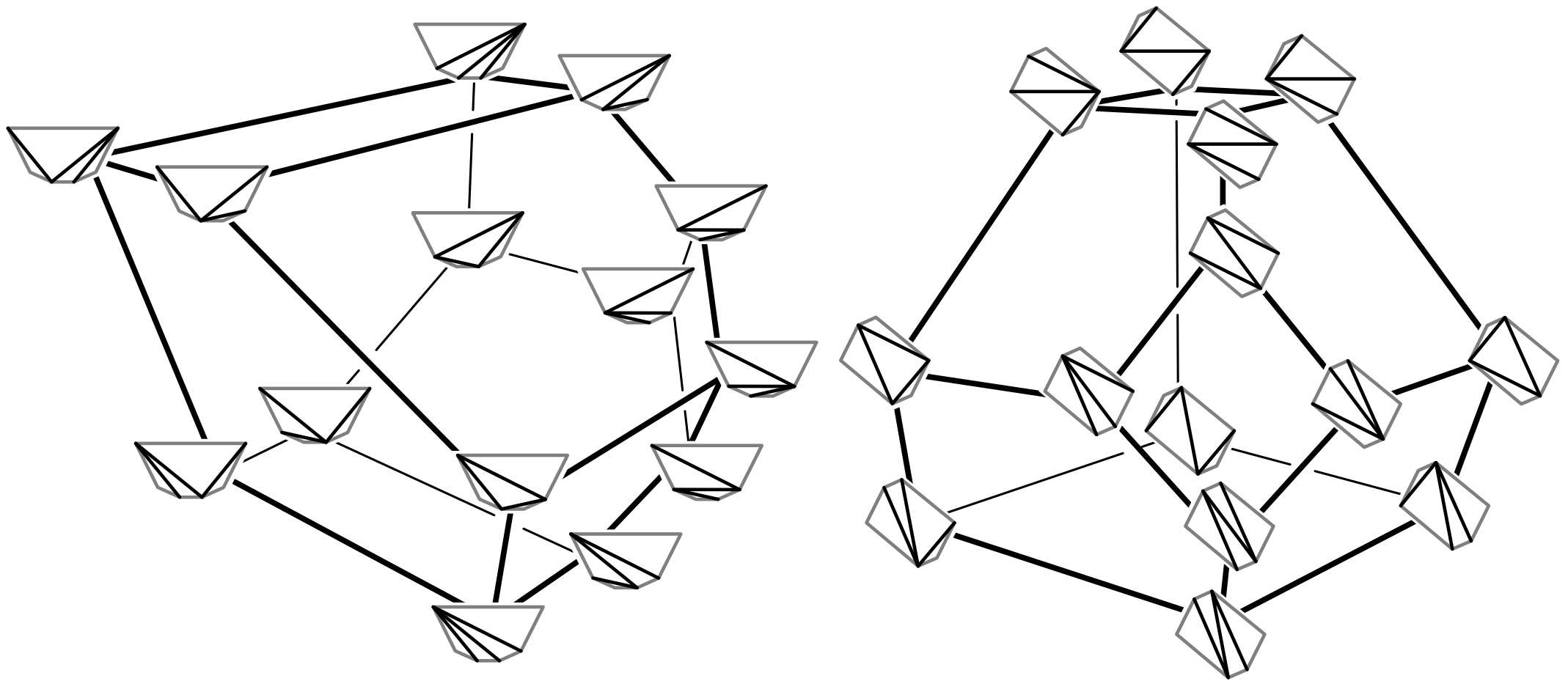




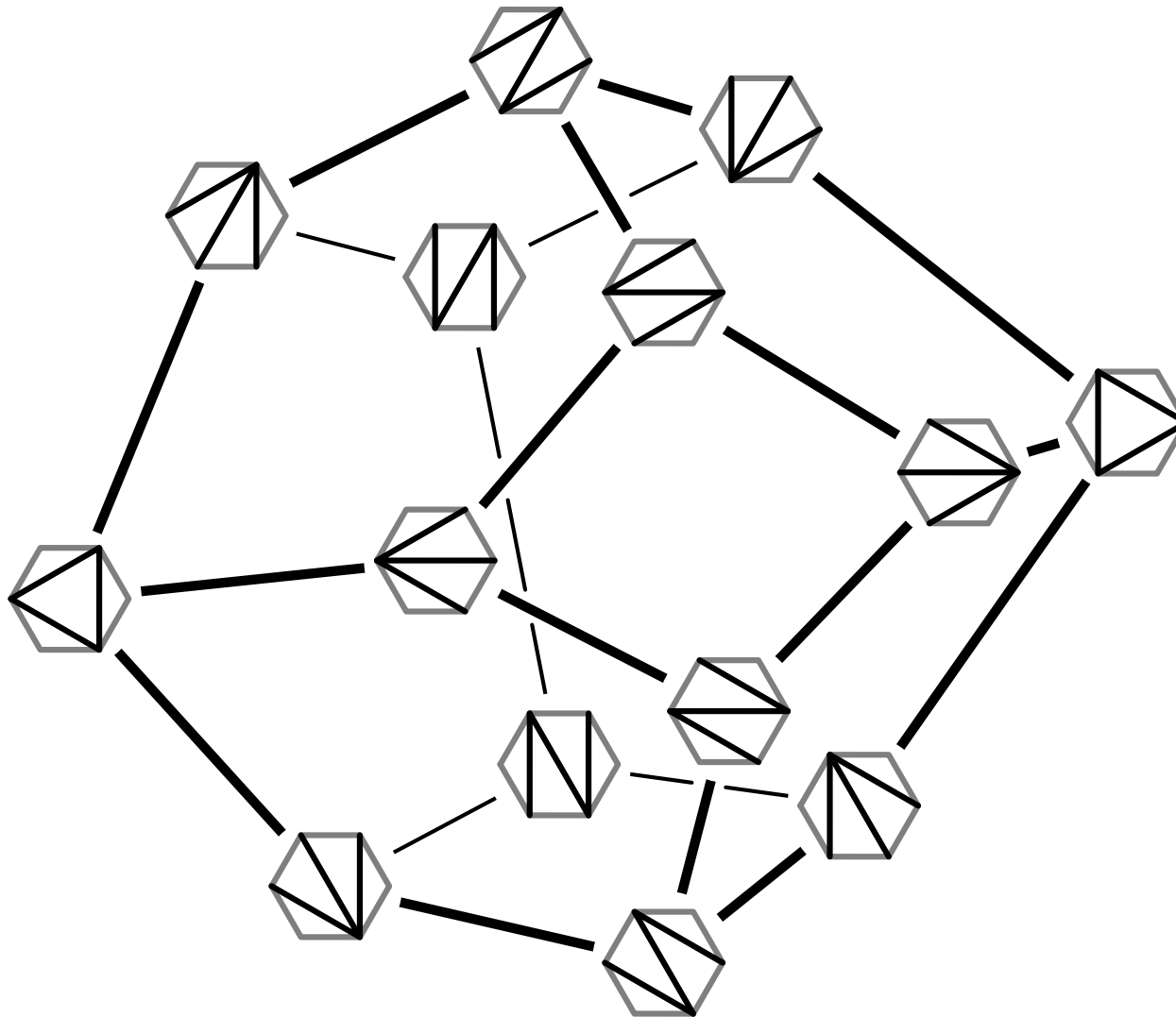
VERTEX BARYCENTERS OF GENERALIZED ASSOCIAHEDRA

Vincent PILAUD (CNRS & LIX, École Polytechnique)
Christian STUMP (Universität Hannover)

ASSOCIAHEDRON
— & —
RELATED STRUCTURES

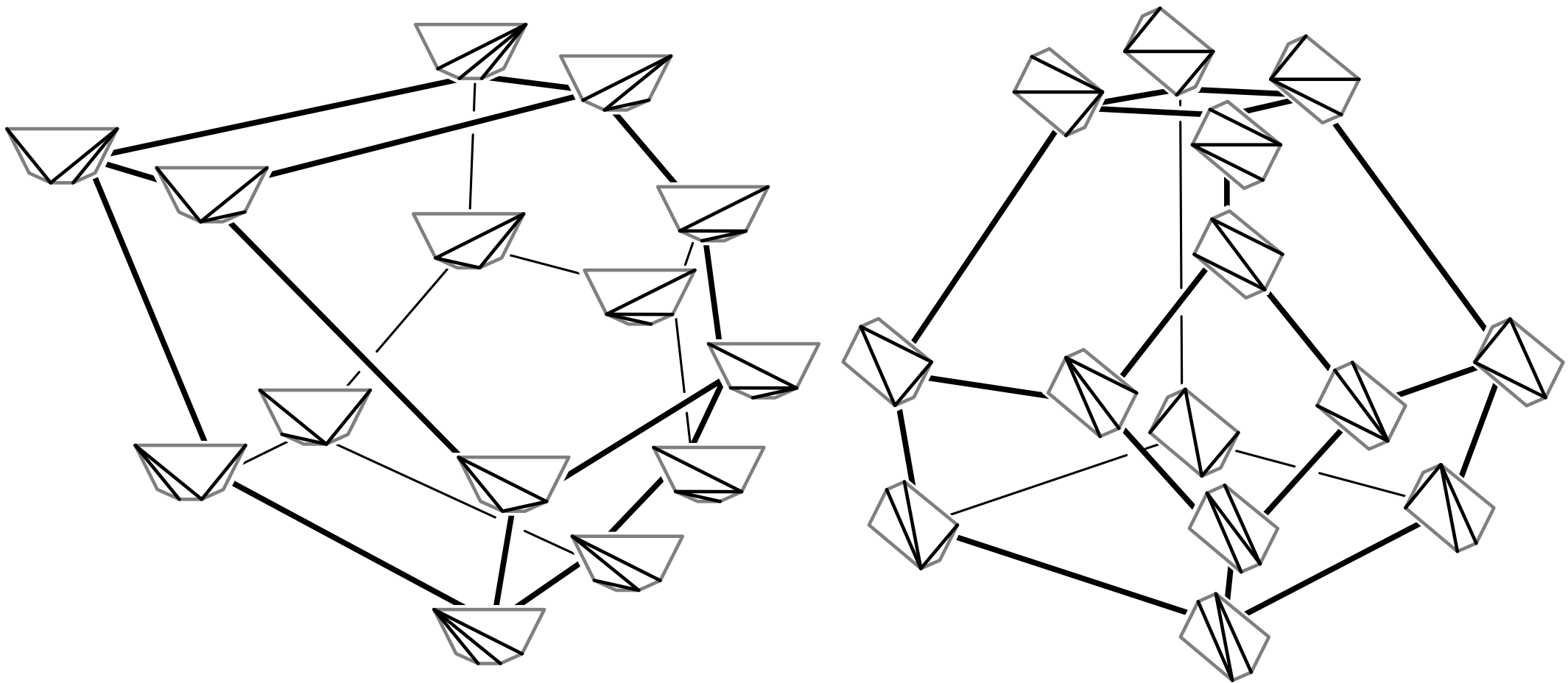
ASSOCIAHEDRON

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex n -gon, ordered by reverse inclusion.



VARIOUS ASSOCIAHEDRA

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex n -gon, ordered by reverse inclusion.

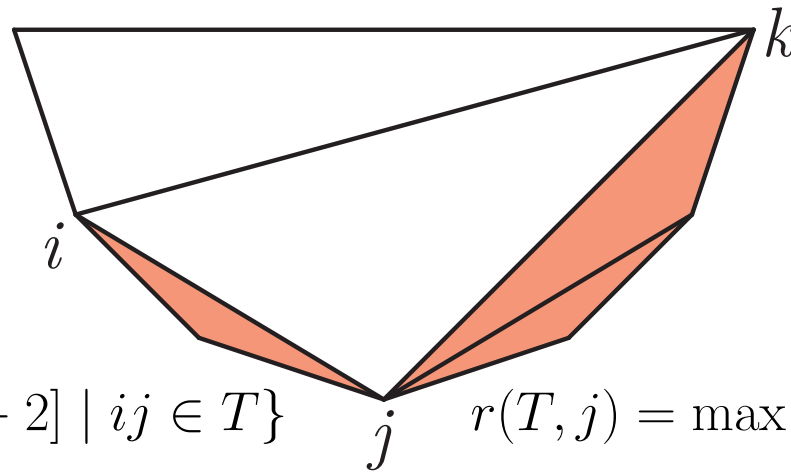


Lee ('89), Gel'fand-Kapranov-Zelevinski ('94), Billera-Filliman-Sturmfels ('90), . . . , Ceballos-Santos-Ziegler ('11)
Loday ('04), Hohlweg-Lange ('07), Hohlweg-Lange-Thomas ('12), P.-Santos ('12), P.-Stump ('12⁺)

LODAY'S ASSOCIAHEDRON

Loday's associahedron = $\text{conv} \{L(T) \mid T \text{ triangulation of the } (n+3)\text{-gon}\}$, where

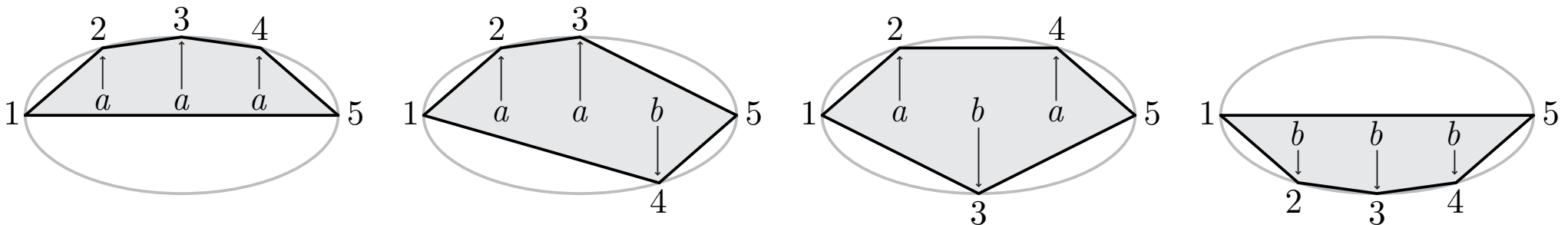
$$L(T) = (\ell(T, j) \cdot r(T, j))_{j \in [n+1]}$$



$$\ell(T, j) = j - \min \{i \in [0, j-2] \mid ij \in T\} \quad r(T, j) = \max \{k \in [j+2, n+2] \mid jk \in T\} - j$$

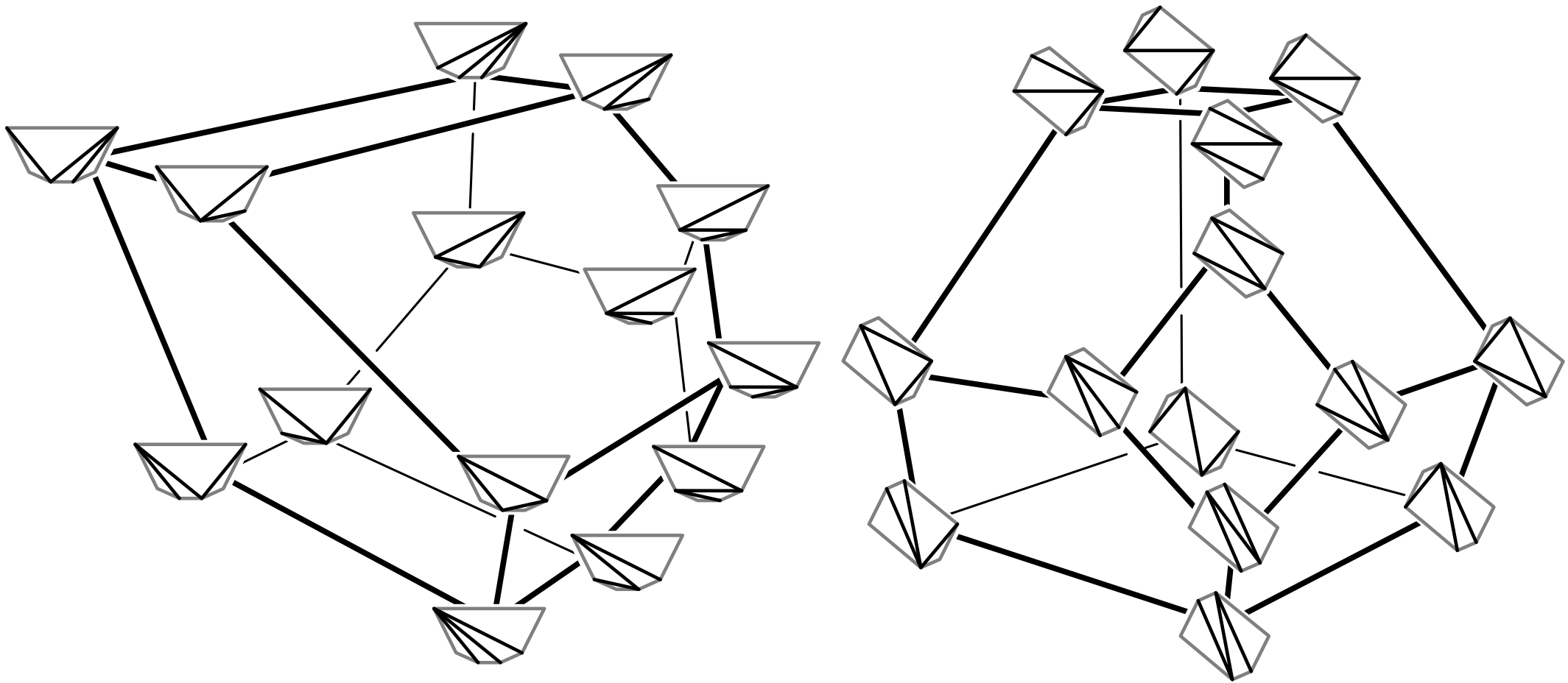
Loday, *Realization of the Stasheff polytope* ('04)

Can also replace this $(n+3)$ -gon by others:



Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

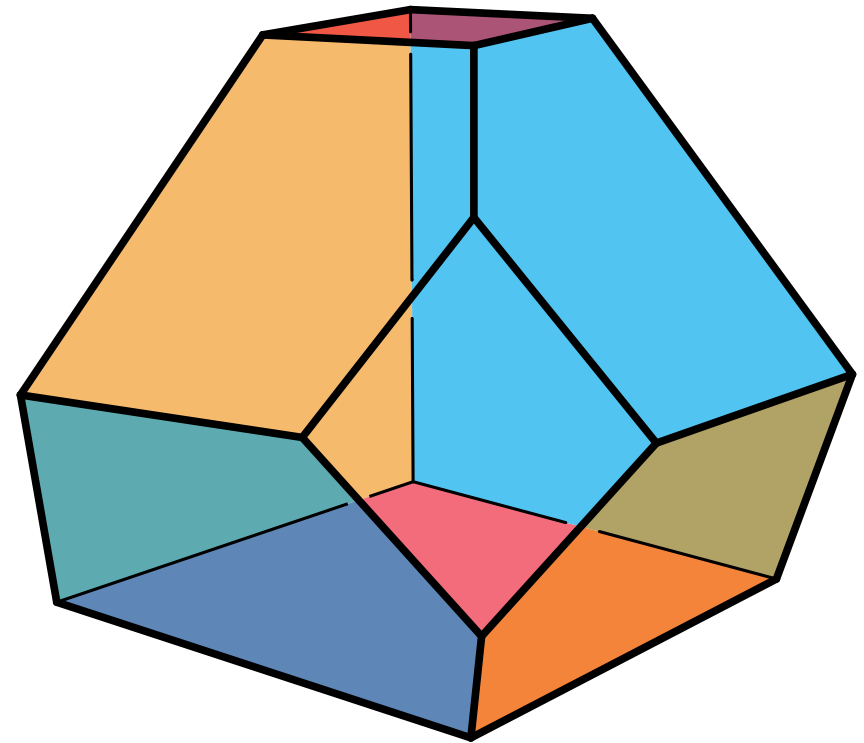
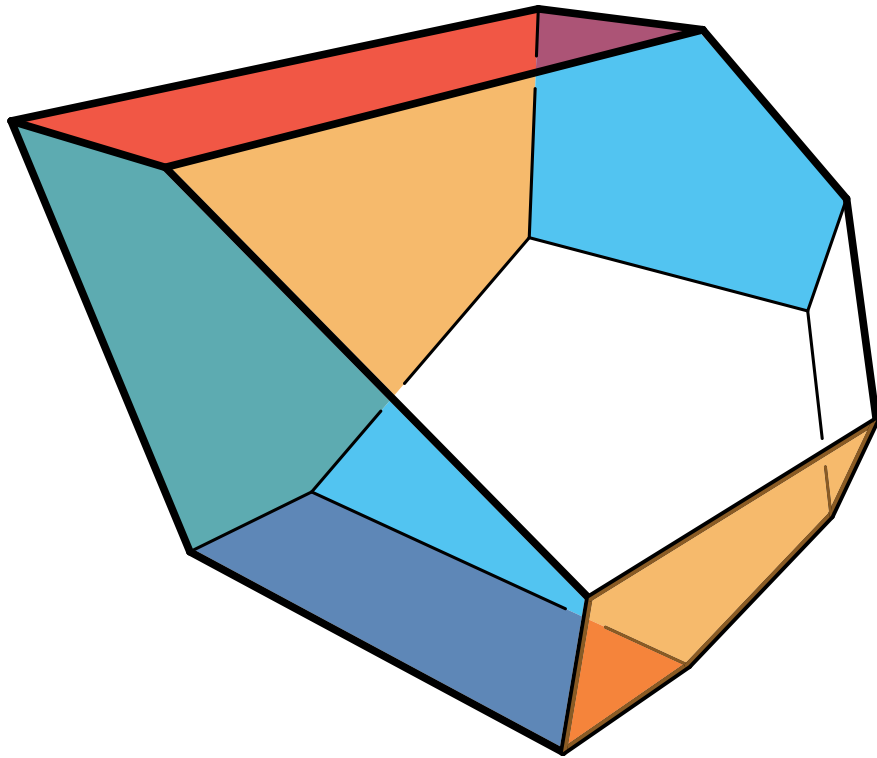
HOHLWEG & LANGE'S ASSOCIAHEDRA



