

# A natural stochastic extension of the sandpile model on a graph

Joint work with Y. Chan and J-F. Marckert

Journées Combinatoires de Bordeaux

13 Février 2014

- 1 Definition and differences with the abelian sandpile model

- 1 Definition and differences with the abelian sandpile model
- 2 A characterisation of the recurrent states

- 1 Definition and differences with the abelian sandpile model
- 2 A characterisation of the recurrent states
- 3 The lacking polynomial

We consider a graph  $G = (V \cup \{s\}, E)$  which is finite, connected, and loop-free. We distinguish a vertex  $s$  called the sink.

# Definition

We consider a graph  $G = (V \cup \{s\}, E)$  which is finite, connected, and loop-free. We distinguish a vertex  $s$  called the sink.

A configuration is a vector  $\eta = (\eta_v, v \in V) \in \mathbb{Z}_+^{|V|}$ . The number  $\eta_v$  represents the number of grains of sand present at the vertex  $v$  in the configuration  $\eta$ .

# Definition

We consider a graph  $G = (V \cup \{s\}, E)$  which is finite, connected, and loop-free. We distinguish a vertex  $s$  called the sink.

A configuration is a vector  $\eta = (\eta_v, v \in V) \in \mathbb{Z}_+^{|V|}$ . The number  $\eta_v$  represents the number of grains of sand present at the vertex  $v$  in the configuration  $\eta$ .

A vertex  $v$  is stable in the configuration  $\eta$  if  $\eta_v \leq d^G(v)$ . A configuration is stable if all its sites are stable.

## Definition (2)

An unstable vertex topples. In the abelian model (ASM), it sends one grain of sand to each of its neighbours. In the stochastic model (SSM), it sends a grain to each neighbour independently with probability  $p$  for some  $p \in (0, 1]$ .

## Definition (2)

An unstable vertex topples. In the abelian model (ASM), it sends one grain of sand to each of its neighbours. In the stochastic model (SSM), it sends a grain to each neighbour independently with probability  $p$  for some  $p \in (0, 1]$ .

The sink never topples, and thus an unstable configuration eventually stabilises. We denote the stabilisation operator  $RS$ .

## Definition (2)

An unstable vertex topples. In the abelian model (ASM), it sends one grain of sand to each of its neighbours. In the stochastic model (SSM), it sends a grain to each neighbour independently with probability  $p$  for some  $p \in (0, 1]$ .

The sink never topples, and thus an unstable configuration eventually stabilises. We denote the stabilisation operator  $RS$ .

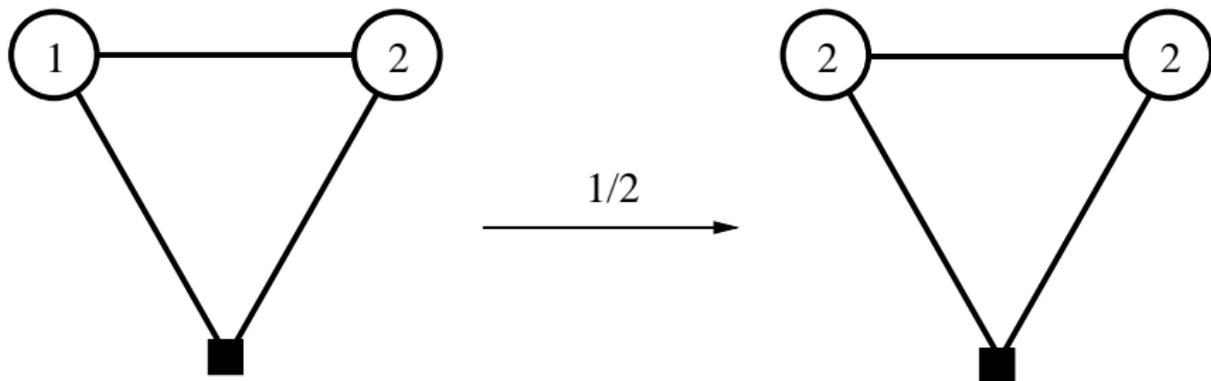
The configuration  $\eta_{\max}$  is defined by  $\eta^{\max}(v) = d^G(v)$  for all  $v \in V$ .

