Some Combinatorial aspects of Nonequilibrium Statistical Mechanics

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Introduction

The statistical mechanics of a system at thermal equilibrium is encoded in the Boltzmann-Gibbs canonical law:

$$P_{\rm eq}(\mathcal{C}) = \frac{{\rm e}^{-E(\mathcal{C})/kT}}{Z}$$

the Partition Function Z being related to the Thermodynamic Free Energy F:

$$F = -kT Log\, Z$$

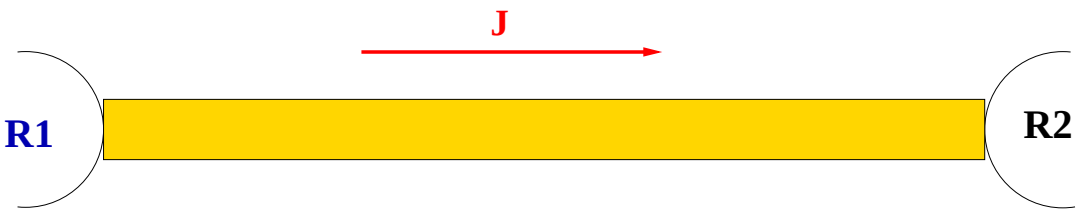
This provides us with a well-defined prescription to analyze systems *at equilibrium*:
(i) Observables are mean values w.r.t. the canonical measure.
(ii) Statistical Mechanics predicts fluctuations (typically Gaussian) that are out of reach of Classical Thermodynamics.

Systems far from equilibrium

No fundamental theory is yet available.

- What are the relevant macroscopic parameters?
- Which functions describe the state of a system?
- Do Universal Laws exist? Can one define Universality Classes?
- Can one postulate a general form for the microscopic measure?
- What do the fluctuations look like ('non-gaussianity')?

Example: Stationary driven systems in contact with reservoirs.



Rare Events and Large Deviations

Let $\epsilon_1, \ldots, \epsilon_N$ be N independent binary variables, $\epsilon_k = \pm 1$, with probability p (resp. $q = 1 - p$). Their sum is denoted by $S_N = \sum_1^N \epsilon_k$.

- The Law of Large Numbers implies that $S_N/N \to p - q$ a.s.
- The Central Limit Theorem implies that $[S_N - N(p-q)]/\sqrt{N}$ converges towards a Gaussian Law.

One can show that for $-1 < r < 1$, in the large N limit,

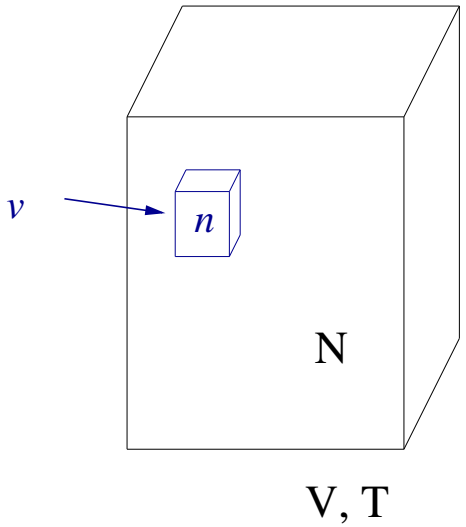
$$\Pr\left(\frac{S_N}{N} = r\right) \sim \mathrm{e}^{-N\,\Phi(r)}$$

where the positive function $\Phi(r)$ vanishes for $r = (p - q)$.

The function $\Phi(r)$ is a Large Deviation Function: it encodes the probability of rare events.

$$\Phi(r) = \frac{1+r}{2} \ln\left(\frac{1+r}{2p}\right) + \frac{1-r}{2} \ln\left(\frac{1-r}{2q}\right)$$

Density fluctuations in a gas



Mean Density $\rho_0 = \frac{N}{V}$

In a volume v s. t. $1 \ll v \ll V$
$\langle \frac{n}{v} \rangle = \rho_0$

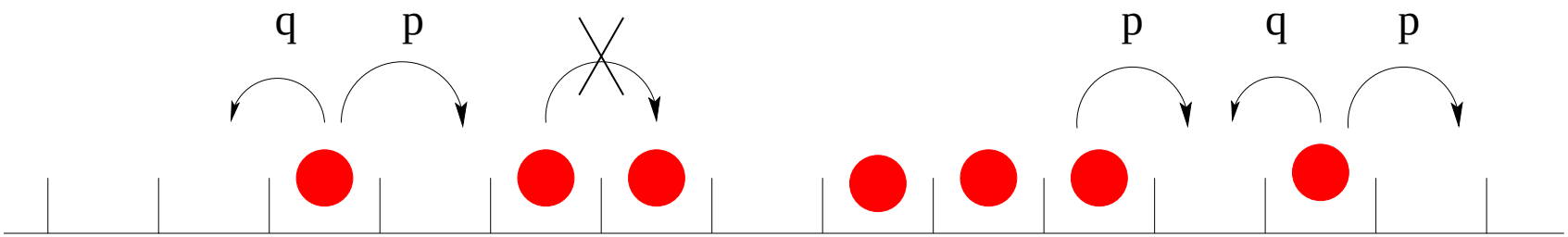
The local density varies around ρ_0 . Typical fluctuations scale as $\sqrt{v/V}$.

The probability of observing large fluctuations is given by

$$\Pr\left(\frac{n}{v} = \rho\right) \sim \mathrm{e}^{-v\,\Phi(\rho)} \quad \text{with} \quad \Phi(\rho_0) = 0$$

This Large Deviation Function for density fluctuations is related to the *free energy per unit volume.*
Large deviation functions *may* play the role of potentials in non-equilibrium statistical mechanics.

Classical Transport in 1d: ASEP



Asymmetric Exclusion Process: A *minimal model* to represent many phenomena.
A paradigm for non-equilibrium Statistical Mechanics.

- EXCLUSION: Hard core-interaction; at most 1 particle per site.
- ASYMMETRIC: External driving; breaks detailed-balance
- PROCESS: Stochastic Markovian dynamics; no Hamiltonian.

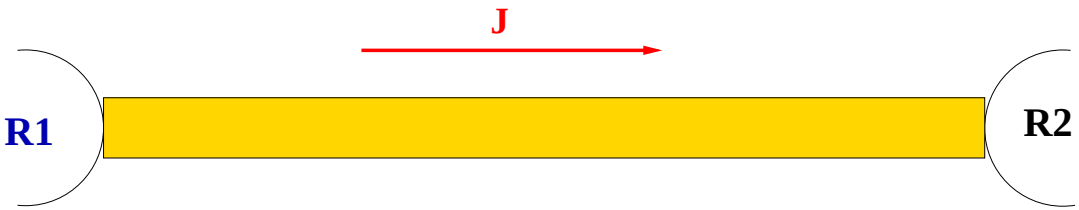
The probability $P_t(\mathcal{C})$ to find the system in the microscopic configuration $\mathcal{C}$ at time t satisfies

$$\frac{dP_t(\mathcal{C})}{dt} = MP_t(\mathcal{C})$$

The Markov Matrix M: transitions rates amongst configurations.

