Tableaux escalier, PASEP et polynômes d’Askey Wilson

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Coworkers

- M. Josuat-Vergès (Université Paris-Sud)
- P. Nadeau (U. Wien)
- L.K. Williams (Berkeley)
- D. Stanton (Minnesota)
Young diagram

\[ \lambda = (\lambda_1, \ldots, \lambda_k) \text{ with } \lambda_1 \geq \ldots \geq \lambda_k \geq 0 \]
\[ \lambda = (7, 5, 5, 3, 0) \]

Length = number of rows + number of columns = \( \lambda_1 + k \)
Young diagram

Number the border: 1, 2, ..., 12
Young diagram

Code the border:  \textit{DEEDDEEDEEEEED}
Example 1

Any $\lambda$ and one 1 per row and per column: Rook placements
The number of RP of length $2n$ is $(2n - 1)!!$. 
Example I

Any $\lambda$ and one 1 per row and per column: Rook placements
The number of RP of length $2n$ is $(2n - 1)!!$. 

```
1
1 1
1 1 6 5
1 1 4 3
1 1 2
```

```
1 2 3 4 5 6 7 8 9 10
```
Example 1

Any $\lambda$ and one 1 per row and per column: Rook placements
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Example 1

Any $\lambda$ and one 1 per row and per column: Rook placements

The number of RP of length $2n$ is $(2n - 1)!!$. 

\[ \begin{array}{cccccccccc}
10 & 9 & 8 & & & & & & & 1 \\
8 & & & & & & & & & 2 \\
7 & & & & & & & & & 1 \\
5 & 6 & 1 & & & & & & & 1 \\
4 & 3 & 1 & & & & & & & 1 \\
3 & & & & & & & & & 1 \\
2 & & & & & & & & & 1 \\
1 & & & & & & & & & 1 \\
\end{array} \]

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]
Any \( \lambda \) and one 1 per row and per column: Rook placements
The number of RP of length \( 2n \) is \((2n - 1)!!\).
Example 1

Any $\lambda$ and one 1 per row and per column: Rook placements

The number of RP of length $2n$ is $(2n - 1)!!$. 
Any $\lambda$ and one 1 per row and per column: Rook placements

The number of RP of length $2n$ is $(2n - 1)!!$.  

Example 1

![Diagram with rook placements and corresponding graph]

1 2 3 4 5 6 7 8 9 10
Example 1

Any $\lambda$ and one 1 per row and per column: Rook placements

The number of RP of length $2n$ is $(2n - 1)!!$.

$$
\sum_{m \in \mathfrak{m}_{2n}} q^{cr(m)} = \langle W | (D + E)^{2n} | V \rangle
$$

where $DE = qED + I$, $\langle W | E = 0$, $D | V \rangle = 0$.

(Touchard-Riordan, Penaud, Flajolet-Noy, Josuat-Vergès . . .)

Moments of $q$-Hermite polynomials
Example II

Leroux (88) 0-1 tableaux: One 1 per column. The number of tableaux of length $n$ is $B_n$ (the $n^{th}$ Bell number)

0-1 tableaux $\leftrightarrow$ Set partitions

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 5 & 4 & 3 \\
3 & 7 & 6 & 9 \\
4 & 8 & 7 & 9
\end{array}
\]

$\leftrightarrow \quad \pi = (1, 3, 4, 8)(2)(5, 6)(7, 9)$

\[
\langle W| (yD + E)^n |V\rangle
\]

with $DE = qED + D$, $\langle W| E = 0, D|V\rangle = |V\rangle$.

Moments of $q$-Charlier polynomials (de Medicis, Stanton, White 95)
Example II

Leroux (88) 0-1 tableaux: One 1 per column.
The number of tableaux of length $n$ is $B_n$ (the $n^{th}$ Bell number)

0-1 tableaux $\leftrightarrow$ Set partitions

$$\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\times & \times & \times & 2 \\
\times & 1 & 5^4 & 3 \\
1 & 7^6 & 9 & 8 \\
\end{array}$$

$\leftrightarrow \pi = (1, 3, 4, 8)(2)(5, 6)(7, 9)$

$$\langle W | (yD + E)^n | V \rangle$$

with $DE = qED + D$, $\langle W | E = 0, D | V \rangle = | V \rangle$.

