

Hook length formulas for integer partitions and planar trees

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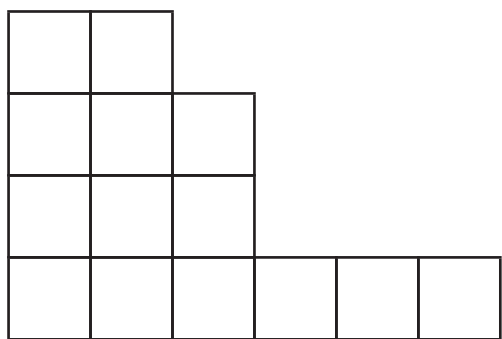
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Hook length formulas for partitions and trees

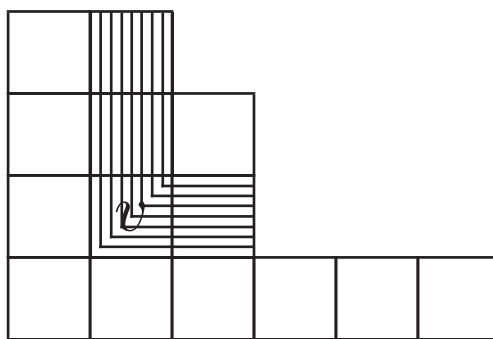
Summary:

- Some well-known examples
- How to discover new hook formulas ?
- The Main Theorem
- Specializations
- Number Theory
- Hook formulas for binary trees

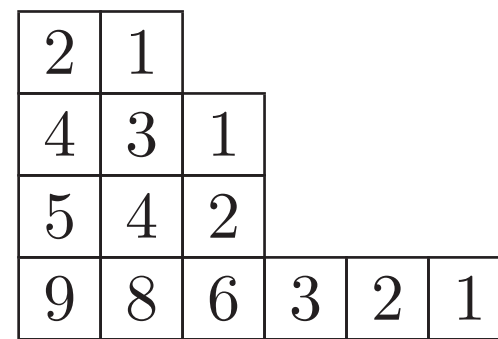
Some well-known examples: Hook length multi-set



Partition
 $\lambda = (6, 3, 3, 2)$



Hook length of v
 $h_v(\lambda) = 4$



Hook lengths
 $\mathcal{H}(\lambda)$

The hook length multi-set of λ is

$$\mathcal{H}(\lambda) = \{2, 1, 4, 3, 1, 5, 4, 2, 9, 8, 6, 3, 2, 1\}$$

Some well-known examples: permutations

f_λ : the number of standard Young tableaux of shape λ

Frame, Robinson and Thrall, 1954

$$f_\lambda = \frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h}$$

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Robinson-Schensted correspondence: $\sum_{\lambda \vdash n} f_\lambda^2 = n!$

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = e^x$$

Some well-known examples: involutions

Robinson-Schensted correspondence: The number of standard Young tableaux of $\{1, 2, \dots, n\}$ is equal to the number of *involutions* of order n .

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h} = e^{x+x^2/2}$$

Some well-known examples: partitions

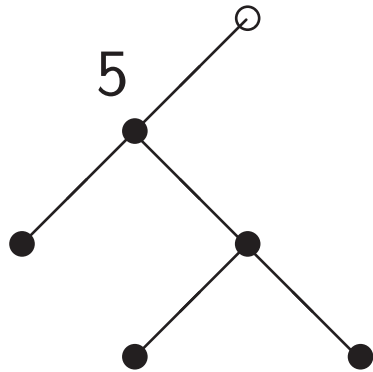
Euler: Generating function for partitions:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} 1 = \prod_{k \geq 1} \frac{1}{1 - x^k}$$

Some well-known examples: binary trees

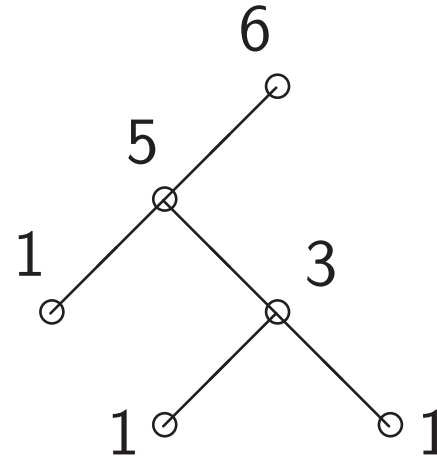
hook length for unlabeled binary trees

T



$$\mathcal{H}_v(T) = 5$$

T



$$\mathcal{H}(T) = \{1, 1, 1, 3, 5, 6\}$$

Some well-known examples: binary trees

f_T : the number of *increasing labeled binary trees*

$$f_T = \frac{n!}{\prod_{h \in \mathcal{H}(T)} h}$$

Some well-known examples: binary trees

Each labeled binary tree with n vertices is in bijection with a permutation of order n

$$\sum_{T \in \mathcal{B}(n)} n! \prod_{v \in T} \frac{1}{h_v} = n!$$

Some well-known examples: binary trees

Each labeled binary tree with n vertices is in bijection with a permutation of order n

$$\sum_{T \in \mathcal{B}(n)} n! \prod_{v \in T} \frac{1}{h_v} = n!$$

Generating function form:

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = \frac{1}{1-x}$$

Some well-known examples: binary trees, Catalan

The number of binary trees with n vertices is equal to the n -th Catalan number

$$\sum_{T \in \mathcal{B}(n)} 1 = \frac{1}{n+1} \binom{2n}{n}$$

Generating function form:

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T)} 1 = \frac{1 - \sqrt{1 - 4x}}{2x}$$

Some well-known examples: binary trees, Postnikov

Postnikov identity (2004)

$$\sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \left(1 + \frac{1}{h_v}\right) = \frac{2^n}{n!} (n+1)^{n-1}$$

Some well-known examples: binary trees, Postnikov

Postnikov identity (2004)

$$\sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \left(1 + \frac{1}{h_v}\right) = \frac{2^n}{n!} (n+1)^{n-1}$$

Lascoux, Du, Liu (2008)

$$\sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \left(x + \frac{1}{h_v}\right) = \frac{1}{(n+1)!} \prod_{k=0}^{n-1} ((n+1+k)x + n+1-k)$$

Some well-known examples: tangent numbers

The tangent number counts the *alternating permutations* (André, 1881), which are in bijection with the labeled complete binary trees.