# The Markov chain

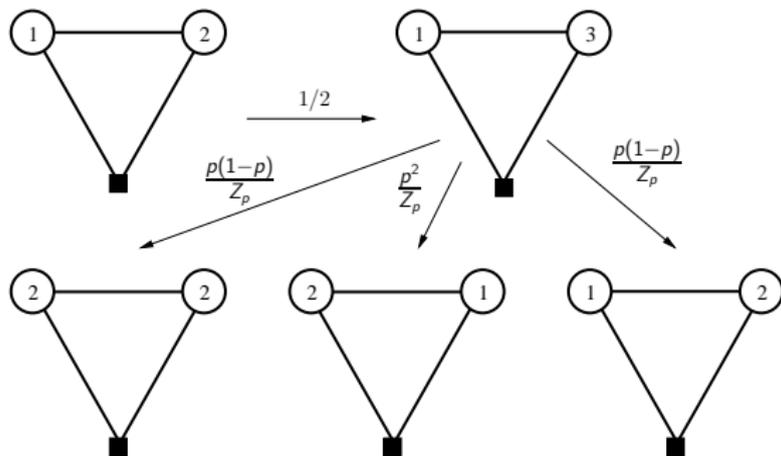
We define a Markov chain structure on the SSM. Let  $\mu$  be a measure with support  $V$ . Let  $(X_i, i \geq 1)$  be i.i.d. random variables with law  $\mu$ . Start from any stable configuration  $\eta_0$ . Given  $\eta_{i-1}$  for any  $i \geq 1$ ,  $\eta_i$  is obtained as follows:

- Add a grain at position  $X_i$  to the configuration  $\eta_{i-1}$ . Let  $\eta'_i$  be the obtained configuration.
- Let  $\eta_i$  be the stabilisation of  $\eta'_i$ , that is  $\eta_i = \text{RS}(\eta'_i)$  (in some cases no toppling is needed).

# An example



# An example



# Recurrent configurations

The set of recurrent configurations for the Markov chain is the unique recurrent class containing the maximal configuration. We denote this set by  $\text{Sto}(G)$ .

# Recurrent configurations

The set of recurrent configurations for the Markov chain is the unique recurrent class containing the maximal configuration. We denote this set by  $\text{Sto}(G)$ .

## Proposition

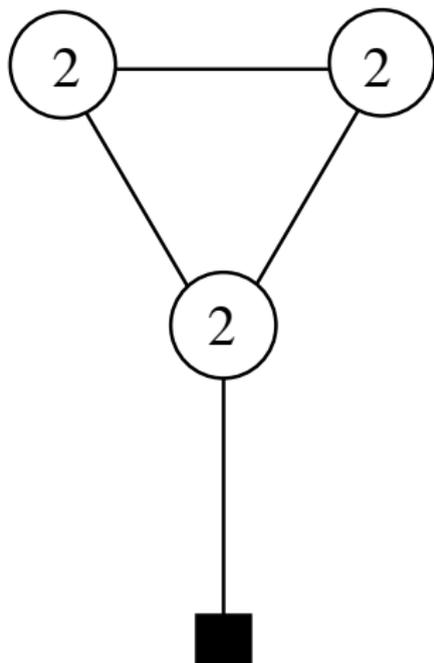
A configuration  $\eta$  is recurrent if and only if there exists a finite sequence of adding of grains and topplings such that  $\eta$  is reached from  $\eta^{max}$  through this sequence.