Moments of $q$-Charlier polynomials (de Medicis, Stanton, White 95)
Permutation tableaux (Postnikov 01, Williams 05)

Totally non negative part of the Grassmanian

Permutation tableau $\mathcal{T}$: a partition $\lambda$ filled with 0’s and 1’s such that:

1. Each column contains at least one 1.
2. There is no 0 which has a 1 above it in the same column and a 1 to its left in the same row.

\[
\begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 \\
1 \\
\end{array}
\quad \quad
\begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 \\
1 \\
\end{array}
\]
Permutation tableaux (Postnikov 01, Williams 05)

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Permutation tableaux (Postnikov 01, Williams 05)

Totally non negative part of the Grassmanian

Permutation tableau $\mathcal{T}$: a partition $\lambda$ filled with 0’s and 1’s such that:

1. Each column contains at least one 1.
2. There is no 0 which has a 1 above it in the same column and a 1 to its left in the same row.

\begin{align*}
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 \\
1 & & & & & & \\
\end{align*}

\begin{align*}
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 \\
1 & & & & & & \\
\end{align*}

Number of permutation tableaux of length $n$ is $n!$
Permutation tableaux and alternative tableaux

Restricted zero: lies below some 1
A permutation tableaux is uniquely defined by its topmost ones and rightmost restricted zeros.

(C. Nadeau 07)
Permutation tableaux and alternative tableaux

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A permutation tableau is uniquely defined by its topmost ones and rightmost restricted zeros.

(C. Nadeau 07)
Permutation tableaux and alternative tableaux

Restricted zero: lies below some 1
A permutation tableaux is uniquely defined by its topmost ones and rightmost restricted zeros.

(C. Nadeau 07)

Alternative tableaux (Viennot 08, Nadeau 09)
Bijections with permutations

Postnikov (00)
Steingrimsson and Williams (06)
Burstein (07)
C. and Nadeau (07)
Viennot (08)

Descents, excedances, crossings, 31-2, RL-min, LR-Max.
Enumeration of PT

- \( u(\mathcal{T}) \): number of unrestricted rows minus one
- \( f(\mathcal{T}) \): number of ones in the first row

\[
\begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & \\
0 & 1 & 1 & 1 & 1 & \\
0 & 0 & 0 & & & \\
1 & & & & & \\
\end{array}
\]

\( f(\mathcal{T}) = 3 \)

\( u(\mathcal{T}) = 4 - 1 = 3 \)

\[
\sum_{\mathcal{T} \text{ length } n+1} x^{u(\mathcal{T})} y^{f(\mathcal{T})} = \prod_{i=0}^{n-1} (x + y + i) = (x + y)_n.
\]

(C. and Nadeau 07)
q-enumeration

\( \text{wt}(\mathcal{T}) \): number of ones minus number of columns

\[
\begin{array}{cccccc}
1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & \\
0 & 1 & 1 & 1 & 1 & \\
\end{array}
\]

\( \text{wt}(\mathcal{T}) = 10 - 7 = 3 \)

\[
E_{k,n}(q) = \sum_{\mathcal{T}} \text{wt}(\mathcal{T}) = q^{n-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]_q^n \binom{n}{i} q^{k-i} + \binom{n}{i-1}
\]

(Williams 05)

q-analogue of Eulerian numbers

\( q = 0 \) Narayana numbers, \( q = -1 \) Binomial numbers.

Moments of q-Laguerre polynomials (Kasraoui, Stanton, Zeng 09)

Tableaux of a given shape (Williams 05, Novelli, Thibon, Williams 08)
Three parameter enumeration

\[ F_n(q, \alpha, \beta) = \sum_{\text{length } n} W(T); \quad W(T) = q^{\text{wt}(T)} \alpha^{-f(T)} \beta^{-u(T)}, \]

\[ F_n(q, \alpha, \beta) = \langle W | (D + E)^{n-1} | V \rangle, \quad \text{where} \]

\[ DE = qED + D + E; \]

\[ \alpha \langle W | E = \langle W |; \quad \beta D | V \rangle = | V \rangle \quad \langle W || V \rangle = 1. \]

(C, Williams 06)

Compute \( F_n(q, \alpha, \beta) \) (Josuat-Vergès 09)
PT and Partially asymmetric exclusion process

Model: \( n \) sites that are empty or occupied
The sites are delimited by \( n + 1 \) positions (\( n - 1 \) positions in between sites, left border and right border).
PT and Partially asymmetric exclusion process

Model: $n$ sites that are empty or occupied
The sites are delimited by $n + 1$ positions ($n - 1$ positions in between sites, left border and right border).

- First a position is chosen at random
PT and Partially asymmetric exclusion process

Model: \( n \) sites that are empty or occupied. The sites are delimited by \( n + 1 \) positions (\( n - 1 \) positions in between sites, left border and right border).

- First a position is chosen at random.
- A particle hops to the right with probability 1.
PT and Partially asymmetric exclusion process

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The sites are delimited by $n + 1$ positions ($n - 1$ positions in between sites, left border and right border).

- First a position is chosen at random
- A particle hops to the right with probability 1
- A particle hops to the left with probability $q$
PT and Partially asymmetric exclusion process

Model: $n$ sites that are empty or occupied. The sites are delimited by $n + 1$ positions ($n - 1$ positions in between sites, left border and right border).