Loday, *Realization of the Stasheff polytope* ('04)

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

HOHLWEG & LANGE'S ASSOCIAHEDRA

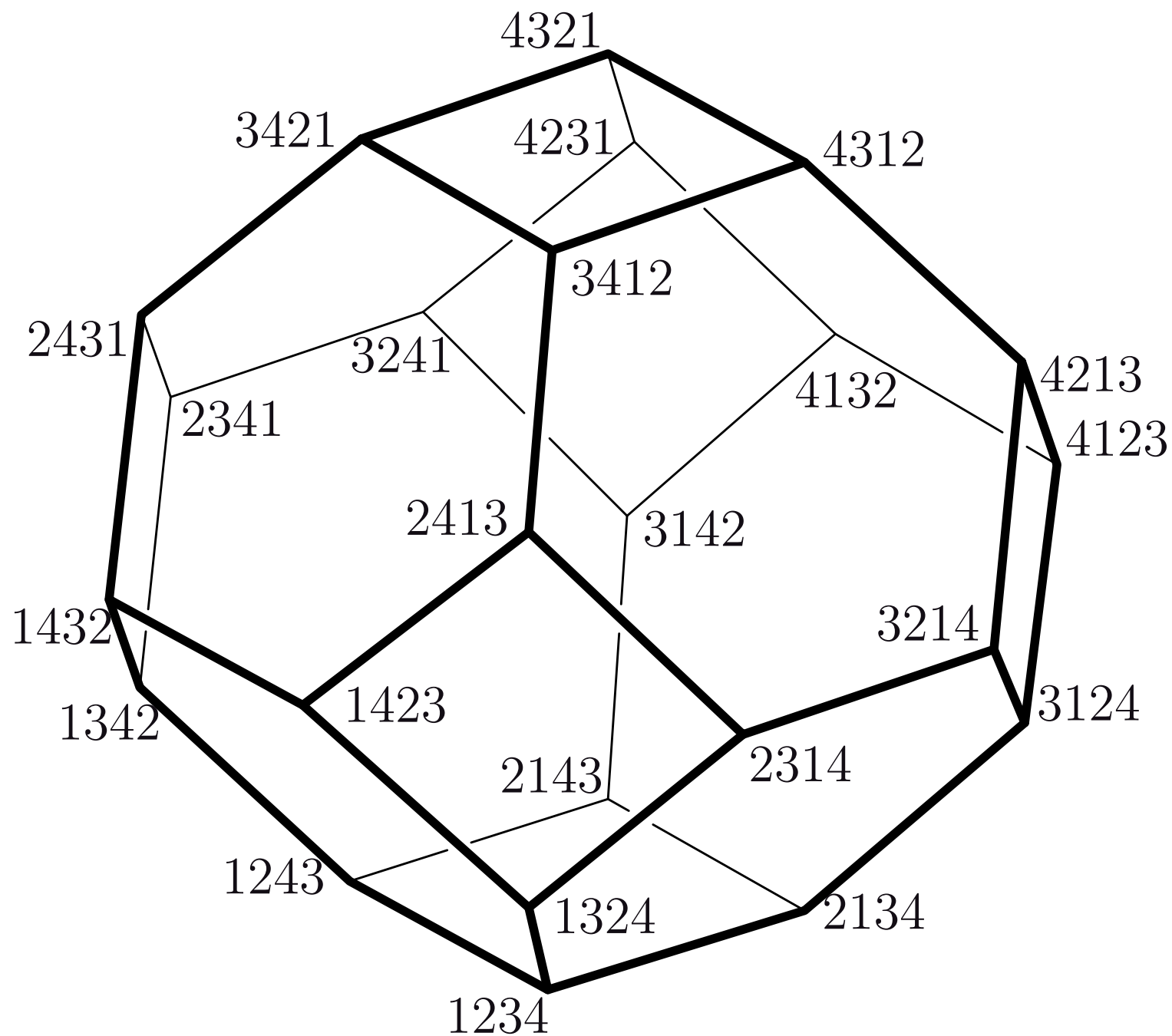


Loday, *Realization of the Stasheff polytope* ('04)

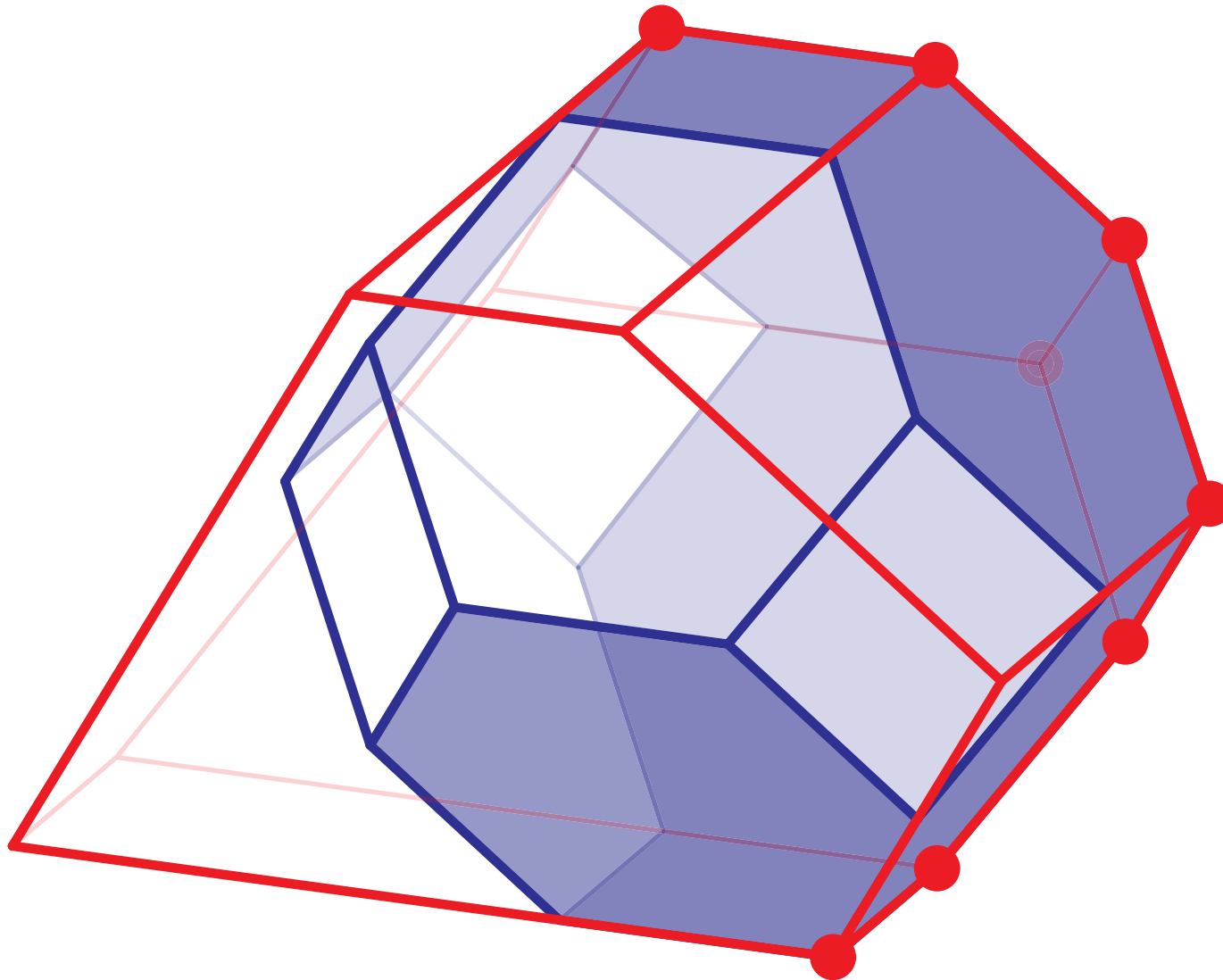
Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

Ceballos-Santos-Ziegler, *Many non-equivalent realizations of the associahedron* ('11⁺)

PERMUTAHEDRON



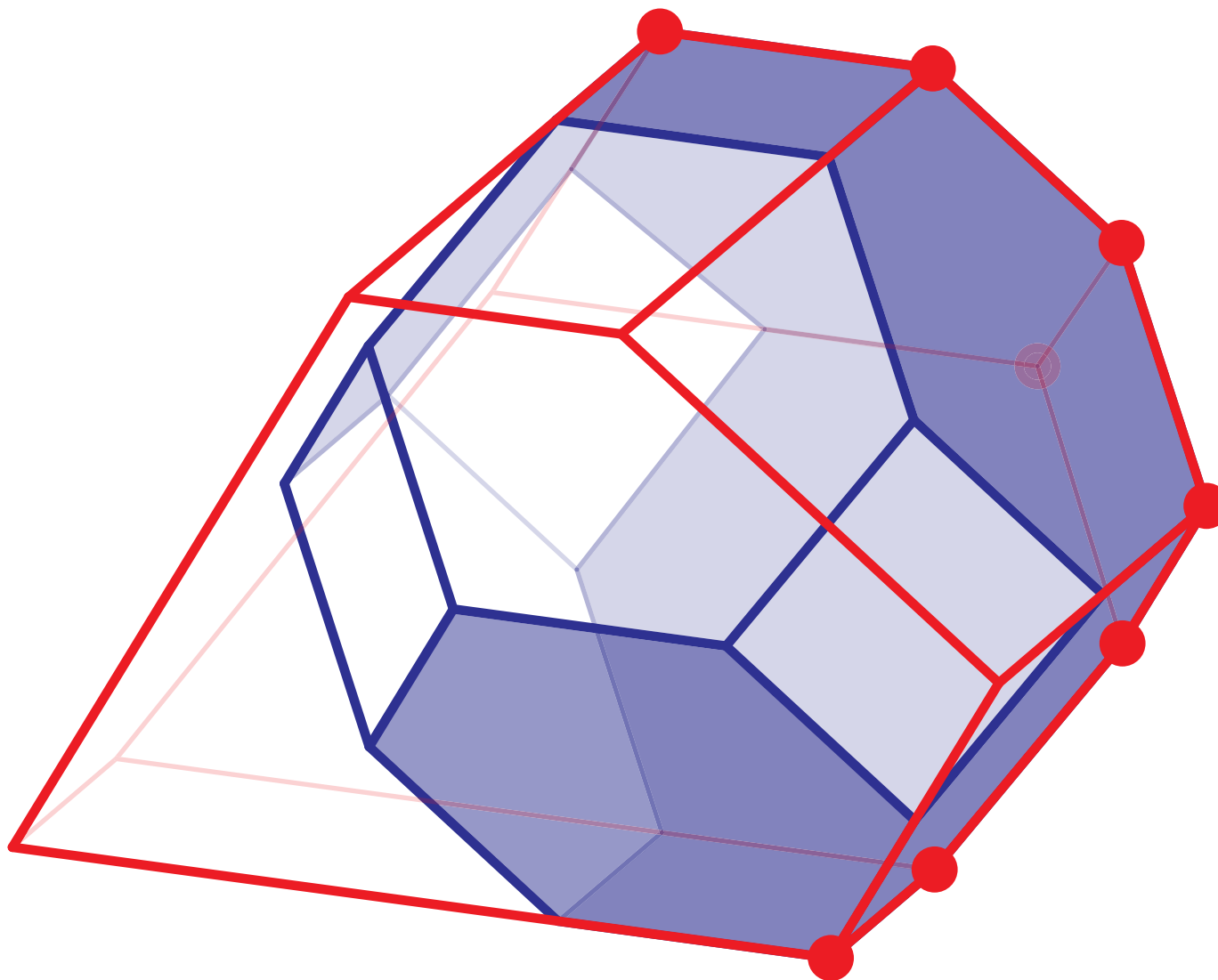
ASSOCIAHEDRA FROM THE PERMUTAHEDRON



Associahedron from permutahedron = remove facets not containing “singletons”.

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

BARYCENTERS



THEOREM. All Hohlweg & Lange's associahedra have the origin for vertex barycenter.

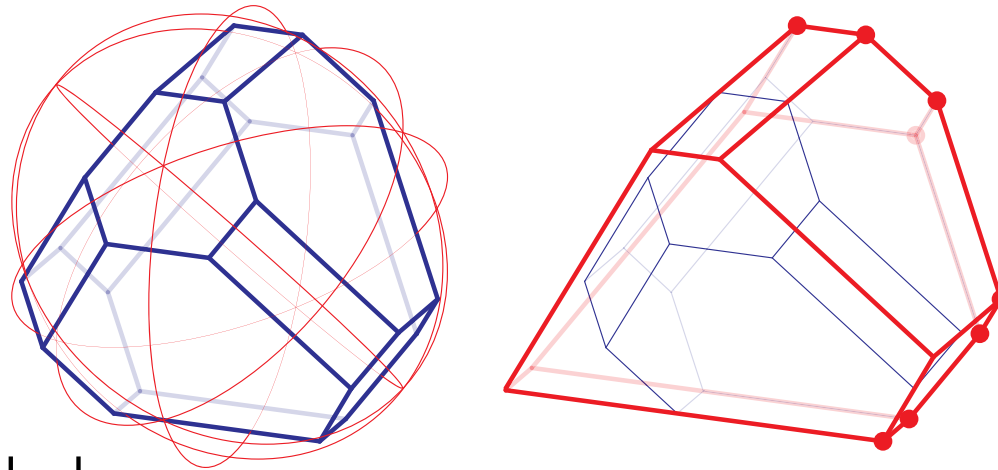
Hohlweg-Lortie-Raymond, *The center of gravity of the associahedron and of the permutahedron are the same* ('07)

BARYCENTERS

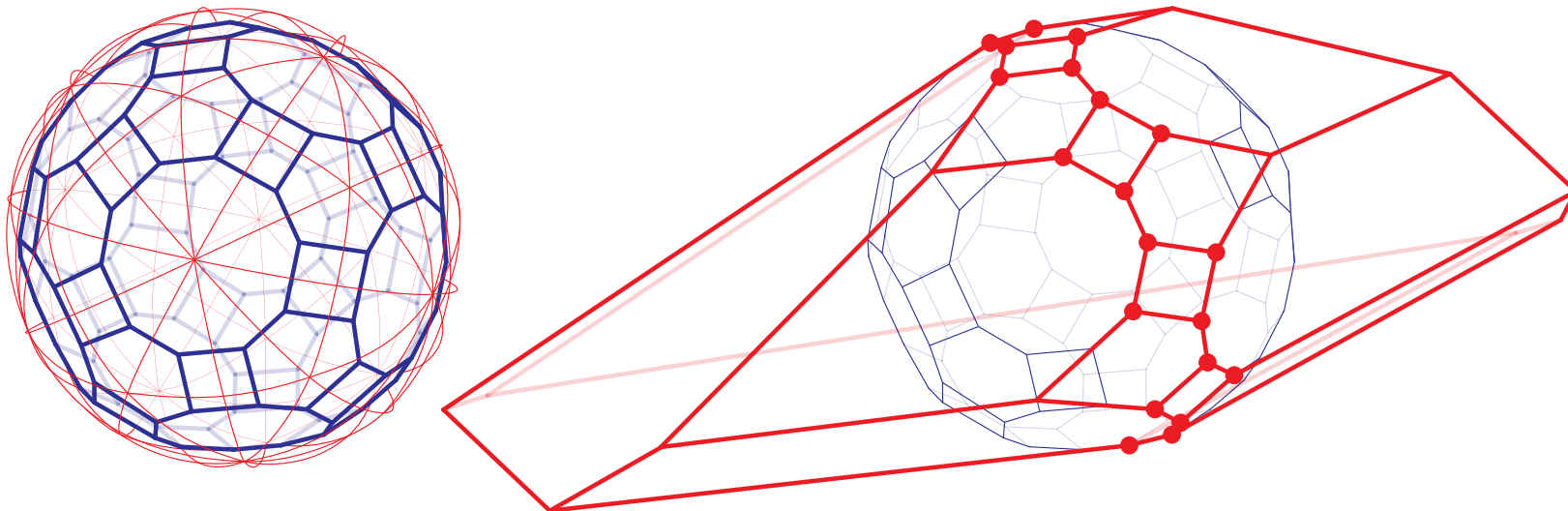
THEOREM. All Hohlweg & Lange's associahedra have the origin for vertex barycenter.

We give an alternative proof of this result, which extends in two directions:

