

Poset-Topological Aspects of Noncrossing Partition Lattices

Henri Mühle

LIAFA (Université Paris Diderot)

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Outline

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology
Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- 1 Introduction
 - Poset Topology
 - Noncrossing Partition Lattices
- 2 Lexicographic Shellability of \mathcal{NC}_W
 - EL-Shellability
 - CL-Shellability
- 3 Lexicographic Shellability and Hurwitz Action
 - Hurwitz Action
- 4 Conclusions

Outline

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology
Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- 1 Introduction
 - Poset Topology
 - Noncrossing Partition Lattices
- 2 Lexicographic Shellability of \mathcal{NC}_W
 - EL-Shellability
 - CL-Shellability
- 3 Lexicographic Shellability and Hurwitz Action
 - Hurwitz Action
- 4 Conclusions

Outline

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- 1 Introduction
 - Poset Topology
 - Noncrossing Partition Lattices
- 2 Lexicographic Shellability of \mathcal{NC}_W
 - EL-Shellability
 - CL-Shellability
- 3 Lexicographic Shellability and Hurwitz Action
 - Hurwitz Action
- 4 Conclusions

Poset Topology

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

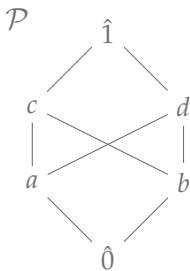
Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

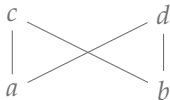
- main task: investigate the (topological) structure of the order complex of a poset
- in particular:
 - determine the homotopy type
 - compute the homology
 - compute bases for the homology
- helpful tools: poset labelings

- bounded poset



- proper part

$\overline{\mathcal{P}}$



Posets

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

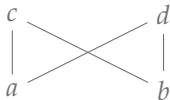
Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

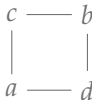
Conclusions

- order complex $\Delta(\mathcal{P})$

$\overline{\mathcal{P}}$

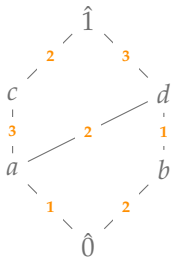


$\Delta(\overline{\mathcal{P}})$



Edge-Labelings

- EL-labeling



Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

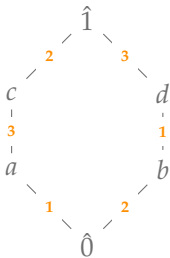
Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

Edge-Labelings

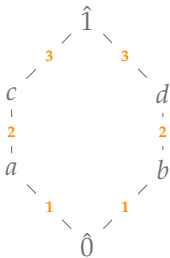
- EL-labeling



not an EL-labeling!

Edge-Labelings

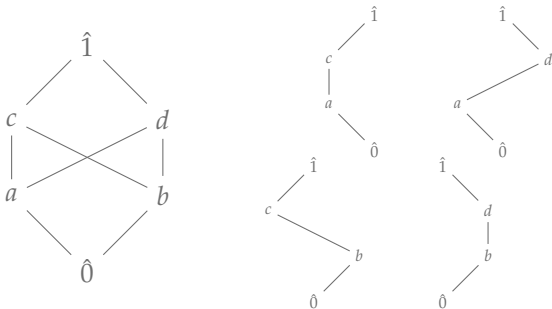
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Chain-Edge-Labelings

- CL-labeling



Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

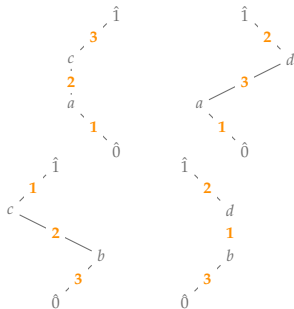
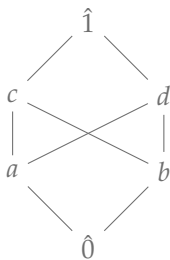
Conclusions

Chain-Edge-Labelings

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

- CL-labeling



Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

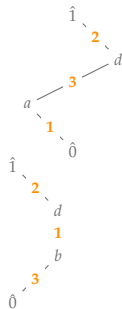
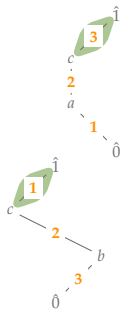
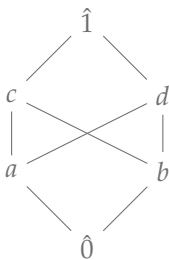
Conclusions

Chain-Edge-Labelings

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

- CL-labeling



Recursive Atom Orders

- **recursive atom order:** total order $a_1 \prec a_2 \prec \cdots \prec a_s$ such that
 - there exists a recursive atom order of $[a_j, \hat{1}]$ such that the first elements of this order are those that cover some $a_i \prec a_j$
 - if $i < j$ and $a_i, a_j \leq y$, then there is some $k < j$ and some $z \leq y$ such that $a_k, a_j \leq z$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

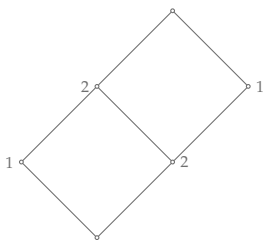
Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

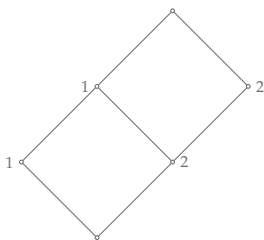
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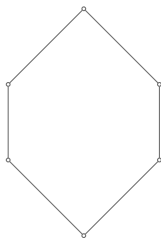
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Recursive Atom Orders

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 - there exists a recursive atom order of $[a_j, \hat{1}]$ such that the first elements of this order are those that cover some $a_i \prec a_j$
 - if $i < j$ and $a_i, a_j \leq y$, then there is some $k < j$ and some $z \leq y$ such that $a_k, a_j \leq z$

Theorem (Björner & Wachs, 1983)

A bounded poset admits an CL-labeling if and only if it admits a recursive atom order.

