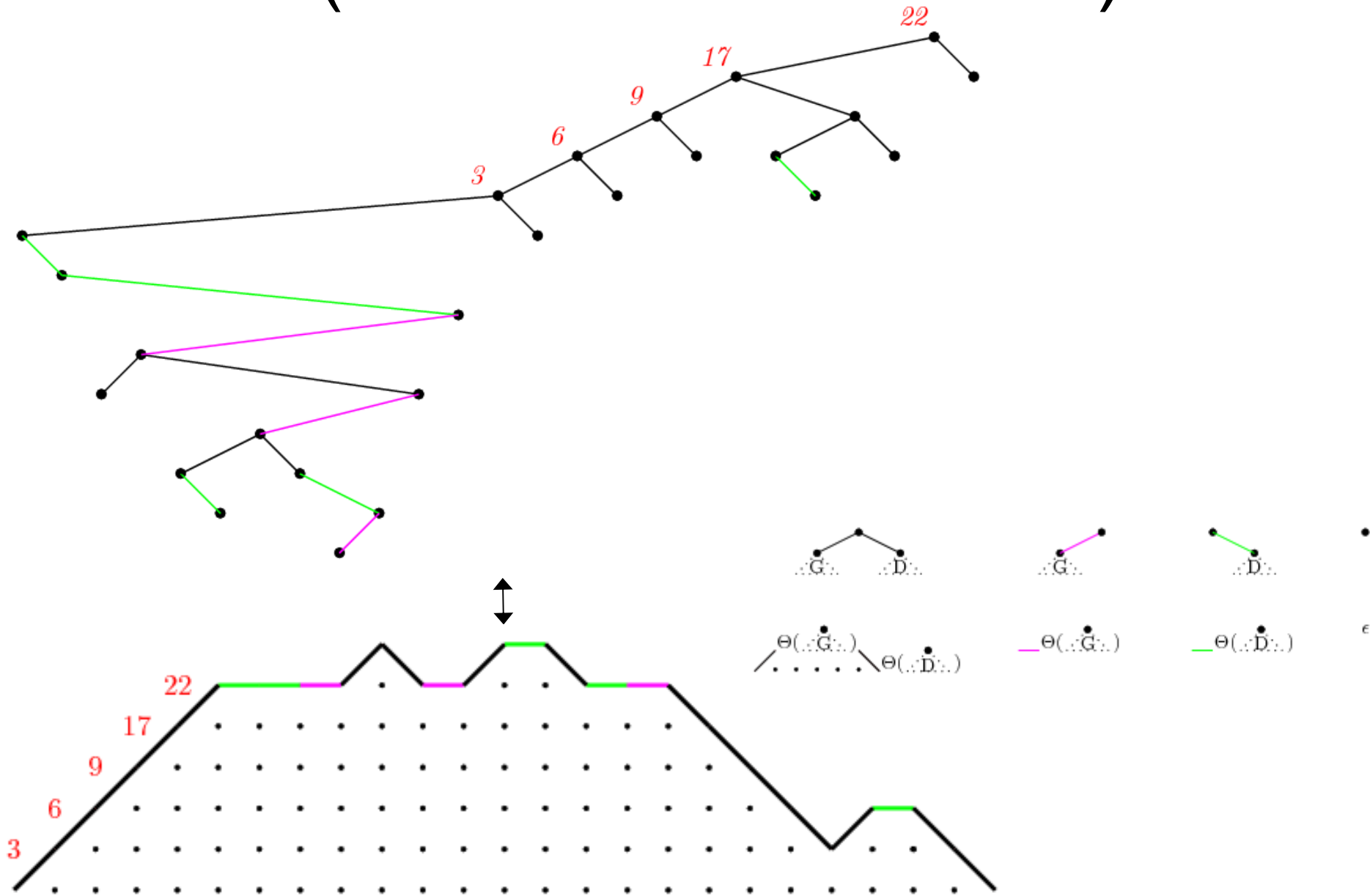
20 19 1 16 8 7 12 11 9 10 14 18 21 17 22 15 23 13 24 4 3 6 5 25 2