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$$\sum_{T \in \mathcal{C}} x^{|T|} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = \tan(x) + \sec(x)$$

\mathcal{C} : complete binary trees

Discover new hook formulas

	<i>Partitions</i>	<i>Trees</i>
Discovering		
Proving		

Discover new hook formulas

	<i>Partitions</i>	<i>Trees</i>
Discovering	Hard	
Proving		

Discover new hook formulas

	<i>Partitions</i>	<i>Trees</i>
Discovering	Hard	Hard
Proving		

Discover new hook formulas

	<i>Partitions</i>	<i>Trees</i>
Discovering	Hard	Hard
Proving	Hard	

Discover new hook formulas

	<i>Partitions</i>	<i>Trees</i>
Discovering	Hard	Hard
Proving	Hard	Easy

Discover new hook formulas

We now introduce an efficient technique for discovering new hook length formulas:

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hook length expansion

Discover new hook formulas: expansion

$\rho(h)$: weight function

$f(x)$: formal power series

They are connected by the relation:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho(h) = f(x)$$

Discover new hook formulas: expansion

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- *generating function* : $\rho \longrightarrow f$

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- *generating function* : $\rho \longrightarrow f$
- *hook length expansion* : $\rho \longleftarrow f$
- *hook length formula* : when both ρ and f have “nice” forms

Discover new hook formulas: algorithm

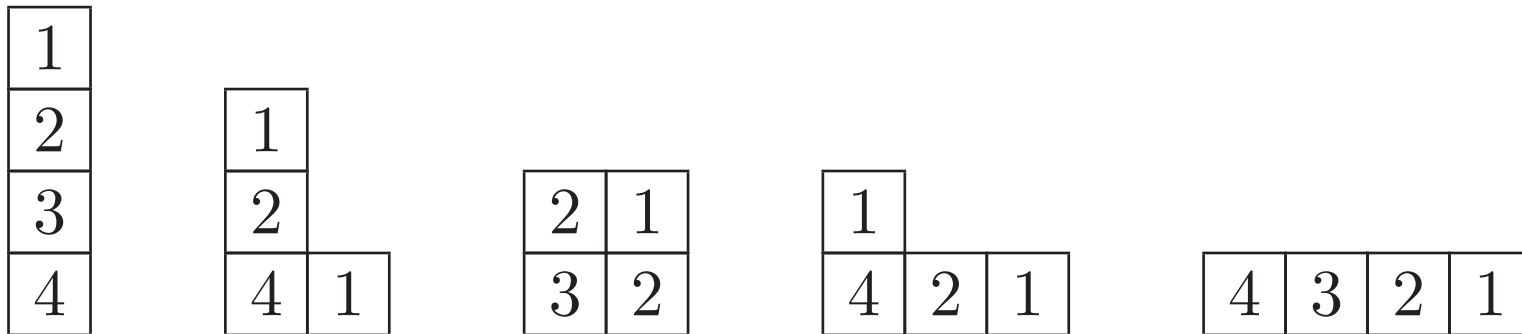
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Discover new hook formulas: algorithm

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- Is there an *algorithm* for computing the hook length expansion ? **Yes**.

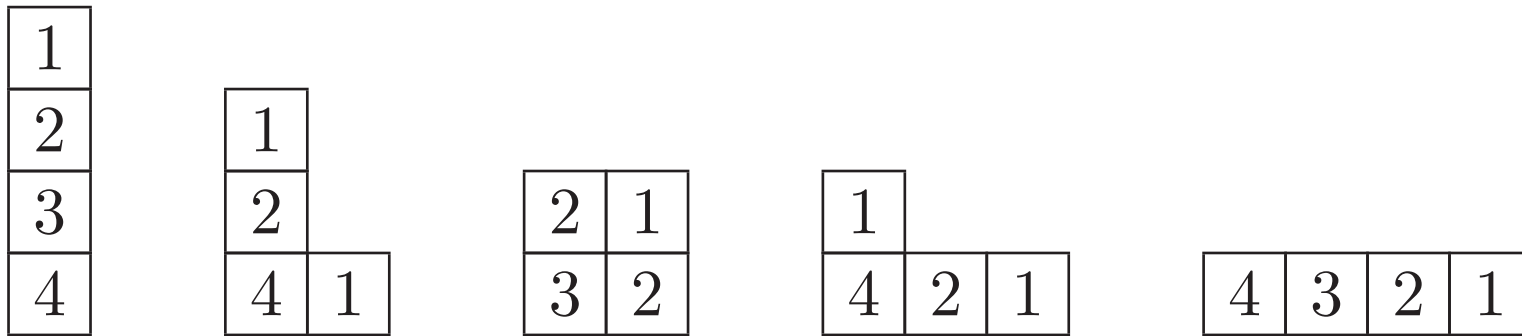
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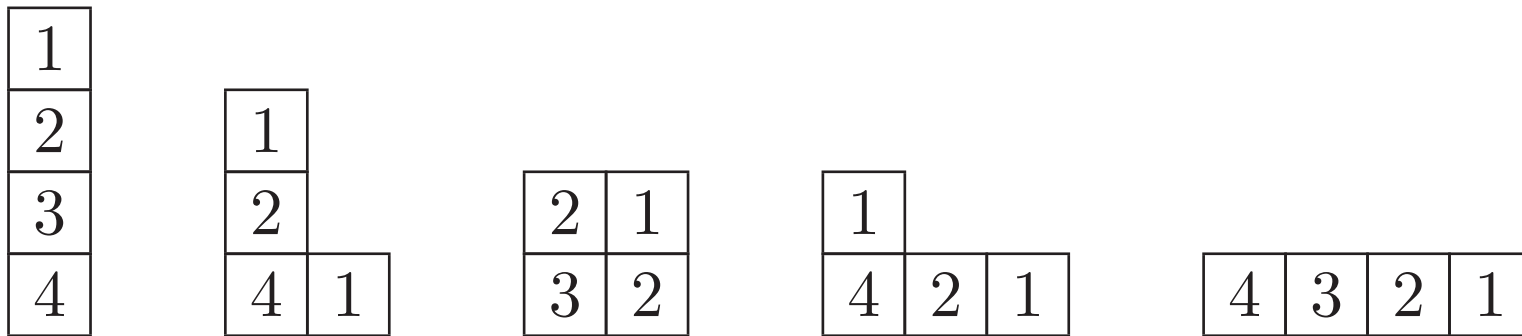
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$$\rho_4 \rho_3 \rho_2 \rho_1 + \rho_4 \rho_2 \rho_1 \rho_1 + \rho_3 \rho_2 \rho_2 \rho_1 + \rho_4 \rho_2 \rho_1 \rho_1 + \rho_4 \rho_3 \rho_2 \rho_1 = f_4$$