Lexicographically Shellable Posets

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- **lexicographically shellable poset**: admits an EL-labeling or a CL-labeling
- if \mathcal{P} is lexicographically shellable, then
 - $\Delta(\overline{\mathcal{P}})$ is shellable,
 - it is homotopic to a wedge of spheres,
 - the dimension of its i -th homology group is given by the number of falling maximal chains of length $i - 2$

Outline

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology
Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- 1 Introduction
 - Poset Topology
 - Noncrossing Partition Lattices
- 2 Lexicographic Shellability of \mathcal{NC}_W
 - EL-Shellability
 - CL-Shellability
- 3 Lexicographic Shellability and Hurwitz Action
 - Hurwitz Action
- 4 Conclusions

Well-Generated Reflection Groups

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- **complex reflection**: unitary transformation that fixes a hyperplane pointwise $\rightsquigarrow T$
- **complex reflection group**: group generated by complex reflections $\rightsquigarrow W$
- **rank**: codimension of fixed space
- **irreducible**: no nontrivial factors
- **well-generated**: irreducible, rank equals minimal number of generators

Well-Generated Reflection Groups

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- **monomial matrix**: one non-zero entry per row and per column
- $G(d, e, n)$.. $(n \times n)$ -monomial matrices, non-zero entries are d -th roots of unity, product is $\frac{d}{e}$ -th root of unity

Theorem (Shephard & Todd, 1954)

A finite group W is a well-generated reflection group if and only if $W \cong G(d, e, n)$ for $d \geq 1$, $e \in \{1, d\}$, or W is one of 26 exceptional groups.

Well-Generated Reflection Groups

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- $G(1, 1, n) \cong A_{n-1} \cong \mathfrak{S}_n$
- $G(2, 1, n) \cong B_n$
- $G(2, 2, n) \cong D_n$
- $G(d, d, 2) \cong I_2(d) \cong \mathfrak{D}_d$
- $G_{23} \cong H_3$
- $G_{28} \cong F_4$
- $G_{30} \cong H_4$
- $G_{35} \cong E_6$
- $G_{36} \cong E_7$
- $G_{37} \cong E_8$

Coxeter Elements

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- **degrees**: certain invariants of W $\rightsquigarrow d_1 \leq \dots \leq d_n$
- **Coxeter number**: highest degree $\rightsquigarrow h$
- **regular element**: has eigenvector that does not lie in any reflection hyperplane
- **Coxeter element**: regular element of order h $\rightsquigarrow c$

Theorem (Lehrer & Springer, 1999)

Coxeter elements exist in well-generated reflection groups.

Absolute Order

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- **absolute length:** length of a minimal T -decomposition $\rightsquigarrow \ell_T$
- **absolute order:** $u \leq_T v$ if and only if
$$\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$$

Noncrossing Partition Lattices

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

- W .. well-generated reflection group

Definition (Brady, 2001; Brady & Watt, 2002; Bessis, 2003; Bessis, 2007)

The **lattice of noncrossing partitions** of W is defined to be the interval $[e, c]_T$ between the identity e and some Coxeter element c of W in absolute order. $\rightsquigarrow \mathcal{NC}_W(c)$

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

Noncrossing Partition Lattices

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- combinatorial definitions
 - type A : Kreweras, 1971
 - type B : Reiner, 1997
 - type D : Athanasiadis & Reiner, 2004
 - type $G(d, d, n)$: Bessis & Corran, 2006
 - type $G(d, 1, n)$: essentially type B

Noncrossing Partition Lattices

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

B-Shellability

CL-Shellability

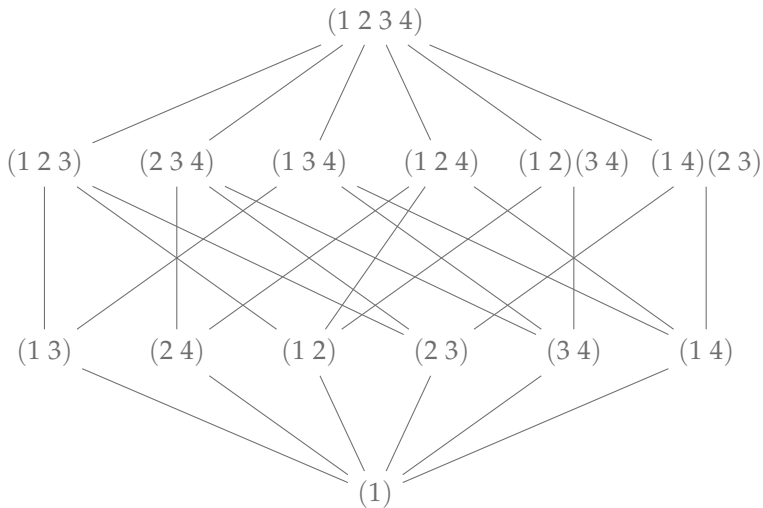
Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

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Example: $\mathcal{NC}_{\mathfrak{S}_4}$



Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

Example: $\mathcal{NC}_{\mathfrak{S}_4}$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

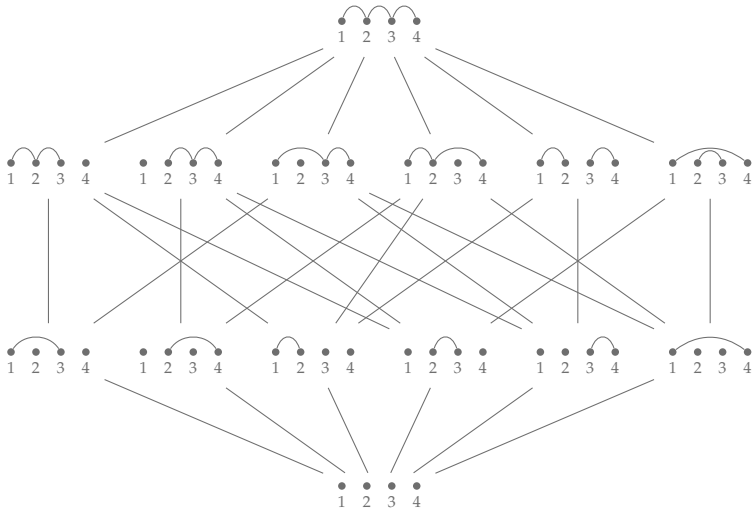
EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions



Properties

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

Theorem (Reiner, Ripoll & Stump, 2014)

For any well-generated reflection group W , and any two Coxeter elements $c, c' \in W$, we have $\mathcal{NC}_W(c) \cong \mathcal{NC}_W(c')$.

Properties

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

Theorem (Kreweras, 1971; Reiner, 1997; Brady, 2001; Brady & Watt, 2002; Bessis, 2003; Athanasiadis & Reiner, 2004; Bessis & Corran, 2006; Bessis, 2007; Brady & Watt, 2008)

\mathcal{NC}_W is indeed a lattice for any well-generated reflection group W .