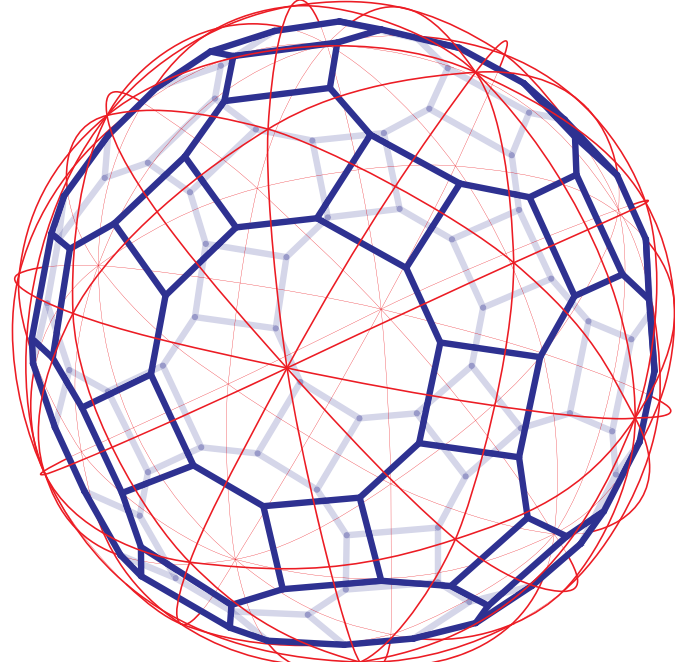
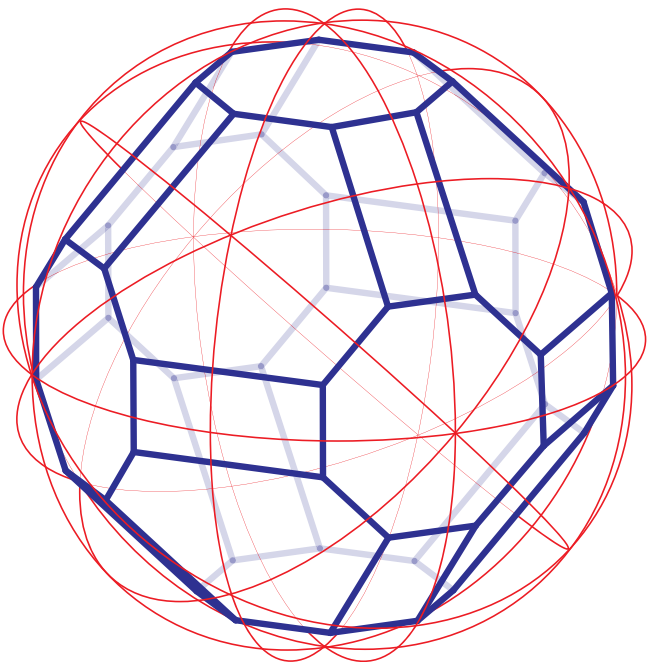
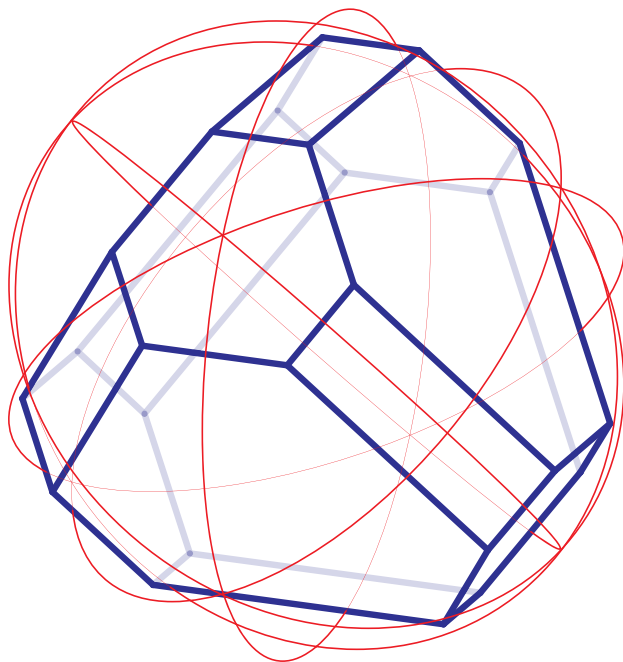
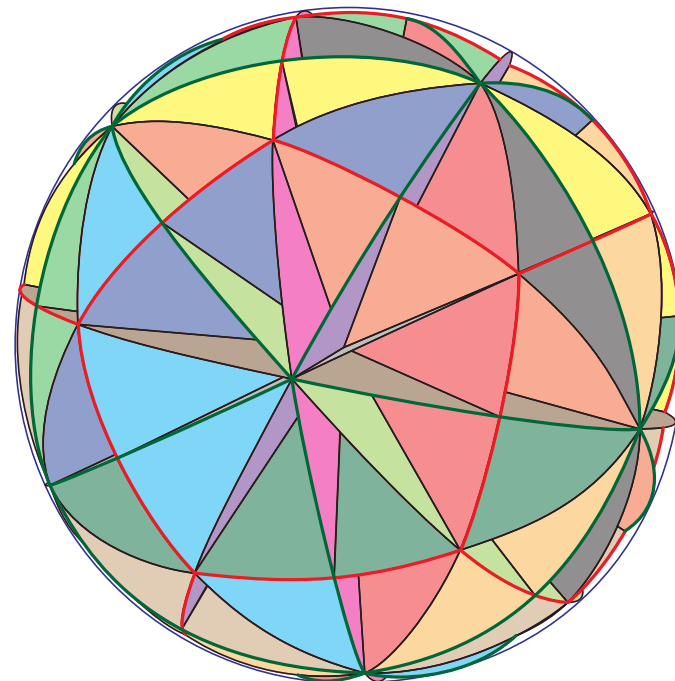
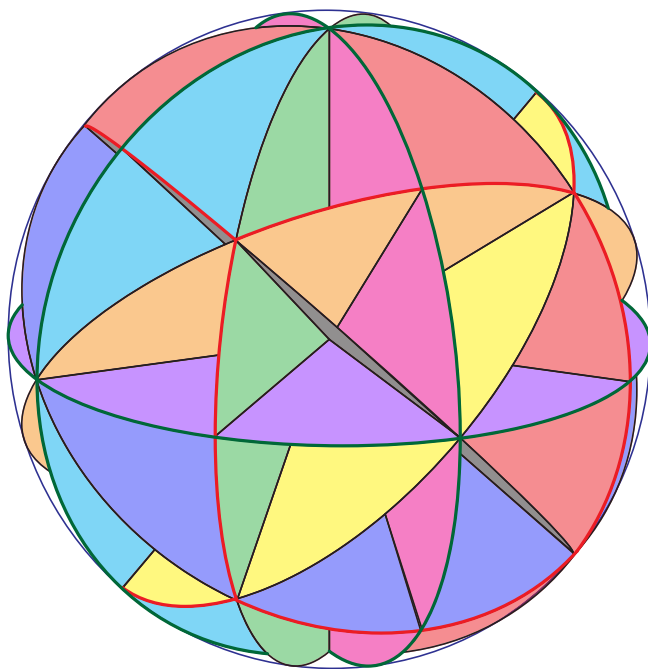
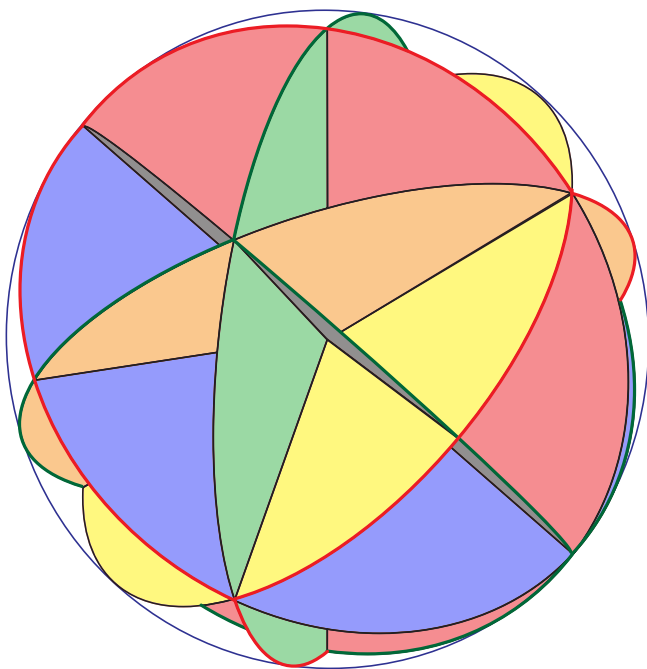
1. Fairly balanced associahedra:



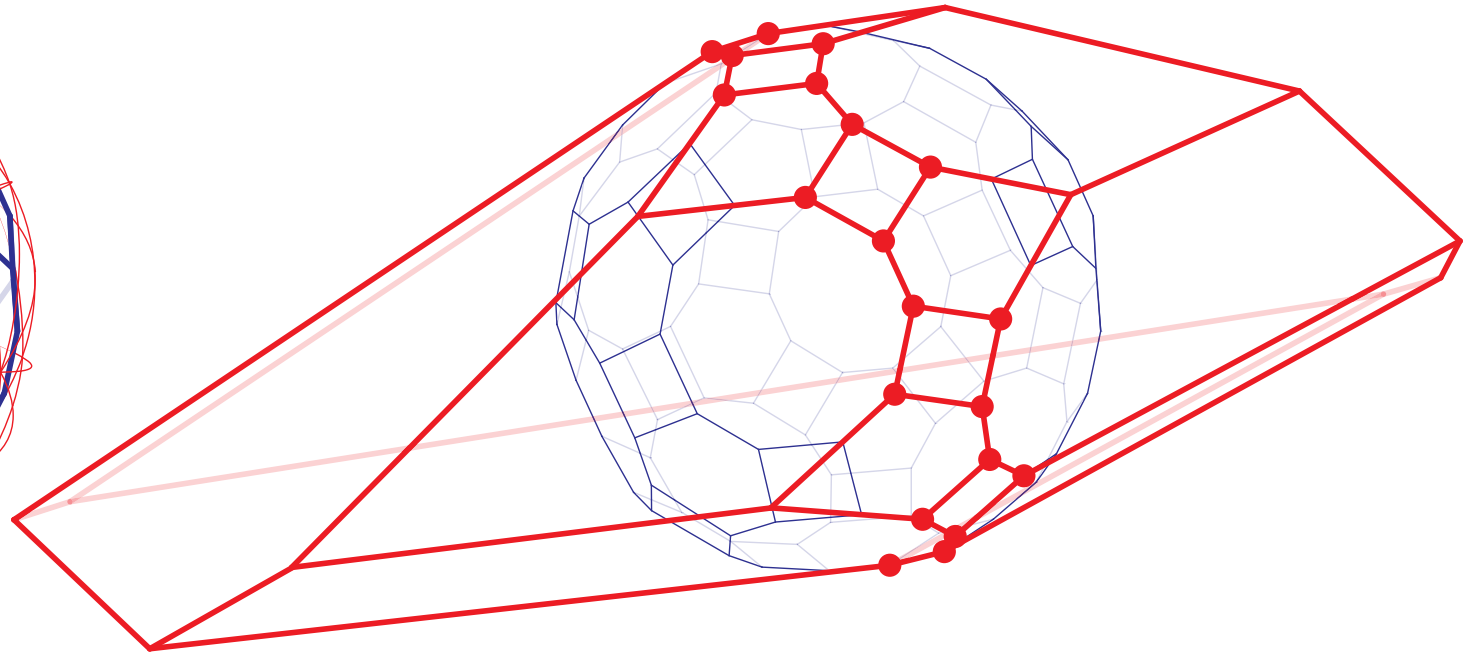
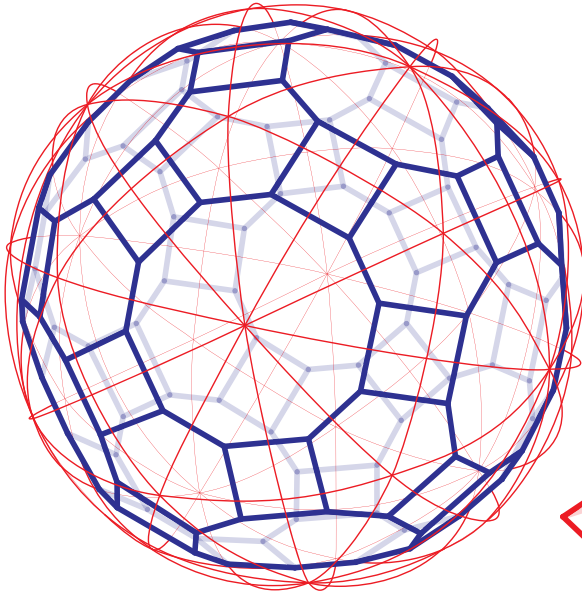
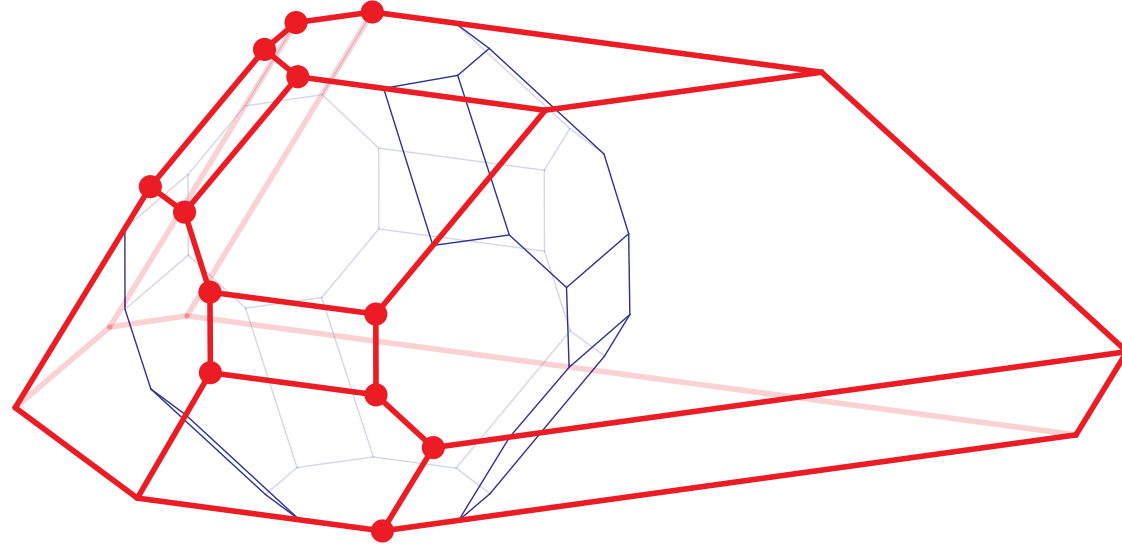
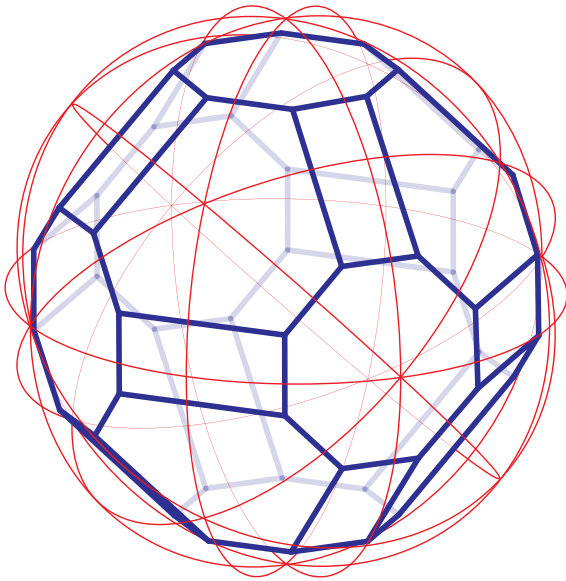
2. Generalized associahedra:



FINITE COXETER GROUPS



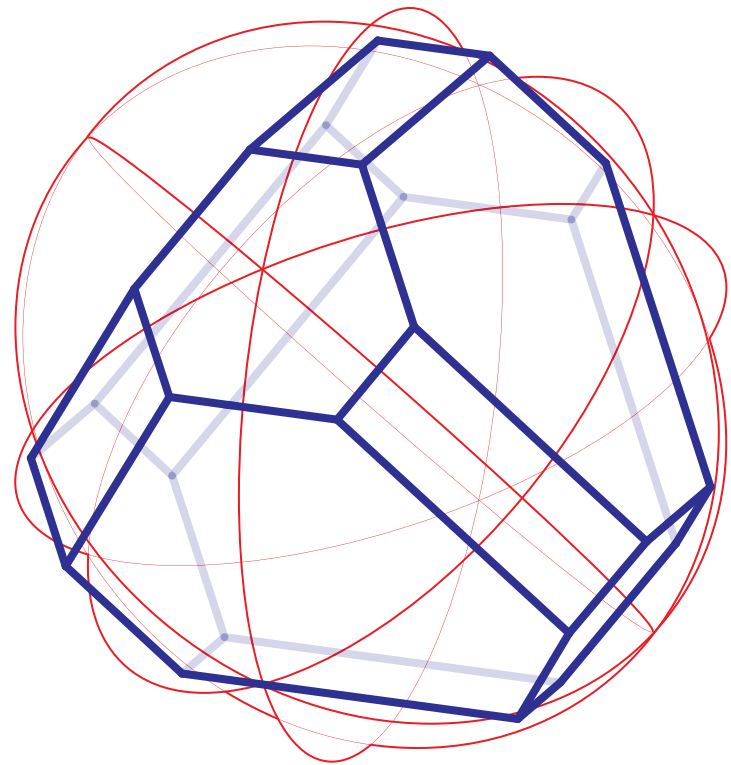
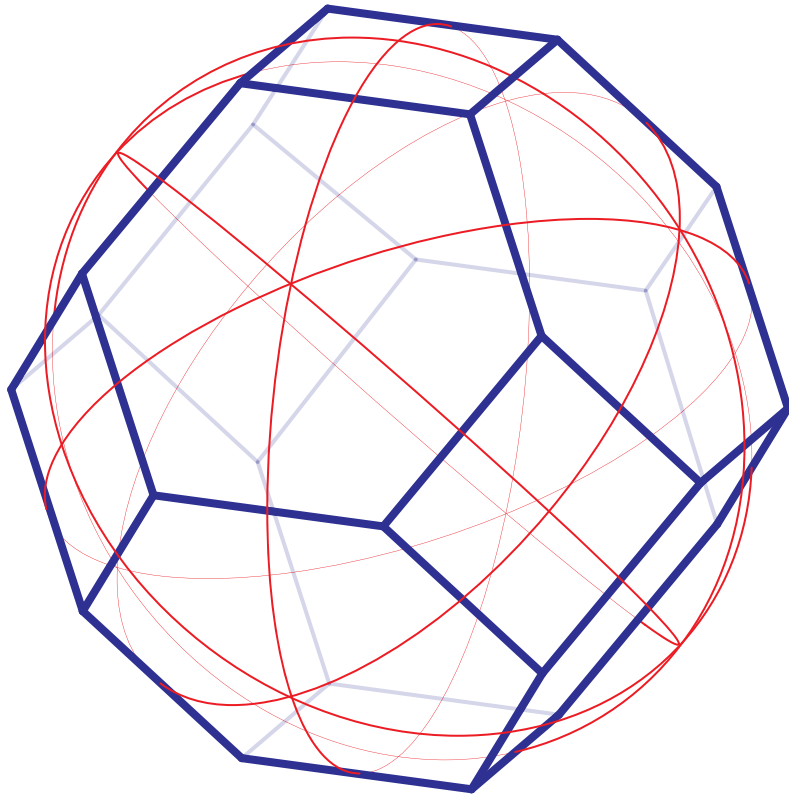
GENERALIZED ASSOCIAHEDRA



Chapoton-Fomin-Zelevinsky, *Polytopal realizations of generalized associahedra* ('02)
Hohlweg-Lange-Thomas, *Permutahedra and generalized associahedra* ('11)

OUR RESULT

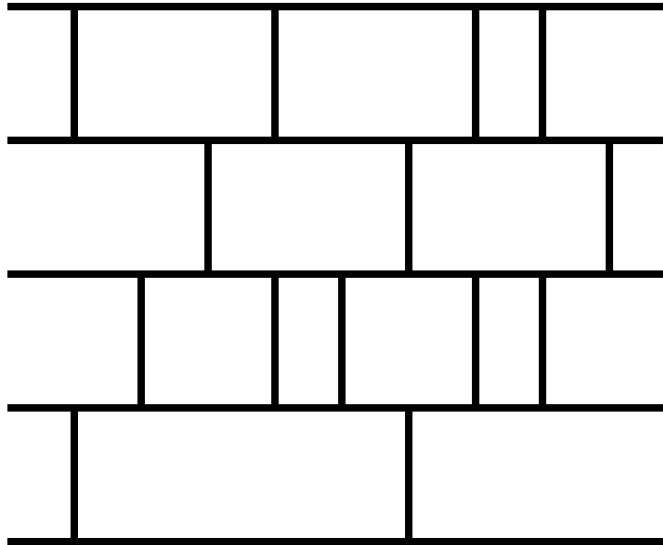
THEOREM. For any finite Coxeter group W ,
any Coxeter element c , any fairly-balanced point u ,
the vertex barycenters of the generalized
associahedron $\text{Asso}_c^u(W)$ and of the permutahedron $\text{Perm}^u(W)$ coincide.



The point u is **fairly balanced** if $w_o(u) = -u$, where w_o is the longest element in W .