Total Current transported through an Open System

A paradigm of a non-equilibrium system



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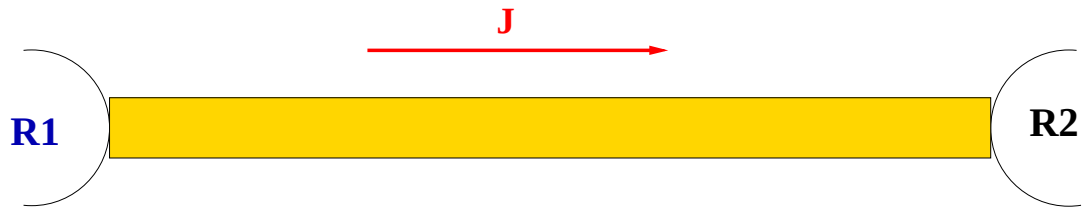
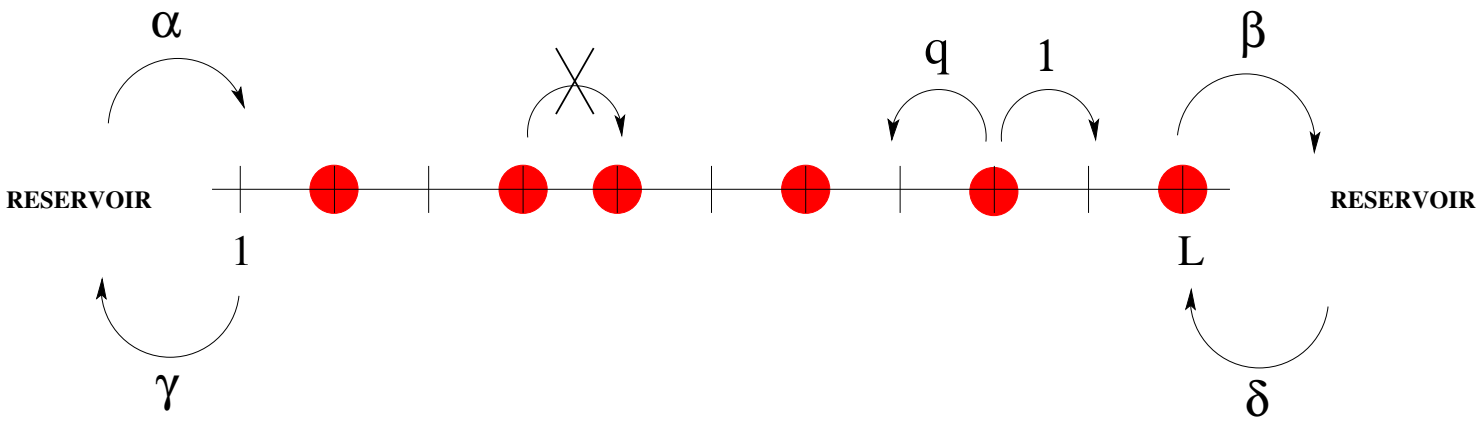


The asymmetric exclusion model with open boundaries



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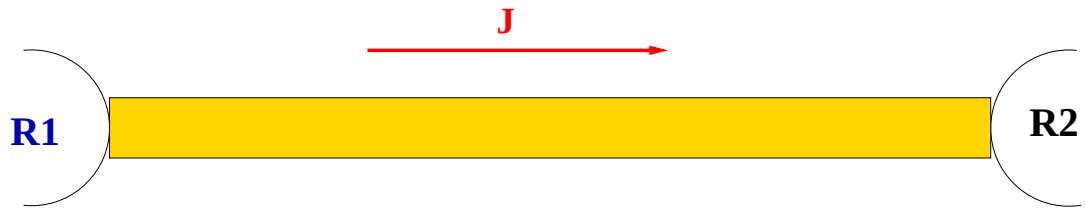

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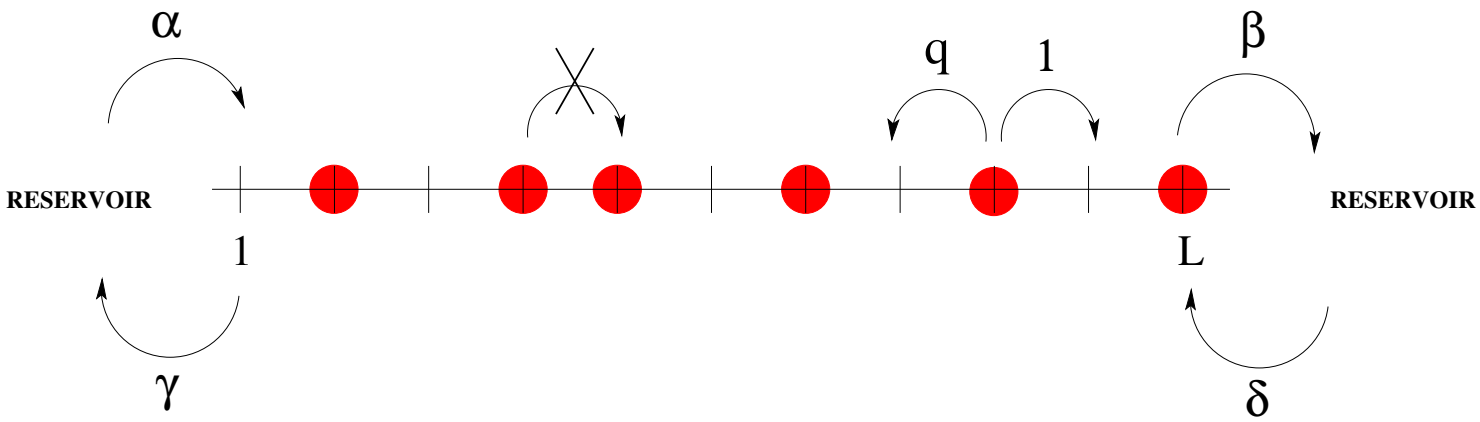


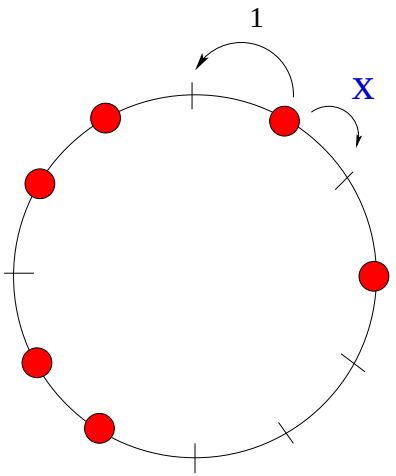
Our aim is to study the statistics of the current and its large deviations starting from the microscopic model.

Outline

1. The Exclusion Process: an Integrable model
2. Fluctuations of the current in an open system
3. Growth and Shapes in 2 and 3 dimensions

1. ASEP on a ring and Bethe Ansatz

Markov Equation for the ASEP on a ring



L SITES

N PARTICLES

$$\Omega = \binom{L}{N}$$

CONFIGURATIONS

x asymmetry parameter

Master Equation for the Probability $P_t(x_1, \ldots, x_N)$ of being in configuration $1 \leq x_1 < \ldots < x_N \leq L$ at time t.

$$\begin{aligned} \frac{\mathrm{dP_t}}{\mathrm{d}t} &= \sum_i{}' \left[P_t(x_1, \ldots, x_i - 1, \ldots, x_N) - P_t(x_1, \ldots, x_i, \ldots x_N)\right] \\ &+ x \sum_i{}' \left[P_t(x_1, \ldots, x_i + 1, \ldots, x_N) - P_t(x_1, \ldots, x_i, \ldots x_N)\right] \\ &= MP\,. \end{aligned}$$

The sum being restricted to admissible configurations.

The Bethe Ansatz

Eigenvector ψ of M written as a linear combination of plane waves, with pseudo-momenta given by $z_1, \ldots z_N$:

$$\psi(x_1, \ldots, x_N) = \sum_{\sigma \in \Sigma_N} \mathcal{A}_\sigma \prod_{i=1}^{N} z_{\sigma(i)}^{x_i}$$

The Bethe Equations provide us with the quantification of the z_i's:

$$z_i^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{x z_i z_j - (1+x) z_i + 1}{x z_i z_j - (1+x) z_j + 1}$$

The corresponding eigenvalue is given by

$$E(z_1, z_2 \ldots z_N) = \sum_{i=1}^{N} \frac{1}{z_i} + x \sum_{i=1}^{N} z_i - N(1+x)\,.$$

Large Deviations of the Current

Total current Y_t, *total distance covered by all the N particles, hopping on a ring of size L, between time 0 and time t.*

WHAT IS THE STATISTICS of Y_t?

One can prove that when $t \to \infty$:

$$\boxed{\left\langle e^{\mu Y_t} \right\rangle \simeq e^{E(\mu) t}}$$

The cumulant generating function $E(\mu)$ is the eigenvalue with maximal real part of the deformed operator $M(\mu)$

$$M(\mu) = M_0 + e^{\mu} M_+ + e^{-\mu} M_-$$

the Markov operator being splitted $M = M_0 + M_+ + M_-$ into positive, negative, null jumps.
Besides, *the large deviation function* $\Phi(j)$ of the current

$$P\left(\frac{Y_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

is related to $E(\mu)$ by a Legendre transform: $\boxed{E(\mu) = \max_j \left(\mu j - \Phi(j)\right)}$

Bethe Ansatz for current statistics

The current statistics is reduced to an eigenvalue problem, solvable by Bethe Ansatz.