Discover new hook formulas: algorithm

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- Is there an *algorithm* for computing the hook length expansion ? **Yes.**



$$\rho_4 \rho_3 \rho_2 \rho_1 + \rho_4 \rho_2 \rho_1 \rho_1 + \rho_3 \rho_2 \rho_2 \rho_1 + \rho_4 \rho_2 \rho_1 \rho_1 + \rho_4 \rho_3 \rho_2 \rho_1 = f_4$$

We can solve ρ_4 when knowing $\rho_1, \rho_2, \rho_3, f_4$, because there is at most one “4” in each partition (linear equation with one variable)

Discover new hook formulas: maple package

Maple package for the hook length expansion

HookExp

Two procedures

hookgen: $\rho \longrightarrow f$

hookexp: $\rho \longleftarrow f$

Discover new hook formulas: permutation

Example : permutations

```
> read("HookExp.mp1") :  
> hookexp(exp(x), 8);
```

$$\left[1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \frac{1}{49}, \frac{1}{64}\right]$$

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = e^x$$

Discover new hook formulas: involution

Example: involutions

```
> hookexp(exp(x+x^2/2), 8);
```

$$\left[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}\right]$$

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h} = e^{x+x^2/2}$$

Discover new hook formulas: interpolation

permutations :
$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = e^x$$

involutions :
$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h} = e^{x+x^2/2}$$

♥♥♥ What about the interpolation

$$e^{x+zx^2/2} \quad ?$$

Discover new hook formulas: interpolation

Try

```
> hookexp(exp(x+z*x^2/2), 8);
```

$$\left[1, \frac{1+z}{4}, \frac{3z+1}{9+3z}, \frac{z^2+6z+1}{16+16z}, \frac{5z^2+10z+1}{5z^2+50z+25}, \right. \\ \left. \frac{z^3+15z^2+15z+1}{120z+36z^2+36}, \frac{7z^3+35z^2+21z+1}{7z^3+147z^2+245z+49} \right]$$

Many binomial coefficients, so that ...

Discover new hook formulas: interpolation

Interpolation between permutations and involutions:

First Conjecture (H., 2008)

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho(z; h) = e^{x + zx^2/2}$$

where

$$\rho(z; n) = \frac{\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} z^k}{n \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} z^k}$$

Latest news on the subject

The First Conjecture has been proved by:

Kevin Carde, Joe Loubert, Aaron Potechin, Adrian Sanborn

under the guidance of Dennis Stanton and Vic Reiner

(the Minnesota school)

Discover new hook formulas: partition

Another example. Euler: generating function for partitions

```
> hookexp(product(1/(1-x^k), k=1..9), 9);
```

```
[1, 1, 1, 1, 1, 1, 1, 1, 1]
```

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} 1 = \prod_{k \geq 1} \frac{1}{1 - x^k}$$

Discover new hook formulas: partition

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♥♥♥ What about $\prod_k (1 - x^k)$?

Discover new hook formulas: partition

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♥♥♥ What about $\prod_k (1 - x^k)$?

♥♥♥ or more generally $\prod_k (1 - x^k)^z$?

Discover new hook formulas: partition

Try it by using HookExp

Discover new hook formulas: partition

Try it by using HookExp

```
> hookexp(product((1-x^k)^z, k=1..7), 7);
```

$$\left[-z, \frac{3-z}{4}, \frac{8-z}{9}, \frac{15-z}{16}, \frac{24-z}{25}, \frac{35-z}{36}, \frac{48-z}{49}\right]$$

Discover new hook formulas: partition

Try it by using HookExp

```
> hookexp(product((1-x^k)^z, k=1..7), 7);  
[-z,  $\frac{3-z}{4}$ ,  $\frac{8-z}{9}$ ,  $\frac{15-z}{16}$ ,  $\frac{24-z}{25}$ ,  $\frac{35-z}{36}$ ,  $\frac{48-z}{49}$ ]
```

We see that the ρ has a very simple expression:

$$\rho(h) = \frac{h^2 - 1 - z}{h^2} = 1 - \frac{z + 1}{h^2}$$

Discover new hook formulas: partition

The previous hook length expansion suggests:

Theorem

$$\sum_{\lambda \in \mathcal{P}} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z+1}{h^2}\right) x = \prod_{k \geq 1} (1 - x^k)^z$$

Discover new hook formulas: proofs

The Russian-Physics Proof

Nekrasov, Okounkov (2003): arXiv: hep-th/0306238, 90 pages

(The last formula is deeply hidden in N-O's paper. See formula (6.12) on page 55)

Discover new hook formulas: proofs

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The “Elementary” Proof

H. (2008): arXiv:0805.1398 [math.CO]

Discover new hook formulas: proof

Dedekind η -function $\eta(x) = x^{1/24} \prod_{m \geq 1} (1 - x^m)$.

Theorem (Macdonald, 1972)

Let $t = 2t' + 1$ be an odd integer. We have

$$\eta(x)^{t^2-1} = c_0 \sum_{(v_0, \dots, v_{t-1})} \prod_{i < j} (v_i - v_j) x^{(v_0^2 + v_1^2 + \dots + v_{t-1}^2)/(2t)},$$

where the sum ranges over all V -codings $(v_0, v_1, \dots, v_{t-1})$ and c_0 is a numerical constant.

Affine root systems and Dedekind's η -function (type A_n)

Discover new hook formulas: proof

Theorem

Let $t = 2t' + 1$ be an odd positive integer. There is a bijection $\phi_V : \lambda \mapsto (v_0, v_1, \dots, v_{t-1})$ which maps each t -core onto a V -coding such that

$$|\lambda| = \frac{1}{2t}(v_0^2 + v_1^2 + \dots + v_{t-1}^2) - \frac{t^2 - 1}{24}$$

and

$$\prod_{v \in \lambda} \left(1 - \frac{t^2}{h_v^2}\right) = \frac{(-1)^{t'}}{1! \cdot 2! \cdot 3! \cdot \dots \cdot (t-1)!} \prod_{0 \leq i < j \leq t-1} (v_i - v_j).$$

Variation of Garvan-Kim-Stanton's bijection (1990): *Cranks and t -cores*

Discover new hook formulas: proof

$$\sum_{\lambda} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) x = \sum_n C_n(z) x^n$$

$$\prod_{k \geq 1} (1 - x^k)^{z-1} = \sum_n D_n(z) x^n$$

The coefficients $C_n(z)$ and $D_n(z)$ are both polynomials in z of degree n . For proving $C_n(z) = D_n(z)$, it suffices to find $n+1$ explicit numerical values z_0, z_1, \dots, z_n such that $C_n(z_i) = D_n(z_i)$ for $0 \leq i \leq n$.