- First a position is chosen at random
- A particle hops to the right with probability 1
- A particle hops to the left with probability $q$
- A particle enters with probability $\alpha$
PT and Partially asymmetric exclusion process

Model: $n$ sites that are empty or occupied
The sites are delimited by $n + 1$ positions ($n - 1$ positions in between sites, left border and right border).

- First a position is chosen at random
- A particle hops to the right with probability 1
- A particle hops to the left with probability $q$
- A particle enters with probability $\alpha$
- A particles leaves with probability $\beta$
Markov chain $n = 2$
Stationary distribution of the PASEP chain

Let $P_{n,\alpha,\beta}^q(\tau)$ be the probability to be in state $\tau = (\tau_1, \ldots, \tau_n)$. Theorem. (Derrida et al. 93) The probability to be in state $\tau = (\tau_1, \ldots, \tau_n)$ is

$$P_n(\tau) = \frac{\langle W | (\prod_{i=1}^n (\tau_i D + (1 - \tau_i) E)) | V \rangle}{Z_n}.$$  

with $Z_n = \langle W | (D + E)^n | V \rangle$, $D$ and $E$ are infinite matrices, $V$ is a column vector, and $W$ is a row vector, such that

$$DE - qED = D + E,$$
$$\beta D | V \rangle = | V \rangle,$$
$$\alpha \langle W | E = \langle W |$$
Permutation tableaux

\[ \tau = (0, 0, 1, 0, 0, 1, 1, 0, 0) \leftrightarrow \lambda(\tau) = \]

**Theorem.** Fix \( \tau = (\tau_1, \ldots, \tau_n) \in \{0, 1\}^n \), and let \( \lambda := \lambda(\tau) \). The probability of finding the PASEP chain in configuration \( \tau \) in the steady state is

\[ \frac{\sum_{\mathcal{T}} W(\mathcal{T})}{Z_n} . \]

where the sum is on the permutation tableaux of shape \( \lambda \).

(C. and Williams 2006)
Theorem. (Derrida et al. 93) The probability to be in state \( \tau = (\tau_1, \ldots, \tau_n) \) is

\[
P_n(\tau) = \frac{\langle W | \prod_{i=1}^{n} (\tau_iD + (1 - \tau_i)E) | V \rangle}{Z_n},
\]

with \( Z_n = \langle W | (D + E)^n | V \rangle \), \( D \) and \( E \) are infinite matrices, \( V \) is a column vector, and \( W \) is a row vector, such that

\[
\begin{align*}
uDE - qED &= D + E \\
\beta D | V \rangle &= | V \rangle + \delta E | V \rangle \\
\alpha \langle W | E &= \langle W | + \gamma \langle W | D
\end{align*}
\]
Theorem.

Fix $\tau = (\tau_1, \ldots, \tau_n) \in \{\bullet, \circ\}^n$, and the probability of finding the PASEP chain in configuration $\tau$ in the steady state is

$$\sum_{\mathcal{T}} \frac{W(\mathcal{T})}{Z_n}.$$

where the sum is on staircase tableaux of type $\tau$.

(C. and Williams 2009)
Staircase tableaux

Staircase shape \((n, n-1, \ldots, 2, 1)\).

Fill some cells with \(\alpha, \beta, \gamma\) or \(\delta\) such that

- Cells of the diagonal are filled
- \(\uparrow\) for \(\alpha\) or \(\gamma\)
- \(\leftarrow\) for \(\beta\) or \(\delta\)
Staircase tableaux

Staircase shape \((n, n-1, \ldots, 2, 1)\).
Fill some cells with \(\alpha, \beta, \gamma\) or \(\delta\) such that

- Cells of the diagonal are filled
- \(\uparrow\) for \(\alpha\) or \(\gamma\)
- \(\leftarrow\) for \(\beta\) or \(\delta\)

Type: read the diagonal, put \(\circ\) if \(\beta\) or \(\gamma\) and \(\bullet\) if \(\alpha\) or \(\delta\).
Staircase tableaux

Staircase shape \((n, n-1, \ldots, 2, 1)\).

