

# Balade de statistiques entre les permutations et les cycles

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## Plan de l'exposé

- Notations, définitions
- Problématique
- Une bijection de S. Elizalde (2011)
- Autres résultats (JLB 2012)
  - Décomposition en tableau de transpositions
  - Bijection préservant excédences, left-to-right maxima (points fixes  $\rightarrow$  points quasi-fixes)
  - Une continuité discrète entre  $S_n$  et  $C_{n+1}$  conservant les excédences
- Deux questions ouvertes pour les descentes

# Notations - Définitions

**Permutation de taille  $n$**  = bijection sur  $[n] = \{1, 2, \dots, n\}$

**Notation linéaire** :  $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$  où  $\sigma_i = \sigma(i)$

$S_n$  = ensemble des permutations de taille  $n$

**Cycle de taille  $n$**  :  $\sigma = \langle i_1, i_2, \dots, i_n \rangle$  si

$\sigma(i_1) = i_2, \sigma(i_2) = i_3, \dots, \sigma(i_{n-1}) = i_n$  et  $\sigma(i_n) = i_1$ .

$C_n$  = ensemble des cycles de taille  $n$

**Pour tout  $\sigma \in S_n$ , il existe une famille unique de cycles à supports disjoints dont le produit est  $\sigma$**

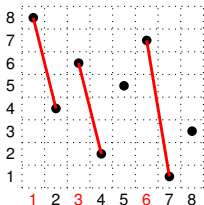
$$\sigma = 62738145 = \langle 2 \rangle \cdot \langle 6, 1 \rangle \cdot \langle 7, 4, 3 \rangle \cdot \langle 8, 5 \rangle$$

# Quelques statistiques

Soit  $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathcal{S}_n$ .

- **Descente** :  $D(\sigma) = \{i \in [n], \sigma_i > \sigma_{i+1}\}$

$\sigma = 84625713$



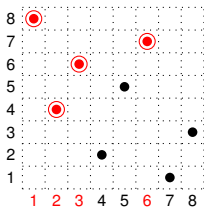
$$D(\sigma) = \{1, 3, 6\}$$

# Quelques statistiques

Soit  $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathcal{S}_n$ .

- **Excédence** :  $E(\sigma) = \{i \in [n], \sigma_i > i\}$

$\sigma = 84625713$



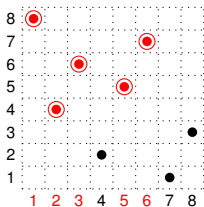
$$E(\sigma) = \{1, 2, 3, 6\}$$

# Quelques statistiques

Soit  $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathcal{S}_n$ .

- **Excédence faible** :  $WE(\sigma) = \{i \in [n], \sigma_i \geq i\}$

$\sigma = 84625713$

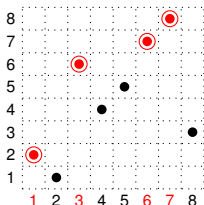


$$WE(\sigma) = \{1, 2, 3, 5, 6\}$$

# Quelques statistiques

Soit  $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathcal{S}_n$ .

- **Left-to-right maximum** :  $L(\sigma) = \{i \in [n], \sigma_i \geq \sigma_j \text{ pour } j \leq i\}$   
 $\sigma = 21645783$



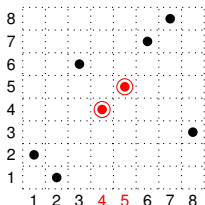
$$L(\sigma) = \{1, 3, 6, 7\}$$

# Quelques statistiques

Soit  $\sigma = \sigma_1\sigma_2\dots\sigma_n \in \mathcal{S}_n$ .

- **Point fixe** :  $F(\sigma) = \{i \in [n], \sigma_i = i\}$

$\sigma = 21645783$



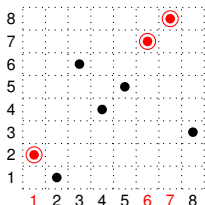
$$F(\sigma) = \{4, 5\}$$



# Quelques statistiques

Soit  $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in S_n$ .

