Symétrie et unimodalité de polynômes de descente via combinatoire de fractions continues

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26 janvier, 2011
Eulerian polynomials

The eulerian polynomials $A_n(t)$ can be defined by

$$
\sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!} = \frac{1 - t}{e^{(t-1)x} - t}.
$$

The first values of $A_n(t) := a_{n,0} + a_{n,1}t + \cdots + a_{n,n-1}t^{n-1}$:

$A_1(t) = 1,$
$A_2(t) = 1 + t,$
$A_3(t) = 1 + 4t + t^2,$
$A_4(t) = 1 + 11t + 11t^2 + t^3,$
$A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4.$
A polynomial $p(x) = a_l x^l + \cdots + a_n x^n$ is symmetric if $a_{l+i} = a_{n-i}$ for $0 \leq i \leq (n-l)/2$, unimodal if there is a $k$ such that

$$a_l \leq a_{l+1} \leq \cdots a_k \geq a_{k+1} \geq \cdots \geq a_n.$$ 

Proposition

The polynomial $A_n(t)$ has only real roots. Hence $A_n(t)$ is both unimodal and symmetric. Moreover

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des} \, \sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc} \, \sigma},$$

where $\mathfrak{S}_n$ is the set of permutations on $\{1, \ldots, n\}$ and

$$\text{des} \, \sigma = \# \{i \in [n-1] | \sigma(i) > \sigma(i+1) \},$$
$$\text{exc} \, \sigma = \# \{i \in [n] | \sigma(i) > i \}.$$
Some well-known facts

Let \( p(x) = \sum_{k=0}^{n} a_k x \) be a polynomial with nonnegative coefficients and with only real zeros. Then

(a) The sequence \((a_0, \ldots, a_n)\) is log-concave: \(a_i^2 \geq a_{i-1} a_{i+1}\).
(b) The sequence \((a_0, \ldots, a_n)\) is unimodal.
(c) Furthermore, if \( p(x) = \sum_{k=0}^{n} a_k x \) is symmetric, then

\[
p(x) = \sum_{i=0}^{n/2} b_i x^i (1 + x)^{n-2i} \quad \text{with} \quad b_i \geq 0.
\]

N.B. (a) \(\implies\) (b) and (c) \(\implies\) (b).
Let $\sigma = \sigma_1 \ldots \sigma_n \in \mathfrak{S}_n$ with $\sigma(0) = \sigma(n + 1) = 0$. An integer $i \in [1, n]$ is

- a peak in $\sigma$ if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$;
- a double descent if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

**Proposition**

Let $a_{n,k}$ be the number of permutations of $[n]$ with $k$ peaks and without double descent. Then

$$A_n(t) = \sum_{k=0}^{\lfloor(n-1)/2\rfloor} a_{n,k} t^k (1 + t)^{n-1-2k}. \quad (1)$$

Example: If $n = 4$, $k = 2$, then $a_{n,k} = 8$:  

2134, 3124, 4123, 1423, 2413, 3412, 1324, 2314
Let \((P, \prec)\) be a partial order of the set \([n] = \{1, 2, \ldots, n\}\) and \(\mathcal{L}(P)\) denote the set of linear extensions of \(P\), i.e. permutations \(w = (w_1, \ldots, w_n)\) of \([n]\) such that if \(i \prec j\) then \(i\) precedes \(j\) in the total order also. For each particular linear extension \(w\), let 
\[d(w) = |\{i: w_i > w_{i+1}\}|\]
 denote the number of so-called descents in \(P\).

**Example:** if \(P\) is an **antichain**, then \(W(P, t)\) is an **Eulerian polynomial**, and has real zeros only.
Conjecture (The Poset or Neggers-Stanley conjecture)

The polynomial \( W_P(t) = \sum_{w \in \mathcal{L}(P)} t^{d(w)+1} \) has real zeros only.

This conjecture was disproved by Brändén for general posets in 2004, and by Stembridge for naturally labelled posets in 2007.