The Bethe Equations are given by

$$z_i^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{x\mathrm{e}^{-\mu} z_i z_j - (1+x) z_i + \mathrm{e}^{\mu}}{x\mathrm{e}^{-\mu} z_i z_j - (1+x) z_j + \mathrm{e}^{\mu}}$$

The eigenvalues of $M(\mu)$ are

$$E(\mu; z_1, z_2 \ldots z_N) = \mathrm{e}^{\mu} \sum_{i=1}^{N} \frac{1}{z_i} + x\mathrm{e}^{-\mu} \sum_{i=1}^{N} z_i - N(1+x)\,.$$

The Bethe equations do not decouple unless $x = 0$ *(This is the TASEP case, solved by B. Derrida and J. L. Lebowitz, 1998).*

General solution (S. Prolhac, 2010)

The generating function $E(\mu)$ is obtained in a parametric form:

$$\mu = -\sum_{k\geq 1} C_k \frac{B^k}{k} \quad \text{and} \quad E = -(1-x)\sum_{k\geq 1} D_k \frac{B^k}{k}$$

C_k and D_k are combinatorial factors enumerating some tree structures. There exists an auxiliary function

$$W_B(z) = \sum_{k\geq 1} \phi_k(z) \frac{B^k}{k}$$

such that C_k and D_k are given by complex integrals along a small contour that encircles 0 :

$$C_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{z} \quad \text{and} \quad D_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{(z+1)^2}$$

The function $W_B(z)$ contains the full information about the statistics of the current.

The function $W_B(z)$ is the solution of a functional Bethe equation:

$$\boxed{W_B(z) = -\ln\left(1 - BF(z)e^{X[W_B](z)}\right)}$$

where

$$\boxed{F(z) = \frac{(1+z)^L}{z^N}}$$

The operator X is a integral operator

$$X[W_B](z_1) = \oint_C \frac{dz_2}{i2\pi\, z_2} W_B(z_2) K(z_1, z_2)$$

with the kernel

$$\boxed{K(z_1, z_2) = 2\sum_{k=1}^{\infty} \frac{x^k}{1-x^k} \left\{ \left(\frac{z_1}{z_2}\right)^k + \left(\frac{z_2}{z_1}\right)^k \right\}}$$

Solving this Functional Bethe Ansatz equation to all orders enables us to calculate cumulant generating function. For $x = 0$, the TASEP result is readily retrieved.

The function $W_B(z)$ also contains information on the 6-vertex model associated with the ASEP.

From the Physics point of view, the solution allows one to

- Classify the different universality classes (KPZ, EW).
- Study the various scaling regimes.
- Investigate the hydrodynamic behaviour.

Cumulants of the Current

- Mean Current: $J = (1-x)\frac{N(L-N)}{L-1} \sim (1-x)L\rho(1-\rho)$ for $L \to \infty$

- Diffusion Constant: $D = (1-x)\frac{2L}{L-1}\sum_{k>0} k^2 \frac{C_L^{N+k}}{C_L^N}\frac{C_L^{N-k}}{C_L^N}\left(\frac{1+x^k}{1-x^k}\right)$

- Third cumulant (Skewness): $\to$ Non Gaussian fluctuations.

$$E_3 \simeq \left(\frac{3}{2} - \frac{8}{3\sqrt{3}}\right)\pi(\rho(1-\rho))^2 L^3$$

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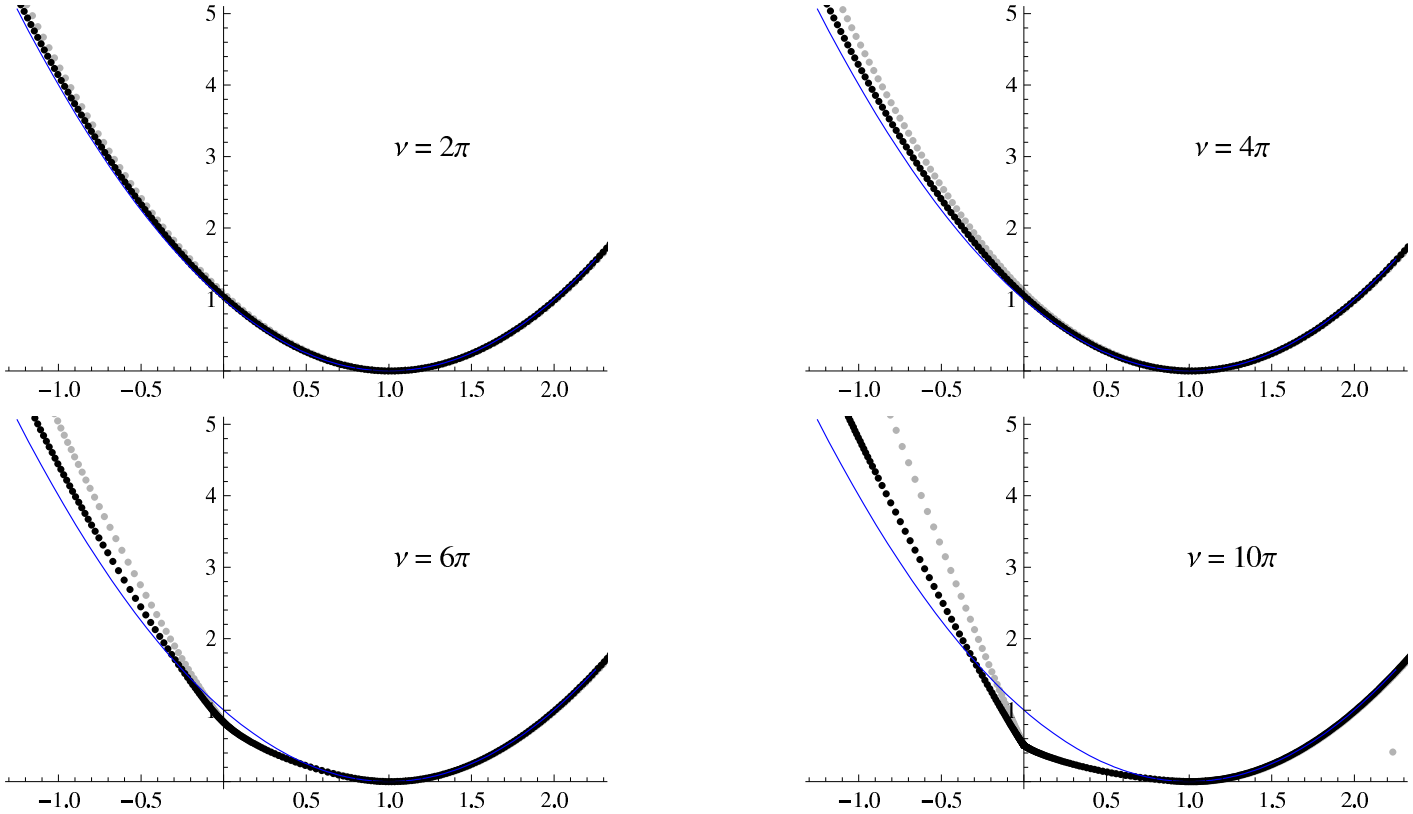
- Third cumulant (Skewness): $\to$ Non Gaussian fluctuations.

$$E_3 \simeq \left(\frac{3}{2} - \frac{8}{3\sqrt{3}}\right)\pi(\rho(1-\rho))^2 L^3$$

$$\begin{aligned}
\frac{E_3}{6L^2} &= \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N-i}C_L^{N+j}C_L^{N-j}}{(C_L^N)^4}(i^2+j^2)\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N+j}C_L^{N-i-j}}{(C_L^N)^3}\frac{i^2+ij+j^2}{2}\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N-i}C_L^{N-j}C_L^{N+i+j}}{(C_L^N)^3}\frac{i^2+ij+j^2}{2}\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\frac{C_L^{N+i}C_L^{N-i}}{(C_L^N)^2}\frac{i^2}{2}\left(\frac{1+x^i}{1-x^i}\right)^2 + \frac{(1-x)N(L-N)}{4(L-1)(2L-1)}\frac{C_{2L}^{2N}}{(C_L^N)^2} \\
&+ \frac{(1-x)N(L-N)}{4(L-1)(2L-1)}\frac{C_{2L}^{2N}}{(C_L^N)^2} - \frac{(1-x)N(L-N)}{6(L-1)(3L-1)}\frac{C_{3L}^{3N}}{(C_L^N)^3}
\end{aligned}$$