- uniform proof only for Coxeter groups

Outline

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology
Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- 1 Introduction
 - Poset Topology
 - Noncrossing Partition Lattices
- 2 Lexicographic Shellability of \mathcal{NC}_W
 - EL-Shellability
 - CL-Shellability
- 3 Lexicographic Shellability and Hurwitz Action
 - Hurwitz Action
- 4 Conclusions

Outline

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology
Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- 1 Introduction
 - Poset Topology
 - Noncrossing Partition Lattices
- 2 Lexicographic Shellability of \mathcal{NC}_W
 - EL-Shellability
 - CL-Shellability
- 3 Lexicographic Shellability and Hurwitz Action
 - Hurwitz Action
- 4 Conclusions

Theorem (Björner & Edelman, 1980; Reiner, 1997)

If $W = A_n$ or $W = B_n$, then \mathcal{NC}_W is EL-shellable for any $n > 0$.

- restrict labeling that comes from the semimodularity of the partition lattice

EL-Shellability

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

Theorem (Athanasiadis, Brady & Watt, 2007)

If W is a Coxeter group, then \mathcal{NC}_W is EL-shellable.

- label (u, v) by $u^{-1}v \in T$
- use **compatible reflection order**
- this is uniform!

Example: $\mathcal{NC}_{\mathfrak{S}_4}$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

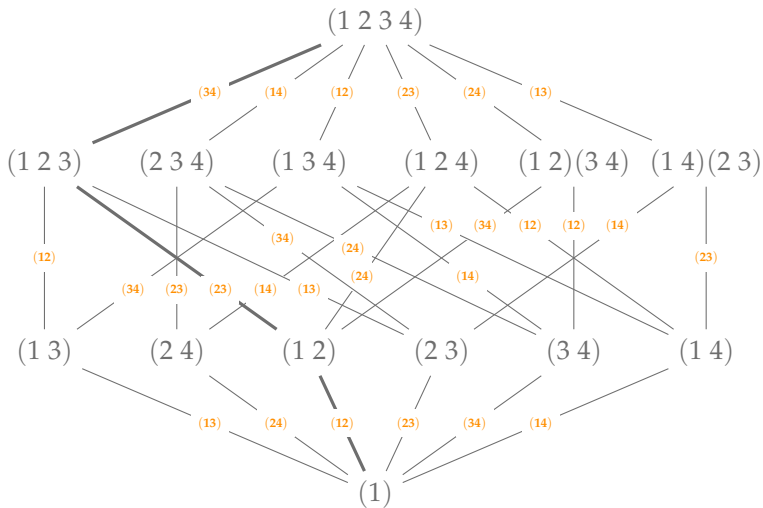
Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions



$$(12) \prec (13) \prec (14) \prec (23) \prec (24) \prec (34)$$

Generalizations?

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- what about well-generated reflection groups that are not Coxeter groups?
 - can we generalize the proof of Athanasiadis, Brady and Watt?

Compatible Reflection Orders

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- W .. Coxeter group
- $c \in W$.. Coxeter element
- **reflection order**: either $t_\alpha \prec t_{a\alpha+b\beta} \prec t_\beta$ or

$$t_\beta \prec t_{a\alpha+b\beta} \prec t_\alpha$$

Definition (Athanasiadis, Brady & Watt, 2007)

A reflection order is **c -compatible** if for any rank-2 subgroup of W , whose simple reflections are s, t , we have $s \prec t$ whenever $st \leq_T c$.

Compatible Reflection Orders

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- W .. well-generated reflection group
- $c \in W$.. Coxeter element
- T_c .. reflections below c

Definition (, 2015)

A total order of T_c is **c -compatible** if for any $w \leq_T c$ with $\ell_T(w) = 2$, there exists a unique rising reduced T -decomposition of w .

Example: $\mathcal{NC}_{\mathfrak{S}_4}$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

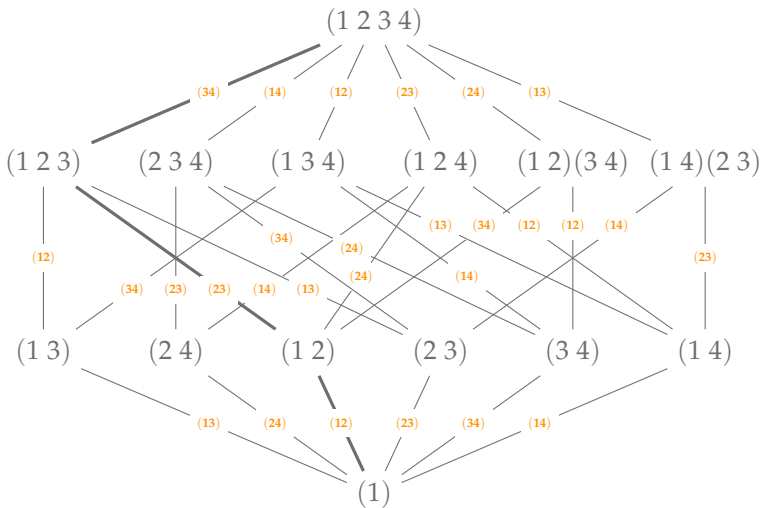
Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions



$$(12) \prec (13) \prec (14) \prec (23) \prec (24) \prec (34)$$

Example: $\mathcal{NC}_{\mathfrak{S}_4}$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

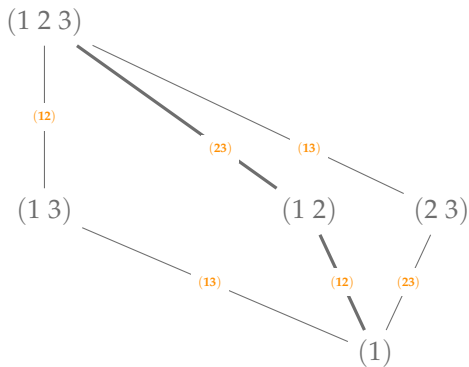
Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions



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Example: $\mathcal{NC}_{\mathfrak{S}_4}$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

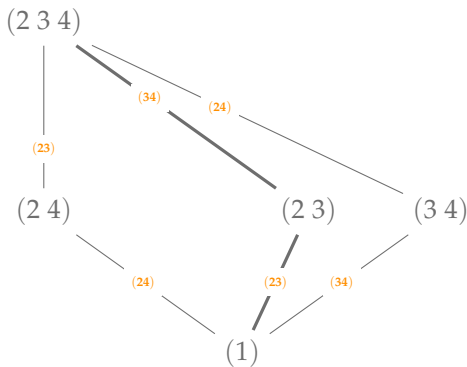
EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions



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Example: $\mathcal{NC}_{\mathfrak{S}_4}$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

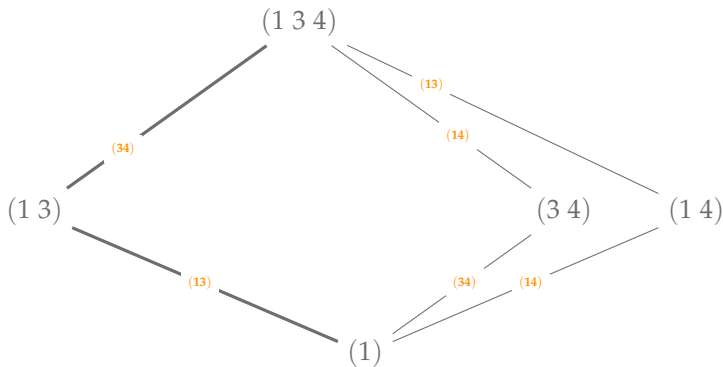
Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions



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Example: $\mathcal{NC}_{\mathfrak{S}_4}$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

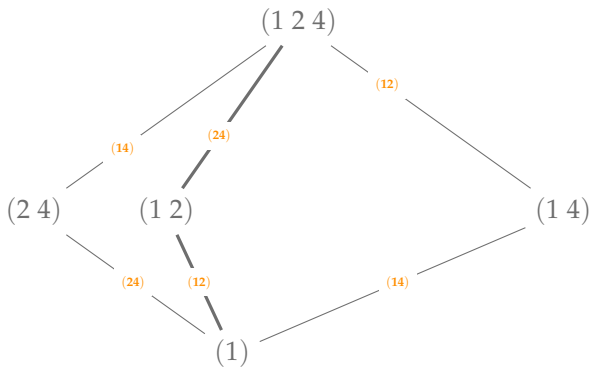
Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions



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Example: $\mathcal{NC}_{\mathfrak{S}_4}$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

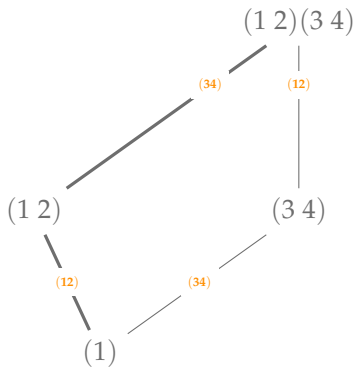
Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions



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Example: $\mathcal{NC}_{\mathfrak{S}_4}$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

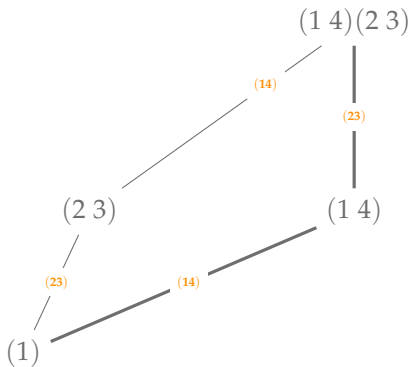
Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions



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Example: $\mathcal{NC}_{\mathfrak{S}_4}$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

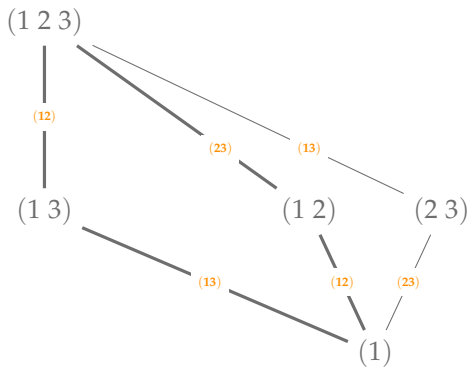
Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions



$$(13) \prec (12) \prec (14) \prec (23) \prec (24) \prec (34)$$

More EL-Shellability

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

Theorem (✂, 2015)

If $W \cong G(d, d, n)$ or if W is one of the exceptional well-generated reflection groups that is not a Coxeter group, then \mathcal{NC}_W is EL-shellable.

- label (u, v) by $u^{-1}v \in T$
- use compatible order for $G(d, d, n)$, and a computer verification for the exceptional types
- not uniform!

Outline

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology
Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- 1 Introduction
 - Poset Topology
 - Noncrossing Partition Lattices
- 2 Lexicographic Shellability of \mathcal{NC}_W
 - EL-Shellability
 - CL-Shellability
- 3 Lexicographic Shellability and Hurwitz Action
 - Hurwitz Action
- 4 Conclusions

Unite the Tribes

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- W .. well-generated reflection group
- $c \in W$.. Coxeter element
- T_c .. reflections below c

Theorem (✂, 2014)

Every c -compatible order of T_c is a recursive atom order of $\mathcal{NC}_W(c)$.

- proof by induction on rank of W
- “almost” uniform!

Unite the Tribes

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

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- $c \in W$.. Coxeter element
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CL-Shellability

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

Proposition (, 2014)

c-compatible orders exist in well-generated reflection groups.

- not uniform!

CL-Shellability

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

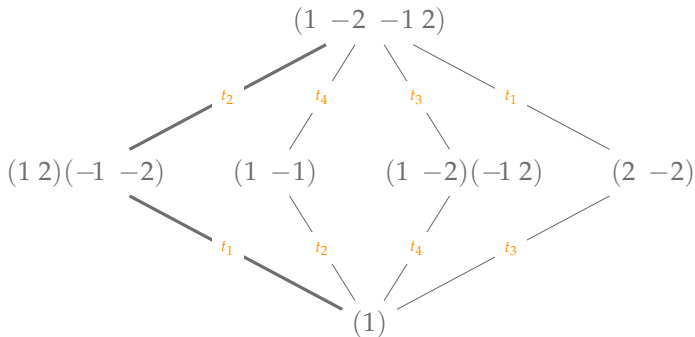
Conclusions

Theorem (, 2014)

If W is a well-generated reflection group, then \mathcal{NC}_W is CL-shellable.

- “almost” uniform!