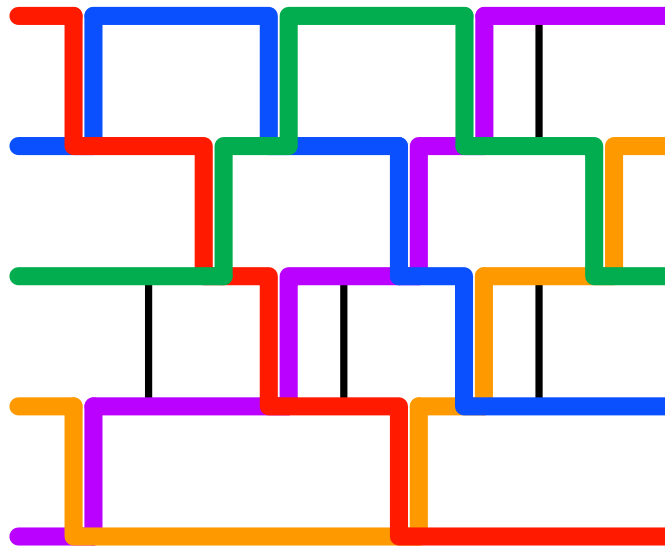
ASSOCIAHEDRON
— & —
SORTING NETWORKS

PRIMITIVE SORTING NETWORKS

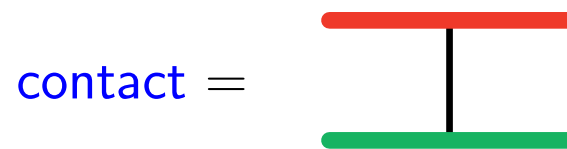
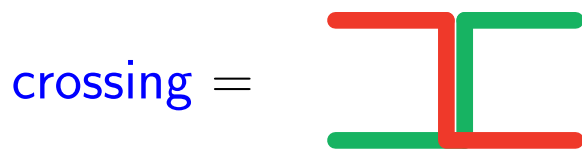


network $\mathcal{N} = n$ horizontal levels and m vertical commutators
bricks of $\mathcal{N} =$ bounded cells

PSEUDOLINE ARRANGEMENTS ON A NETWORK



pseudoline = abscissa-monotone path

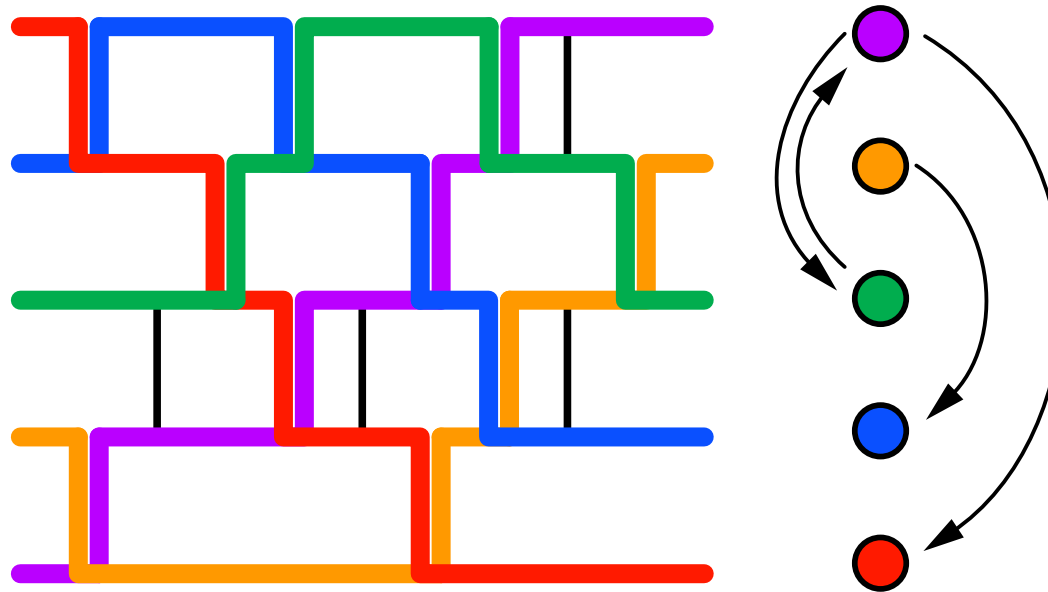


pseudoline arrangement (with contacts) = n pseudolines supported by $\mathcal{N}$ which have pairwise exactly one crossing, possibly some contacts, and no other intersection

CONTACT GRAPH OF A PSEUDOLINE ARRANGEMENT

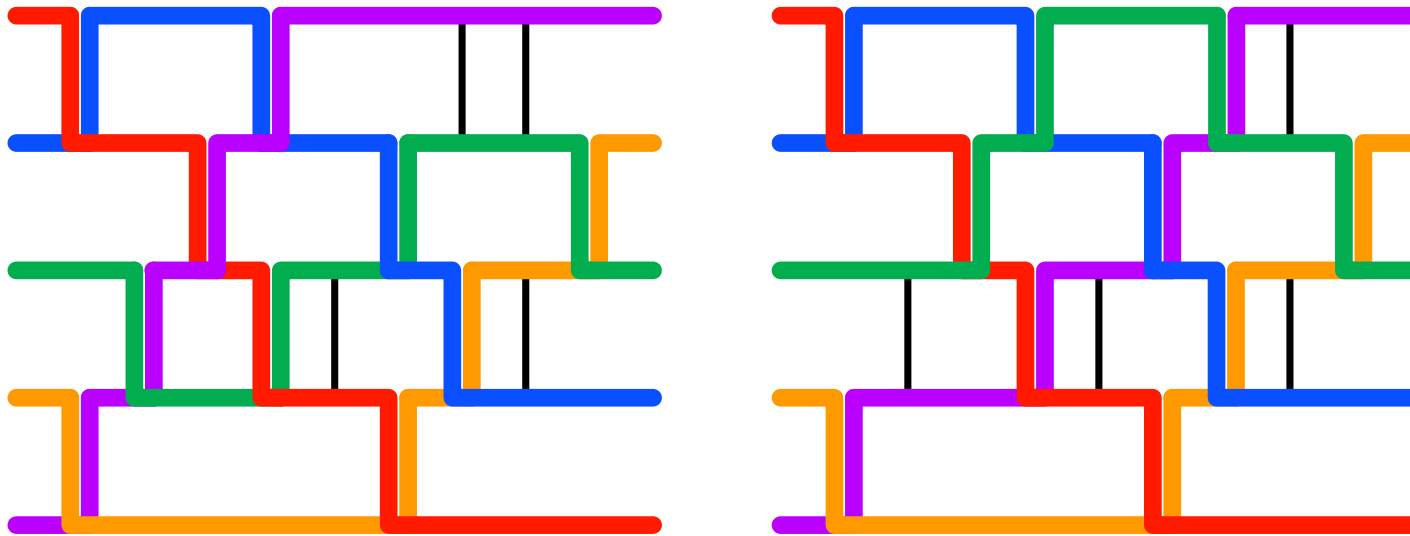
contact graph $\Lambda^\#$ of a pseudoline arrangement $\Lambda =$

- a node for each pseudoline of Λ , and
- an arc for each contact of Λ oriented from top to bottom



FLIPS

flip = exchange an arbitrary contact with the corresponding crossing



Combinatorial and geometric properties of the graph of flips $G(\mathcal{N})$?

Knutson-Miller, *Subword complexes in Coxeter groups* ('04)

P.-Pocchiola, *Multitriangulations, pseudotriangulations and sorting networks* ('12)

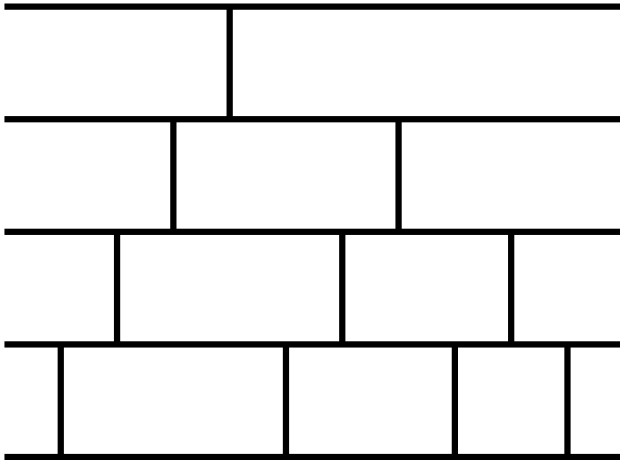
P.-Santos, *The brick polytope of a sorting network* ('12)

Ceballos-Labbé-Stump, *Subword complexes, cluster complexes, and generalized multi-associahedra* ('12⁺)

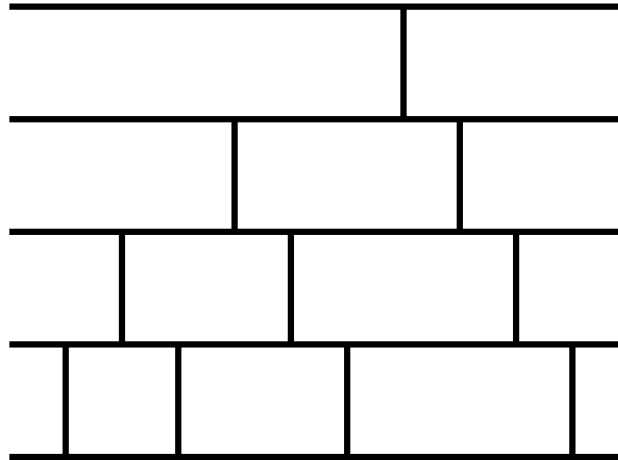
P.-Stump, *Brick polytopes of spherical subword complexes: a new approach to generalized associahedra* ('12⁺)

P.-Stump, *EL-labelings and canonical spanning trees for subword complexes* ('12⁺)

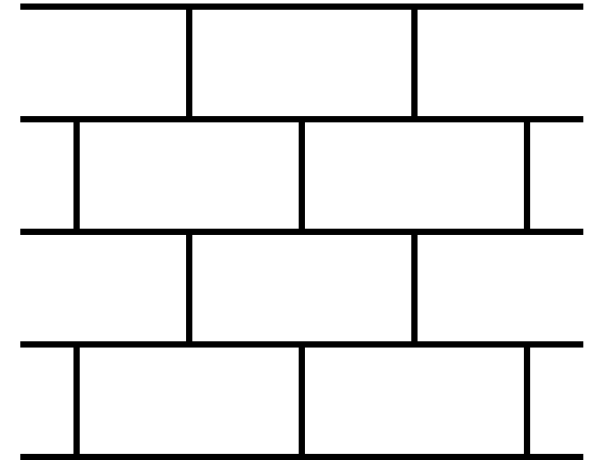
MINIMAL SORTING NETWORKS



bubble sort



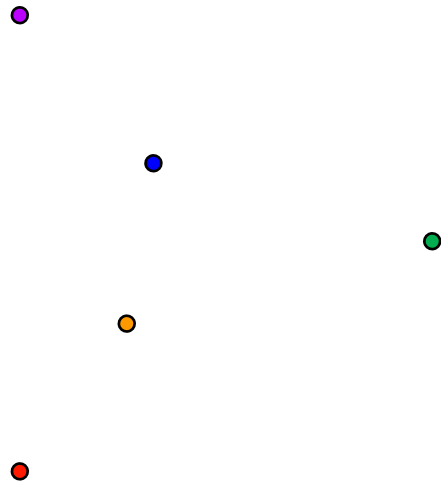
insertion sort



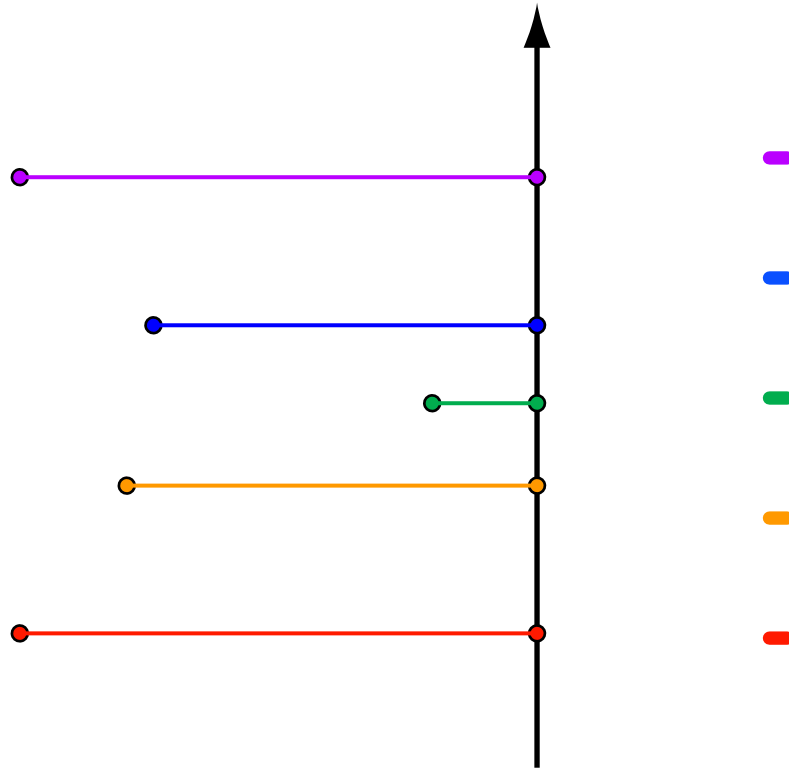
even-odd sort

Knuth, *The art of Computer Programming*, Vol. 3 Sorting and Searching ('97)