- **Point quasi-fixe** :  $QF(\sigma) = \{i \in [n], \sigma_i = i + 1\}$   
 $\sigma = 21645783$



$$QF(\sigma) = \{1, 6, 7\}$$

## Question:

Existence de bijections entre  $S_n$  et  $C_{n+1}$  qui préservent (ou transportent) les ensembles précédents?



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## Descent sets of cyclic permutations

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### ABSTRACT

We present a bijection between cyclic permutations of  $\{1, 2, \dots, n+1\}$  and permutations of  $\{1, 2, \dots, n\}$  that preserves the descent set of the first  $n$  entries and the set of weak excedances. This non-trivial bijection involves a Foata-like transformation on the cyclic notation of the permutation, followed by certain conjugations. We also give an alternate derivation of the consequent result about the equidistribution of descent sets using work of Gessel and Reutenauer. Finally, we prove a conjecture of the author in [S. Elizalde, The number of permutations realized by a shift, *SIAM J. Discrete Math.* 23 (2009) 765–786] and a conjecture of Eriksen, Freij and Wästlund.

## Theorem (Elizalde 2011)

For every  $n$  there is a bijection  $\varphi : C_{n+1} \rightarrow S_n$  such that if  $\pi \in C_{n+1}$  and  $\sigma = \varphi(\pi)$ , then

$$D(\pi) \cap [n-1] = D(\sigma)$$

**Table 1**

The images by  $\varphi$  of all elements in  $C_5$ .

| $\pi \in C_5$           | $\sigma = \varphi(\pi) \in S_4$ | $D(\pi) \cap [3] = D(\sigma)$ |                         |      |           |
|-------------------------|---------------------------------|-------------------------------|-------------------------|------|-----------|
| (1, 2, 3, 4, 5) = 23451 | 1234                            | $\emptyset$                   | (2, 3, 1, 4, 5) = 43152 | 3214 | (1, 2)    |
| (2, 1, 3, 4, 5) = 31452 | 2134                            | {1}                           | (2, 4, 3, 1, 5) = 54132 | 4213 |           |
| (3, 2, 1, 4, 5) = 41253 | 3124                            |                               | (4, 2, 3, 1, 5) = 53124 | 4312 |           |
| (4, 3, 2, 1, 5) = 51234 | 4123                            |                               | (1, 4, 2, 3, 5) = 43521 | 3241 | (1, 3)    |
| (1, 3, 2, 4, 5) = 34251 | 1324                            | {2}                           | (2, 1, 4, 3, 5) = 41532 | 2143 |           |
| (1, 4, 3, 2, 5) = 45231 | 1423                            |                               | (2, 3, 4, 1, 5) = 53412 | 4231 |           |
| (3, 1, 2, 4, 5) = 24153 | 2314                            |                               | (3, 4, 2, 1, 5) = 51423 | 4132 |           |
| (3, 1, 4, 2, 5) = 45123 | 3412                            |                               | (4, 2, 1, 3, 5) = 31524 | 3142 |           |
| (4, 3, 1, 2, 5) = 25134 | 2413                            |                               | (1, 3, 4, 2, 5) = 35421 | 1432 | (2, 3)    |
| (1, 2, 4, 3, 5) = 24531 | 1243                            | {3}                           | (3, 4, 1, 2, 5) = 25413 | 2431 |           |
| (2, 4, 1, 3, 5) = 34512 | 1342                            |                               | (4, 1, 3, 2, 5) = 35214 | 3421 |           |
| (4, 1, 2, 3, 5) = 23514 | 2341                            |                               | (3, 2, 4, 1, 5) = 54213 | 4321 | (1, 2, 3) |

## Idée de la démonstration

### Construction algorithmique de $\sigma \in S_n$ à partir du cycle $\pi \in C_{n+1}$

Repeat the following steps for  $i = 1, 2, \dots, r - 1$ :

- Let  $z$  be the rightmost entry of  $\Gamma_i$ . If  $P(z, z + 1)$  or  $P(z, z - 1)$  holds, let  $\varepsilon \in \{-1, 1\}$  be such that  $P(z, z + \varepsilon)$  holds and  $\tilde{\sigma}(z + \varepsilon)$  is largest.
- Repeat for as long as  $P(z, z + \varepsilon)$  holds:
  - I. Switch  $z$  and  $z + \varepsilon$  in the cycle form of  $\tilde{\sigma}$ .
  - II. If the last switch did not involve the leftmost entry of  $\Gamma_i$ , let  $x$  and  $y$  be the elements preceding the switched entries. If  $|x - y| = 1$ , switch  $x$  and  $y$  in the cycle form of  $\tilde{\sigma}$ , and repeat step II.
  - III. Let  $z := z + \varepsilon$  (the new rightmost entry of  $\Gamma_i$ ).