True for every $z = t^2$ with odd integer t , since

$$\prod_{v \in \lambda} \left(1 - \frac{t^2}{h_v^2}\right) = 0$$

for every partition λ which is not a t -core.

Main Theorem: $1 + x^k$

Partition with distinct parts - shift Young tableaux

$$\prod_k (1 + x^k)$$

Thrall, 1952

The number of standard shifted Young tableaux is given by

$$\frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h}$$

Main Theorem: $1 + x^k$

```
> hooktype := "PAD";  
> hookexp(product( 1+x^k, k=1..9), 9);
```

```
[1,  $\frac{1}{2}$ ,  $\frac{2}{3}$ ,  $\frac{3}{8}$ ,  $\frac{18}{29}$ ,  $\frac{52}{113}$ ,  $\frac{43539}{71974}$ ,  $\frac{50712791}{136184240}$ ,  $\frac{224560049745548}{376968863190753}$ ]
```

No formula !

Main Theorem: $1 + x^k$

```
> hooktype := "PAD";  
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```

$$\left[1, \frac{1}{2}, \frac{2}{3}, \frac{3}{8}, \frac{18}{29}, \frac{52}{113}, \frac{43539}{71974}, \frac{50712791}{136184240}, \frac{224560049745548}{376968863190753}\right]$$

No formula !

```
> hooktype := "PA";  
> hookexp(product( 1+x^k, k=1..14), 14);
```

$$\left[1, \frac{1}{2}, 1, \frac{7}{8}, 1, \frac{17}{18}, 1, \frac{31}{32}, 1, \frac{49}{50}, 1, \frac{71}{72}, 1, \frac{97}{98}\right]$$

Main Theorem: $1 + x^k$

We have

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda), h \text{ even}} \left(1 - \frac{2}{h^2}\right) = \prod_{k \geq 1} (1 + x^k).$$

Main Theorem: $1 + x^k$

We have

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda), h \text{ even}} \left(1 - \frac{2}{h^2}\right) = \prod_{k \geq 1} (1 + x^k).$$

Compare with ($z = 1$ in N-O formula):

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{2}{h^2}\right) = \prod_{k \geq 1} (1 - x^k).$$

Main Theorem: $1 + x^k$

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 - No, with the right-hand side by `hookexp(←—)`, because no “nice” expansion for

$$\prod \frac{1}{1 + x^k} \quad \text{or} \quad \prod (1 + x^k)^z.$$

Main Theorem: $1 + x^k$

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$$\prod \frac{1}{1 + x^k} \quad \text{or} \quad \prod (1 + x^k)^z.$$

- Yes, with the left-hand side by `hookgen(→—)`.

Main Theorem: $1 + x^k$ variation

We have just seen:

$$\rho = \left[1, \frac{1}{2}, 1, \frac{7}{8}, 1, \frac{17}{18}, 1, \frac{31}{32}, 1, \frac{49}{50}, 1, \frac{71}{72}, 1, \frac{97}{98}\right] \longrightarrow \prod_{k \geq 1} (1 + x^k).$$

Try the following variations of ρ with [hookgen](#):

$$\left[1, 1 - \frac{z}{2}, 1, 1 - \frac{z}{8}, 1, 1 - \frac{z}{18}, 1, 1 - \frac{z}{32}, 1, 1 - \frac{z}{50}, 1\right]$$

$$[1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1]$$

$$[1, 1, z, 1, 1, z, 1, 1, z, 1, 1, z, 1, 1, z]$$

Main Theorem: $1 + x^k$ variation

> ...

$$\left[1, 1 - \frac{z}{2}, 1, 1 - \frac{z}{8}, 1, 1 - \frac{z}{18}, 1, 1 - \frac{z}{32}, 1, 1 - \frac{z}{50}, 1\right]$$

> hookgen(%): etamake(%, x, 10): simplify(%);

$$\prod_{k \geq 1} \frac{(1 - x^{2k})^z}{1 - x^k}$$

When $z = 1$

$$\prod_{k \geq 1} \frac{(1 - x^{2k})^z}{1 - x^k} = \prod_{k \geq 1} \frac{1 - x^{2k}}{1 - x^k} = \prod_{k \geq 1} (1 + x^k)$$

Main Theorem: $1 + x^k$ variation

```
> r:=n-> if n mod 3=0 then -1 else 1 fi:  
> [seq(r(i), i=1..17)];
```

[1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1]