# Differences between the ASM and the SSM

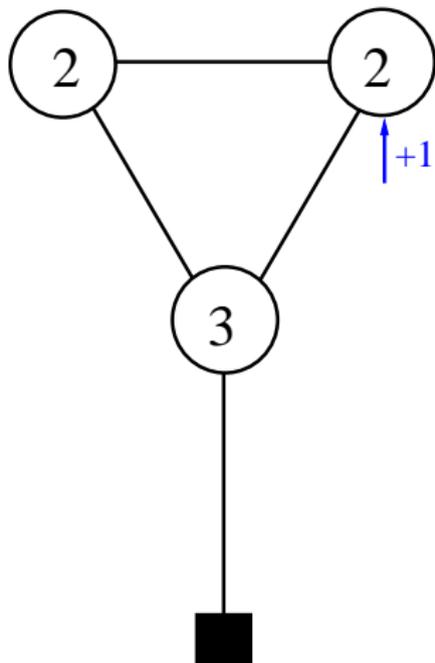
The set of recurrent configurations is not the same.

# Differences between the ASM and the SSM

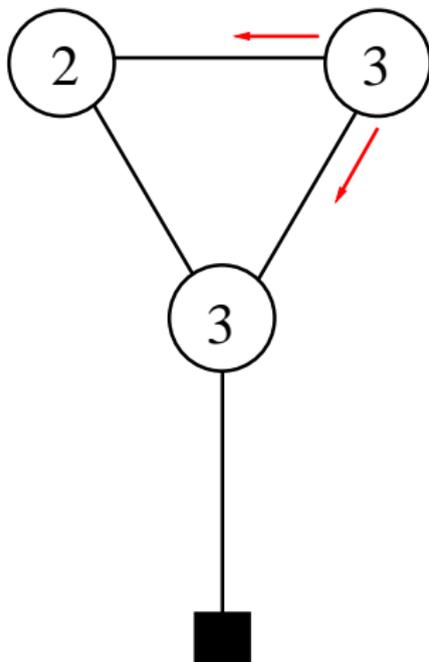
The set of recurrent configurations is not the same.



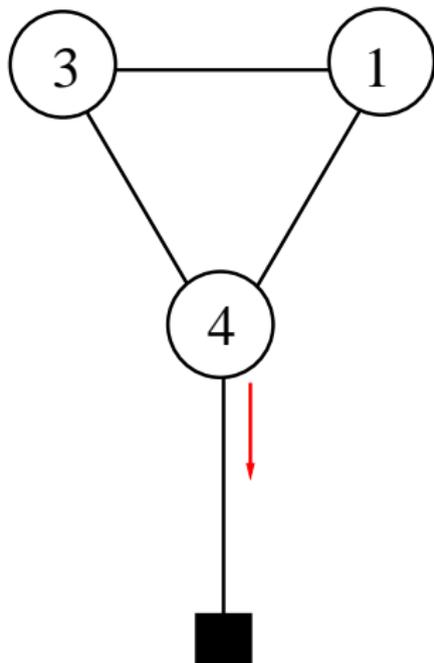
# Differences between the ASM and the SSM



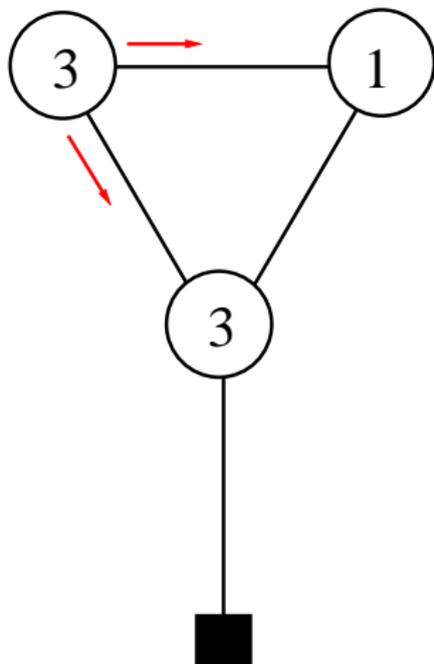
# Differences between the ASM and the SSM