Phase transition (weak asymmetry)



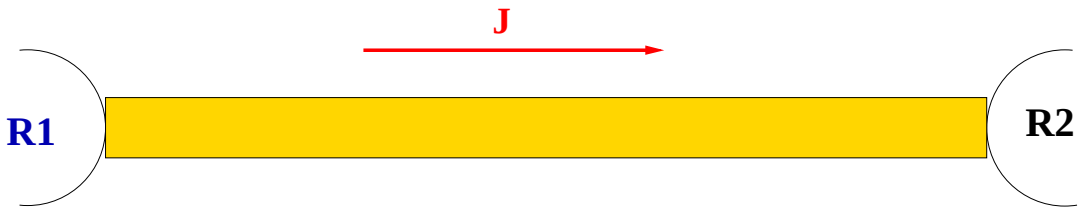
$$E\left(\frac{\mu}{L}\right) \simeq \frac{\rho(1-\rho)(\mu^2+\mu\nu)}{L} - \frac{\rho(1-\rho)\mu^2\nu}{2L^2} + \frac{1}{L^2}\psi[\rho(1-\rho)(\mu^2+\mu\nu)]$$

$$\text{with} \quad \psi(z) = \sum_{k=1}^{\infty} \frac{B_{2k-2}}{k!(k-1)!} z^k$$

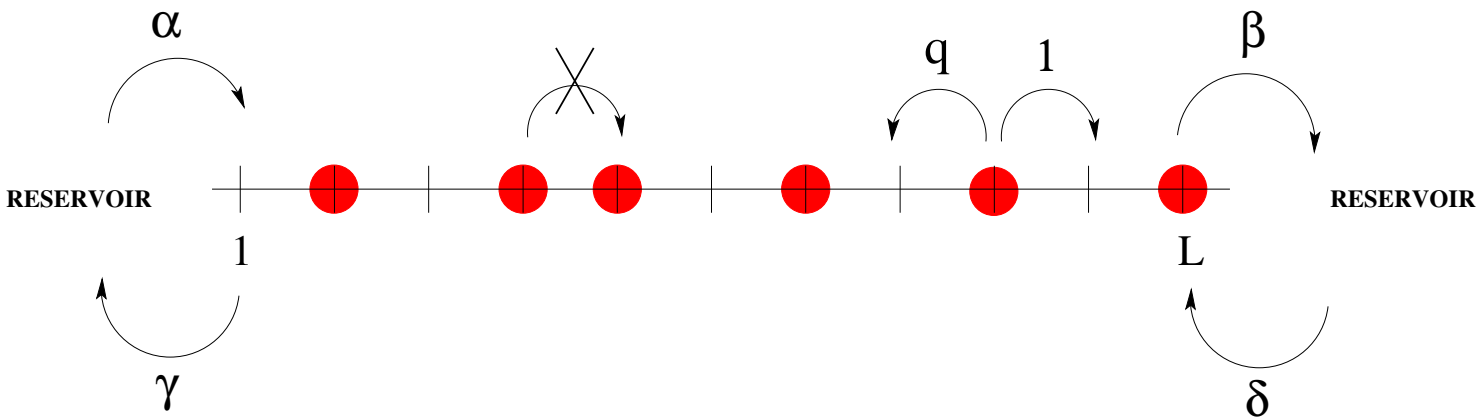
2. Current Fluctuations in the open ASEP

The Current in the Open System

The fundamental paradigm

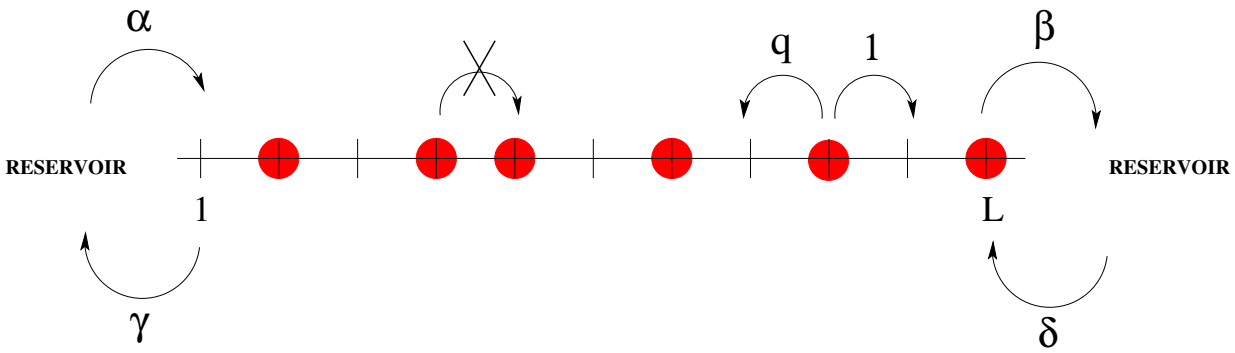


The asymmetric exclusion model with open boundaries



NB: the asymmetry parameter in now denoted by q.

Matrix Ansatz for ASEP (DEHP, 1993)



The stationary probability of a configuration $\mathcal{C}$ is given by

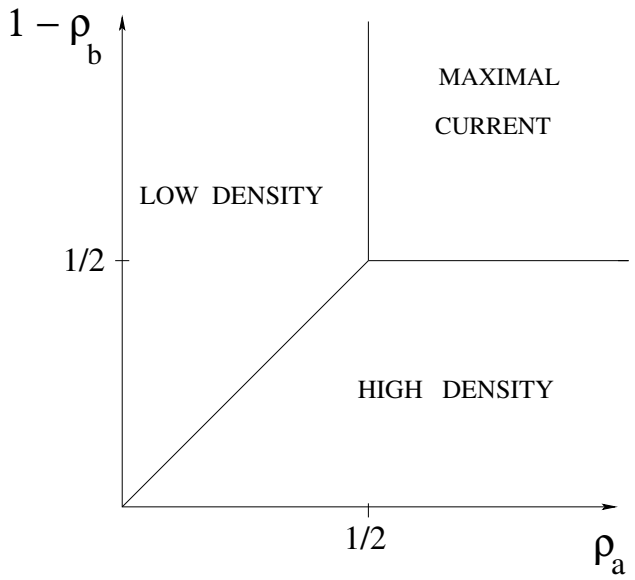
$$P(\mathcal{C}) = \frac{1}{Z_L} \langle W | \prod_{i=1}^{L} (\tau_i D + (1-\tau_i) E) | V \rangle$$

where $\tau_i = 1$ (or 0) if the site i is occupied (or empty) and the normalization constant is $Z_L = \langle W | (D+E)^L | V \rangle$

The operators D and E, the vectors $\langle W|$ and $|V\rangle$ satisfy

$$\begin{aligned} D\,E - qED &= (1-q)(D+E) \\ (\beta\, D - \delta\, E)\,|V\rangle &= |V\rangle \\ \langle W|(\alpha\, E - \gamma\, D) &= \langle W| \end{aligned}$$

The Phase Diagram



$\rho_a = \frac{1}{a_+ + 1}$: effective left reservoir density.

$\rho_b = \frac{b_+}{b_+ + 1}$: effective right reservoir density.

$$a_\pm = \frac{(1 - q - \alpha + \gamma) \pm \sqrt{(1 - q - \alpha + \gamma)^2 + 4\alpha\gamma}}{2\alpha}$$

$$b_\pm = \frac{(1 - q - \beta + \delta) \pm \sqrt{(1 - q - \beta + \delta)^2 + 4\beta\delta}}{2\beta}$$

Representations of the quadratic algebra

The algebra encodes combinatorial recursion relations between systems of different sizes.

Infinite dimensional Representation:

$D = 1 + d$ where d is a q-destruction operator.