Conjecture (consequences of the POSET conjecture, still open)

(1) \( W_P(t) = \sum_{w \in \mathcal{L}(P)} t^{d(w)+1} \) is log-concave.

(2) \( W_P(t) = \sum_{w \in \mathcal{L}(P)} t^{d(w)+1} \) is unimodal.

(3) If \( W_P(t) = \sum_{w \in \mathcal{L}(P)} t^{d(w)+1} \) is symmetric of center \( d/2 \), then there are \( w_{n,k} \geq 0 \) such that

\[
W_P(t) = \sum_{k=0}^{d/2} w_{n,k} t^k (1 + t)^{d-2k}.
\]
The descent polynomial of involutions

Let $\mathcal{I}_n$ be the set of involutions on $[n]$ and consider

$$H_n(t) := \sum_{\sigma \in \mathcal{I}_n} t^{\text{des} \sigma} = h_{n,0} + h_{n,1}t + \cdots + h_{n,n-1}t^{n-1}.$$ 

- **Strehl** (1981) proved Dumont’s conjecture: $\{h_{n,k}\}$ is symmetric.
- **Brenti** (2004) conjectured that $\{h_{n,k}\}$ is log-concave.
- **Guo-Z.** (2006) proved that $\{h_{n,k}\}$ is unimodal.

**Conjecture (Guo-Z., 2006, still open)**

$$H_n(t) = \sum_{k=0}^{\lfloor(n-1)/2\rfloor} a_{n,k} t^k (1 + t)^{n-1-2k} \quad a_{n,k} \in \mathbb{N}.$$
Problems related to continued fractions

Let $\sigma = \sigma(1) \ldots \sigma(n) \in \mathcal{S}_n$, define the statistics

$$(2-31)\sigma = \#\{(i, j) | 1 \leq i < j \leq n - 1, \ \sigma(j + 1) < \sigma(i) < \sigma(j)\},$$

$$(13-2)\sigma = \#\{(i, j) | 2 \leq i < j \leq n, \ \sigma(i - 1) < \sigma(j) < \sigma(i)\}.$$ 

In 2008, Brändén (after the works of Postnikov, Williams and Corteel) considered the refined Eulerian polynomials:

$$A_n(p, q, t) := \sum_{\sigma \in \mathcal{S}_n} p^{(13-2)\sigma} q^{(2-31)\sigma} t^{\text{des} \sigma}$$

and proved, modifying the Foata-Strehl action, the identity:

$$A_n(p, q, t) := \sum_{k=0}^{\lfloor(n-1)/2\rfloor} a_{n,k}(p, q)t^k(1 + t)^{n-1-2k},$$

where $2^{n-1-2k}a_{n,k}(p, q) \in \mathbb{N}[p, q]$. 
Conjecture (Brändén)

The polynomial $a_{n,k}(p, q)$ has a factor $(p + q)^k$ for all $0 \leq k \leq \lfloor (n - 1)/2 \rfloor$.

The first values of $a_{n,k}(p, q)$ are given by $a_{n,0}(p, q) = 1$ for $1 \leq n \leq 5$ and

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<td>$(p + q)[(p + q)^2 + 2(p + q) + 3]$</td>
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Derangement analogue

Let $\mathcal{D}_n$ be the set of *derangements* in $\mathfrak{S}_n$. Consider the derangement analogue of the Eulerian polynomials:

$$B_n(t) = \sum_{\sigma \in \mathcal{D}_n} t^{\text{exc} \sigma} = B_{n,1} t + B_{n,2} t^2 + \cdots + B_{n,n-1} t^{n-1}.$$ 

Then

$$\sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!} = \frac{1 - t}{e^{tx} - te^x}.$$ 

By analytical method, one can show

$$B_n(t) = \sum_{i=1}^{[n/2]} b_{n,k} t^k (1 + t)^{n-2k} \quad b_{n,k} \in \mathbb{N}. \quad (2)$$