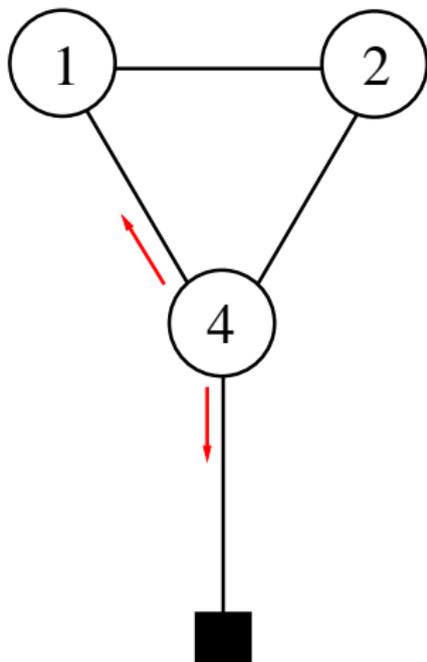
# Differences between the ASM and the SSM



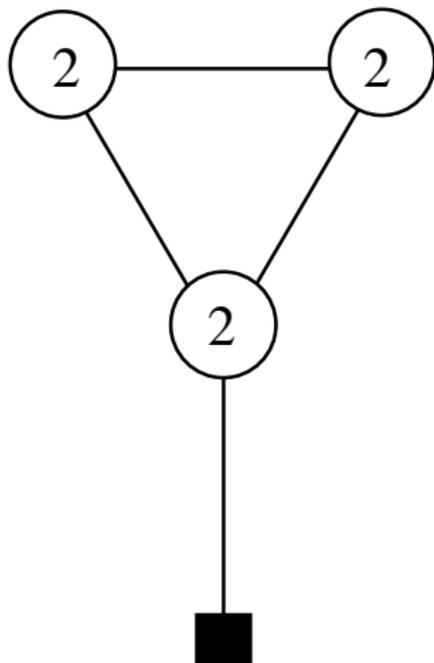
# Differences between the ASM and the SSM



# Differences between the ASM and the SSM

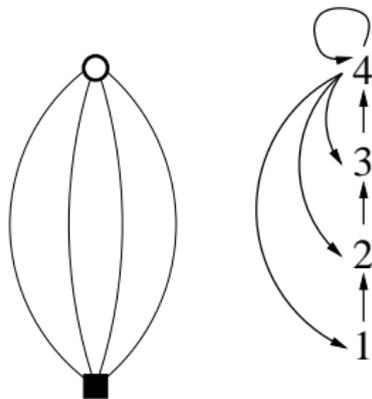


# Differences between the ASM and the SSM



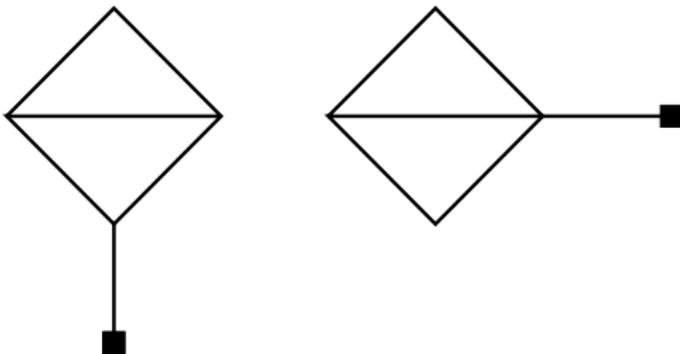
# Differences between the ASM and the SSM (2)

The invariant distribution on the SSM is not the uniform measure.



# Differences between the ASM and the SSM (3)

Changing the place of the sink can change the number of recurrent configurations for the SSM.



# Characterisation of the recurrent configurations

An orientation of  $G$  is an orientation of each edge. For a configuration  $\eta$ , define the *lacking number* of  $\eta$  at  $v$  by  $I_{\eta}^G(v) = d^G(v) - \eta_v$ .

# Characterisation of the recurrent configurations

An orientation of  $G$  is an orientation of each edge. For a configuration  $\eta$ , define the *lacking number* of  $\eta$  at  $v$  by  $l_{\eta}^G(v) = d^G(v) - \eta_v$ .