```
> hookgen(%): etamake(%, x, 17): simplify(%);
```

$$\prod_{k \geq 1} \frac{(1 - x^{12k})^3 (1 - x^{3k})^6}{(1 - x^{6k})^9 (1 - x^k)}$$

Main Theorem

The previous and many other experimentations suggest:

Main Theorem (H. 2008)

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{tyz}{h^2} \right) = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{(1 - (yx^t)^k)^{t-z} (1 - x^k)}$$

$$\mathcal{H}_t(\lambda) = \{h \mid h \in \mathcal{H}(\lambda), h \equiv 0(\text{mod } t)\}.$$

Main Theorem: fields of interest

This work has some links with the following fields:

- General Mathematical Community: Euler, Jacobi, Gauss
- High Energy Physics Theory: Nekrasov, Okounkov
- Lie Algebra and Representation Theory: Macdonald, Dyson, Kostant, Milne, Schlosser, Bessenrodt
- Modular Forms and Number Theory: Ramanujan, Lehmer, Ono
- q-Series, Combinatorics: Andrews, Stanton, Stanley
- Symmetric Functions: Cauchy, Schur, Lascoux
- Algorithm, Computer Algebra: RSK, Krattenthaler (rate), Garvan (qseries), Rubey, Sloane
- Plane Trees: Viennot, Foata, Schützenberger, Strehl, Gessel, Postnikov

Main Theorem: Proof

Proof.

Let t be a positive integer. The **Littlewood decomposition** maps a partition λ to $(\mu; \lambda^0, \lambda^1, \dots, \lambda^{t-1})$ such that

(P1) μ is a t -core and $\lambda^0, \lambda^1, \dots, \lambda^{t-1}$ are partitions;

(P2) $|\lambda| = |\mu| + t(|\lambda^0| + |\lambda^1| + \dots + |\lambda^{t-1}|)$;

(P3) $\{h/t \mid h \in \mathcal{H}_t(\lambda)\} = \mathcal{H}(\lambda^0) \cup \mathcal{H}(\lambda^1) \cup \dots \cup \mathcal{H}(\lambda^{t-1})$.

The vector $(\lambda^0, \lambda^1, \dots, \lambda^{t-1})$ is usually called the t -quotient of the partition λ .

Main Theorem: Specializations

The Main Theorem has so many specializations:

- the Jacobi triple product identity \rightarrow
- the Gauss identity \rightarrow
- the Nekrasov-Okounkov formula
- the generating function for partitions
- the Macdonald identity for $A_\ell^{(a)}$
- the classical hook length formula
- the marked hook formula \rightarrow
- the generating function for t -cores
- the t -core analogues of the hook formula
- the t -core analogues of the marked hook formula
- ...

Main Theorem: Specializations

Why it has so many specializations?

- It contains 3 variables t, y, z
- We can give special values to t, y, z
- Compare the coefficients of the minimal terms
- Compare the coefficients of the maximal terms

Specializations, Jacobi + Gauss

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{tyz}{h^2} \right) = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{(1 - (yx^t)^k)^{t-z} (1 - x^k)}$$

$t = 1, y = 1, z = 4$:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_1(\lambda)} \left(1 - \frac{4}{h^2} \right) = \prod_{k \geq 1} (1 - x^k)^3$$

$t = 2, y = 1, z = 2$:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_2(\lambda)} \left(1 - \frac{4}{h^2} \right) = \prod_{k \geq 1} \frac{(1 - x^{2k})^2}{1 - x^k}$$

Specializations, Jacobi + Gauss

Simplify

$$J := \prod_{h \in \mathcal{H}_1(\lambda)} \left(1 - \frac{4}{h^2}\right) \quad \text{and} \quad G := \prod_{h \in \mathcal{H}_2(\lambda)} \left(1 - \frac{4}{h^2}\right)$$

- If a partition λ contains one box v whose hook length is $h_v = 2$, then

$$J = G = 0.$$

- Otherwise λ must be a *staircase partition*

1			
3	1		
5	3	1	
7	5	3	1

Specializations, Jacobi + Gauss

-3			
$\frac{5}{9}$	-3		
$\frac{21}{25}$	$\frac{5}{9}$	-3	
$\frac{45}{49}$	$\frac{21}{25}$	$\frac{5}{9}$	-3

J

1			
1	1		
1	1	1	
1	1	1	1

G

$$J = \left(\frac{(2m-1)^2 - 4}{(2m-1)^2} \right)^1 \cdots \left(\frac{5}{9} \right)^{m-1} \left(\frac{-3}{1} \right)^m = (-1)^m (2m+1)$$

$$G = 1$$

Specializations, Jacobi + Gauss

The Main Theorem unifies Jacobi and Gauss identities.

$t = 1, y = 1, z = 4:$

Jacobi

$$\prod_{m \geq 1} (1 - x^m)^3 = \sum_{m \geq 0} (-1)^m (2m + 1) x^{m(m+1)/2}$$

$t = 2, y = 1, z = 2:$

Gauss

$$\prod_{m \geq 1} \frac{(1 - x^{2m})^2}{1 - x^m} = \sum_{m \geq 0} x^{m(m+1)/2}$$

Specializations, t -cores

Let $\{z = t \text{ or } y = 0\}$, we get the well known formula:

$$\sum_{\lambda: t\text{-cores}} x^{|\lambda|} = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{1 - x^k}$$

Specializations, t -cores

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♥♥♥ What about

$$\prod_{k \geq 1} \frac{(1 + x^{tk})^t}{1 - x^k} ?$$

Specializations, t -cores

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♥♥♥ What about

$$\prod_{k \geq 1} \frac{(1 + x^{tk})^t}{1 - x^k} ?$$

♥♥♥ How to generalize it ?

Specializations, t -cores

First, try `hookexp` (\leftarrow):

```
> hookexp( product( (1+x^k)/(1-x^k), k=1..9), 9);  
[2, 1, 1, 1, 1, 1, 1, 1, 1]
```

Specializations, t -cores

First, try `hookexp` (\leftarrow):