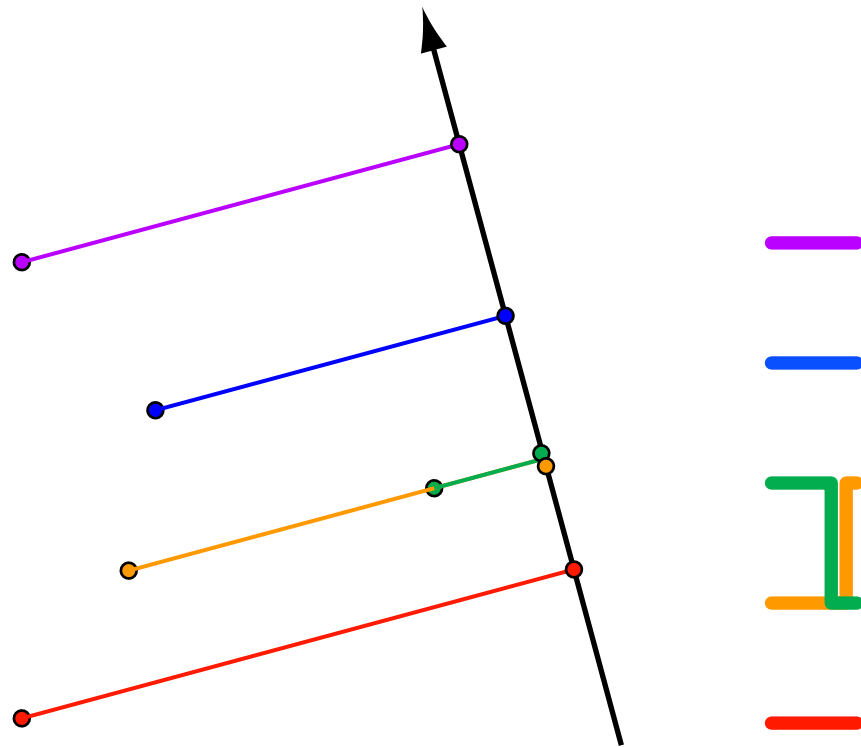
POINT SETS & MINIMAL SORTING NETWORKS



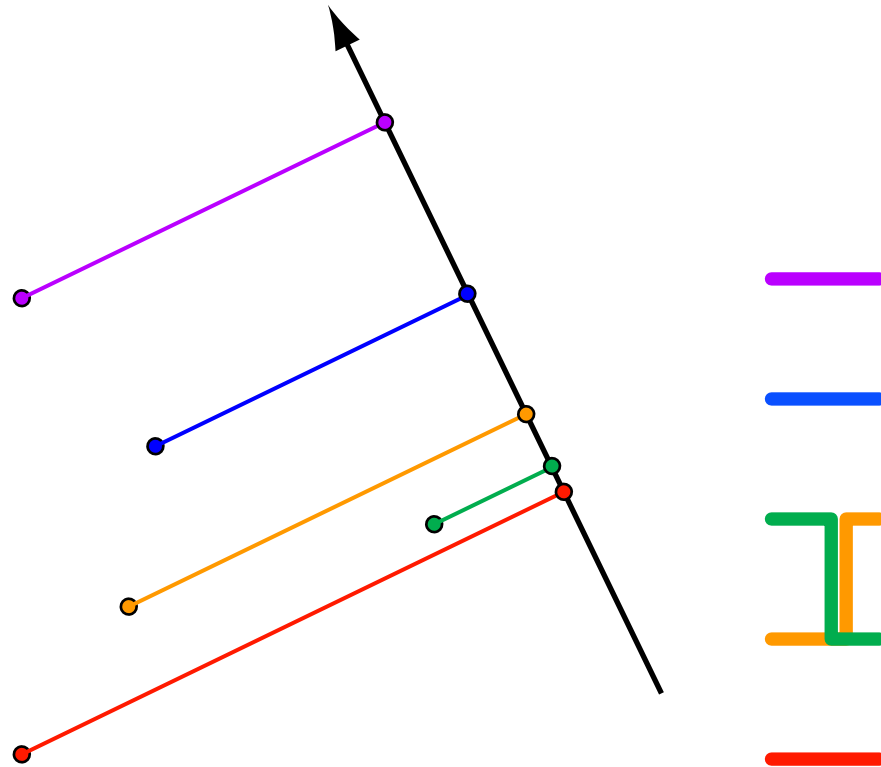
POINT SETS & MINIMAL SORTING NETWORKS



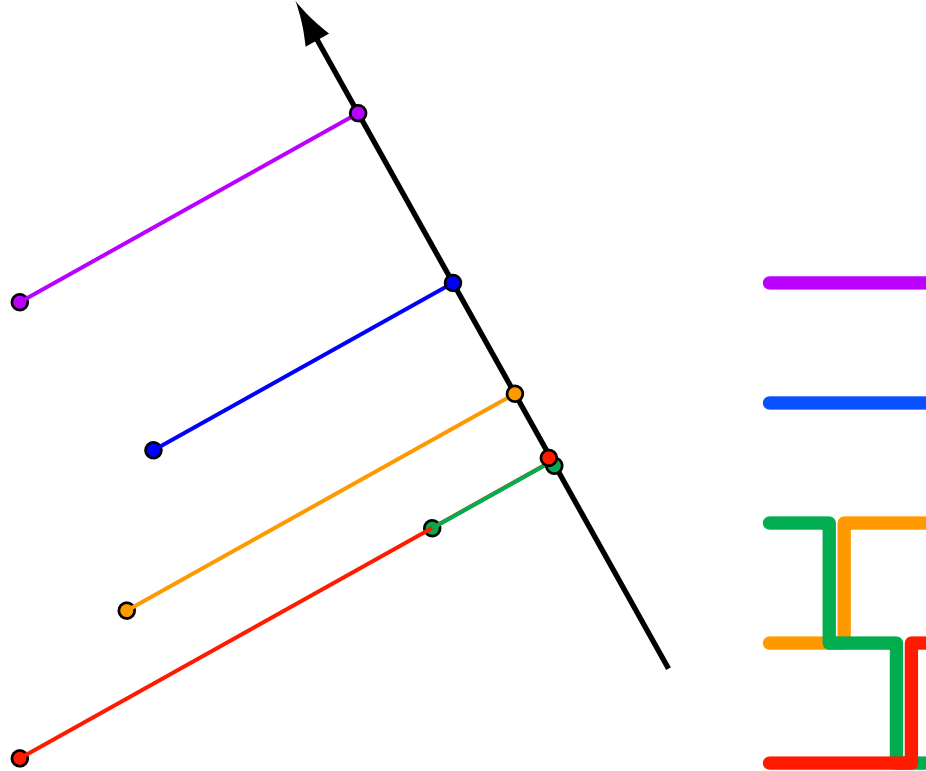
POINT SETS & MINIMAL SORTING NETWORKS



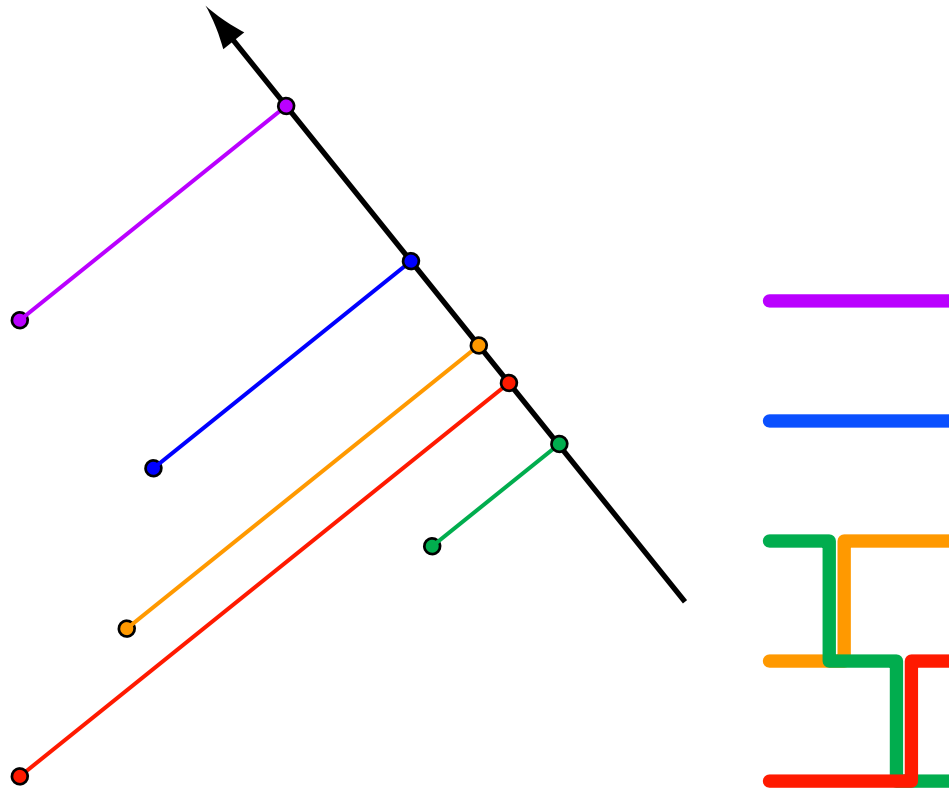
POINT SETS & MINIMAL SORTING NETWORKS



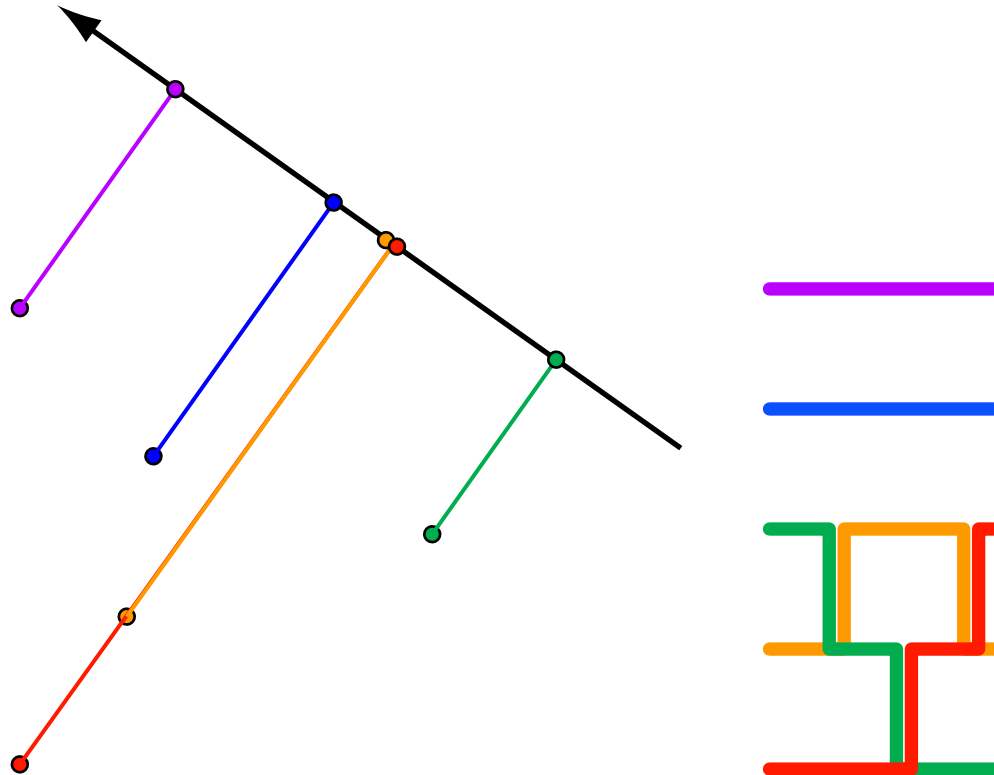
POINT SETS & MINIMAL SORTING NETWORKS



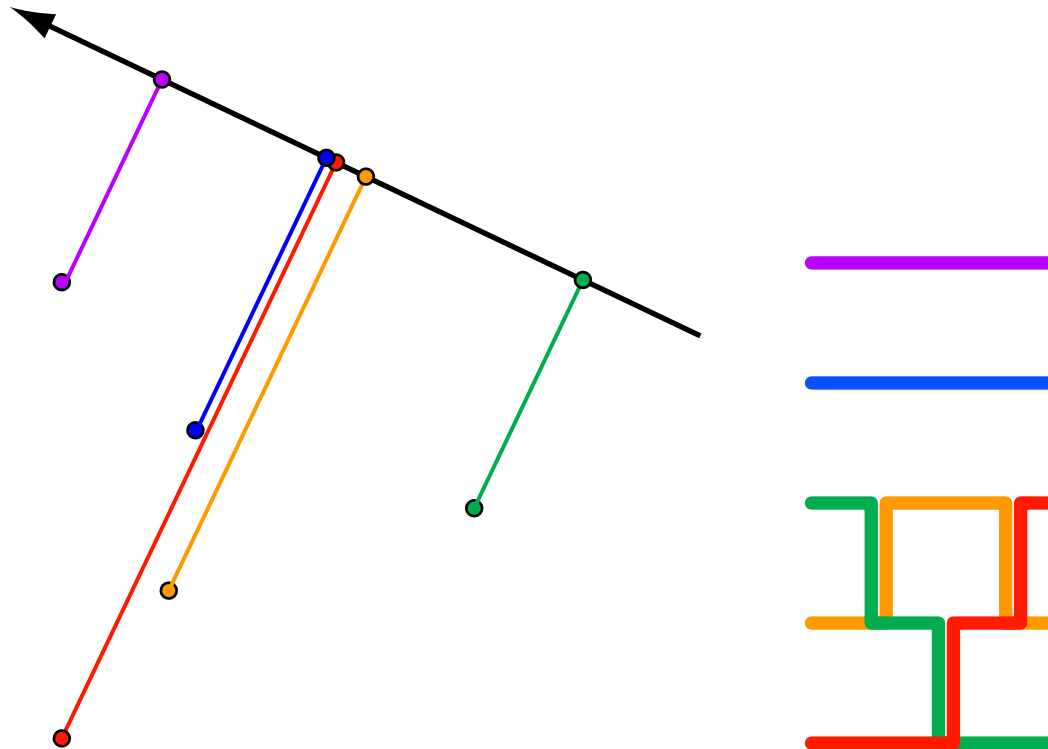
POINT SETS & MINIMAL SORTING NETWORKS



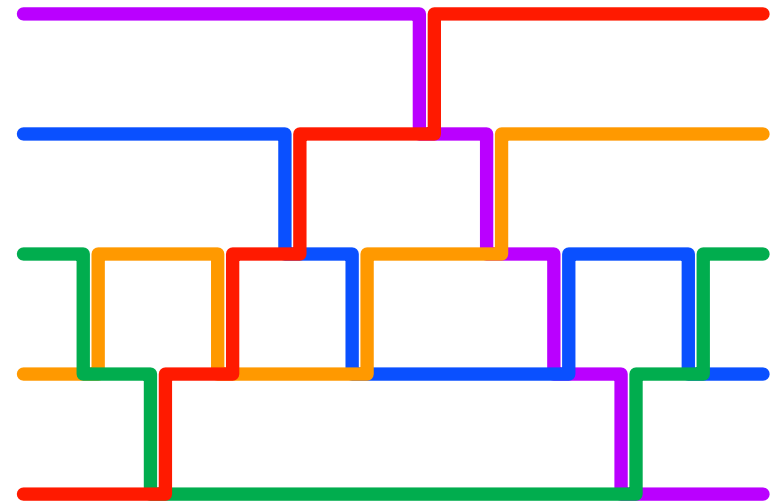
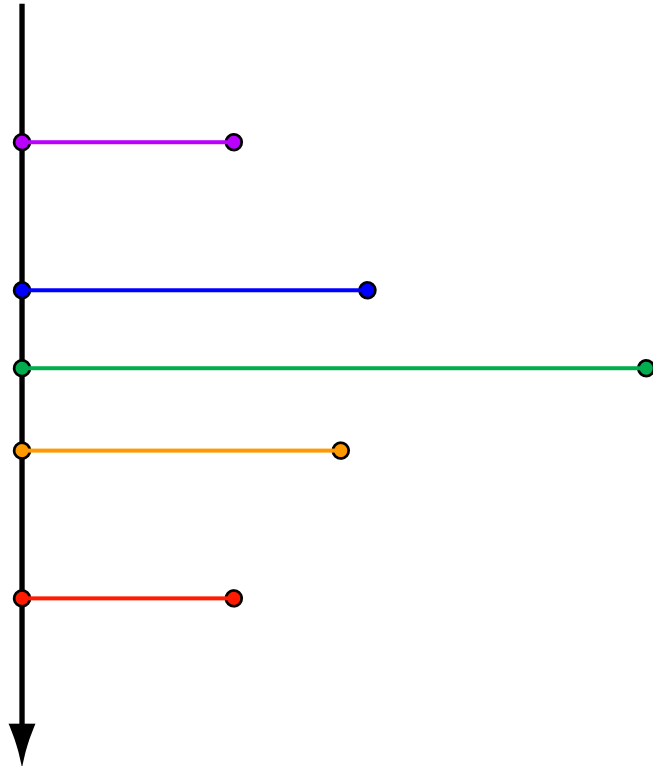
POINT SETS & MINIMAL SORTING NETWORKS



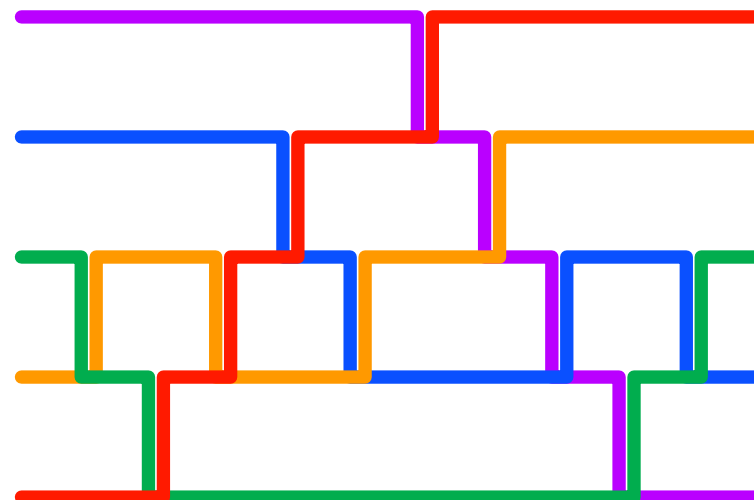
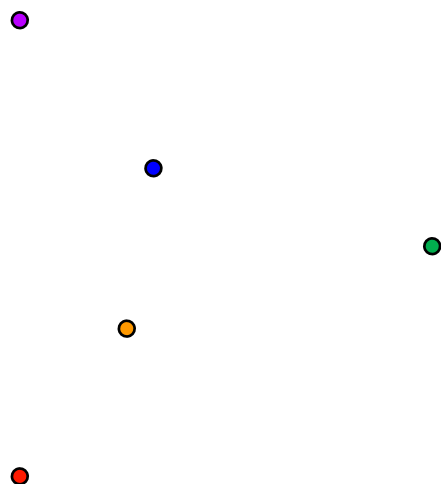
POINT SETS & MINIMAL SORTING NETWORKS



POINT SETS & MINIMAL SORTING NETWORKS



POINT SETS & MINIMAL SORTING NETWORKS



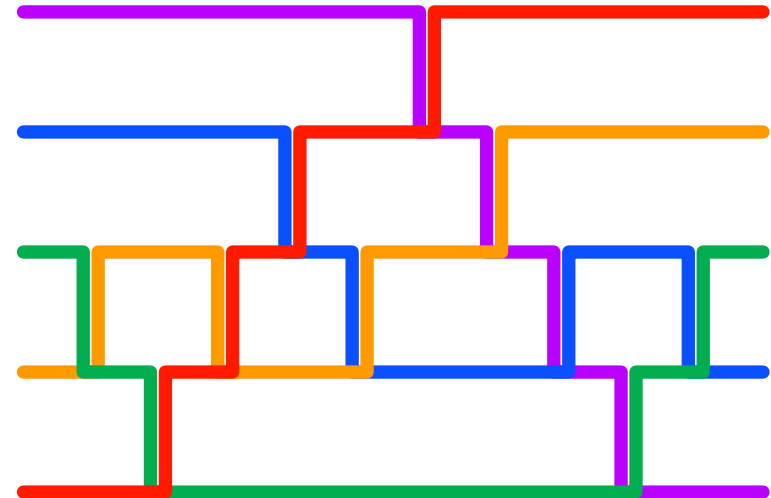
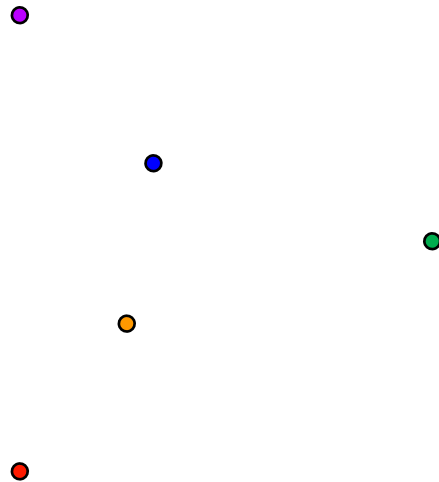
n points in $\mathbb{R}^2 \implies$ minimal primitive sorting network with n levels

point $\longleftrightarrow$ pseudoline

edge $\longleftrightarrow$ crossing

boundary edge $\longleftrightarrow$ external crossing

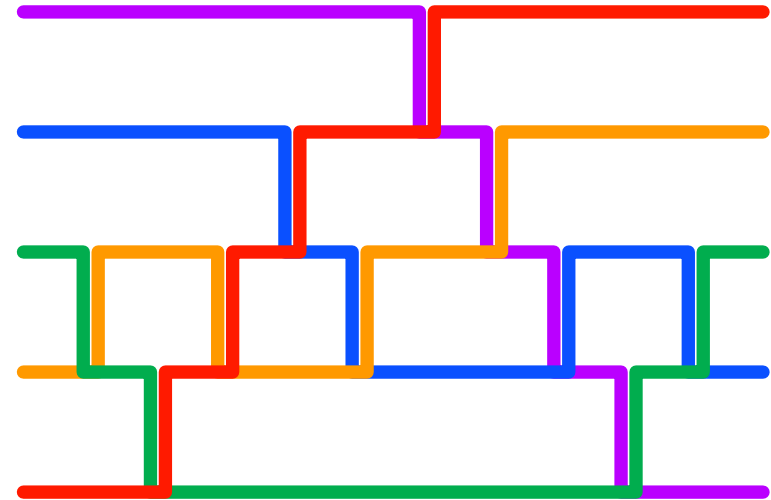
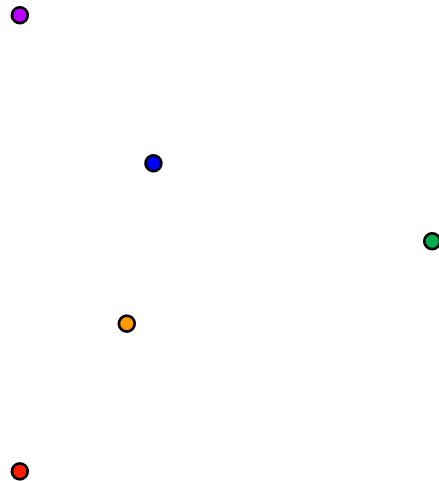
POINT SETS & MINIMAL SORTING NETWORKS



n points in $\mathbb{R}^2 \implies$ minimal primitive sorting network with n levels

not all minimal primitive sorting networks correspond to points sets of $\mathbb{R}^2$
 $\implies$ realizability problems

POINT SETS & MINIMAL SORTING NETWORKS



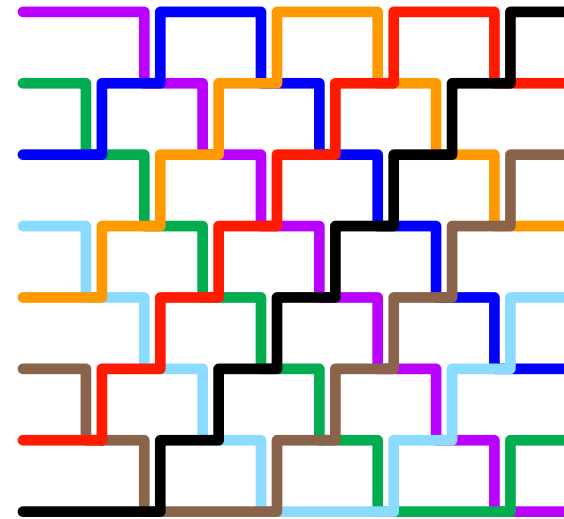
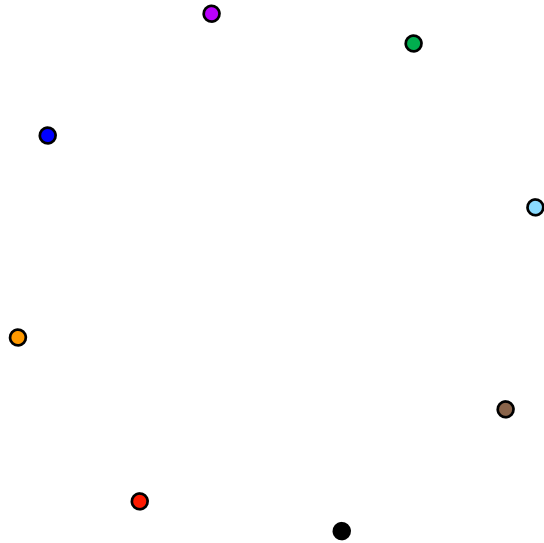
Goodmann-Pollack, *On the combinatorial classification of nondegenerate configurations in the plane* ('80)

Knuth, *Axioms and Hulls* ('92)

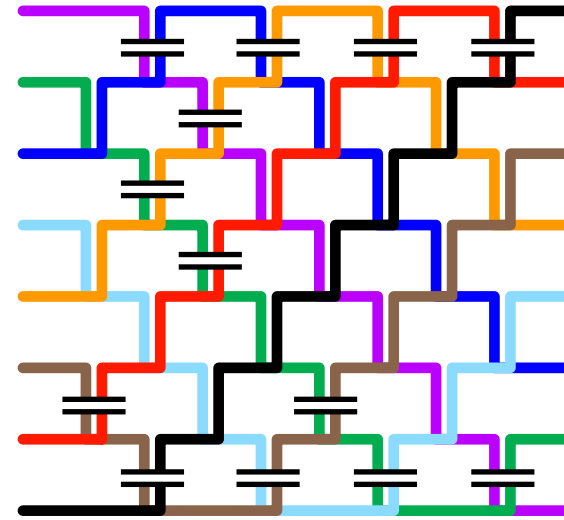
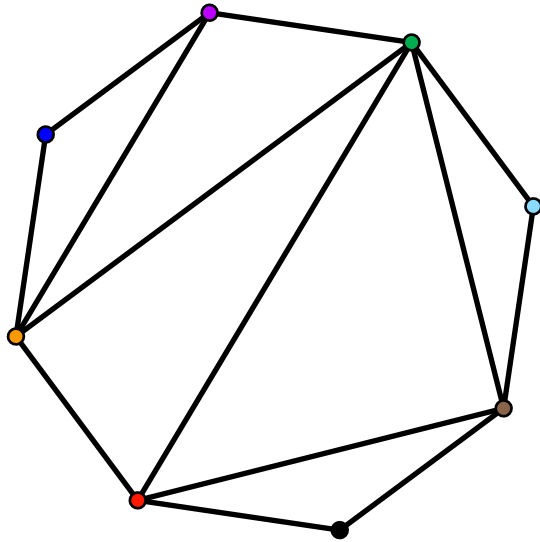
Björner-Las Vergnas-Sturmfels-White-Ziegler, *Oriented Matroids* ('99)

Bokowski, *Computational oriented matroids* ('06)

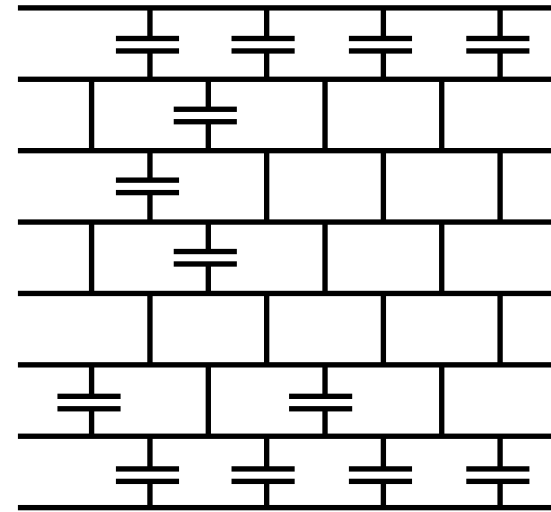
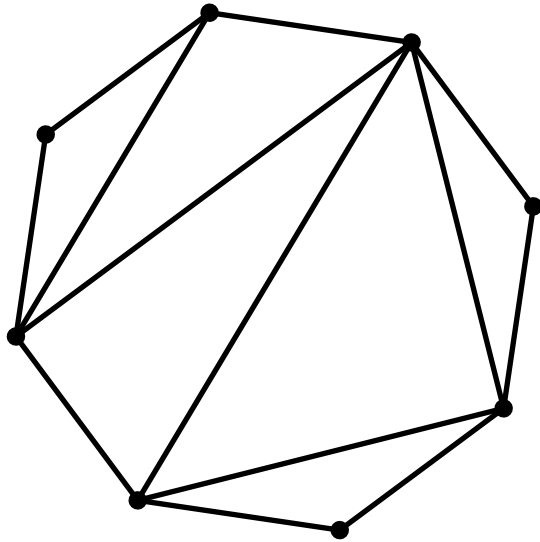
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