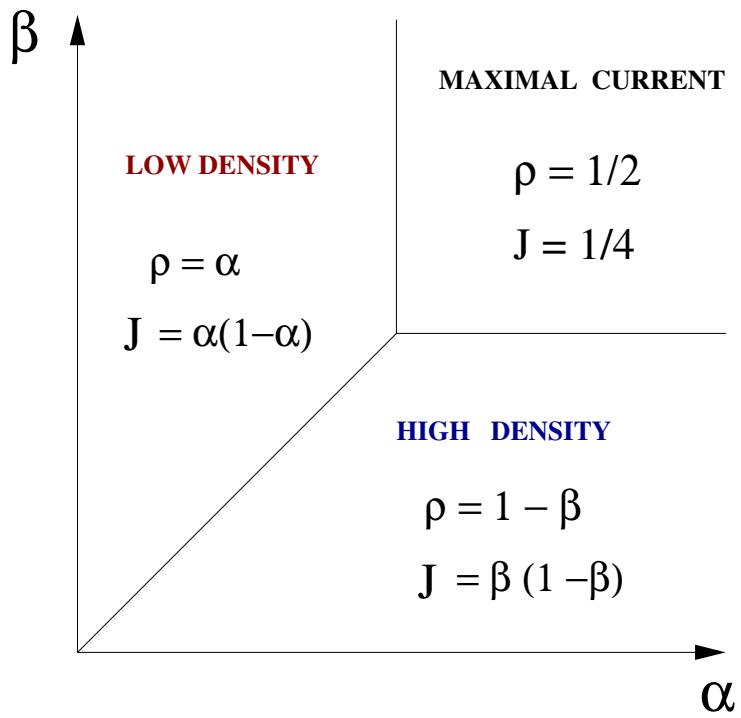
$E = 1 + e$ where e is a q-creation operator.

$$d = \begin{pmatrix} 0 & \sqrt{1-q} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{1-q^2} & 0 & \dots \\ 0 & 0 & 0 & \sqrt{1-q^3} & \dots \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad e = d^\dagger$$

The matrix Ansatz allows one to calculate Stationary State Properties (currents, correlations, fluctuations) and to derive the Phase Diagram in the infinite size limit.

TASEP CASE (q =0)

In the TASEP case, the matrices d and e are shift-operators. (The algebra becomes simpler).



Total Current

The observable Y_t counts the total number of particles exchanged between the system and the left reservoir between times 0 and t.

Hence, $Y_{t+dt} = Y_t + y$ with

- $y = +1$ if a particle enters at site 1 (at rate α),
- $y = -1$ if a particle exits from 1 (at rate γ)
- $y = 0$ if no particle exchange with the left reservoir has occurred during dt.

Statistical properties of Y_t:

- Average current: $J(q, \alpha, \beta, \gamma, \delta, L) = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t}$
  It can be calculated by the steady-state matrix Ansatz $J = \frac{Z_{L-1}}{Z_L}$.
- Current fluctuations: $\Delta(q, \alpha, \beta, \gamma, \delta, L) = \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t}$
  The fluctuations of the total current. It does not depend on the stationary measure only.
- Cumulant Generating Function: $\left\langle e^{\mu Y_t} \right\rangle \simeq e^{E(\mu)t}$ for $t \to \infty$ $E(\mu)$ encodes the statistical properties of the total current.

Structure of the solution I

For arbitrary values of q and $(\alpha, \beta, \gamma, \delta)$, and for any system size L the parametric representation of $E(\mu)$ is given by

$$\begin{aligned} \mu &= -\sum_{k=1}^{\infty} C_k(q;\alpha,\beta,\gamma,\delta,L)\frac{B^k}{2k} \\ E &= -\sum_{k=1}^{\infty} D_k(q;\alpha,\beta,\gamma,\delta,L)\frac{B^k}{2k} \end{aligned}$$

The coefficients C_k and D_k are given by contour integrals in the complex plane:

$$C_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi}\frac{\phi_k(z)}{z} \quad \text{and} \quad D_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi}\frac{\phi_k(z)}{(z+1)^2}$$

There exists an auxiliary function

$$W_B(z) = \sum_{k\geq 1} \phi_k(z)\frac{B^k}{k}$$

that contains the full information about the statistics of the current.

Structure of the solution II

This auxiliary function $W_B(z)$ solves a functional Bethe equation:

$$\boxed{W_B(z) = -\ln\left(1 - BF(z)e^{X[W_B](z)}\right)}$$

- The operator X is a integral operator

$$X[W_B](z_1) = \oint_{\mathcal{C}} \frac{dz_2}{\imath 2\pi\, z_2} W_B(z_2) K\left(\frac{z_1}{z_2}\right)$$

$$\boxed{\text{with kernel} \quad K(z) = 2\sum_{k=1}^{\infty} \frac{q^k}{1-q^k}\left\{z^k + z^{-k}\right\}}$$

- The function $F(z)$ is given by

$$\boxed{F(z) = \frac{(1+z)^L(1+z^{-1})^L(z^2)_\infty(z^{-2})_\infty}{(a_+z)_\infty(a_+z^{-1})_\infty(a_-z)_\infty(a_-z^{-1})_\infty(b_+z)_\infty(b_+z^{-1})_\infty(b_-z)_\infty(b_-z^{-1})_\infty}}$$

where $(x)_\infty = \prod_{k=0}^{\infty}(1-q^k x)$ and $a_\pm$, $b_\pm$ depend on the boundary rates.

- The complex contour $\mathcal{C}$ encircles 0, $q^k a_+, q^k a_-, q^k b_+, q^k b_-$ for $k \geq 0$.

Discussion

- These results are of *combinatorial nature: valid for arbitrary values of the parameters and for any system sizes with no restrictions.*
- *Average-Current:*

$$J = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = (1-q)\frac{D_1}{C_1} = (1-q)\frac{\oint_\Gamma \frac{dz}{2\,i\,\pi}\frac{F(z)}{z}}{\oint_\Gamma \frac{dz}{2\,i\,\pi}\frac{F(z)}{(z+1)^2}}$$

(cf. T. Sasamoto, 1999.)

- *Diffusion Constant:*

$$\Delta = \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = (1-q)\frac{D_1 C_2 - D_2 C_1}{2C_1^3}$$

where C_2 and D_2 are obtained using

$$\phi_1(z) = \frac{F(z)}{2} \quad \textit{and} \quad \phi_2(z) = \frac{F(z)}{2}\left(F(z) + \oint_\Gamma \frac{dz_2 F(z_2) K(z/z_2)}{2\imath\pi z_2}\right)$$

(cf. the TASEP case: B. Derrida, M. R. Evans, K. M., 1995)

Asymptotic behaviour

- Maximal Current Phase:

$$\mu = -\frac{L^{-1/2}}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(2k)!}{k!k^{(k+3/2)}} B^k$$

$$\mathcal{E} - \frac{1-q}{4}\mu = -\frac{(1-q)L^{-3/2}}{16\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(2k)!}{k!k^{(k+5/2)}} B^k$$

- Low Density (and High Density) Phases:
  Dominant singularity at a_+: $\phi_k(z) \sim F^k(z)$. By Lagrange Inversion:

$$E(\mu) = (1-q)(1-\rho_a)\frac{e^{\mu}-1}{e^{\mu}+(1-\rho_a)/\rho_a}$$

(cf de Gier and Essler, 2011).

Current Large Deviation Function:

$$\Phi(j) = (1-q)\left\{\rho_a - r + r(1-r)\ln\left(\frac{1-\rho_a}{\rho_a}\frac{r}{1-r}\right)\right\}$$

where the current j is parametrized as $j = (1-q)r(1-r)$.
Matches the predictions of Macroscopic Fluctuation Theory, as observed by T. Bodineau and B. Derrida.