Example: B_2



$$t_1 \prec t_3 \prec t_4 \prec t_2$$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

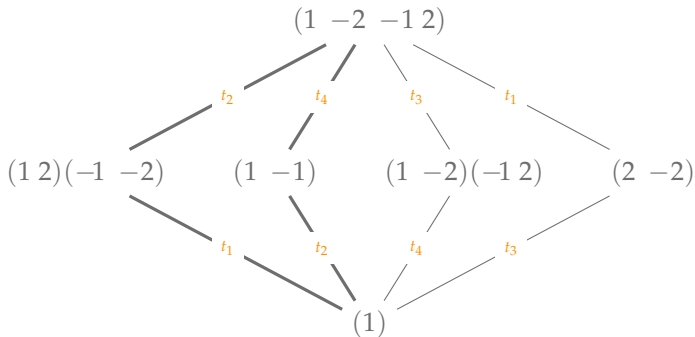
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

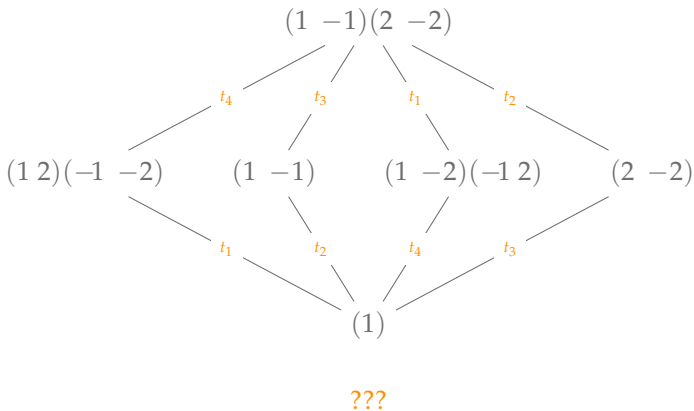
Conclusions

Example: B_2



$$t_1 \prec t_2 \prec t_3 \prec t_4$$

Example: B_2



Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

Outline

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology
Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- 1 Introduction
 - Poset Topology
 - Noncrossing Partition Lattices
- 2 Lexicographic Shellability of \mathcal{NC}_W
 - EL-Shellability
 - CL-Shellability
- 3 Lexicographic Shellability and Hurwitz Action
 - Hurwitz Action
- 4 Conclusions

Outline

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology
Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- 1 Introduction
 - Poset Topology
 - Noncrossing Partition Lattices
- 2 Lexicographic Shellability of \mathcal{NC}_W
 - EL-Shellability
 - CL-Shellability
- 3 Lexicographic Shellability and Hurwitz Action
 - Hurwitz Action
- 4 Conclusions

Hurwitz Action

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- **braid group:** $\mathfrak{B}_k = \langle \sigma_1, \dots, \sigma_{k-1} \mid (\sigma_i \sigma_{i+1})^3 = (\sigma_i \sigma_j)^2 = e, \text{ for } |j - i| > 1 \rangle$
- minimal T -decompositions of $w \in W \quad \rightsquigarrow \text{Red}_T(w)$
- $\mathfrak{B}_{\ell_T(w)}$ acts on $\text{Red}_T(w)$ by
$$\sigma_i \cdot (w_1 \cdots w_k) = w_1 \cdots w_{i-1} (w_i w_{i+1} w_i^{-1}) w_i w_{i+2} \cdots w_k$$
$$\rightsquigarrow \text{Hurwitz move}$$

Hurwitz Action

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

Theorem (Deligne, 1974; Bessis & Corran, 2006; Bessis, 2007)

For any well-generated reflection group W , and any Coxeter element $c \in W$, the group $\mathfrak{B}_{\ell_T(w)}$ acts transitively on $\text{Red}_T(w)$, whenever $w \leq_T c$.

- uniform proof only for Coxeter groups

A Connection

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

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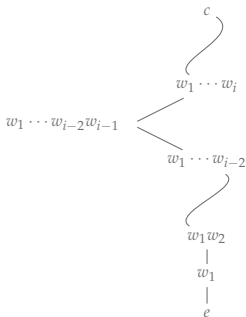
Theorem (, 2015)

If \mathfrak{B}_2 acts transitively on $\text{Red}_T(w)$ for every $w \leq_T c$ with $\ell_T(w) = 2$, and $\mathcal{NC}_W(c)$ is lexicographically shellable, then \mathfrak{B}_n acts transitively on $\text{Red}_T(c)$.

Sketch of Proof

- minimal T -decompositions of c correspond to maximal chains in $\mathcal{NC}_W(c)$
- Hurwitz moves correspond to “taking detours”

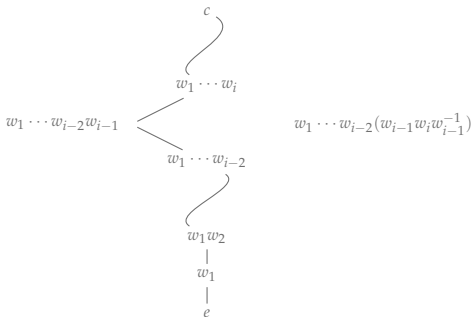
$$c = w_1 w_2 \cdots w_{i-2} w_{i-1} w_i w_{i+1} \cdots w_n$$



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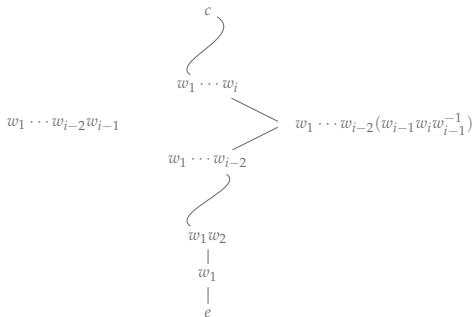
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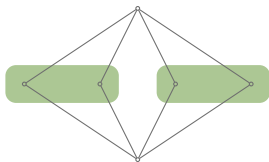
$$c = w_1 w_2 \cdots w_{i-2} (w_{i-1} w_i w_{i-1}^{-1}) w_{i-1} w_{i+1} \cdots w_n$$



Sketch of Proof

- non-transitivity can be caused by two scenarios

rank-2 violation



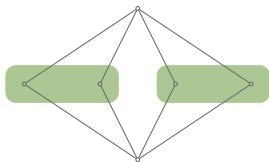
large “gaps”



Sketch of Proof

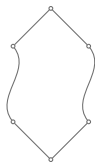
- non-transitivity can be caused by two scenarios

rank-2 violation



Contradicts rank-2 assumption!

large "gaps"



Contradicts shellability assumption!