No combinatorial interpretation for $b_{n,k}$ seems to be known hitherto.
(p, q)-analogue of the (-1)-evaluation of $A_n(t)$ and $B_n(t)$

The Euler numbers $E_n$ are defined by

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \tan x + \sec x.$$ 

The Euler number $E_n$ counts the alternating permutations in $\mathcal{S}_n$. It is well known that

$$A_{2n}(-1) = 0, \quad A_{2n+1}(-1) = (-1)^n E_{2n+1},$$
$$B_{2n}(-1) = (-1)^n E_{2n}, \quad B_{2n+1}(-1) = 0,$$ (3)

It follows that $a_{2n+1,n} = E_{2n+1}$ and $b_{2n,n} = E_{2n}$. Some $q$-analogues of (3) have been recently studied by Foata-Han, Josuat-Vergès.
Two continued fractions of Euler, Rogers

\[ \sum_{n=0}^{\infty} (n + 1)!x^n = \frac{1}{1 - 2x - \frac{1 \cdot 2x^2}{1 - 4x - \frac{2 \cdot 3x^2}{1 - 6x - \cdots}}}, \]

\[ \sum_{n=0}^{\infty} n!x^n = \frac{1}{1 - x - \frac{1^2 x^2}{1 - 3x - \frac{2^2 x^2}{1 - 5x - \cdots}}}. \]
Consider the following Motzkin path $\gamma$:

The weight is $w(\gamma) = b_0^2 b_1 \lambda_1^2 \lambda_2$. Denote by $\mathcal{M}_n$ the set of Motzkin paths of length $n \geq 1$. Then

$$1 + \sum_{n \geq 1} \sum_{\gamma \in \mathcal{M}_n} w(\gamma) x^n = \frac{1}{1 - b_0 x - \frac{\lambda_1 x^2}{1 - b_1 x - \frac{\lambda_2 x^2}{1 - \ldots}}}.$$ (4)
Laguerre history: $h = (\gamma; p_1, \ldots, p_n)$ with respect to the valuation
\[ a_k = k + 1, \quad b_k = 2k + 2 \quad \text{for} \quad k \geq 0; \quad c_k = k + 1 \quad \text{for} \quad k \geq 1. \]
or $\lambda_k := a_{k-1} c_k = k(k + 1)$ for $k \geq 1$. The number of Laguerre histories of length $n$ is $(n + 1)!$.

Restricted Laguerre history: $h = (\gamma; p_1, \ldots, p_n)$ with respect to the valuation
\[ a_k = k + 1, \quad b_k = 2k + 1 \quad \text{for} \quad k \geq 0; \quad c_k = k \quad \text{for} \quad k \geq 1. \]
or $\lambda_k := a_{k-1} c_k = k^2$ for $k \geq 1$. The number of Laguerre histories of length $n$ is $n!$.

**Theorem**

There are **Bijections between $\mathfrak{S}_n$ and the Laguerre histories of length $n$.**
Main results

Definition

For $\sigma \in \mathfrak{S}_n$, let $\sigma(0) = \sigma(n + 1) = 0$. Then any entry $\sigma(i)$ ($i \in [n]$) can be classified according to one of the four cases:

- a peak if $\sigma(i - 1) < \sigma(i)$ and $\sigma(i) > \sigma(i + 1)$;
- a valley if $\sigma(i - 1) > \sigma(i)$ and $\sigma(i) < \sigma(i + 1)$;
- a double ascent if $\sigma(i - 1) < \sigma(i)$ and $\sigma(i) < \sigma(i + 1)$;
- a double descent if $\sigma(i - 1) > \sigma(i)$ and $\sigma(i) > \sigma(i + 1)$.