```
> hookexp( product( (1+x^k)/(1-x^k), k=1..9), 9);  
[2, 1, 1, 1, 1, 1, 1, 1, 1]
```

We have

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} 2^{\#\{h \in \mathcal{H}(\lambda), h=t\}} = \prod_{k \geq 1} \frac{(1+x^{tk})^t}{1-x^k}.$$

Specializations, t -cores

First, try `hookexp` (\leftarrow):

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Specializations, t -cores

First, try `hookexp` (\longleftarrow):

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} 2^{\#\{h \in \mathcal{H}(\lambda), h=t\}} = \prod_{k \geq 1} \frac{(1 + x^{tk})^t}{1 - x^k}.$$

Then, try `hookgen` (\longrightarrow):

Theorem (H. 2008)

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} y^{\#\{h \in \mathcal{H}(\lambda), h=t\}} = \prod_{k \geq 1} \frac{(1 + (y - 1)x^{tk})^t}{1 - x^k}$$

Special cases: $y = 0, 1, 2$

Latest news on the subject

A. Velingker, E. Clader, Y. Kemper, M. Wage, D. Collins, S. Wolfe
were working on these new hook length formulas and found
interesting applications on Modular Forms and Number Theory
under the guidance of Ken Ono

(the Wisconsin school)

Specializations, marked hook formula

- $\{z = -b/y, y \rightarrow 0\}$ in Main Theorem:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{tb}{h^2} = e^{bx^t} \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{1 - x^k}$$

- Compare the coefficients of $b^n x^{tn}$:

$$\sum_{\lambda \vdash tn, \# \mathcal{H}_t(\lambda) = n} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^2} = \frac{1}{t^n n!}$$

- $t = 1$:

$$\sum_{\lambda \vdash n} f_{\lambda}^2 = n!$$

Specializations, marked hook formula

- Compare the coefficients of $(-z)^{n-1} x^{nt} y^n$

$$\sum_{\lambda \vdash nt, \# \mathcal{H}_t(\lambda) = n} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}_t(\lambda)} h^2 = \frac{3n - 3 + 2t}{2(n-1)!}$$

- $t = 1$:

Marked hook formula

$$\sum_{\lambda \vdash n} f_{\lambda}^2 \sum_{h \in \mathcal{H}(\lambda)} h^2 = \frac{n(3n-1)}{2} n!$$

Specializations, marked hook formula

- Direct marked-RSK proof ? Not yet
- Generalizations ? Yes

Specializations, marked hook formula

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- Generalizations ? Yes

Specializations, marked hook formula

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} h^2 = \frac{3n - 1}{2(n - 1)!}$$

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} h^4 = \frac{40n^2 - 75n + 41}{6(n - 1)!}$$

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} h^6 = \frac{1050n^3 - 4060n^2 + 5586n - 2552}{24(n - 1)!}$$

Second Conjecture (H. 2008)

$$P_k(n) = (n - 1)! \sum_{\lambda \vdash n} \left(\prod_{v \in \lambda} \frac{1}{h_v^2} \right) \left(\sum_{u \in \lambda} h_u^{2k} \right)$$

is a polynomial in n of degree k .

Latest news on the subject

The Second Conjecture has been proved by
Richard Stanley and Greta Panova

Tewodros Amdeberhan slightly simplified Stanley's proof

(the MIT school)

Latest news on the subject: Okada

- Stanley proved:

$$\frac{1}{n!} \sum_{\lambda \vdash n} f_{\lambda}^2 F(h^2 : h \in \mathcal{H}(\lambda))$$

is a polynomial in n , where F is any symmetric function.

- Okada Conjecture:

$$\frac{1}{n!} \sum_{\lambda \vdash n} f_{\lambda}^2 \sum_{h \in \mathcal{H}(\lambda)} \prod_{i=1}^r (h^2 - i^2) = \frac{1}{2(r+1)^2} \binom{2r}{r} \binom{2r+2}{r+1} \prod_{j=0}^r (n-j)$$

Latest news on the subject: Okada

- Okada Conjecture implies:

$$\begin{aligned} & \frac{1}{n!} \sum_{\lambda \vdash n} f_{\lambda}^2 \sum_{h \in \mathcal{H}(\lambda)} h^{2k} \\ &= \sum_{i=0}^k T(k+1, i+1) \frac{1}{2(i+1)^2} \binom{2i}{i} \binom{2i+2}{i+1} \prod_{j=0}^i (n-j) \end{aligned}$$

where $T(k, i)$ are the central factorial numbers.

- Panova has proved the Okada Conjecture.

Specializations, Power sum

- $\{y = 1; \text{ compare the coefficients of } z\}$ in Main Theorem

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \sum_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^2} = \frac{1}{t} \prod_{m \geq 1} \frac{1}{1 - x^m} \sum_{k \geq 1} \frac{x^{tk}}{k(1 - x^{tk})}.$$

- $t = 1$:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \sum_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = \prod_{m \geq 1} \frac{1}{1 - x^m} \sum_{k \geq 1} \frac{x^k}{k(1 - x^k)}.$$

Specializations, Power sum

Direct proof. By using an elegant result on multi-sets of hook lengths and multi-sets of partition parts.

It is amusing to see that this result is rediscovered periodically:

- Stanley (1972, partial)
- Kirdar, Skyrme (1982, partial)
- Elder (1984, partial)
- Hoare (1986, partial)
- Bessenrodt (1998)
- Bacher, Manivel (2002)
- H., Bessenrodt (2009)
- Shin, Zeng (2009)

Specializations, Power sum

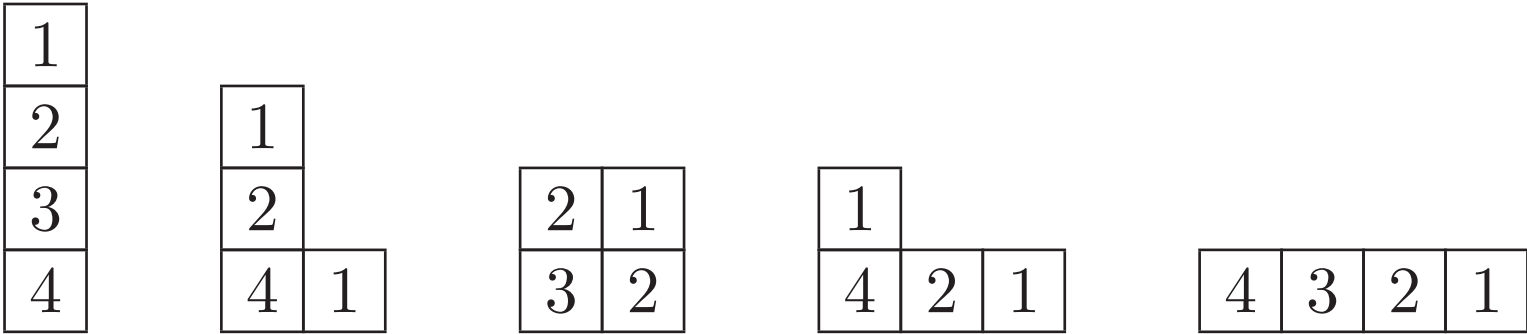
Each hook length h can be split into $h = a + l + 1$, where a is the *arm length* and l the *leg length*. The ordered pair (a, l) is called a *hook type*.

Theorem (Bessenrodt 1998; Bacher-Manivel 2002)

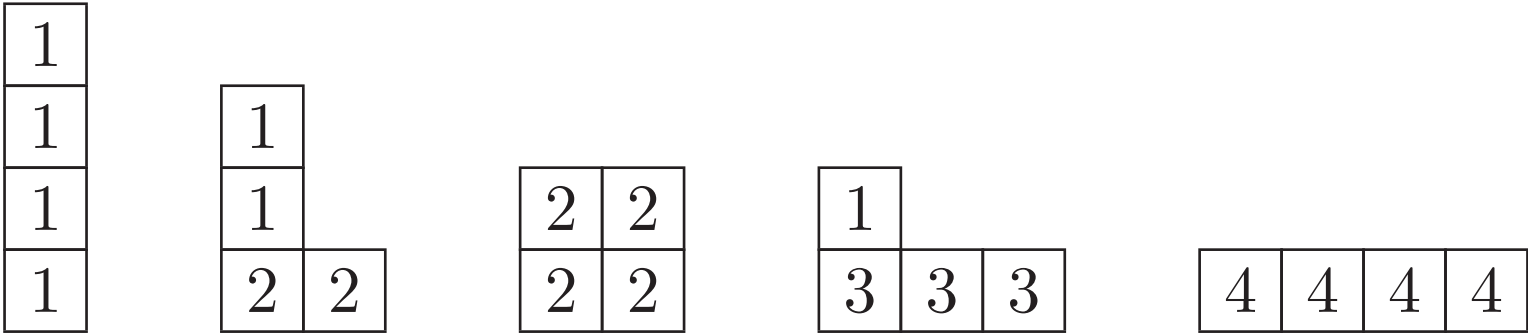
Let $n \geq k \geq 1$ be two integers. Then, for every positive $j < k$, the total number of occurrences of the part k among all partitions of n is equal to the number of boxes, whose hook type is $(j, k - j - 1)$.

Specializations, Power sum

Example:



hook lengths h_v