Define  $\varphi(\pi) = \tilde{\sigma}$ .

$$\pi = \langle l_1, \dots, l_2, \dots, l_k, \dots, n + 1 \rangle \in C_{n+1} \longrightarrow$$
$$\langle l_1 \dots \rangle \cdot \langle l_2 \dots \rangle \cdots \langle l_k \dots \rangle \in S_n \xrightarrow{\text{Algorithm}} \sigma \in S_n$$

$$\pi = \langle 11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21 \rangle \in C_{21}$$

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$$\sigma = \langle 11, 4, 10, 1, 7 \rangle \cdot \langle 16, 9, 3, 5, 12 \rangle \cdot \langle 20, 2, 6, 14, 18, 8, 13, 19, 15, 17 \rangle$$



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### Comportement des descentes

- $\pi(1) = 7 > \pi(2) = 6$  et on a:  $\sigma(1) = 7 > \sigma(2) = 6$   
(Descente préservée)

$$\pi = \langle 11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21 \rangle \in C_{21}$$

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- $\pi(7) = 16 > \pi(6) = 14$  alors que  $\sigma(7) = 11 < \sigma(6) = 14$   
(Descente non préservée)

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- $\pi(7) = 16 > \pi(6) = 14$  alors que  $\sigma(7) = 11 < \sigma(6) = 14$   
(Descente non préservée)

### Algorithme

$$\sigma = \langle 11, 4, 10, 1, 6 \rangle \cdot \langle 16, 9, 3, 5, 12 \rangle \cdot \langle 20, 2, 7, 14, 18, 8, 13, 19, 15, 17 \rangle$$

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- $\pi(1) = 7 > \pi(2) = 6$  et on a:  $\sigma(1) = 7 > \sigma(2) = 6$   
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### Algorithme

$$\sigma = \langle 11, 4, 10, 1, 6 \rangle \cdot \langle 16, 9, 3, 5, 12 \rangle \cdot \langle 20, 2, 7, 14, 18, 8, 13, 19, 15, 17 \rangle$$

Attention!  $\sigma(1) = 6 < \sigma(2) = 7$

$$\sigma = \langle 11, 4, 10, 1, 6 \rangle \cdot \langle 16, 9, 3, 5, 12 \rangle \cdot \langle 20, 2, 7, 14, 18, 8, 13, 19, 15, 17 \rangle$$

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Finalemment:

$$\begin{array}{l} \pi = 7 \ 6 \ 5 \ 10 \ 12 \ 14 \ 16 \ 13 \ 3 \ 1 \ 4 \ 20 \ 19 \ 18 \ 16 \ 9 \ 21 \ 8 \ 15 \ 2 \ 11 \in C_{21} \\ \sigma = 7 \ 6 \ 5 \ 9 \ 11 \ 13 \ 15 \ 12 \ 3 \ 1 \ 4 \ 19 \ 18 \ 17 \ 16 \ 10 \ 20 \ 8 \ 14 \ 2 \in S_{20} \end{array}$$

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$$\begin{array}{l} \pi = 7 \ 6 \ 5 \ 10 \ 12 \ 14 \ 16 \ 13 \ 3 \ 1 \ 4 \ 20 \ 19 \ 18 \ 16 \ 9 \ 21 \ 8 \ 15 \ 2 \ 11 \in C_{21} \\ \sigma = 7 \ 6 \ 5 \ 9 \ 11 \ 13 \ 15 \ 12 \ 3 \ 1 \ 4 \ 19 \ 18 \ 17 \ 16 \ 10 \ 20 \ 8 \ 14 \ 2 \in S_{20} \end{array}$$

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## Question :

Existence de bijections de  $S_n$  vers  $C_{n+1}$  qui préservent d'autres statistiques?