More Compatibility

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

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More Compatibility

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- W .. well-generated reflection group
- $c \in W$.. Coxeter element
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Outline

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology
Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability
CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

- 1 Introduction
 - Poset Topology
 - Noncrossing Partition Lattices
- 2 Lexicographic Shellability of \mathcal{NC}_W
 - EL-Shellability
 - CL-Shellability
- 3 Lexicographic Shellability and Hurwitz Action
 - Hurwitz Action
- 4 Conclusions

Conclusions

- c -compatible reflection orders are good!
- for Coxeter groups, they allow uniform solutions for:
 - lattice property of \mathcal{NC}_W
 - lexicographic shellability of \mathcal{NC}_W
 - bases of homology of $\Delta(\overline{\mathcal{NC}_W})$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

Conclusions

- c -compatible reflection orders are good!
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 - lattice property of \mathcal{NC}_W
 - lexicographic shellability of \mathcal{NC}_W
 - bases of homology of $\Delta(\overline{\mathcal{NC}_W})$

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

Conclusions

Conclusions

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Problem

Give a uniform description of a c -compatible order of the reflections below c for all well-generated reflection groups, and some (uniform) choice of Coxeter element c .

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Introduction

Poset Topology

Noncrossing
Partition Lattices

Lexicographic
Shellability of
 \mathcal{NC}_W

EL-Shellability

CL-Shellability

Lexicographic
Shellability
and Hurwitz
Action

Hurwitz Action

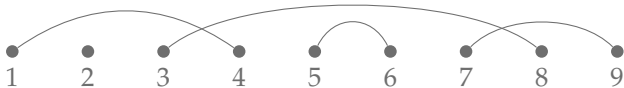
Conclusions

Merci Beaucoup.

Noncrossing Partitions

- noncrossing (set) partition

$\rightsquigarrow NC_n$



Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Noncrossing
Partitions

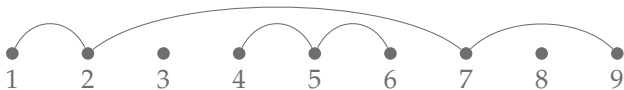
Generated
Groups with
Conjugation
Closed
Generating
Sets

The Braid
Group of a
Reflection
Group

Noncrossing Partitions

- noncrossing (set) partition

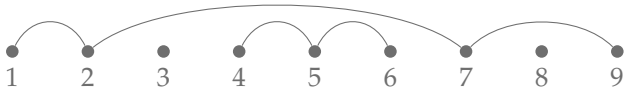
$\rightsquigarrow NC_n$



Noncrossing Partitions

- refinement order: join blocks

$\rightsquigarrow \leq_{\text{ref}}$



Noncrossing Partitions

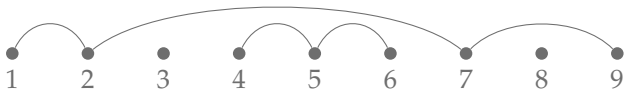
- refinement order: join blocks

$\rightsquigarrow \leq_{\text{ref}}$



A Map to Permutations

- group blocks into cycles



Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

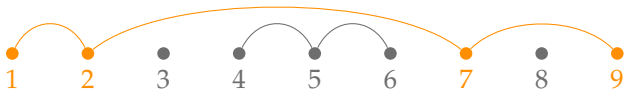
Noncrossing
Partitions

Generated
Groups with
Conjugation
Closed
Generating
Sets

The Braid
Group of a
Reflection
Group

A Map to Permutations

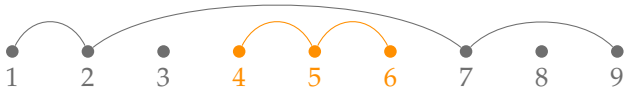
- group blocks into cycles



(1279)

A Map to Permutations

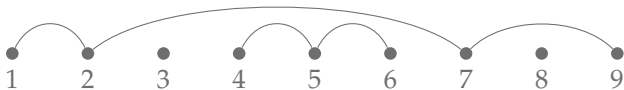
- group blocks into cycles



$$(1279)(456)$$

A Map to Permutations

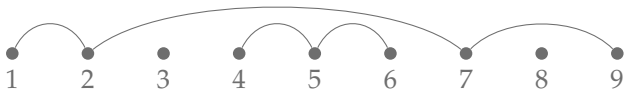
- group blocks into cycles



$$(1279)(456)$$

A Map to Permutations

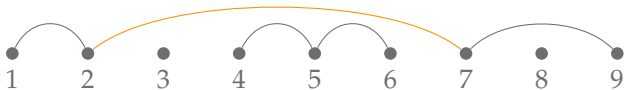
- cover relations correspond to transpositions



$$(1279)(456)$$

A Map to Permutations

- cover relations correspond to transpositions



$$(1279)(456)$$

A Map to Permutations

- cover relations correspond to transpositions



(1245679)

A Map to Permutations

- cover relations correspond to transpositions



$$(1245679) = (1279)(456) \cdot (26)$$

A Symmetric Group Object

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Noncrossing
Partitions

Generated
Groups with
Conjugation
Closed
Generating
Sets

The Braid
Group of a
Reflection
Group

- T .. transpositions in \mathfrak{S}_n
- e .. identity permutation
- c .. some long cycle

Theorem (Biane, 1997)

For $n > 0$, the poset $\mathcal{NC}_n = (\text{NC}_n, \leq_{\text{ref}})$ is isomorphic to the interval $[e, c]$ in the Cayley graph of (\mathfrak{S}_n, T) .

- use this connection as a starting point to generalize \mathcal{NC}_n to all well-generated reflection groups

A More General Setting

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Noncrossing
Partitions

Generated
Groups with
Conjugation
Closed
Generating
Sets

The Braid
Group of a
Reflection
Group

- **generated group**: group G with a distinguished generating set A $\rightsquigarrow (G, A)$
- define a word length $\rightsquigarrow \ell_A$
- partial orders:
$$u \leq v \quad \text{if and only if} \quad \ell_A(v) = \ell_A(u) + \ell_A(u^{-1}v)$$
- A closed under conjugation \rightsquigarrow well-defined Hurwitz action

A More General Setting

Poset-
Topological
Aspects of
Noncrossing
Partition
Lattices

Henri Mühle

Noncrossing
Partitions

Generated
Groups with
Conjugation
Closed
Generating
Sets

The Braid
Group of a
Reflection
Group

- **generated group**: group G with a distinguished generating set A $\rightsquigarrow (G, A)$
- define a word length $\rightsquigarrow \ell_A$
- partial orders:
$$u \geq v \quad \text{if and only if} \quad \ell_A(u) = \ell_A(v) + \ell_A(uv^{-1})$$
- A closed under conjugation \rightsquigarrow well-defined Hurwitz action

More General Results

Poset-
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Aspects of
Noncrossing
Partition
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Henri Mühle

Noncrossing
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- (G, A) .. generated group, A closed under conjugation
- $x \in G$.. some element
- A_x .. generators below x

Theorem (✂, 2015)

Suppose that \mathfrak{B}_2 acts transitively on $\text{Red}_T(A)g$, whenever $\ell_A(g) = 2$. If $[e, x]_A$ is lexicographically shellable, then $\mathfrak{B}_{\ell_A(x)}$ acts transitively on $\text{Red}_T(A)x$.