part lengths p_v

Specializations, Power sum

Let $m_k(\lambda)$ denote the number of parts in λ equal to k .

Corollary

$$\sum_{\lambda \vdash n} \sum_{h \in \mathcal{H}(\lambda)} h^\beta = \sum_{\lambda \vdash n} \sum_{k \geq 1} k^{\beta+1} m_k(\lambda).$$

Specializations, Power sum

$$\sum_{n \geq 1} q^n \sum_{\lambda \vdash n} \sum_{k \geq 1} k^\beta m_k(\lambda) = \prod_{m \geq 1} \frac{1}{1 - q^m} \times \sum_{k \geq 1} k^\beta \frac{q^k}{1 - q^k}.$$

Specializations, Power sum

Finally we obtain:

Theorem

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \sum_{h \in \mathcal{H}(\lambda)} h^\beta = \prod_{m \geq 1} \frac{1}{1 - q^m} \times \sum_{k \geq 1} k^{\beta+1} \frac{q^k}{1 - q^k}.$$

(Specialization: $\beta = -2$)

From equidistribution to symmetry distribution

$$\sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{h_v} = \sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{p_v}$$

$$7x + 6x^2 + 3x^3 + 4x^4$$

Theorem (H., Bessenrodt, 2009)

$$\sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{h_v} y^{p_v} = \sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{p_v} y^{h_v}$$

From equidistribution to symmetry distribution

For example, the joint distribution of h_v and p_v for the partitions of 4 is reproduced in the following tableau, which is symmetric.

$p \setminus h$	1	2	3	4	Σ
1	3	2	1	1	7
2	2	2	1	1	6
3	1	1	0	1	3
4	1	1	1	1	4
Σ	7	6	3	4	20

From equidistribution to symmetry distribution

$$A(q) = 1/(q; q)_m;$$

$$B(q) = \begin{bmatrix} l + a \\ a \end{bmatrix}_q;$$

$$C(q) = \frac{1}{(1 - q^{m+a+1})(1 - q^{m+a+2}) \dots} = \frac{(q; q)_{m+a}}{(q; q)_\infty};$$

$$D(q) = q^{(m+1)(l+1)+a}.$$

$$A(q)B(q)C(q)D(q) = \frac{(q; q)_a}{(q; q)_\infty} \begin{bmatrix} l + a \\ a \end{bmatrix}_q \begin{bmatrix} m + a \\ a \end{bmatrix}_q q^{(m+1)(l+1)+a}$$

Number Theory, Corollary

Corollary [$y = 1$]. *We have*

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(1 - \frac{tz}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^{tk})^z}{1 - x^k}.$$

Number Theory, Discrete interpolation

Discrete interpolation :

$$\sum_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}_1(\lambda)} \left(1 - \frac{36}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^k)^{36}}{1 - x^k};$$

$$\sum_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}_2(\lambda)} \left(1 - \frac{36}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^{2k})^{18}}{1 - x^k};$$

$$\sum_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}_3(\lambda)} \left(1 - \frac{36}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^{3k})^{12}}{1 - x^k};$$

$$\sum_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}_6(\lambda)} \left(1 - \frac{36}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^{6k})^6}{1 - x^k},$$

where each sum is over all 6-cores λ .

Third Conjecture (H. 2008)

Let n, s, t be positive integers such that $t \neq 4, 10$ and $s \mid t$.
Then the coefficient of x^n in

$$\prod_{k \geq 1} \frac{(1 - x^{sk})^{t^2/s}}{1 - x^k}$$

is equal to zero, if and only if the coefficient of x^n in

$$\prod_{k \geq 1} \frac{(1 - x^{tk})^t}{1 - x^k}$$

is also equal to zero.

Number Theory, $t = 2$

$t = 2$: Third conjecture is true.

Jacobi

$$\prod_{m \geq 1} \frac{(1 - x^m)^4}{(1 - x^m)} = \sum_{m \geq 0} (-1)^m (2m + 1) x^{m(m+1)/2}$$

Gauss

$$\prod_{m \geq 1} \frac{(1 - x^{2m})^2}{1 - x^m} = \sum_{m \geq 0} x^{m(m+1)/2}$$

Number Theory, Ramanujan and Lehmer

Ramanujan τ -function is defined by

$$\begin{aligned} x \prod_{m \geq 1} (1 - x^m)^{24} &= \sum_{n \geq 1} \tau(n) x^n \\ &= x - 24x^2 + 252x^3 - 1472x^4 + 4830x^5 - 6048x^6 + \dots \end{aligned}$$

Conjecture (Lehmer)

For each n we have $\tau(n) \neq 0$.

Specializations, $t = 5$

$t = 5$: Third conjecture becomes

Third Conjecture ($t = 5$)

Then the coefficient of x^n in

$$\prod_{k \geq 1} \frac{(1 - x^k)^{25}}{1 - x^k} = \prod_{k \geq 1} (1 - x^k)^{24}$$

is equal to zero, if and only if the coefficient of x^n in

$$\prod_{k \geq 1} \frac{(1 - x^{5k})^5}{1 - x^k}$$

is also equal to zero.

Granville and Ono ($t = 5$)

$$\prod_{k \geq 1} \frac{(1 - x^{5k})^5}{1 - x^k} = \sum_{n \geq 0} \alpha(n) x^n.$$

Then for each n we have $\alpha(n) \neq 0$.

Specializations, $t = 5$

$t = 5$: Third conjecture becomes Lehmer conjecture.

Number Theory, $t = 3$

$t = 3$:

$$\begin{aligned}\prod_{m \geq 1} (1 - x^m)^8 &= \sum_{n \geq 0} a(n)x^n \\ &= 1 - 8x + 20x^2 - 70x^4 + 64x^5 + 56x^6 - 125x^8 + \\ &\quad \dots - 20482x^{220} + 24050x^{224} - 21624x^{225} + \dots\end{aligned}$$

$$\begin{aligned}\prod_{m \geq 1} \frac{(1 - x^{3m})^3}{1 - x^m} &= \sum_{b \geq 0} b(n)x^n \\ &= 1 + x + 2x^2 + 2x^4 + x^5 + 2x^6 + x^8 + \\ &\quad \dots + 2x^{220} + 2x^{224} + 3x^{225} + \dots\end{aligned}$$

Number Theory, $t = 3$, Theorem

$t = 3$: Third conjecture is true.

Theorem (H., Ono, 2008)

$$a(n) = 0 \quad \text{if and only if} \quad b(n) = 0$$

Modular form, arithmetic

Binary trees: tangent numbers

The tangent number counts the *alternating permutations* (André, 1881), or the labeled complete binary trees.

Binary trees: tangent numbers

The tangent number counts the *alternating permutations* (André, 1881), or the labeled complete binary trees.