Fill some cells with \(\alpha, \beta, \gamma\) or \(\delta\) such that

- Cells of the diagonal are filled
- \(\uparrow\) for \(\alpha\) or \(\gamma\)
- \(\leftarrow\) for \(\beta\) or \(\delta\)
Weight of a staircase tableau

\[
\begin{array}{|c|cccc|}
\hline
\text{under} \backslash \text{right} & \alpha & \beta & \gamma & \delta \\
\hline
\alpha & u & u & u & q \\
\beta & q & u & q & q \\
\gamma & q & u & q & q \\
\delta & u & u & u & q \\
\hline
\end{array}
\]

\[
W(T) = \alpha^2 \beta^2 \gamma^3 \delta^2 q^9 u^8.
\]
Staircase tableau of type \( \bullet \bullet \)

\[
\begin{array}{cccccccc}
 u & \alpha & q & \delta & q & \delta & u & \alpha \\
\alpha & \alpha & \delta & \delta & \delta & \delta & \alpha & \alpha & \alpha & \alpha & \alpha \\
\end{array}
\]

Probabilité de \( \bullet \bullet \)

\[
\frac{\alpha^2 u + \delta^2 q + \alpha \delta (u + q + \alpha + \beta + \gamma + \delta)}{Z_2}
\]
Staircase and alternative tableaux

No $\gamma$, No $\delta$

\[\begin{array}{cccc}
\beta & \alpha & \alpha \\
\alpha & \alpha & & \\
\alpha & & \beta \\
\beta & \alpha & & \\
\beta & & \beta & \\
\beta & & & \\
\alpha & & & \\
\end{array}\]
Staircase and alternative tableaux

No $\gamma$, No $\delta$
Staircase and alternative tableaux

No $\gamma$, No $\delta$

\[\begin{array}{cccc}
\beta & \alpha & \alpha & \alpha \\
\alpha & \alpha & & \\
\beta & \alpha & & \\
\beta & & & \\
\beta & & & \\
\alpha & & & \\
\end{array}\]

\[\begin{array}{cc}
\beta & \alpha \\
\alpha & \\
\beta & \\
\end{array}\]

$\leftrightarrow$
Staircase and alternative tableaux

No $\gamma$, No $\delta$
Matrix Ansatz and staircase tableaux

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Staircase tableau of length \(n + 1\). First diagonal:

- \(\alpha \leftrightarrow \gamma\)
- \(\beta\) : take off the first line \(\beta u^n\)
- \(\delta\) : take off the first line \(\delta q^n\)

\[
\alpha < W|EX|V > = \gamma < W|DX|V > + (\alpha \beta u^n - \gamma \delta q^n) < W|X|V >
\]

\(X \in \{D, E\}^n\)
Matrix Ansatz and staircase tableaux

\[ < W|XDEY|V > = q < W|XEDY|V > + (\alpha \beta u^{n+m+1} - \gamma \delta q^{n+m+1}) < W|X(D + E)Y|V > \]

\[ \beta < W|XD|V > = \delta < W|XE|V > + (\alpha \beta u^{n} - \gamma \delta q^{n}) < W|X|V > \]

\[ X \in \{D, E\}^{n}, \ Y \in \{D, E\}^{m}. \]

Proof by induction
Enumeration of staircase tableaux

There exist $2^n n!$ staircase tableaux of type $\tau$
There exist $4^n n!$ staircase tableaux of size $n$

$u = q = 1,$

$$Z_n = \prod_{i=0}^{n-1} (\alpha + \beta + \gamma + \delta + i(\alpha + \gamma)(\beta + \delta))$$
Bijection

(4, 5, 1, 6, 2, 3, 7)

First sign : type, second sign : label of the beginning of the path
(− − 4, − + 5, + − 1, + + 6, + + 2, − + 3, − − 7)
Enumeration formula \( u = 1 \)

**Theorem.** (C. Stanton, Williams, 2010)

\[
Z_n(y, a, b, c, d, q) = (abcd; q)_n \left( \frac{\alpha \beta}{1-q} \right)_n \frac{1}{2^n} \sum_{k=0}^{n} \frac{(ab, acy, ad; q)_k}{(abcd; q)_k} q^k
\]

\[
\times \sum_{j=0}^{k} q^{-(k-j)^2} (a^2 y)^{j-k} \frac{(1 + y + q^{k-j} ay + q^{j-k}/a)^n}{(q, q^{2j-2k+1}/a^2/y; q)_{k-j}(q, a^2 y q^{1-2j+2k}; q)_j}
\]

\[
\alpha = \frac{1 - q}{1 - ac + a + c}, \quad \beta = \frac{1 - q}{1 - bd + b + d},
\]

\[
\gamma = \frac{(1 - q)ac}{1 + ac + a + c}, \quad \delta = \frac{(1 - q)bd}{1 + bd + b + d}
\]

Moments of the Askey Wilson polynomials

\[
\frac{(1 - q)^n}{2^n i^n \prod_{j=0}^{n-1} (\alpha \beta - \gamma \delta q^j)} Z_n(-1, ai, bi, ci, di, q)
\]
Open problems

- Symmetry $\alpha \leftrightarrow \gamma$, $\beta \leftrightarrow \delta$, $q \leftrightarrow u$.

- Enumeration of staircase tableaux
- Combinatorics of staircase tableaux? What does $q$ count?
- Lifted chain on staircase tableaux?
- Link with the double Grassmanian?
- Hopf algebra?
Merci

Merci!