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Proposition (, 2015)

If there exists a x -compatible order of A_x , then \mathfrak{B}_2 acts transitively on $\text{Red}_T(A)g$, whenever $\ell_A(g) = 2$.

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Theorem (, 2015)

Every x -compatible order of A_x is a recursive atom order of $[e, x]_A$.

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- how frequently do generated groups with conjugation closed generating sets appear?
- how frequently do x -compatible orders exist in these groups?

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The Braid
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- **braid group**: fundamental group of complement of reflection hyperplanes $\rightsquigarrow \mathfrak{B}(W)$

- $\mathfrak{B}_n = \mathfrak{B}(A_{n-1})$

- group presentation:

$$W = \langle T \mid t_i^{\varepsilon_i} = e, R \rangle$$
$$\downarrow$$
$$\mathfrak{B}(W) = \langle T \mid R \rangle$$

- consider $(\mathfrak{B}(W), \leq_T)$
 - in particular, intervals $[e, c^m]_T$ for some $m > 0$

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Example: \mathfrak{B}_3

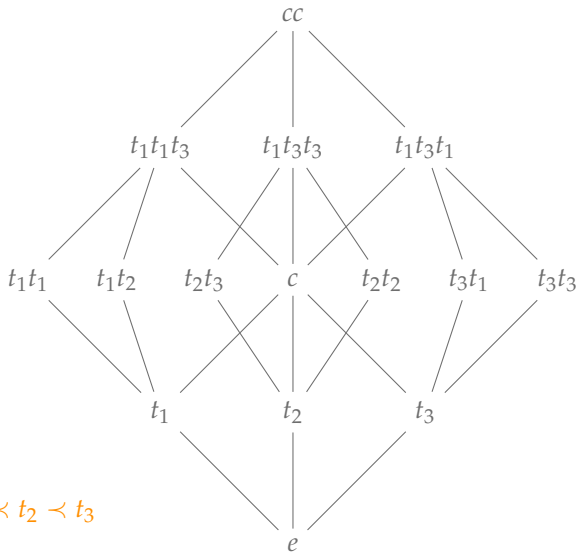
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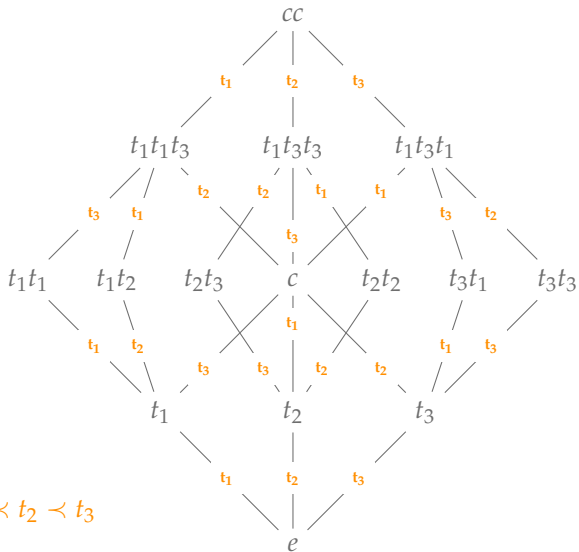
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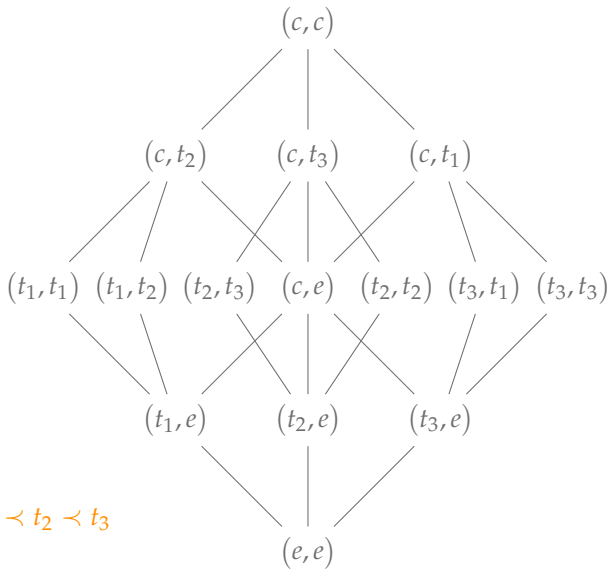
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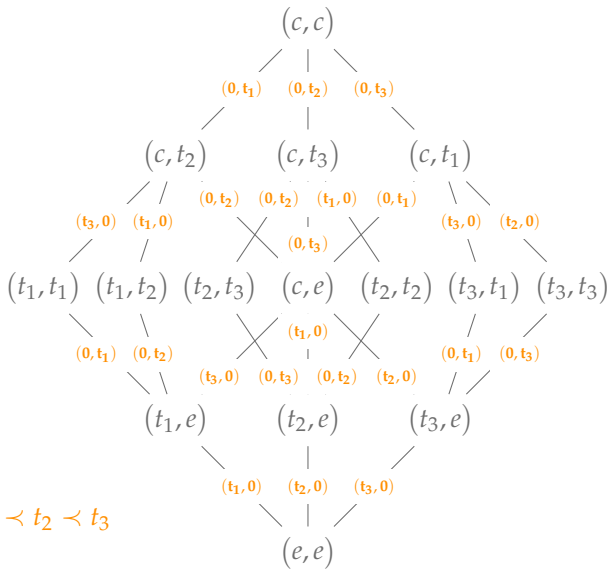
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$$t_1 \prec t_2 \prec t_3$$

Example: \mathfrak{B}_3



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Example: $\mathcal{NC}_{A_2}^{[2]}$