Let peak$^*\sigma$ (resp. valley$^*\sigma$, da$^*\sigma$, dd$^*\sigma$) denote the number of peaks (resp. valleys, double ascents, double descents) in $\sigma$. Clearly we have peak$^*\sigma = \text{valley}^*\sigma + 1$. 
The generalized Eulerian polynomial defined by

$$A_n(p, q, t, u, v, w) = \sum_{\sigma \in S_n} p^{(2-13)\sigma} q^{(31-2)\sigma} t^{\text{des} \sigma} u^{\text{da}^* \sigma} v^{\text{dd}^* \sigma} w^{\text{valley}^* \sigma}.$$

Define the polynomial

$$a_{n,k}(p, q) = \sum_{\sigma \in S_{n,k}} p^{(2-13)\sigma} q^{(31-2)\sigma},$$

where $S_{n,k}$ is the subset of permutations $\sigma \in S_n$ with exactly $k$ valleys and without double descents.
Theorem (Shin-Z.)

We have the expansion formula

\[ A_n(p, q, t, u, v, w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p, q)(tw)^k(u + vt)^{n-1-2k}. \] (5)

Moreover, for all \(0 \leq k \leq \lfloor (n-1)/2 \rfloor\), the following divisibility holds

\[(p + q)^k \mid a_{n,k}(p, q).\] (6)

The above Theorem generalizes Brändén’s result and (6) solves his conjecture.
Definition

For $\sigma \in \mathcal{S}_n$, a value $x = \sigma(i)$ ($i \in [n]$) is called

- a cyclic peak if $i = \sigma^{-1}(x) < x$ and $x > \sigma(x)$;
- a cyclic valley if $i = \sigma^{-1}(x) > x$ and $x < \sigma(x)$;
- a double excedance if $i = \sigma^{-1}(x) < x$ and $x < \sigma(x)$;
- a double drop if $i = \sigma^{-1}(x) > x$ and $x > \sigma(x)$;
- a fixed point if $x = \sigma(x)$.

Let $\text{cpeak } \sigma$ (resp. $\text{cvalley } \sigma$, $\text{cda } \sigma$, $\text{cdd } \sigma$, $\text{fix } \sigma$) be the number of cyclic peaks (resp. valleys, double excedances, double drops, fixed points) in $\sigma$. 
For $\sigma \in \mathfrak{S}_n$ define the crossing and nesting numbers by

\[
cros(\sigma) = \sum_{i=1}^{n} \# \{ j \in [n] | j < i \leq \sigma(j) < \sigma(i) \text{ or } j > i > \sigma(j) > \sigma(i) \},
\]

\[
nest(\sigma) = \sum_{i=1}^{n} \# \{ j \in [n] | i < j \leq \sigma(j) < \sigma(i) \text{ or } i > j > \sigma(j) > \sigma(i) \},
\]

Diagram of $\sigma = 9 \ 3 \ 7 \ 4 \ 6 \ 10 \ 5 \ 8 \ 1 \ 2$: $\text{cros } \sigma = 4$ and $\text{nest } \sigma = 7$. 
Definition

For $\sigma \in \mathfrak{S}_n$, let $\sigma(0) = 0$ and $\sigma(n + 1) = n + 1$. The corresponding number of peaks, valleys, double ascents, and double descents of permutation $\sigma$ is denoted by

$$\text{peak } \sigma, \quad \text{valley } \sigma, \quad \text{da } \sigma, \quad \text{dd } \sigma.$$ 

Moreover, a double ascent $\sigma(i)$ of $\sigma$ ($i \in [n]$) is said to be a foremaximum if $\sigma(i)$ is a left-to-right maximum of $\sigma$, i.e., $\sigma(j) < \sigma(i)$ for all $1 \leq i < j$. Denote the number of foremaxima of $\sigma$ by $f_{\text{max}} \sigma$.

For instance, $f_{\text{max}}(42157368) = 2$. Note that $\text{peak } \sigma = \text{valley } \sigma$ for any $\sigma \in \mathfrak{S}_n$. 
Theorem (Shin-Z.)