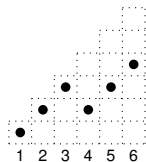
## Décomposition en produit de transpositions :

Soit  $\sigma \in S_n$ .  $\sigma = \langle p_1, 1 \rangle \cdot \langle p_2, 2 \rangle \cdot \langle p_3, 3 \rangle \cdots \langle p_n, n \rangle$   
où  $1 \leq p_i \leq i \leq n$

**Bijection :**  $S_n \longrightarrow T_n = [1] \times [2] \times \cdots \times [n]$

**Tableau de transpositions :**  $p_1 p_2 \dots p_n \in T_n$ .

**Exemple :**  $\sigma = 1\ 4\ 5\ 6\ 3\ 2 = \langle 1, 1 \rangle \cdot \langle 2, 2 \rangle \cdot \langle 3, 3 \rangle \cdot \langle 2, 4 \rangle \cdot \langle 3, 5 \rangle \cdot \langle 4, 6 \rangle$



**Tableau de transpositions :**  $1\ 2\ 3\ 2\ 3\ 4 \in T_6$

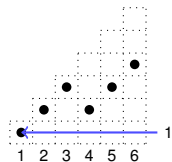
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**Tableau de transpositions :**  $1\ 2\ 3\ 2\ 3\ 4 \in T_6$



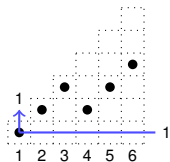
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**Tableau de transpositions :**  $1\ 2\ 3\ 2\ 3\ 4 \in T_6$

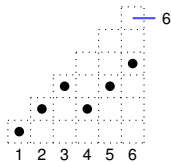
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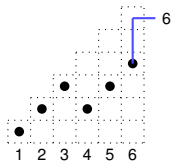
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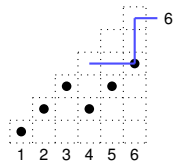
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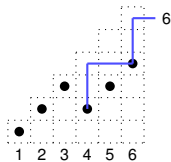
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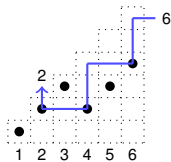
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**Tableau de transpositions :**  $1\ 2\ 3\ 2\ 3\ 4 \in T_6$

## Construction du tableau de transpositions en temps linéaire

$$\sigma = 1\ 4\ 5\ 6\ 3\ 2 \longleftrightarrow \sigma^{-1} = 1\ 6\ 5\ 2\ 3\ 4 \longleftrightarrow p_1, p_2, p_3, p_4, p_5, 4 \in T_6$$

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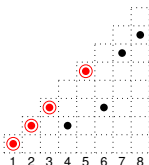
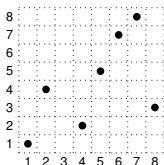
Après 5 itérations

$$\sigma = 1 \ 2 \ 3 \ 4 \ 5 \ 6 \longleftrightarrow \sigma^{-1} = 1 \ 2 \ 3 \ 4 \ 5 \ 6 \longleftrightarrow 1, 2, 3, 2, 3, 4 \in T_6$$

Soit  $\sigma \in \mathcal{S}_n$  et  $p_1 p_2 \dots p_n$  son tableau de transpositions

### Lemme 1 (B. 2012):

- $\sigma$  est un cycle  $\iff$  son tableau de transpositions contient un unique point fixe.
- Le nombre de cycles de  $\sigma$  est égal au nombre de points fixes dans le tableau de transpositions.

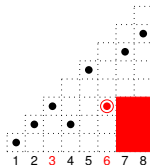
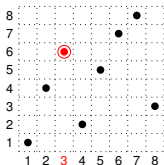


$$1, 4, 6, 2, 5, 7, 8, 3 = \langle 1 \rangle \langle 4, 2 \rangle \langle 5 \rangle \langle 8, 3, 6, 7 \rangle \leftrightarrow \mathbf{1, 2, 3, 2, 5, 3, 6, 7}$$

Soit  $\sigma \in S_n$  et  $p_1 p_2 \dots p_n$  son tableau de transpositions

## Lemme 2 :

- $k$  est un left-to-right maximum de  $\sigma \iff$  il existe  $i \geq k$  tel que  $p_i = k$ , et  $p_j > k$  pour  $j > i$ .