```
> hooktype:="CBT": # Complete Binary Trees  
> hookexp(tan(x)+sec(x), 9);
```

$$\left[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\right]$$

Binary trees: tangent numbers

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$$\sum_{T \in \mathcal{C}} x^{|T|} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = \tan(x) + \sec(x)$$

\mathcal{C} : complete binary trees

tangent numbers

```
> hooktype:="BT": # Binary Trees  
> hookexp(tan(x)+sec(x), 8);
```

$$\left[1, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}\right]$$

tangent numbers

```
> hooktype:="BT": # Binary Trees  
> hookexp(tan(x)+sec(x), 8);
```

$$\left[1, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}\right]$$

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T), h \geq 2} \frac{1}{2h} = \tan(x) + \sec(x)$$

tangent numbers

```
> hooktype:="BT": # Binary Trees  
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```

$$\left[1, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}\right]$$

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T), h \geq 2} \frac{1}{2h} = \tan(x) + \sec(x)$$

The tangent number counts *André permutations* (Foata, Schützenberger, Strehl, 1973).

Hook length formulas for plane trees, T1

Theorem T1 [$a = 0, z = 1, \text{ all permutations}$]. *We have*

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = \frac{1}{1-x}.$$

Hook length formulas for plane trees, T2

Theorem T2 [$a \rightarrow \infty, z = 1$, Catalan number]. *We have*

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T)} 1 = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Hook length formulas for plane trees, T3

Theorem T3 [$a = 1, z = 1$, Postnikov]. *We have*

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{h \in \mathcal{H}(T)} \left(1 + \frac{1}{h}\right) = \sum_{n \geq 0} (n+1)^{n-1} \frac{(2x)^n}{n!}.$$

Hook length formulas for plane trees, T4

Theorem T4 [$z = 1$, left-hand side extension of Postnikov identity (Lascoux, Du-Lu)]. *We have*

$$\sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \left(a + \frac{1}{h_v} \right) = \frac{1}{(n+1)!} \prod_{k=0}^{n-1} ((n+1+k)a + n+1-k).$$

Hook length formulas for plane trees, T5

Theorem T5 [$a = 1$, right-hand side extension of Postnikov identity (Han, 2008)]. *We have*

$$\sum_{T \in \mathcal{B}} x^{|T|} \prod_{v \in T} \frac{(z + h)^{h-1}}{h(2z + h - 1)^{h-2}} = \sum_{n \geq 0} z(z + n)^{n-1} \frac{(2x)^n}{n!}.$$

Hook length formulas for plane trees

Theorem TX(H., 2008)

We have

$$\begin{aligned} \sum_{T \in \mathcal{B}(n)} \prod_{h \in \mathcal{H}(T)} \frac{\prod_{i=1}^{h-1} (za + z + (2h - i)a + i)}{2h \prod_{i=1}^{h-2} (2za + 2z + (2h - 2 - i)a + i)} \\ = \frac{z(a+1)}{n!} \prod_{i=1}^{n-1} (za + z + (2n - i)a + i). \end{aligned}$$

Hook length formulas for plane trees

Theorem TX(H., 2008)

We have

$$\sum_{T \in \mathcal{B}(n)} \prod_{h \in \mathcal{H}(T)} \frac{\prod_{i=1}^{h-1} (za + z + (2h - i)a + i)}{2h \prod_{i=1}^{h-2} (2za + 2z + (2h - 2 - i)a + i)}$$
$$= \frac{z(a+1)}{n!} \prod_{i=1}^{n-1} (za + z + (2n - i)a + i).$$

♥♥♥ Theorem TX unifies a lot of well known hook formulas by taking special values of a and z , including Theorems T1, T2, T3, T4 and T5.

Latest news on the subject

L. Yang, B. Sagan, W. Chen, O. Gao, P. Guo, N. Eriksen
have found generalizations and other proofs of certain
hook length formulas for plane trees

- Discovering hook length formulas by an expansion technique
- New hook length formulas for binary trees
- Yet another generalization of Postnikov's hook length formula for binary trees
- Some conjectures and open problems on partition hook lengths
- An explicit expansion formula for the powers of the Euler product in terms of partition hook lengths (arXiv exclusive)
- The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications
- (with K. Ono) Hook lengths and 3-cores
- Hook lengths and shifted parts of partitions
- (with K. Ji) Combining hook length formulas and BG-ranks for partitions via the Littlewood decomposition
- (with Ch. Bessenrodt) Symmetry distribution between hook length and part length for partitions

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- H. Shin, J. Zeng, An involution for symmetry of hook length and part length of partitions
- N. Eriksen, Combinatorial proofs for some forest hook length identities

Merci !

`www-irma.u-strasbg.fr/~guoniu/hook`