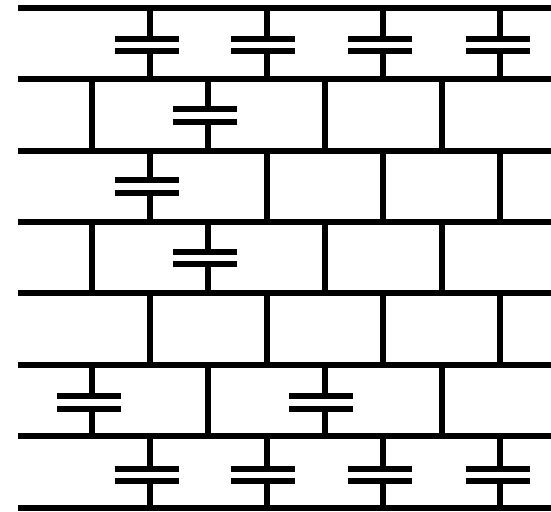
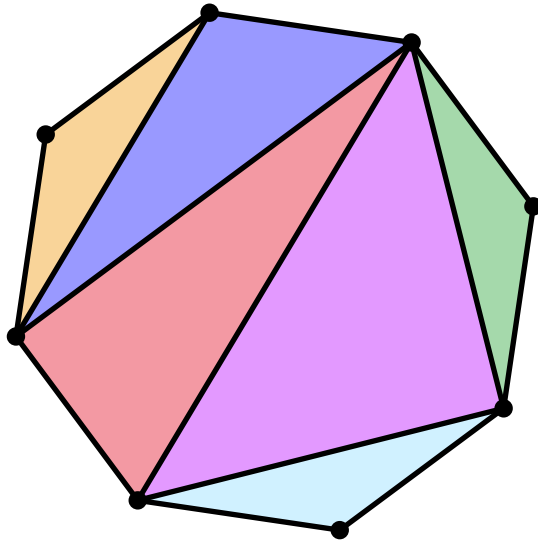
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



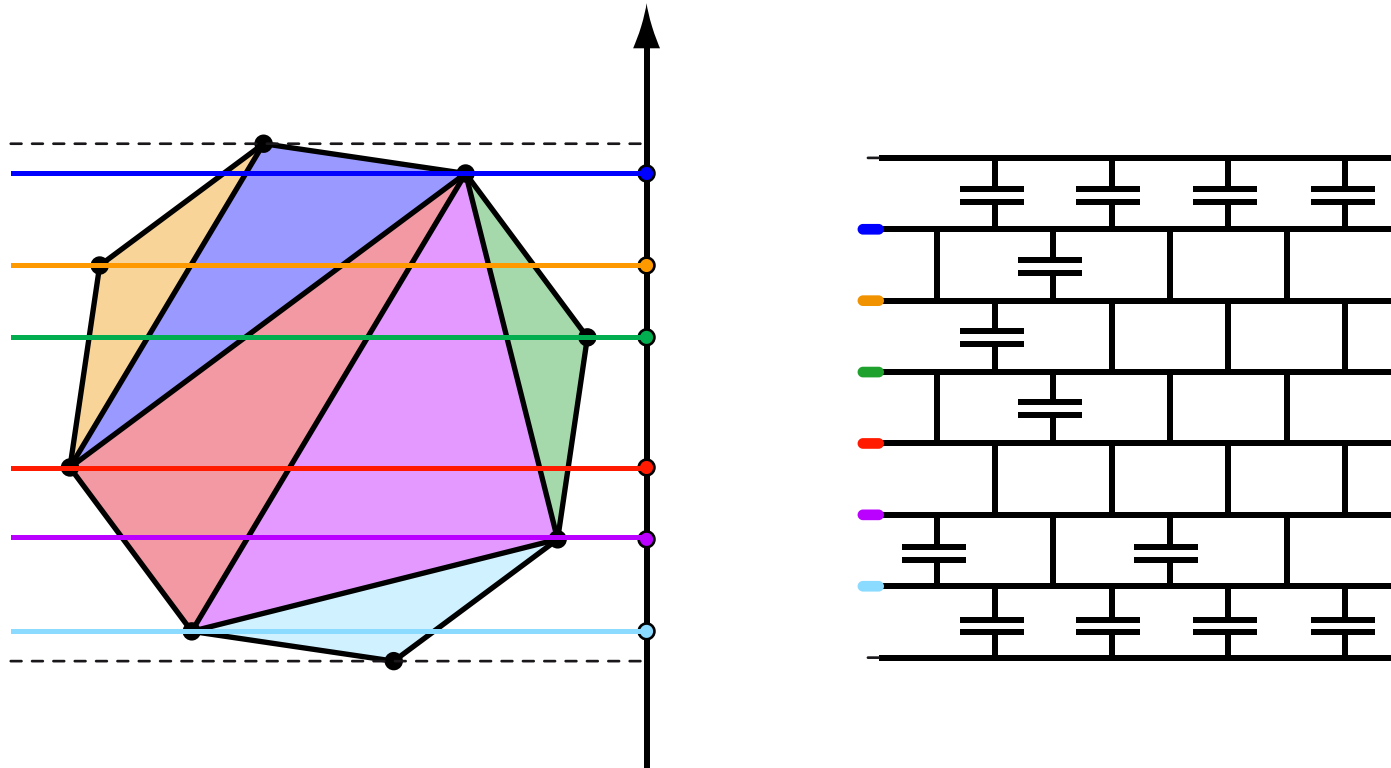
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



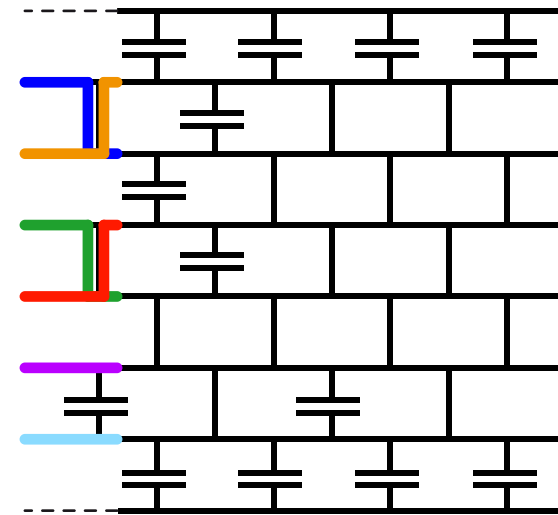
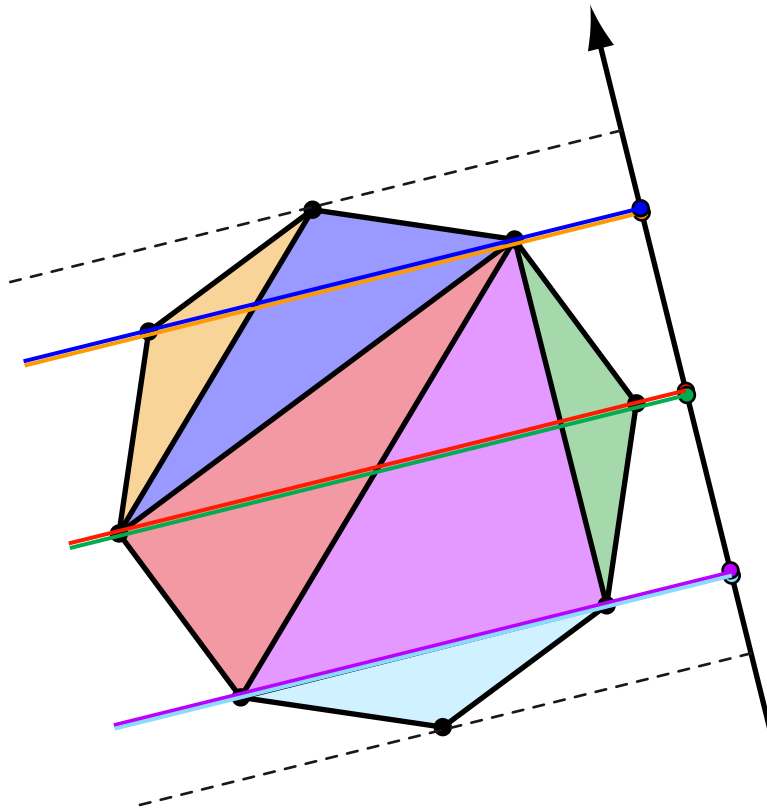
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



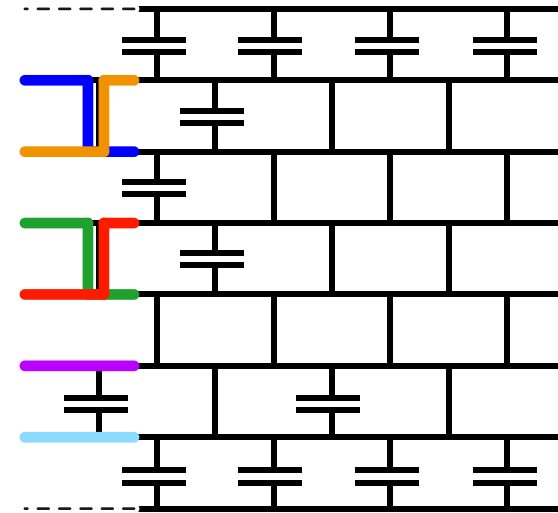
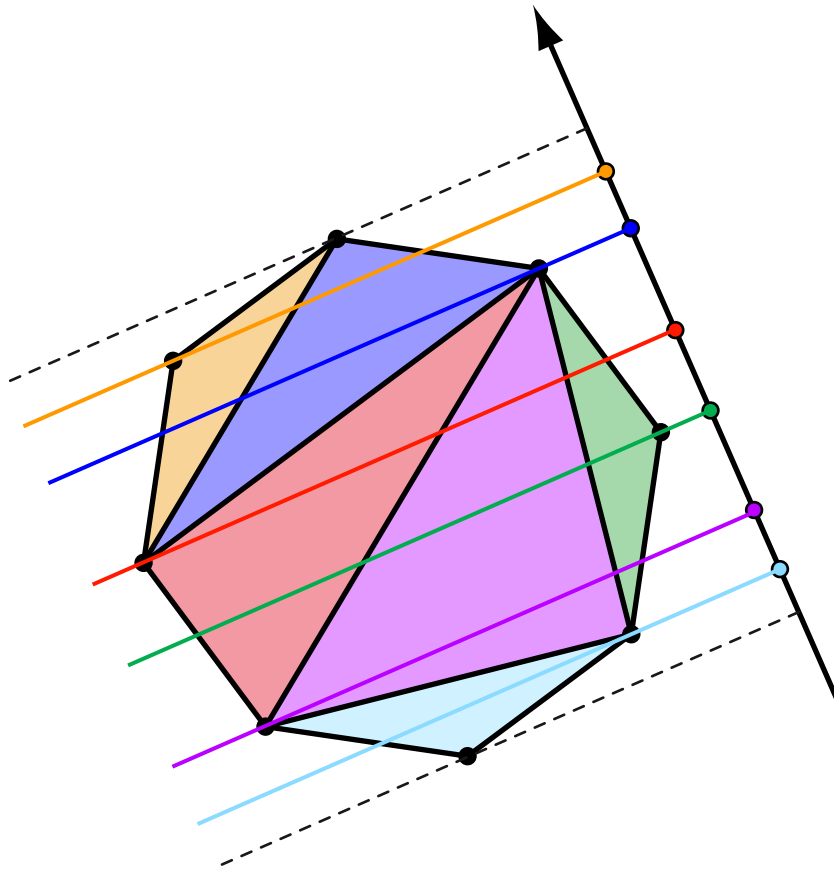
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



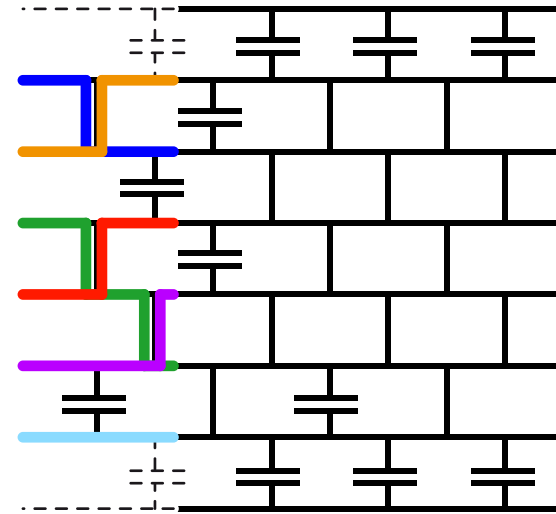
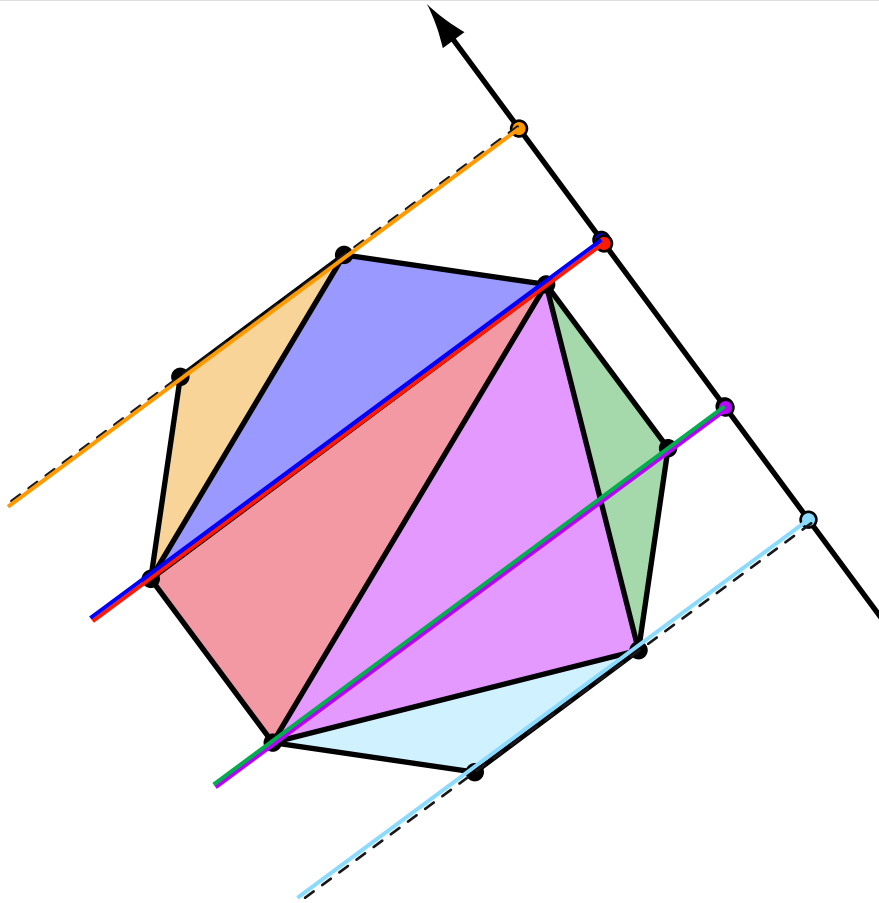
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



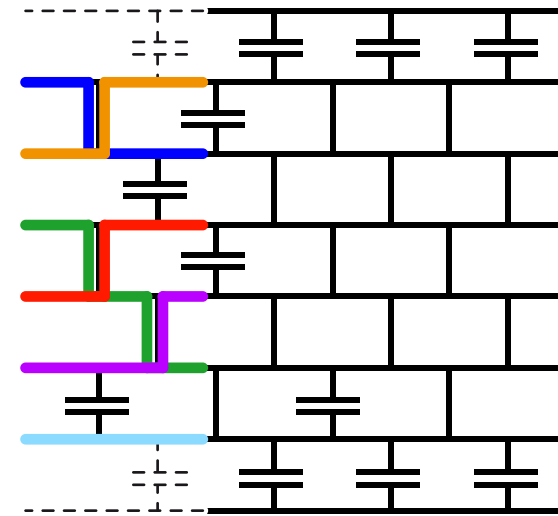
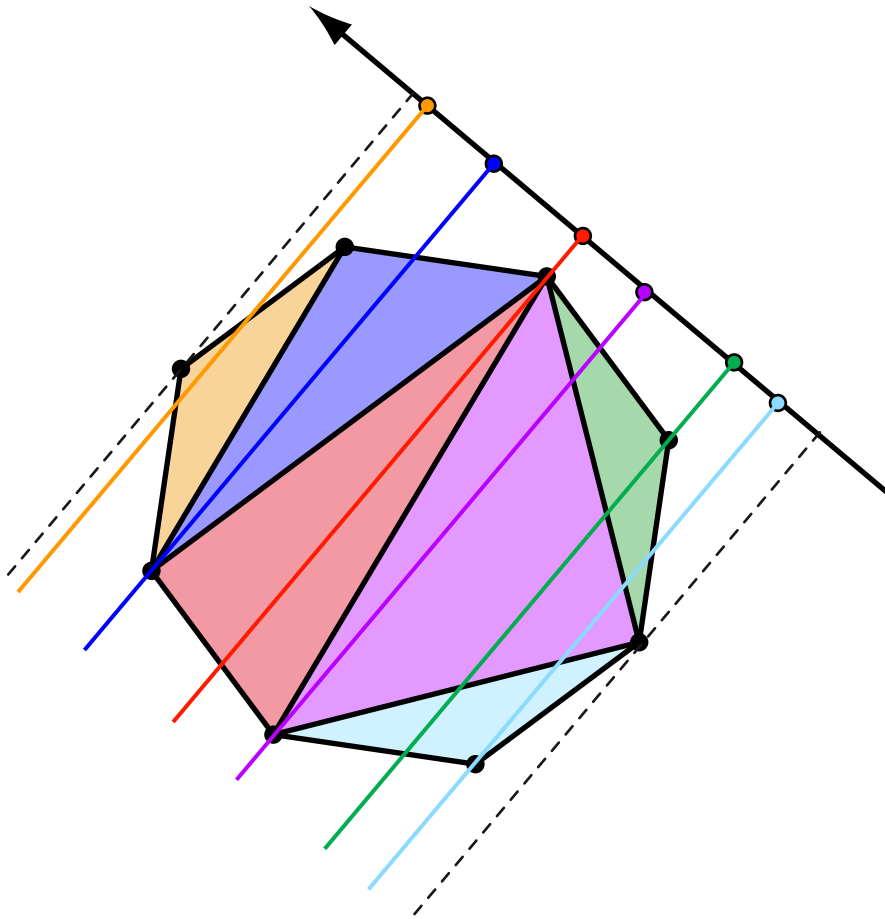
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



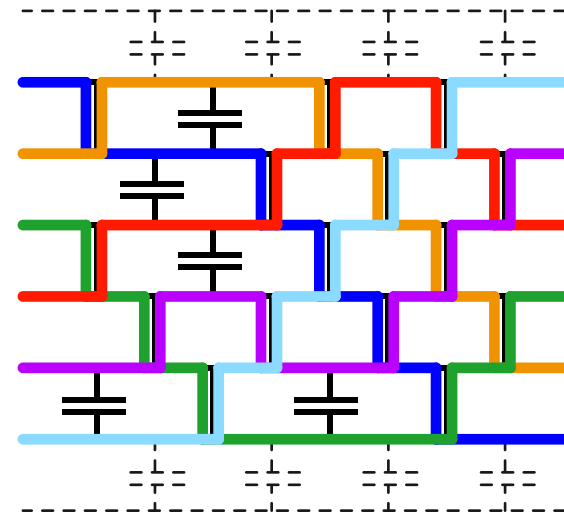
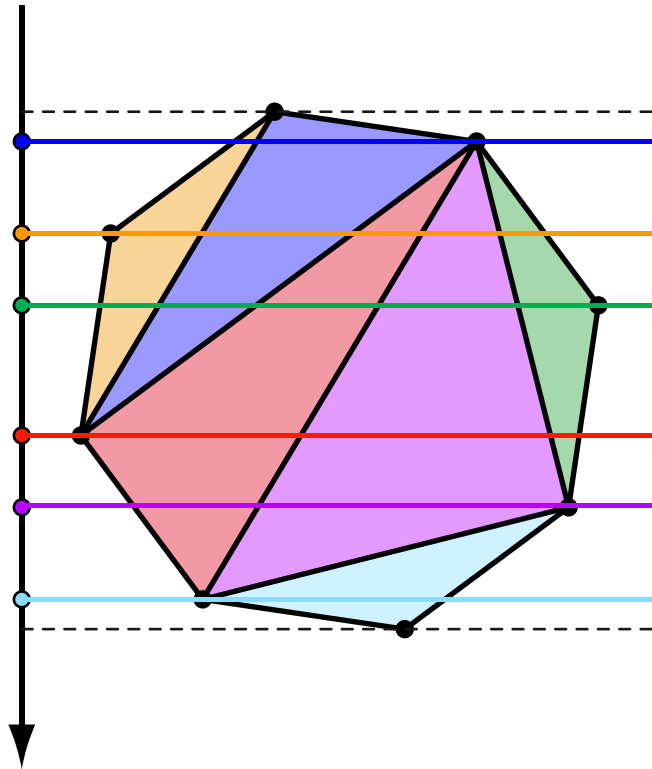
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