## Definition

We say that a configuration  $\eta$  on  $G$  is compatible with an orientation  $O$  of  $G$ , and write  $\eta \in \text{comp}(O)$ , if it is stable and

$$\forall v \in V, l_{\eta}^G(v) + 1 \leq \text{Out}_O^G(v).$$

# Characterisation of the recurrent configurations

An orientation of  $G$  is an orientation of each edge. For a configuration  $\eta$ , define the *lacking number* of  $\eta$  at  $v$  by  $I_\eta^G(v) = d^G(v) - \eta_v$ .

## Definition

We say that a configuration  $\eta$  on  $G$  is compatible with an orientation  $O$  of  $G$ , and write  $\eta \in \text{comp}(O)$ , if it is stable and

$$\forall v \in V, I_\eta^G(v) + 1 \leq \text{Out}_O^G(v).$$

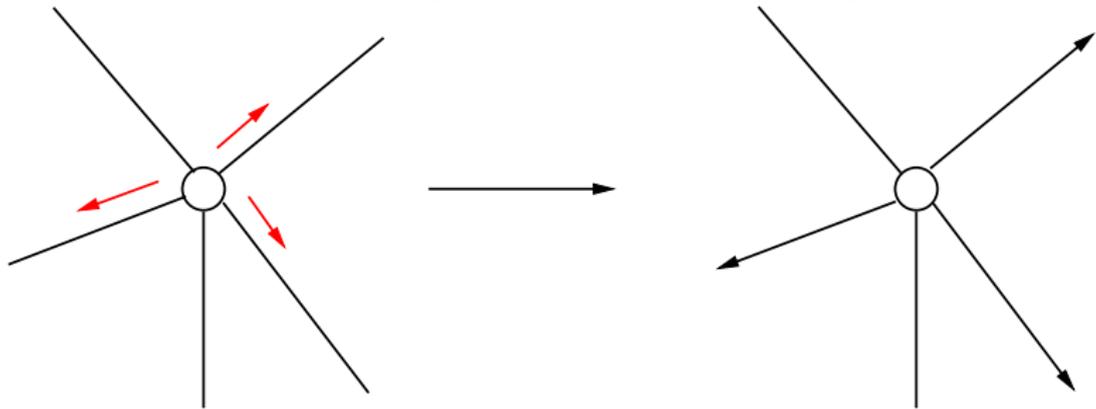
## Theorem

We have

$$\text{Sto}(G) = \bigcup_O \text{comp}(O).$$

# Interpretation

The orientations of the edges “mimic” the topplings.



# Characterisation of the recurrent configurations (2)

## Theorem

A (stable) configuration  $\eta$  is recurrent (for the SSM) if and only if

$$\forall A \subseteq V, |E(G_A)| \leq \eta(A) - |A|.$$

# Characterisation of the recurrent configurations (2)

## Theorem

A (stable) configuration  $\eta$  is recurrent (for the SSM) if and only if

$$\forall A \subseteq V, |E(G_A)| \leq \eta(A) - |A|.$$

## Theorem

A (stable) configuration  $\eta$  is recurrent for the ASM if and only if there exists an orientation  $O$  of  $G$  with no directed cycles such that  $\eta \in \text{comp}(O)$ .

# The lacking polynomial - definition

For a configuration  $\eta$ , write  $\ell(\eta) = \sum_v I_\eta^G(v)$  for the total number of grains lacking in  $\eta$ .