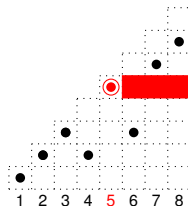
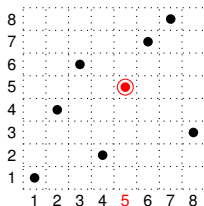


$$1, 4, \mathbf{6}, 2, 5, 7, 8, 3 \leftrightarrow 1, 2, 3, 2, 5, \mathbf{3}, 6, 7$$

Soit  $\sigma \in \mathcal{S}_n$  et  $p_1 p_2 \dots p_n$  son tableau de transpositions

### Lemme 3 :

- $k$  est un point fixe de  $\sigma \iff$ 
  - $p_k = k$ , et
  - il n'existe pas  $i > k$  tel que  $p_i = k$ .

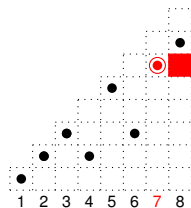
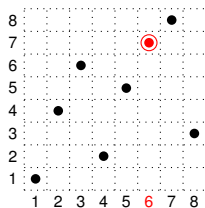


$$1, 4, 6, 2, 5, 7, 8, 3 \leftrightarrow 1, 2, 3, 2, 5, 3, 6, 7$$

Soit  $\sigma \in \mathcal{S}_n$  et  $p_1 p_2 \dots p_n$  son tableau de transpositions

### Lemme 4 :

- $k$  est un point quasi-fixe de  $\sigma \iff$ 
  - $p_{k+1} = k$ , et
  - il n'existe pas  $i > k + 1$  tel que  $p_i = k$ .

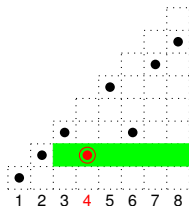
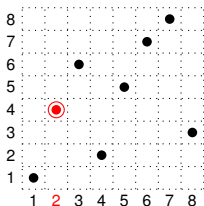


$$1, 4, 6, 2, 5, 7, 8, 3 \leftrightarrow 1, 2, 3, 2, 5, 3, 6, 7$$

Soit  $\sigma \in S_n$  et  $p_1 p_2 \dots p_n$  son tableau de transpositions

### Lemme 5 :

$k$  est une excédence de  $\sigma \iff$  il existe  $i > k$  tel que  $p_i = k$ .



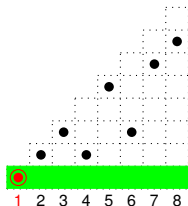
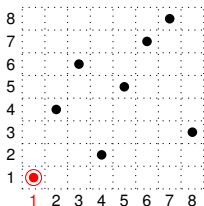
$$1, 4, 6, 2, 5, 7, 8, 3 \leftrightarrow 1, 2, 3, 2, 5, 3, 6, 7$$



Soit  $\sigma \in \mathcal{S}_n$  et  $p_1 p_2 \dots p_n$  son tableau de transpositions

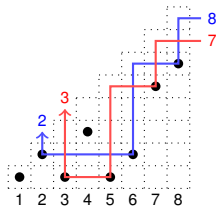
### Lemme 6 :

$k$  est une excédence faible de  $\sigma \iff$  il existe  $i \geq k$  tel que  $p_i = k$ .



$$1, 4, 6, 2, 5, 7, 8, 3 \leftrightarrow 1, 2, 3, 2, 5, 3, 6, 7$$

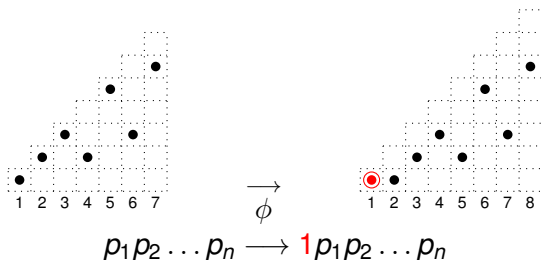
## Et pour les descentes? $\sigma = 5 6 4 1 7 8 3 2$



Soit  $\phi$  l'application de  $S_n$  vers  $C_{n+1}$  définie par:

$$\phi(\sigma) = \langle 1, 1 \rangle \cdot \langle p_1, 2 \rangle \cdot \langle p_2, 3 \rangle \cdots \langle p_n, n+1 \rangle,$$

où le tableau de transpositions de  $\sigma$  est  $p_1 p_2 \dots p_n$ .