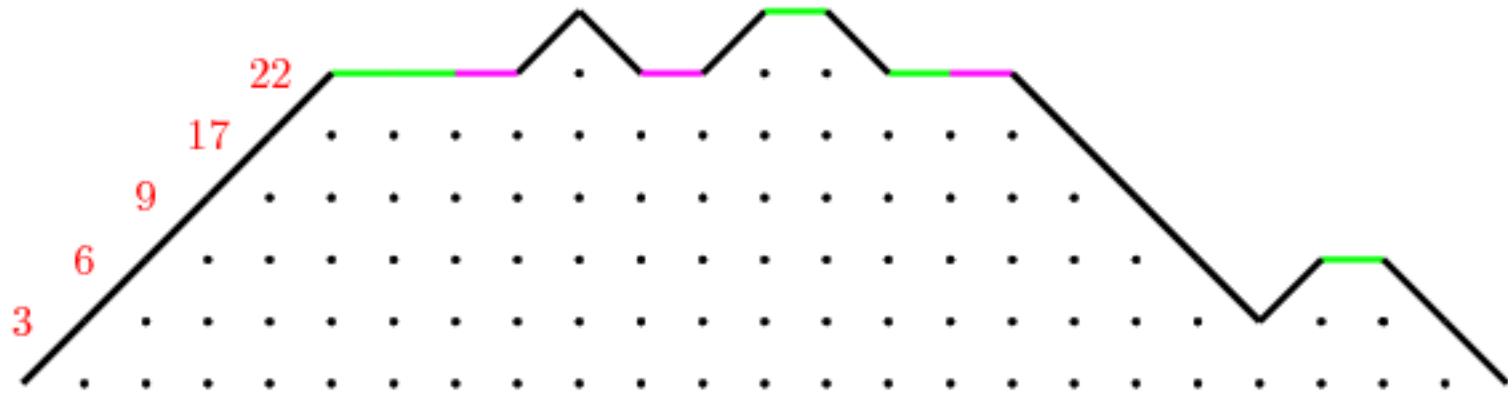
# Id(**S** $\circ$ **r** $\circ$ **c** $\circ$ **S** $\circ$ **c** $\circ$ **S**)



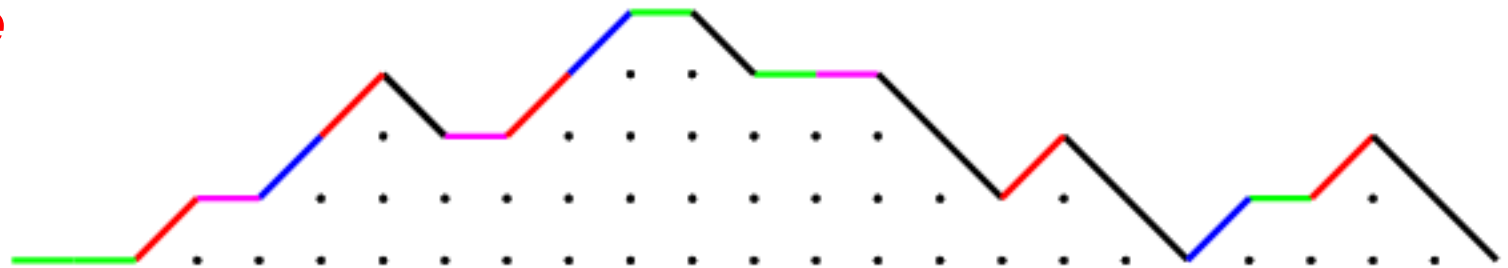
# Id(**S** ◦ **r** ◦ **c** ◦ **S** ◦ **c** ◦ **S**)



$\text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{c} \circ \mathbf{S} \circ \mathbf{c} \circ \mathbf{S})$



$\begin{array}{l} \diagup \cdot \\ \diagdown \cdot \end{array} \quad e$   
 $\begin{array}{l} \diagup \cdot \\ \diagdown \cdot \end{array} \quad \text{en } e$