## Definition

The *lacking polynomial*  $L_G$  of a graph  $G$  is the generating function of the recurrent configurations on  $G$ , with  $x$  conjugate to the number of lacking particles in the configuration:

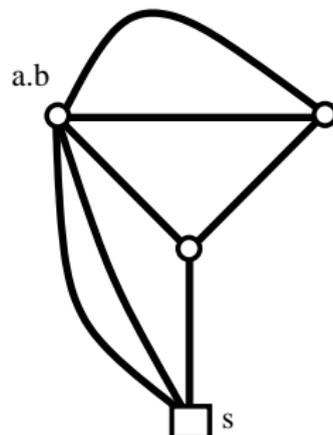
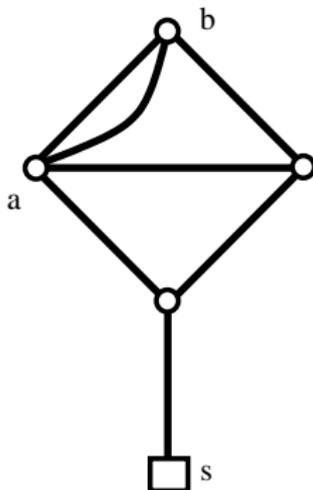
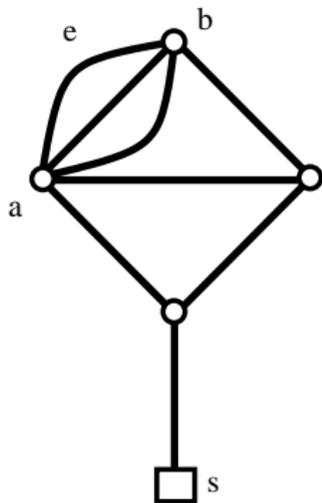
$$L_G(x) = \sum_{\eta \in \text{Sto}(G)} x^{\ell(\eta)}.$$

## Definition

Let  $G = (V \cup \{s\}, E)$  be a graph, and consider an edge  $e = \{x, y\} \in E$ , with  $x, y \neq s$ .

- 1 **Edge deletion.** The graph  $G \setminus e$  is the graph  $G$  with  $e$  removed, i.e.  $G \setminus e = (V \cup \{s\}, E \setminus \{e\})$ .
- 2 **General edge contraction.** Define the graph  $G.e$  as follows:
  - If  $e$  is simple, then  $G.e$  is  $G$  with  $e$  contracted, i.e.  $G.e = (V \cup \{x.y, s\} \setminus \{x, y\}, E \setminus \{e\})$ , where edges adjacent in  $G$  to either  $x$  or  $y$  are now connected to  $x.y$  instead.
  - If  $e$  has multiplicity  $k \geq 2$ , contract one of these edges as above, and replace the other  $k - 1$  edges with  $k - 1$  edges  $\{x.y, s\}$ .

# Example



# The recurrence relation

## Theorem

Let  $e$  be an edge of  $E$  which is neither a bridge (i.e. removing  $e$  doesn't disconnect the graph), nor connected to the sink. Then

$$L_G(x) = xL_{G \setminus e}(x) + L_{G.e}(x).$$

## Remark

The recurrence relation for the Tutte polynomial  $T_G$  is given by

$$T_G(x, y) = T_{G \setminus e}(x, y) + T_{G \cdot e}(x, y).$$

## Remark

The recurrence relation for the Tutte polynomial  $T_G$  is given by

$$T_G(x, y) = T_{G \setminus e}(x, y) + T_{G \cdot e}(x, y).$$

## Theorem (López 97, Cori - Le Borgne 03)

For the ASM, the level polynomial  $R_G$  satisfies the following specification:

$$R_G(y) = T_G(1, y).$$

## Proposition

- ① If  $V = \{u\}$  and there are  $k$  edges between  $u$  and  $s$ , then

$$L_G(x) = \sum_{i=0}^{k-1} x^i.$$

- ② If  $T$  is a tree attached to the rest of  $G$  at some vertex  $v$ , then  $L_G = L_{G \setminus T}$ .
- ③ If we can write  $G$  as the union of connected graphs  $G_i = (V_i \cup \{s\}, E_i)$ ,  $i = 1, \dots, k$ , so that the  $V_i$  are mutually disjoint, then

$$L_G(x) = \prod_{i=1}^k L_{G_i}(x).$$

- Study the invariant distribution: seems very complicated.
- Generalise to directed graphs.
- Study the lacking polynomial in more detail: link with the Tutte polynomial, what does it count?

Thank you for your attention