The TASEP case

Here $q = \gamma = \delta = 0$ and (α, β) are arbitrary.
The parametric representation of $E(\mu)$ is

$$\mu = -\sum_{k=1}^{\infty} C_k(\alpha,\beta)\frac{B^k}{2k}$$

$$E = -\sum_{k=1}^{\infty} D_k(\alpha,\beta)\frac{B^k}{2k}$$

with

$$C_k(\alpha,\beta) = \oint_{\{0,a,b\}} \frac{dz}{2i\pi}\frac{F(z)^k}{z} \quad \text{and} \quad D_k(\alpha,\beta) = \oint_{\{0,a,b\}} \frac{dz}{2i\pi}\frac{F(z)^k}{(1+z)^2}$$

where

$$F(z) = \frac{-(1+z)^{2L}(1-z^2)^2}{z^L(1-az)(z-a)(1-bz)(z-b)}\,, \qquad a = \frac{1-\alpha}{\alpha}\,, \qquad b = \frac{1-\beta}{\beta}$$

Special case of TASEP

In the case $\alpha = \beta = 1$, a parametric representation of the cumulant generating function $E(\mu)$:

$$\mu = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)]!}{[k(L+1)]!\,[k(L+2)]!} \frac{B^k}{2k} ,$$

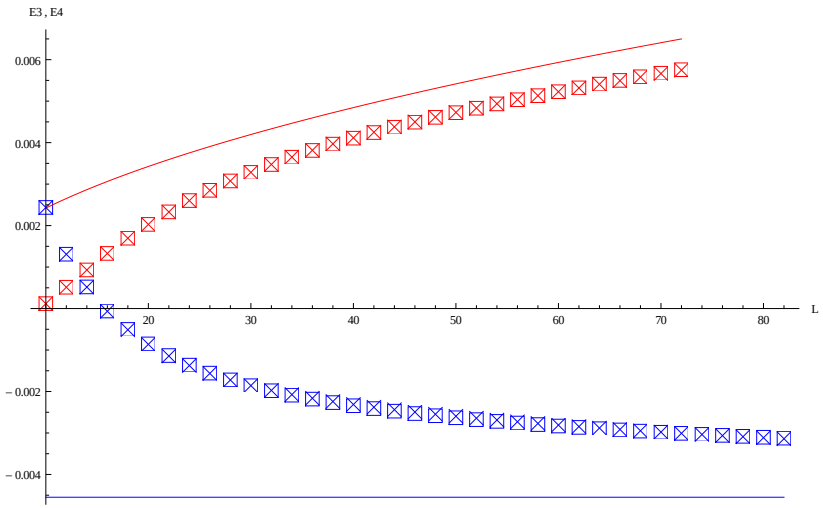
$$E = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)-2]!}{[k(L+1)-1]!\,[k(L+2)-1]!} \frac{B^k}{2k} .$$

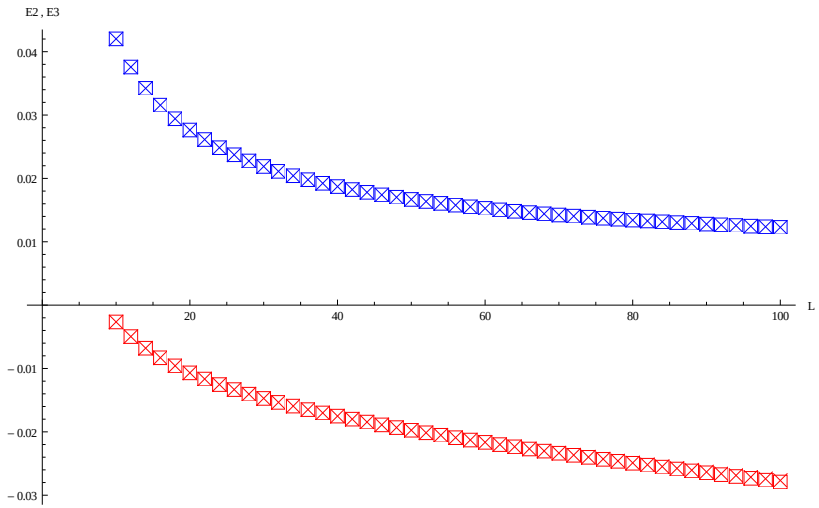
First cumulants of the current

- Mean Value : $J = \frac{L+2}{2(2L+1)}$
- Variance : $\Delta = \frac{3}{2} \frac{(4L+1)![L!(L+2)!]^2}{[(2L+1)!]^3(2L+3)!}$
- Skewness :
  $E_3 = 12 \frac{[(L+1)!]^2[(L+2)!]^4}{(2L+1)[(2L+2)!]^3} \left\{ 9 \frac{(L+1)!(L+2)!(4L+2)!(4L+4)!}{(2L+1)![(2L+2)!]^2[(2L+4)!]^2} - 20 \frac{(6L+4)!}{(3L+2)!(3L+6)!} \right\}$

  For large systems: $E_3 \to \frac{2187-1280\sqrt{3}}{10368}\pi \sim -0.0090978...$

Numerical results (DMRG)





Left: **Max. Current** ($q = 0.5$, $a_+ = b_+ = 0.65$, $a_- = b_- = 0.6$), Third and Fourth cumulant.

Right: **High Density** ($q = 0.5$, $a_+ = 0.28$, $b_+ = 1.15$, $a_- = -0.48$ and $b_- = -0.27$), Second and Third cumulant.

A. Lazarescu and K. Mallick, J. Phys. A 44, 315001 (2011).
M. Gorissen, A. Lazarescu, K. M. and C. Vanderzande, PRL **109** 170601 (2012)

Current Statistics: Mathematical Framework

These three mutually exclusive types of transitions lead to a splitting the Markov operator:

$$M = M_0 + M_+ + M_-$$

- M_0 corresponds to transitions that do not modify the value of Y.
- M_+ are transitions that increment Y by 1: a particle enters the system from the left reservoir.
- M_- encodes rates in which Y decreases by 1, if a particle exits the system from the left reservoir (does not happen in the simplest TASEP case).

The cumulant-generating function $E(\mu)$ when $t \to \infty$, $\left\langle \mathrm{e}^{\mu Y_t} \right\rangle \simeq \mathrm{e}^{E(\mu)t}$, is the dominant eigenvalue of the deformed matrix

$$M(\mu) = M_0 + \mathrm{e}^{\mu} M_+ + \mathrm{e}^{-\mu} M_-$$

The current statistics is reduced to an eigenvalue problem, solvable by Bethe Ansatz, or by Matrix Ansatz.

Generalized Matrix Ansatz

One can prove that the dominant eigenvector of the deformed matrix $M(\mu)$ is given by the following matrix product representation:

$$F_\mu(\mathcal{C}) = \frac{1}{Z_L^{(k)}} \langle W_k | \prod_{i=1}^{L} (\tau_i D_k + (1-\tau_i) E_k) \, | V_k \rangle + \mathcal{O}\left(\mu^{k+1}\right)$$

The matrices D_k and E_k are constructed recursively (knowing D_1 and E_1)

$$\begin{aligned} D_{k+1} &= (1 \otimes 1 + d \otimes e) \otimes D_k + (1 \otimes d + d \otimes 1) \otimes E_k \\ E_{k+1} &= (1 \otimes 1 + e \otimes d) \otimes E_k + (e \otimes 1 + 1 \otimes e) \otimes D_k \end{aligned}$$

The boundary vectors $\langle W_k|$ and $|V_k\rangle$ are also obtained recursively:

$$|V_k\rangle = |\beta\rangle |\tilde{V}\rangle |V_{k-1}\rangle \quad \text{and} \quad \langle W_k| = \langle W^\mu | \langle \tilde{W}^\mu | \langle W_{k-1}|$$

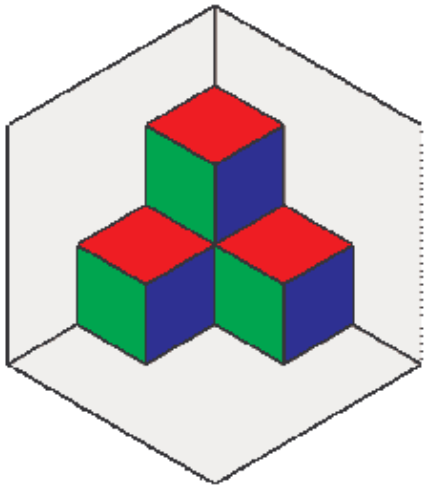
$$[\beta(1-d) - \delta(1-e)] \, |\tilde{V}\rangle = 0$$