Permutations  $k$ -triangles  
combinant  $k-1$  symétries

# Résultats expérimentaux : $k=3$

1 2 5 14 42 132 429 1430 4862 : **S**, ScS, ScScS, SciScS, SciS, ScSciS, SciSciS

1 2 6 20 72 272 1064 4272 17504 : **SrcScS**, SiScS

1 2 6 20 75 306 1359 6538 33990 : SrcSciS, SiSciS

1 2 6 22 90 394 1806 8558 41586 : **SScS**, SrciScS

1 2 6 22 91 408 1938 9614 49335 : **SS**, ScSS, SciSS, SrciS, ScSrciS, SrSrciS, SciSrciS, SriSrciS

1 2 6 22 91 408 1938 9614 49335 : **SrS**, ScSrS, SciSrS, SriS, ScSriS, SciSriS

1 2 6 22 92 422 2074 10754 58202 : **SrcS**, ScSrcS, SrSrcS, SciSrcS, SriSrcS, SiS, ScSiS, SciSiS

1 2 6 22 94 446 2292 12516 71720 : **SSciS**, SrciSciS

1 2 6 24 112 580 3244 19248 119687 : **SrSiS**, SriSiS

1 2 6 24 114 606 3494 21426 137901 : **SSS**, SrciSS

1 2 6 24 114 606 3494 21426 137901 : **SrSS**, SrciSS

1 2 6 24 114 608 3538 22040 145020 : SrcSrS, SiSrS

1 2 6 24 114 612 3602 22780 152635 : **SSiS**, SrciSiS

1 2 6 24 114 616 **3675** 23732 163460 : SrcSriS, SiSriS

1 2 6 24 115 618 **3595** 22184 143339 : SSrciS, SrciSrcS

1 2 6 24 115 620 3639 22790 150229 : **SrScS**, SriScS

1 2 6 24 115 624 3716 23786 161286 : **SSrcS**, SrciSrcS

1 2 6 24 115 628 3781 **24544** 169027 : **SSriS**, SrciSriS

1 2 6 24 116 634 3784 **24168** 162971 : SrcSrciS, SiSrciS

1 2 6 24 116 636 3814 24464 165371 : **SSrS**, SrciSrS

1 2 6 24 116 636 3814 24464 165371 : **SrSrS**, SriSrS

1 2 6 24 116 636 3822 24650 168172 : SrcSS, SiSS

1 2 6 24 116 636 3824 24694 168811 : SrcSrcS, SiSrcS

1 2 6 24 116 638 3855 25032 172106 : SrcSiS, SiSiS

1 2 6 24 118 674 4305 29958 223032 : SrciS, SriSciS

1 2 6 24 119 692 4533 32516 250400 : SrSriS, SriSriS

| $k$        | 1 | 2 | 3   | 4   |
|------------|---|---|-----|-----|
| #séquences | 1 | 3 | 23  | 180 |
| #ensembles | 1 | 4 | 26  | 187 |
|            |   | = | =   | =   |
|            |   | 3 | 23  | 180 |
|            |   | 1 | 1   | 1   |
|            |   |   | 2   | 2   |
|            |   |   |     | 4   |
|            | 😊 | 😊 | +1! | 😞   |

# En attendant 2033

Bijection [M. Albert, M. Bouvel] entre

$\text{Id}(\mathbf{S} \circ \mathbf{S} \circ \alpha_2 \circ \mathbf{S} \circ \dots \circ \mathbf{S} \circ \alpha_{k-1} \circ \mathbf{S})$  et

$\text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S} \circ \alpha_2 \circ \mathbf{S} \circ \dots \circ \mathbf{S} \circ \alpha_{k-1} \circ \mathbf{S})$

avec  $\alpha_i \in \{id, \mathbf{r}\} \forall 2 \leq i \leq k-1$

Équidistribution : (taille,) ensemble des descentes,  
positions des saillants supérieurs gauche et droit,  
statistique de Zeilberger

## Permutation Patterns '13

La 11<sup>e</sup> édition se tiendra à l'Université Paris  
Diderot (Paris, France) du 1<sup>er</sup> au 5 juillet 2013

Merci