There is a bijection $\Phi$ on $\mathfrak{S}_n$ such that for all $\sigma \in \mathfrak{S}_n$ we have

$$(\text{nest}, \text{cros}, \text{drop}, \text{cda}, \text{cdd}, \text{cvalley}, \text{fix})\sigma = (2\text{-}31, 31\text{-}2, \text{des}, \text{da} - \text{fmax}, \text{dd}, \text{valley}, \text{fmax})\Phi(\sigma).$$

In particular, $\Phi$ transforms $(\text{drop}, \text{nest}, \text{cros})$ to $(\text{des}, 2\text{-}31, 31\text{-}2)$.

- Corteel gave a similar bijection $\Phi_C$ on $\mathfrak{S}_n$ which transforms $(\text{n} \text{d} \text{e} \text{s}, 31\text{-}2, 2\text{-}31)$ to $(\text{w} \text{e} \text{x}, \text{nest}, \text{cros})$,
- Steingríómsson-Williams gave another one which transforms $(\text{n} \text{d} \text{e} \text{s}, 31\text{-}2, 2\text{-}31)$ to $(n + 1 - \text{w} \text{e} \text{x}, \text{nest}, \text{cros})$. 
Note that \( \text{wex} = n - \text{drop} \).

**Corollary**

We have

\[
\sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest} \sigma} q^{\text{cros} \sigma} t^{\text{wex} \sigma} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p, q)t^{k+1}(1 + t)^{n-1-2k}.
\]
Consider the common enumerator polynomial

\[ B_n(p, q, t, u, v, w, y) := \sum_{\sigma \in S_n} p^{\text{nest } \sigma} q^{\text{cros } \sigma} t^{\text{drop } \sigma} u^{\text{cda } \sigma} v^{\text{cdd } \sigma} w^{\text{cvalley } \sigma} y^{\text{fix } \sigma} \]  
(7)

\[ = \sum_{\sigma \in S_n} p^{(2-31) \sigma} q^{(31-2) \sigma} t^{\text{des } \sigma} u^{\text{da } \sigma} v^{\text{dd } \sigma} w^{\text{valley } \sigma} y^{\text{fmax } \sigma}. \]  
(8)

When \( u = 0 \) and \( t = v = 1 \) we can write

\[ B_n(p, q, 1, 0, 1, w, y) := \sum_{k, j \geq 0} b_{n,k,j}(p, q) w^k y^d, \]  
(9)

where \( b_{n,k,j}(p, q) \) is a polynomial in \( p \) and \( q \) with non negative integral coefficients.
Let
\[ S_{n,k,j} = \{ \sigma \in S_n : \text{cvalley} \sigma = k, \ \text{fix} \sigma = j, \ \text{cda} \sigma = 0 \} \]
\[ S_{n,k,j}^* = \{ \sigma \in S_n : \text{valley} \sigma = k, \ \text{fmax} \sigma = j, \ \text{da} \sigma = j \} \]
We derive from the above Theorem the following result.

**Corollary**

We have
\[ b_{n,k,d}(p, q) = \sum_{\sigma \in S_{n,k,j}} p^{\text{nest} \sigma} q^{\text{cros} \sigma} = \sum_{\sigma \in S_{n,k,j}^*} p^{(2-31) \sigma} q^{(31-2) \sigma}. \]

In particular, when \( j = 0 \), we obtain
\[ b_{n,k,0}(p, q) = \sum_{\sigma \in S_{n,k,0}} p^{\text{nest} \sigma} q^{\text{cros} \sigma} = \sum_{\sigma \in S_{n,k,0}^*} p^{(2-31) \sigma} q^{(31-2) \sigma}. \]
Theorem

\[ B_n(p, q, t, u, v, w, y) = \sum_{j=0}^{n} y^j \sum_{k=0}^{\left\lfloor (n-d)/2 \right\rfloor} b_{n, k, j}(p, q)(tw)^k(qu + tv)^{n-j-2k}. \]

Setting \( y = 0 \) in the above Theorem we obtain the following \((p, q)\)-analogue of \((-1)\)-evaluation of Eulerian polynomials.