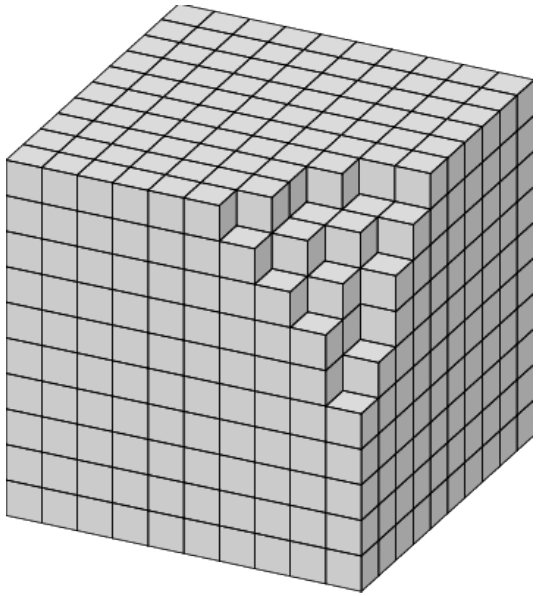
$$\langle W^\mu | [\alpha(1 + \mathrm{e}^{\mu} \, e) - \gamma(1 + \mathrm{e}^{-\mu} \, d)] = (1-q) \langle W^\mu |$$

$$\langle \tilde{W}^\mu | [\alpha(1 - \mathrm{e}^{\mu} \, e) - \gamma(1 - \mathrm{e}^{-\mu} \, d)] = 0$$

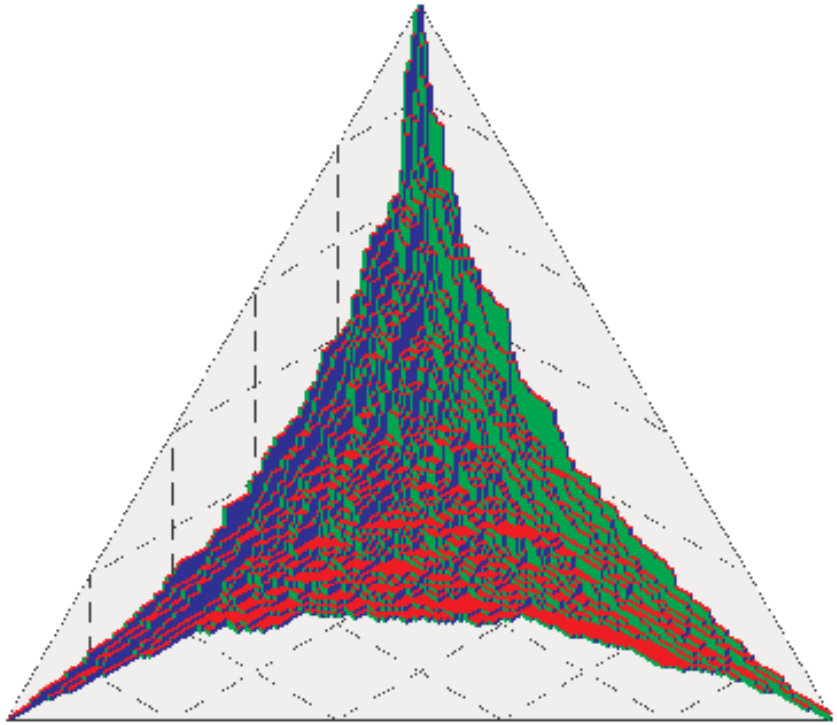
3. Shapes of Growth

Corner Growth/Melting in three dimensions





Corner Growth in three dimensions



3d Plane Partitions with Uniform Measure

In a plane partition, the height satisfies

$$0 \leq h(k+1,l) \leq h(k,l) \quad \text{and} \quad 0 \leq h(k,l+1) \leq h(k,l)$$

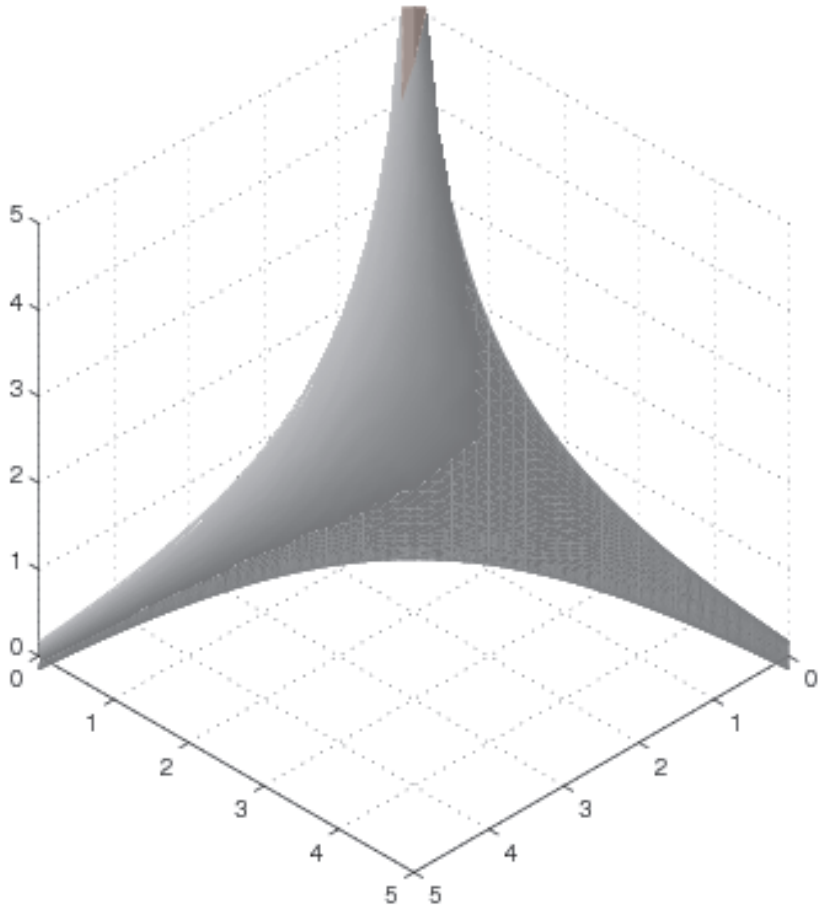
Consider plane partitions of total volume N, chosen randomly with uniform probability measure. The limiting shape of the corner crystal was found by R. Cerf and R. Kenyon, and also by A. Okounkov (ca 2000).

The limiting surface is $\left(\frac{\zeta(3)}{4}\right)^{-1/3} S_0$ (after rescaling the coordinates by $N^{1/3}$), S_0 being given by the parametric equations

$$S_0 = \{(f(A,B,C) - \ln A,\ f(A,B,C) - \ln B,\ f(A,B,C) - \ln C)\}$$

A, B, C are strictly positive and

$$f(A,B,C) = \frac{1}{4\pi^2} \int_{[0,2\pi]} \int_{[0,2\pi]} \ln \left| A + B\mathrm{e}^{\mathrm{i}\theta} + \mathrm{Ce}^{\mathrm{i}\phi} \right| d\theta d\phi$$


5
4
3
2
1
0
0
1
2
3
4
5
0
1
2
3
4
5

Growing Plane Partitions

What is the limit shape for a Growing Plane Partition (Glauber dynamics with non-zero magnetic field)?

We have conjectured the following governing equation for the interface profile:

$$z_t = \frac{z_x}{z_x - 1} \frac{z_y}{z_y - 1} \left[1 - \frac{1}{z_x + z_y} \right]$$

- We seek an equation of the form $z_t = F(z_x, z_y)$ involving only first derivatives.
- The correct 3d equation has to reduce to the 2d equation of motion on the boundaries $x = 0$ or $y = 0$. In 2 dimensions, the dynamics of an interface $\zeta(\eta)$ obeys

$$\zeta_t = \frac{\zeta_\eta}{\zeta_\eta - 1}$$

- The equation $z_t = F(z_x, z_y)$ must be invariant under exchange of any pair of coordinates:

$$F\left(\frac{1}{a}, \frac{1}{b}\right) = -\frac{1}{a} F\left(a, -\frac{a}{b}\right)$$