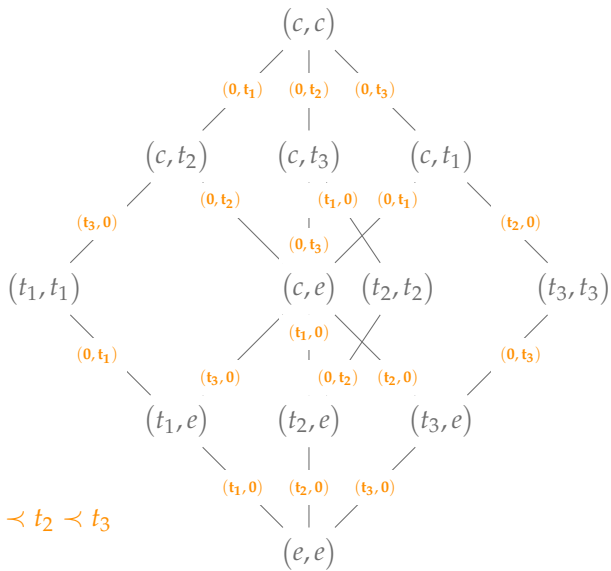
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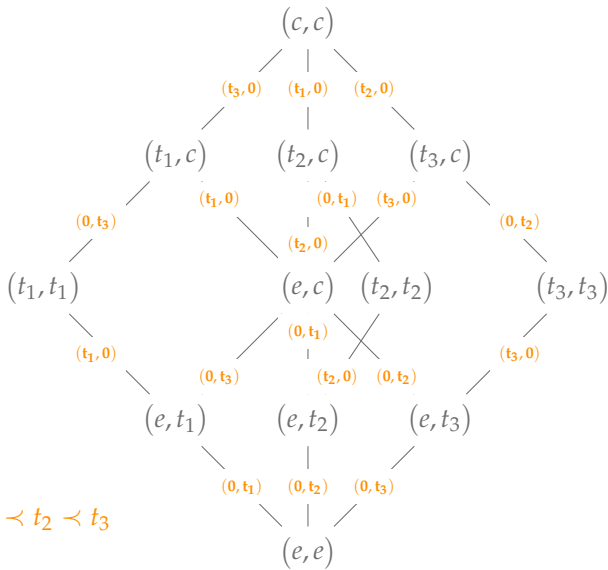
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Multichains of Noncrossing Partitions

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- elements in $\text{NC}_W^{[m]}$ are m -multichains in \mathcal{NC}_W
 - but: different partial order than Armstrong's m -divisible noncrossing partitions!

The Poset of m -Multichains

- $\mathcal{P} = (P, \leq)$.. some poset
- m -multichain: (x_1, x_2, \dots, x_m) with $x_1 \leq x_2 \leq \dots \leq x_m$
 $\rightsquigarrow P^{[m]}$
- poset of m -multichains: $(P^{[m]}, \leq)$
 $\rightsquigarrow \mathcal{P}^{[m]}$

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Topological
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Noncrossing
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Groups with
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Sets

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- poset of m -multichains: $(P^{[m]}, \leq)$ $\rightsquigarrow \mathcal{P}^{[m]}$

Theorem (✂, 2014)

Let $\mathcal{P} = (P, \leq)$ be a bounded poset, and let $\mathcal{P}^{[m]}$ denote its poset of m -multichains, ordered componentwise by \leq . Then, \mathcal{P} is lexicographically shellable if and only if $\mathcal{P}^{[m]}$ is lexicographically shellable for every $m > 0$.

Example: $\mathcal{NC}_{A_2}^{[2]}$

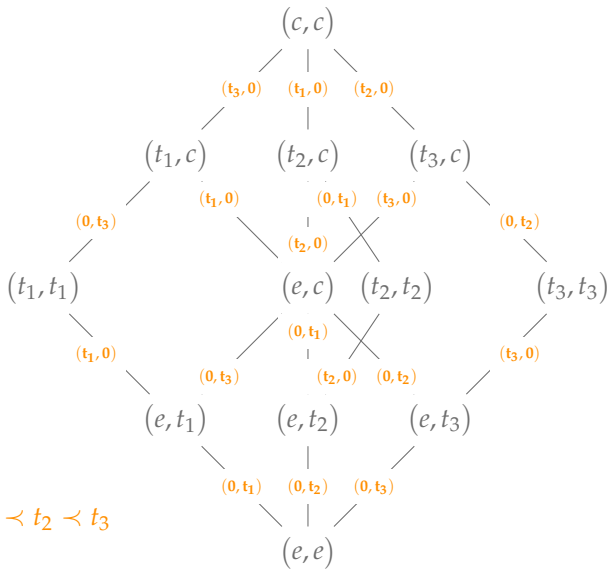
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Aspects of
Noncrossing
Partition
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Groups with
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Sets

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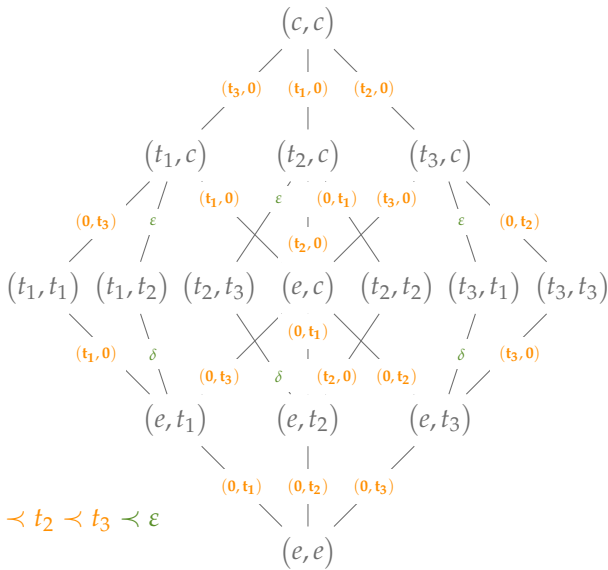
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Groups with
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Closed
Generating
Sets

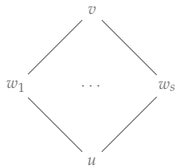
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Rank-2 Intervals

- rank-2 intervals of $[e, c^m]_T$ in $\mathfrak{B}(W)$ are of one of the two forms
 - \rightsquigarrow rank-2 transitivity of the Hurwitz action
 - \rightsquigarrow proving lexicographic shellability yields Hurwitz transitivity “for free”!

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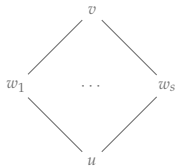


$$\ell_T(u^{-1}v) = 2$$

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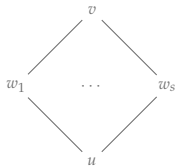


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