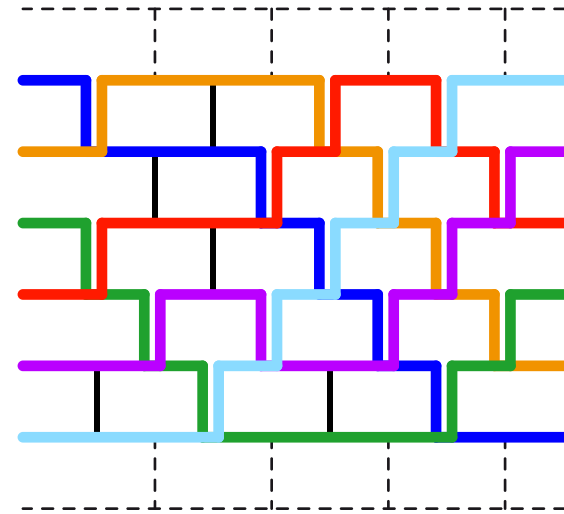
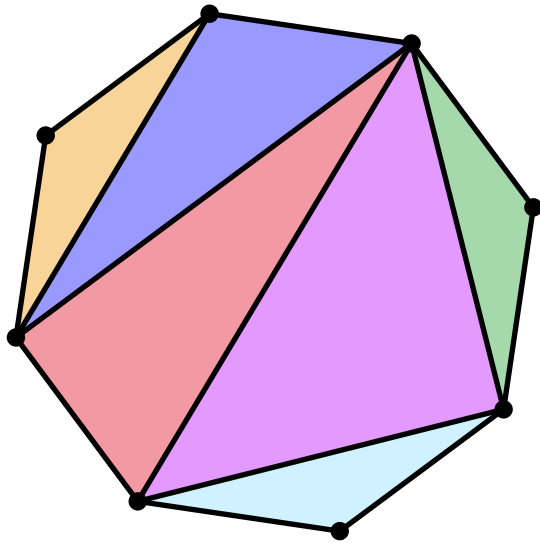
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



TRIANGULATIONS & ALTERNATING SORTING NETWORKS



TRIANGULATIONS & ALTERNATING SORTING NETWORKS



triangulation of the n -gon $\longleftrightarrow$ pseudoline arrangement

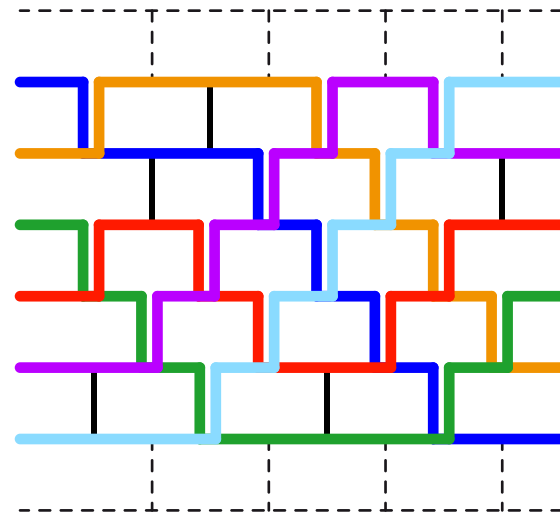
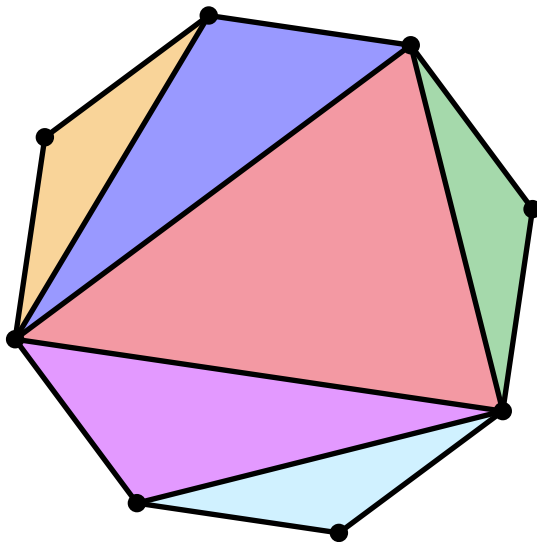
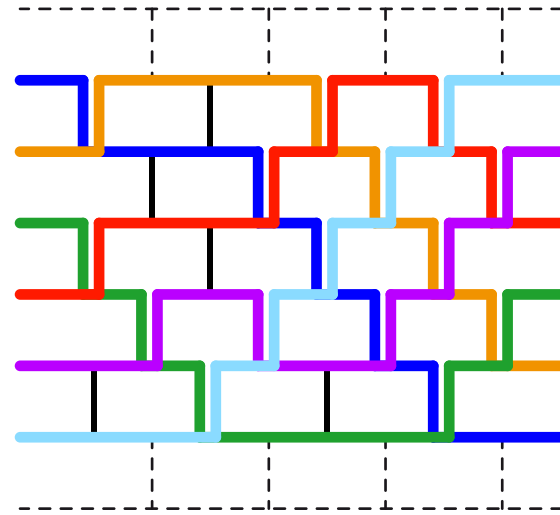
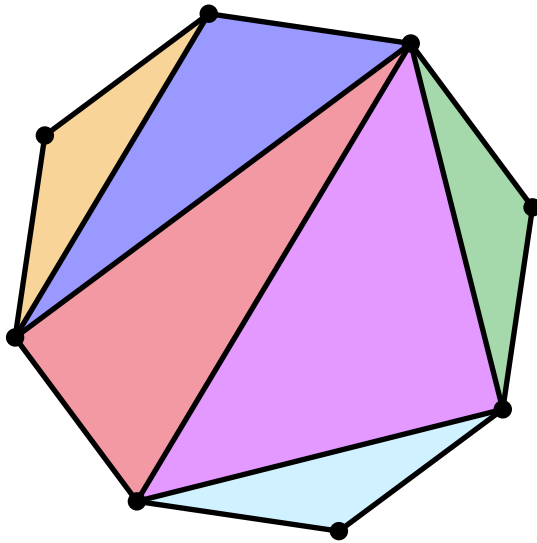
triangle $\longleftrightarrow$ pseudoline

edge $\longleftrightarrow$ contact point

common bisector $\longleftrightarrow$ crossing point

dual binary tree $\longleftrightarrow$ contact graph

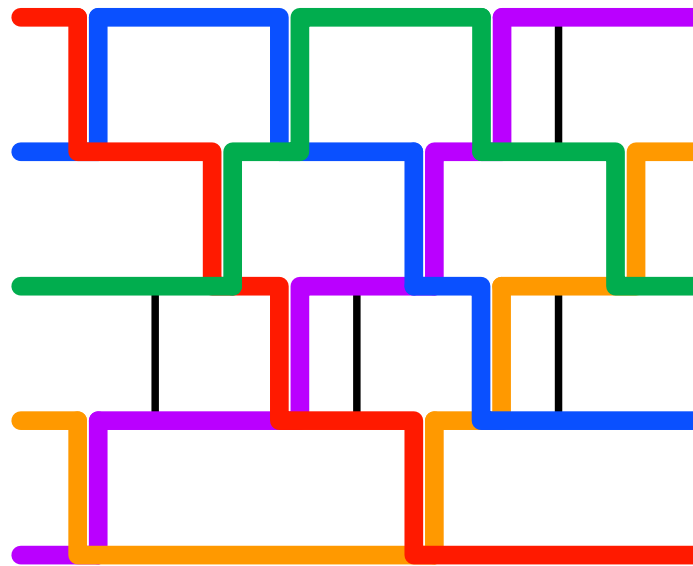
FLIPS



ASSOCIAHEDRON
— & —
BRICK POLYTOPE

BRICK POLYTOPE

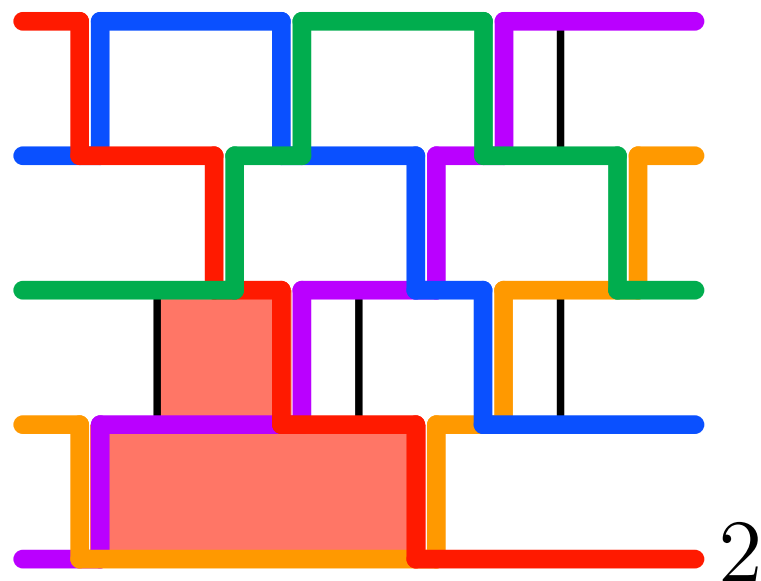
Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $b(\Lambda) \in \mathbb{R}^n$
 $b(\Lambda)_j =$ number of bricks of $\mathcal{N}$ below the j th pseudoline of Λ



Brick polytope $\mathcal{B}(\mathcal{N}) = \text{conv} \{b(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\}$

BRICK POLYTOPE

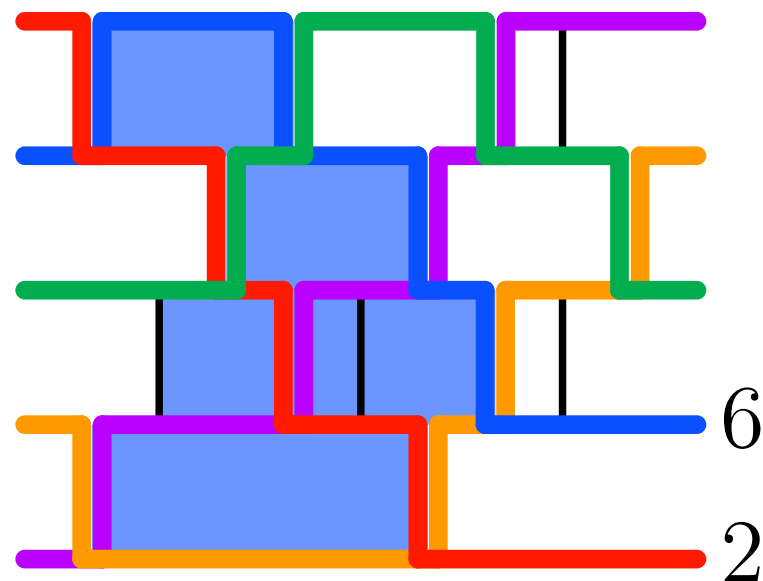
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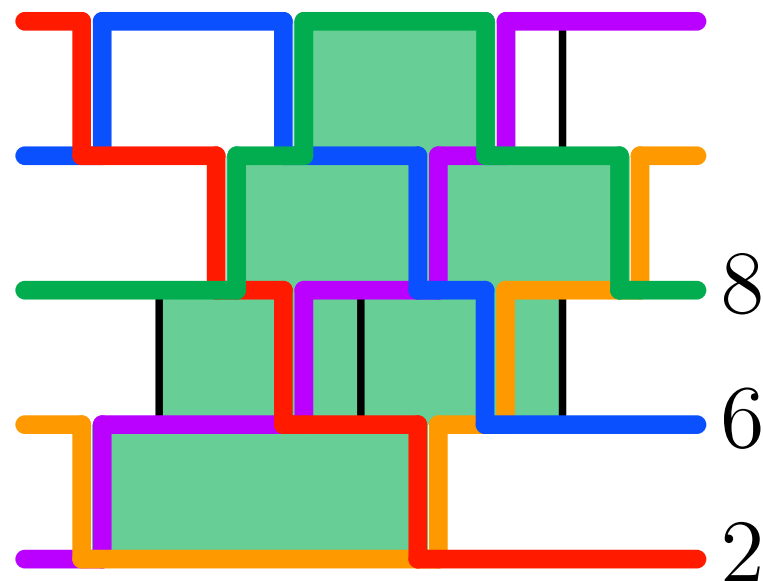
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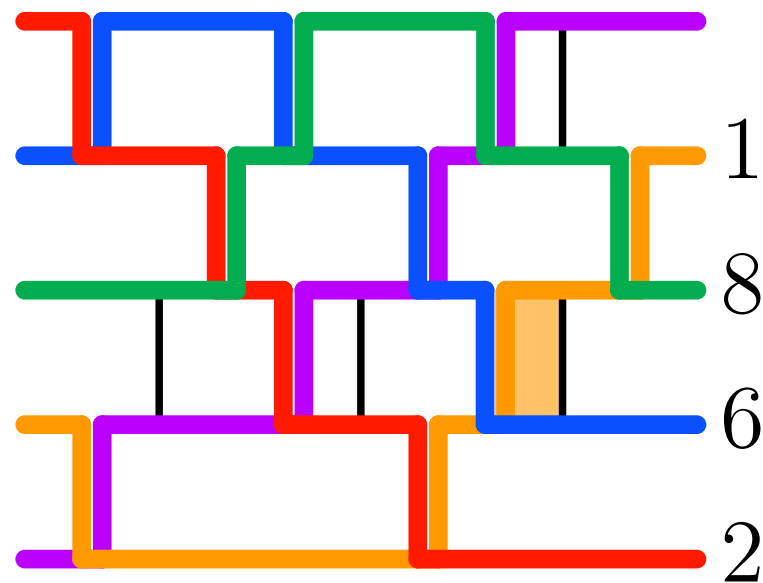
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BRICK POLYTOPE

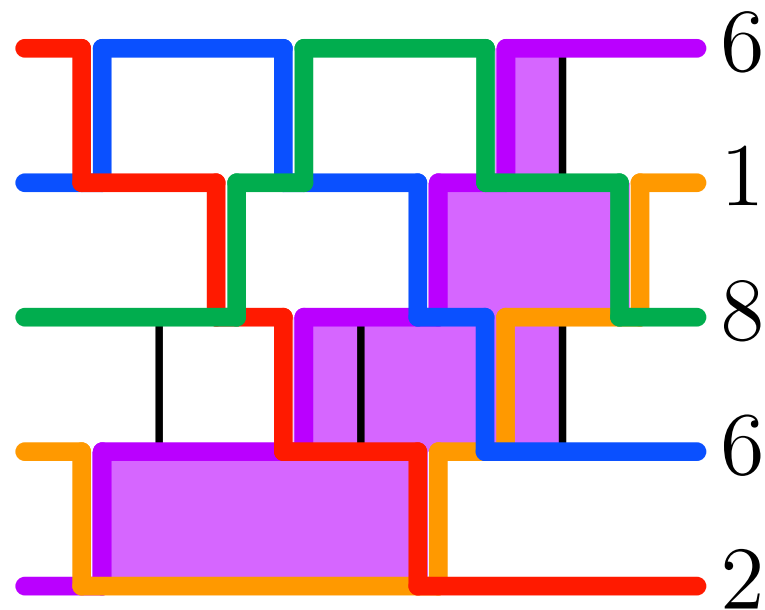
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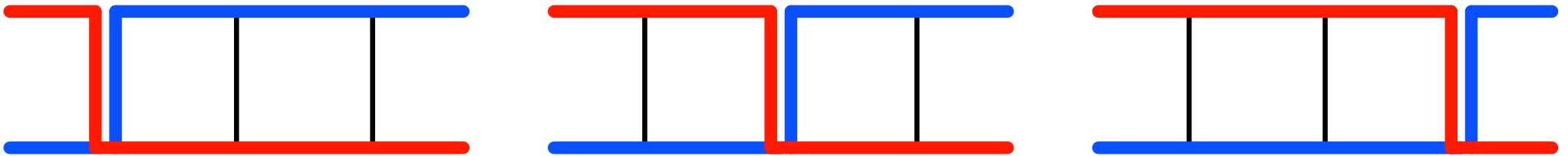
Brick polytope $\mathcal{B}(\mathcal{N}) = \text{conv} \{b(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\}$

BRICK POLYTOPE

$\mathcal{X}_m$ = network with two levels and m commutators

graph of flips $G(\mathcal{X}_m)$ = complete graph K_m

$$\text{brick polytope } \mathcal{B}(\mathcal{X}_m) = \text{conv} \left\{ \binom{m-i}{i-1} \mid i \in [m] \right\} = \left[\binom{m-1}{0}, \binom{0}{m-1} \right]$$

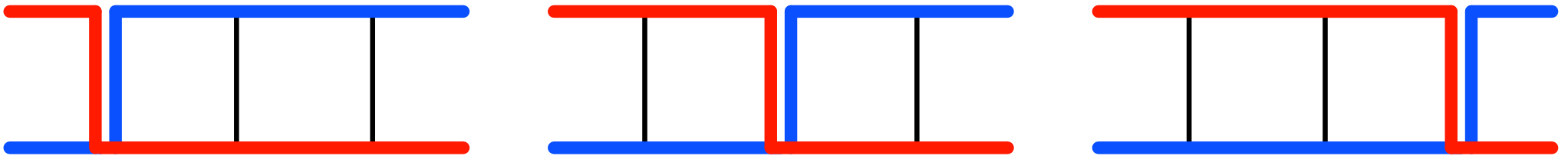


BRICK POLYTOPE

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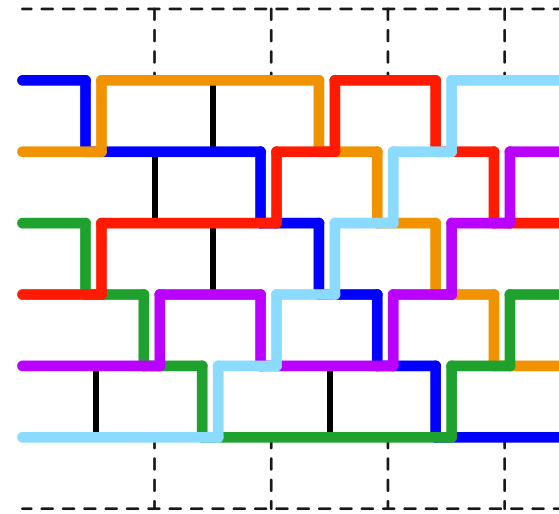
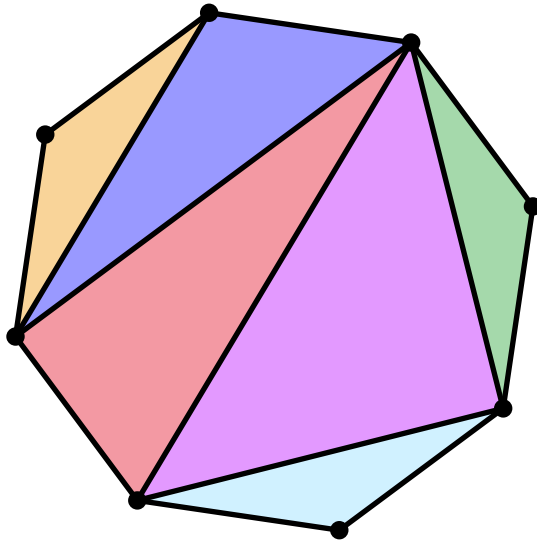
$$\text{brick polytope } \mathcal{B}(\mathcal{X}_m) = \text{conv} \left\{ \binom{m-i}{i-1} \mid i \in [m] \right\} = \left[\binom{m-1}{0}, \binom{0}{m-1} \right]$$



The brick vector $b(\Lambda)$ is a vertex of $\mathcal{B}(\mathcal{N}) \iff$ the contact graph $\Lambda^\#$ is acyclic
 The graph of the brick polytope $\mathcal{B}(\mathcal{N})$ is a subgraph of the flip graph $G(\mathcal{N})$

The graph of the brick polytope $\mathcal{B}(\mathcal{N})$ coincides with the graph of flips $G(\mathcal{N})$
 $\iff$ the contact graphs of the pseudoline arrangements supported by $\mathcal{N}$ are forests