Corollary

\[ \sum_{\sigma \in \mathcal{D}_n} p^{\text{nest} \sigma} q^{\text{cros} \sigma} t^{\text{exc} \sigma} = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{n, k, 0}(p, q)t^k(1 + qt)^{n-2k}. \]
Let $\mathcal{A}_n$ be the set of alternating permutations: $\sigma_1 > \sigma_2 < \ldots$

Theorem (Shin-Z.)

For $n \geq 1$, we have

$$\sum_{\sigma \in \mathcal{S}_n} (-1)^{\text{wex} \sigma} p^{\text{nest} \sigma} q^{\text{cros} \sigma}$$

$$= \begin{cases} 0 & \text{if } n \text{ is even}, \\ (-1)^{n+1} \frac{n+1}{2} \sum_{\sigma \in \mathcal{A}_n} p^{(2-13)\sigma} q^{(31-2)\sigma} & \text{if } n \text{ is odd}; \end{cases}$$

$$\sum_{\sigma \in \mathcal{D}_n} (-1/q)^{\text{exc} \sigma} p^{\text{nest} \sigma} q^{\text{cros} \sigma}$$

$$= \begin{cases} (-1/q)^{n/2} \sum_{\sigma \in \mathcal{A}_n} p^{(2-31)\sigma} q^{(31-2)\sigma} & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

The $p = 1$ case is due to Josuat-Vergès.
Sketch of the proof

\[
\sum_{n \geq 1} A_n(p, q, t, u, v, w)x^{n-1}
\]

\[
= \frac{1}{1 - (u + tv)[1]_{p,q}x - \frac{[1]_{p,q}[2]_{p,q}twx^2}{1 - (u + tv)[2]_{p,q}x - \frac{[2]_{p,q}[3]_{p,q}twx^2}{\ldots}}}
\]

For \(0 \leq k \leq \lfloor (n - 1)/2 \rfloor\), let \(a_{n,k}(p, q, t, u, v)\) be the coefficient of \(w^k\) in \(A_n(p, q, t, u, v, w)\), i.e.,

\[
A_n(p, q, t, u, v, w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p, q, t, u, v)w^k. \quad (10)
\]
Substituting \( x \leftarrow \frac{x}{(u+tv)} \) and \( w \leftarrow \frac{w(u+tv)^2}{t} \) in the above formula

\[
\sum_{n \geq 1} \sum_{k=0}^{[(n-1)/2]} \frac{a_{n,k}(p, q, t, u, v)}{t^k(u + tv)^{n-1-2k}} w^k x^{n-1} = \frac{1}{1 - [1]_{p,q}x - \frac{[1]_{p,q}[2]_{p,q}w x^2}{1 - [2]_{p,q}x - \frac{[2]_{p,q}[3]_{p,q}w x^2}{1 - [3]_{p,q}x - \frac{[3]_{p,q}[4]_{p,q}w x^2}{\ldots}}}}.
\]
The coefficient of $w^k x^{n-1}$ in the left-hand side is a polynomial in $p$ and $q$ with nonnegative integral coefficients. If we denote this coefficient by

$$P_{n,k}(p, q) := \frac{a_{n,k}(p, q, t, u, v)}{t^k(u + tv)^{n-1-2k}},$$

then

$$P_{n,k}(p, q) = a_{n,k}(p, q, 1, 1, 0) = a_{n,k}(p, q, 0, 1) = a_{n,k}(p, q).$$

Finally, as

$$(p + q) \mid [n]_{p,q}[n + 1]_{p,q} \quad \text{for all } n \geq 1,$$

each $w$ appears with a factor $(p + q)$ in the expansion of the continued fraction, so the polynomial $P_{n,k}(p, q)$ is divisible by $(p + q)^k$. 
Merci !