Solution of the PDE

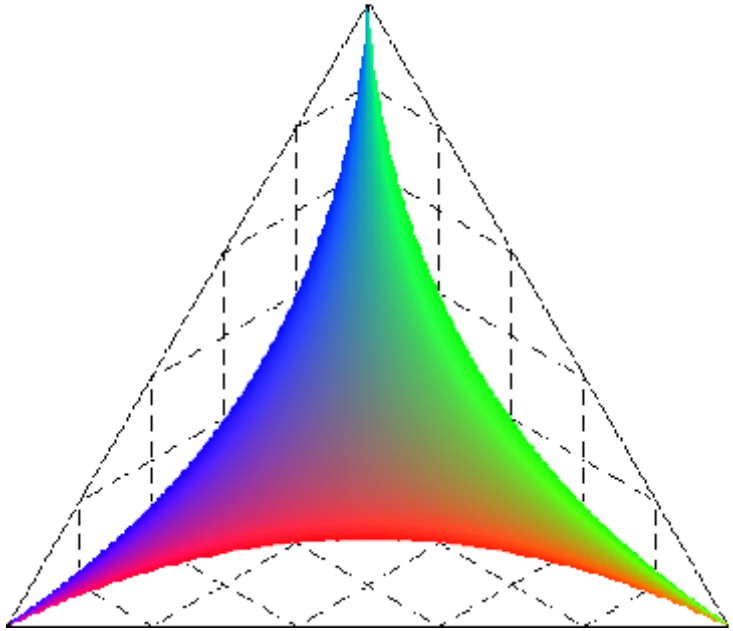
Using the method of characteristics, the conjectured equation can explicitly be solved:

$$\frac{x}{t} = A(q,r) \quad \frac{y}{t} = B(q,r) \quad \frac{z}{t} = C(q,r)$$

with

$$\begin{aligned}
A &= \frac{r^2}{(r-1)(q-1)(q+r)}\left[\frac{1}{q-1}+\frac{1}{q+r}\right] \\
B &= \frac{q^2}{(r-1)(q-1)(q+r)}\left[\frac{1}{r-1}+\frac{1}{q+r}\right] \\
C &= \frac{q^2 r^2}{(r-1)(q-1)(q+r)}\left[\frac{1}{q-1}+\frac{1}{r-1}\right]
\end{aligned}$$

Some numerical tests



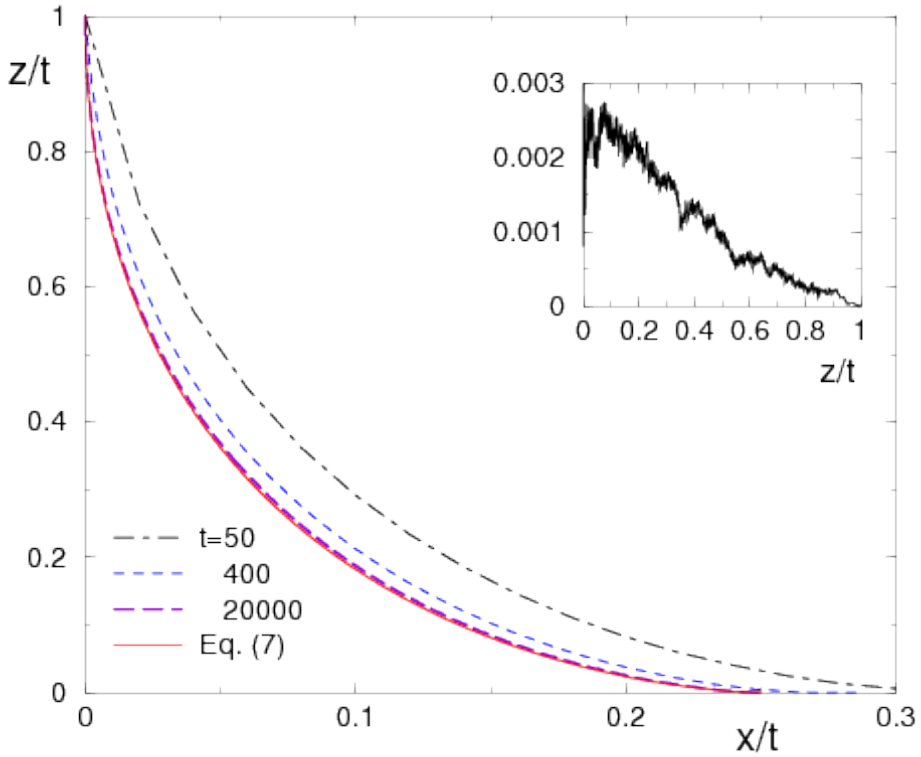
This surface cuts the ground plane $z = 0$ along the curve $\sqrt{x} + \sqrt{y} = \sqrt{t}$. The intersection with the ray $x = y = z$ is predicted to be $\frac{t}{8}$. Numerical simulations give the speed 0.1261...
The volume can be calculated exactly and it grows as

$$V/t^3 = v = \frac{3\pi^2}{2^{11}} = 0.014457\ldots$$

The numerical measurement gives $v \approx 0.01472(3)$, within 1.8% of the prediction.

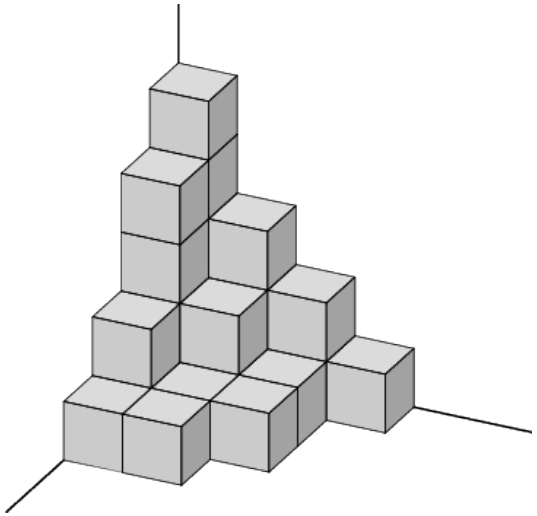
The intersection with the plane $x = y$ is given by

$$\frac{x}{t} = \frac{1}{2}\frac{z}{t} - \frac{3}{4}\left(\frac{z}{t}\right)^{2/3} + \frac{1}{4}$$

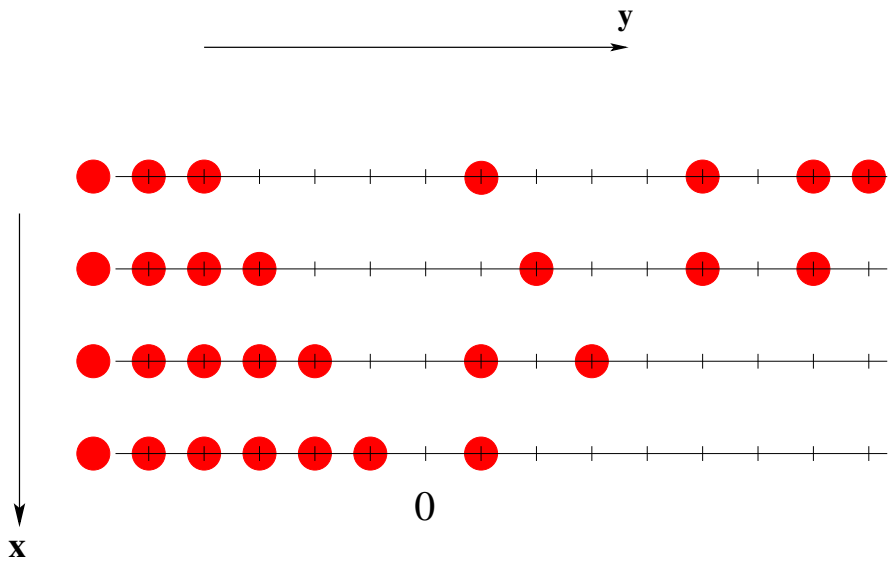


(Jason Olejarz, Sid Redner, P. K. and K. M: PRL **108** 016102 (2012))

Equivalence with particle systems



Equivalent to family of coupled exclusion processes:



Conclusion

Exact solutions of the asymmetric exclusion process are paradigms for the behaviour of systems far from equilibrium in low dimensions: Dynamical phase transitions, Non-Gibbsean measures, Large deviations, Fluctuations Theorems...

Large deviation functions (LDF) appear as the right generalization of the thermodynamic potentials: convex, optimized at the stationary state, and non-analytic features can be interpreted as phase transitions. The LDF's are very likely to play a key-role in the future of non-equilibrium statistical mechanics.

Interacting Particle Processes also represent Growth Models and can used to investigate properties of dimer tilings, Young tableaux and plane partitions.
M. Evans, P. Ferrari, O. Golinelli, M. Gorissen, P. Krapivsky, A. Lazarescu, J. Olejarz, S. Prolhac, S. Redner and C. Vanderzande