ALTERNATING NETWORKS & ASSOCIAHEDRA



triangulation of the n -gon $\longleftrightarrow$ pseudoline arrangement

triangle $\longleftrightarrow$ pseudoline

edge $\longleftrightarrow$ contact point

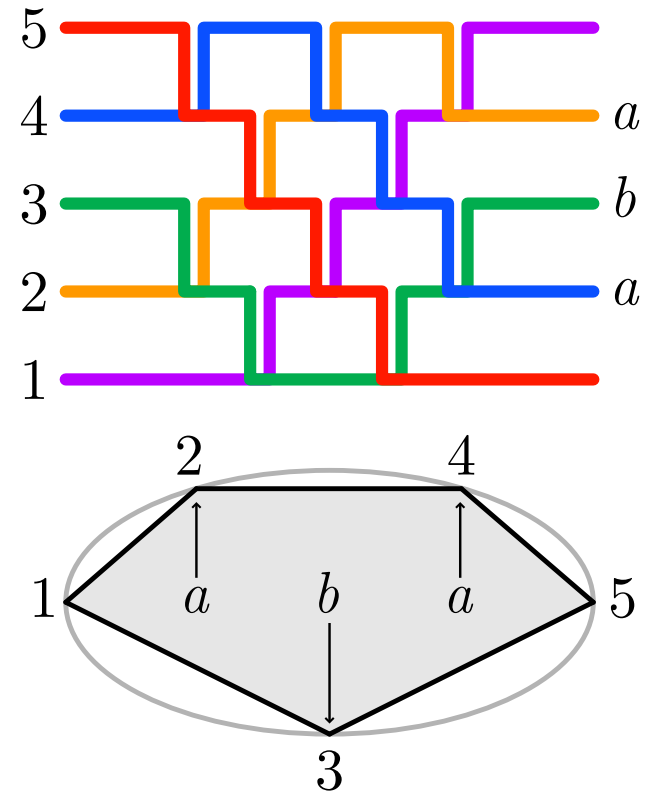
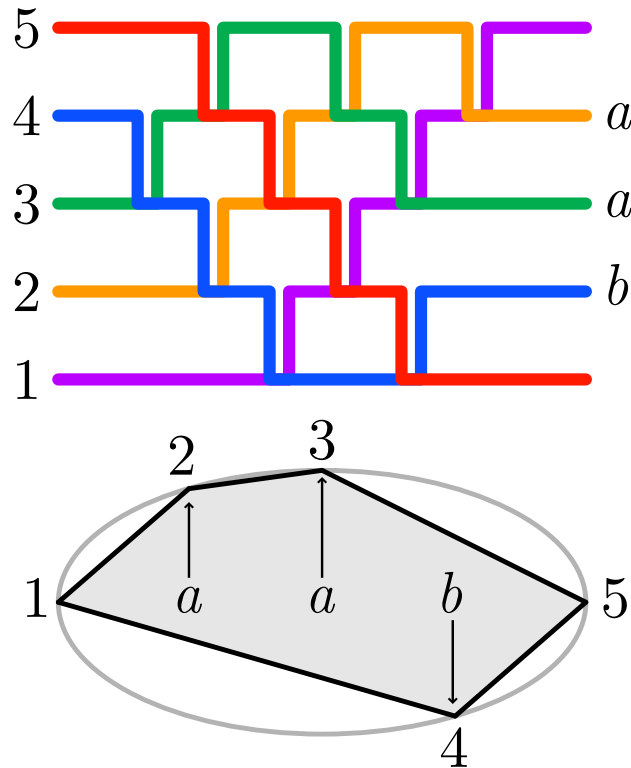
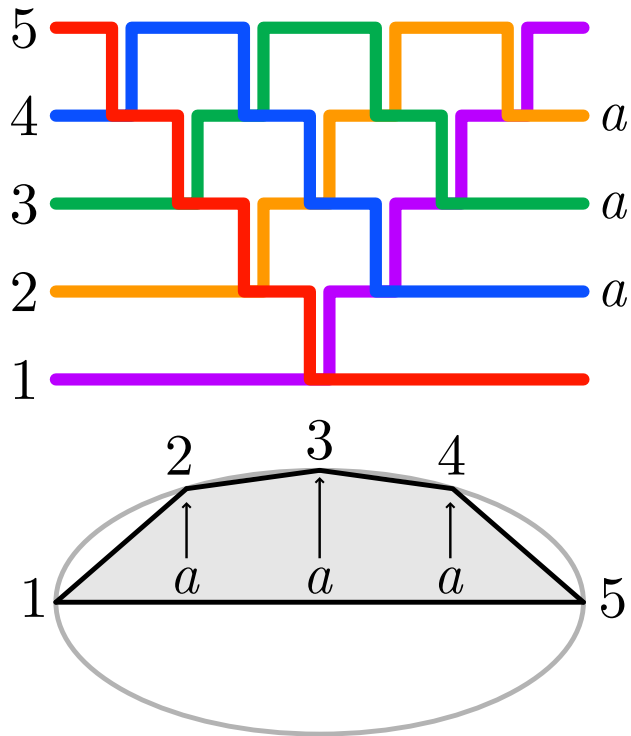
common bisector $\longleftrightarrow$ crossing point

dual binary tree $\longleftrightarrow$ contact graph

The brick polytope is an associahedron.

ALTERNATING NETWORKS & ASSOCIAHEDRA

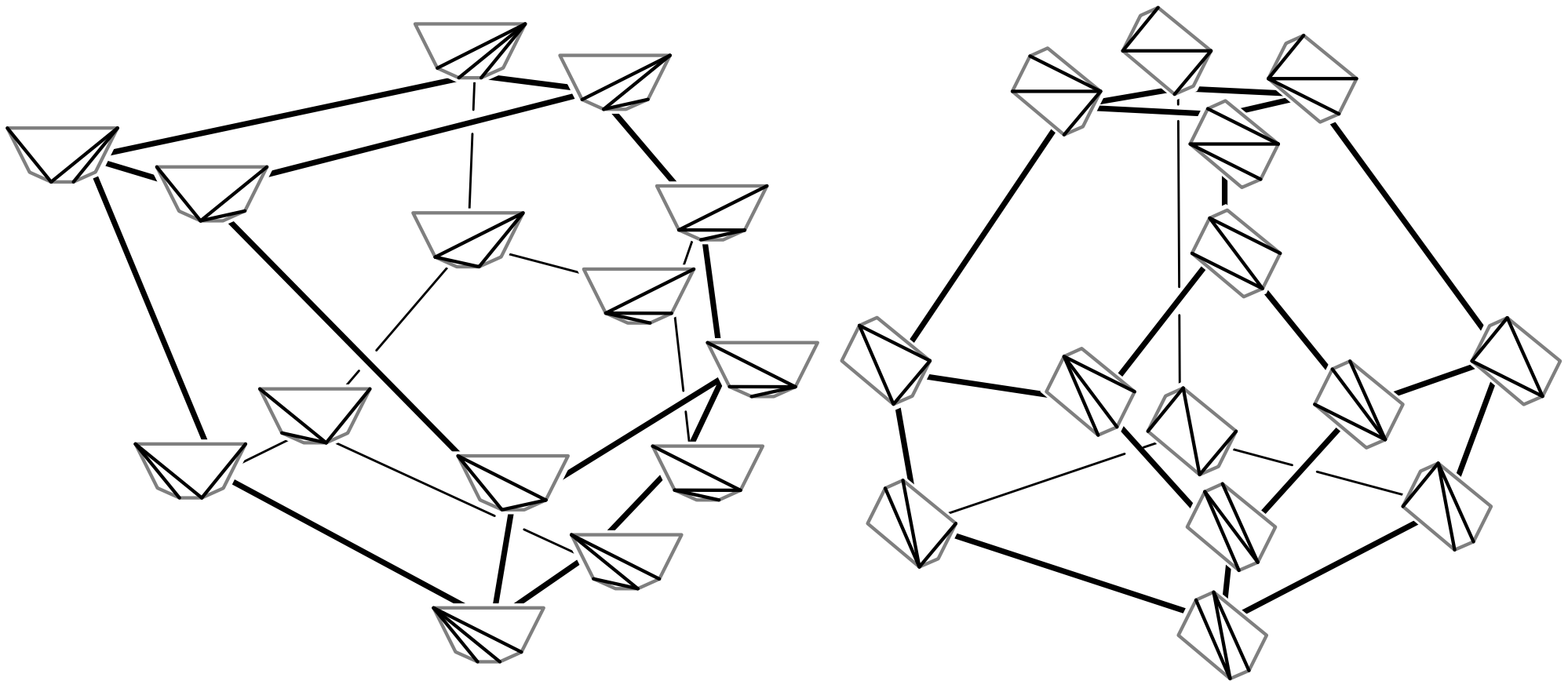
for $x \in \{a, b\}^{n-2}$, define a reduced alternating network $\mathcal{N}_x$ and a polygon $\mathcal{P}_x$



Pseudoline arrangements on $\mathcal{N}_x^1 \longleftrightarrow$ triangulations of the polygon $\mathcal{P}_x$.

ALTERNATING NETWORKS & ASSOCIAHEDRA

For any word $x \in \{a, b\}^{n-2}$, the brick polytope $\mathcal{B}_x = \mathcal{B}(\mathcal{N}_x^1)$ is an associahedron.

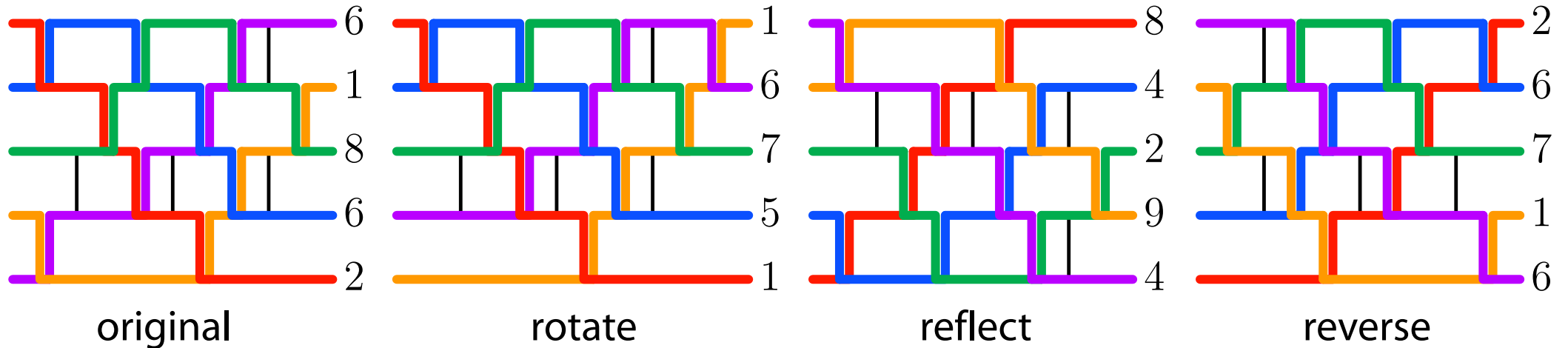


Up to a translation Ω_x , the brick polytope $\mathcal{B}_x$ coincides with the associahedron Asso_x of Hohlweg and Lange.

ASSOCIAHEDRON
— & —
BARYCENTER

THREE OPERATIONS

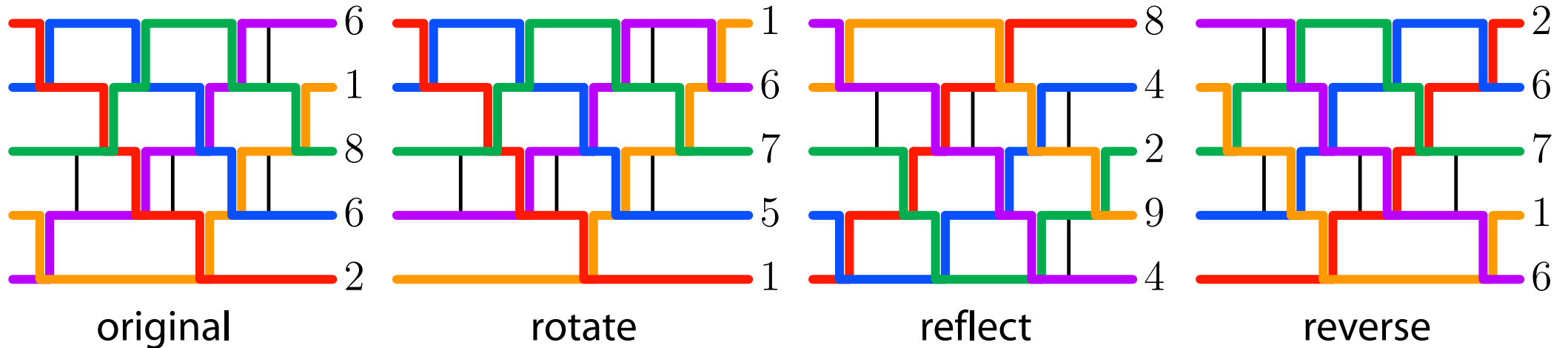
Evolution of the brick vector $b_{\mathcal{N}}(\Lambda)$ under three operations:



1. Rotate: $b_{\mathcal{N}^\circ}(\Lambda^\circ) - b_{\mathcal{N}}(\Lambda) \in \omega_i + \mathbb{R}(e_{i+1} - e_i)$
2. Reflect: $b_{\mathcal{N}^\downarrow}(\Lambda^\downarrow) = \#\{\text{bricks of } \mathcal{N}\} \cdot \mathbb{1} - (b_{\mathcal{N}}(\Lambda))^{\leftarrow\rightarrow}$
3. Reverse: $b_{\mathcal{N}^{\leftarrow\rightarrow}}(\Lambda^{\leftarrow\rightarrow}) = (b_{\mathcal{N}}(\Lambda))^{\leftarrow\rightarrow}$

THREE OPERATIONS

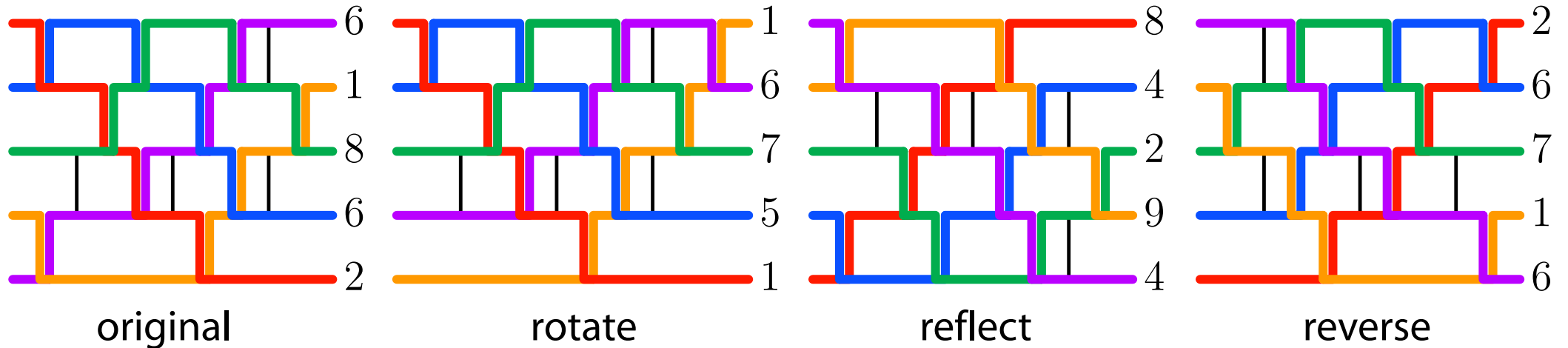
Evolution of the translated brick vector $\bar{b}_x(\Lambda) = b_x(\Lambda) - \Omega_x$ under three operations:



1. Rotate: $\bar{b}_{x^\circ}(\Lambda^\circ) - \bar{b}_x(\Lambda) \in \mathbb{R}(e_{i+1} - e_i)$
2. Reflect: $\bar{b}_{x^\downarrow}(\Lambda^\downarrow) = -(\bar{b}_x(\Lambda))^{\leftarrow}$
3. Reverse: $\bar{b}_{x^\leftarrow}(\Lambda^\leftarrow) = (\bar{b}_x(\Lambda))^{\leftarrow}$

THREE OPERATIONS

Evolution of the translated brick vector $\bar{b}_x(\Lambda) = b_x(\Lambda) - \Omega_x$ under three operations:

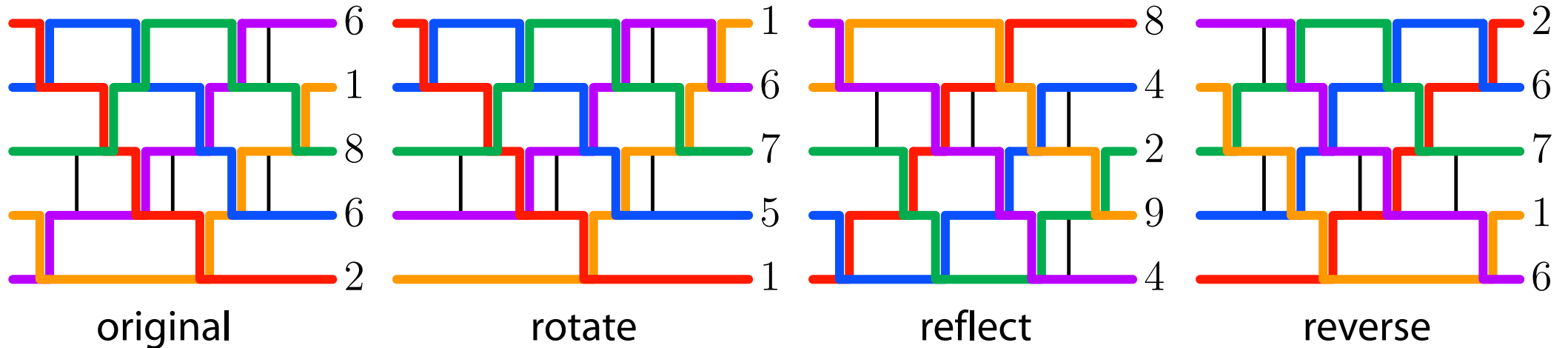


1. Rotate: $\bar{b}_{x^\circ}(\Lambda^\circ) - \bar{b}_x(\Lambda) \in \mathbb{R}(e_{i+1} - e_i)$

All associahedra A_{SSO_x} have the same barycenter

THREE OPERATIONS

Evolution of the translated brick vector $\bar{b}_x(\Lambda) = b_x(\Lambda) - \Omega_x$ under three operations:



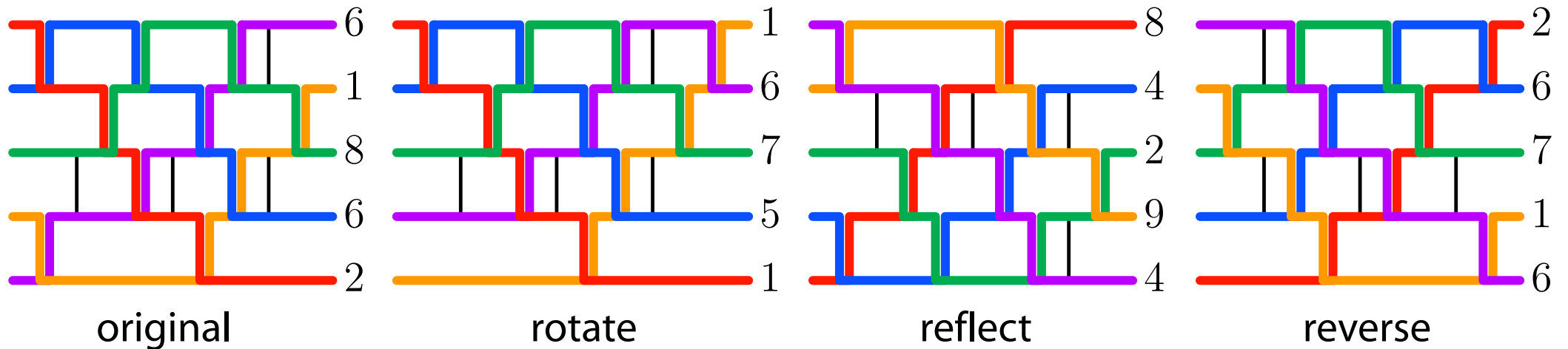
2. Reflect: $\bar{b}_{x\downarrow}(\Lambda^\uparrow) = -(\bar{b}_x(\Lambda))^{\leftrightarrow}$

3. Reverse: $\bar{b}_{x\leftrightarrow}(\Lambda^{\leftrightarrow}) = (\bar{b}_x(\Lambda))^{\leftrightarrow}$

The barycenter of the superposition of the vertices of $\text{ASSO}_{x\downarrow}$ and $\text{ASSO}_{x\leftrightarrow}$ is the origin

THREE OPERATIONS

Evolution of the translated brick vector $\bar{b}_x(\Lambda) = b_x(\Lambda) - \Omega_x$ under three operations:



All associahedra Asso_x have the same barycenter

The barycenter of the superposition of the vertices of $\text{Asso}_{x \updownarrow}$ and $\text{Asso}_{x \leftrightarrow}$ is the origin

THEOREM. All associahedra Asso_x have vertex barycenter at the origin

...and the same method works for fairly balanced and generalized associahedra.

THANK YOU