## Théorème (B. 2012)

$\phi$  est une Bijection de  $S_n$  vers  $C_{n+1}$  telle que

$$WE(\sigma) = WE(\phi(\sigma)) = E(\phi(\sigma))$$

$$L(\sigma) = L(\phi(\sigma)) \quad \text{et} \quad F(\sigma) = QF(\phi(\sigma))$$

## Corollaire

Bijection de  $D_n$  vers  $C_{n+1}^q$  telle que  $E(\sigma) = E(\phi(\sigma))$  et  $L(\sigma) = L(\phi(\sigma))$

| $\sigma$   | $T(\sigma)$ | $\phi(\sigma)$ | $T(\phi(\sigma))$ | $L(\sigma)$   | $F(\sigma)$   | $WE(\sigma)$  |
|------------|-------------|----------------|-------------------|---------------|---------------|---------------|
| <b>123</b> | 123         | <b>2341</b>    | 1123              | $\{1, 2, 3\}$ | $\{1, 2, 3\}$ | $\{1, 2, 3\}$ |
| <b>132</b> | 122         | <b>2413</b>    | 1122              | $\{1, 2\}$    | $\{1\}$       | $\{1, 2\}$    |
| <b>213</b> | 113         | <b>3142</b>    | 1113              | $\{1, 3\}$    | $\{3\}$       | $\{1, 3\}$    |
| <b>231</b> | 112         | <b>3421</b>    | 1112              | $\{1, 2\}$    | $\emptyset$   | $\{1, 2\}$    |
| <b>312</b> | 111         | <b>4123</b>    | 1111              | $\{1\}$       | $\emptyset$   | $\{1\}$       |
| <b>321</b> | 121         | <b>4312</b>    | 1121              | $\{1\}$       | $\{2\}$       | $\{1, 2\}$    |

**Table:** La bijection  $\phi$  de  $S_3$  vers  $C_4$ .

## Continuité discrète entre $S_n$ et $C_{n+1}$

### Définition de $S_n(k)$ , pour $1 \leq k \leq n$

- (a)  $\sigma(n) \geq k$ ; et,  
(b) tous les entiers de l'intervalle  $[k, n]$  appartiennent à un même cycle de  $\sigma$ .

**Exemple :**  $54327186 = \langle 3 \rangle \langle 4, 2 \rangle \langle 8, 6, 1, 5, 7 \rangle \in S_8(5)$

- Pour  $k = 1$ , on a:  $S_n(1) = C_n$ ,
- Pour  $k = n$ , on a:  $S_n(n) = S_{n-1} \cdot n$ .

## Une bijection entre $S_n(k)$ et $S_n(k - 1)$

### Définition

Pour  $k \in \{2, \dots, n\}$ , on définit  $h_k: S_n(k) \rightarrow S_n(k - 1)$ :

$$h_k(\sigma) = \begin{cases} \sigma & \text{si } \sigma \in S_n(k - 1) \\ \sigma \cdot \langle \sigma^{-1}(k - 1), n \rangle & \text{sinon.} \end{cases}$$

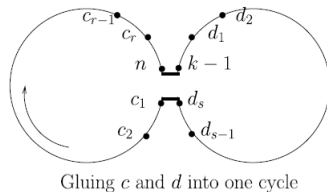
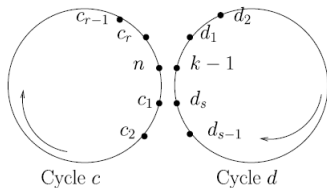
### Théorème (B. 2012)

Pour  $k \in \{2, \dots, n\}$ ,  $h_k: S_n(k) \rightarrow S_n(k - 1)$  est une bijection

Un résultat de Elizalde pour les descentes  
 Et pour d'autres statistiques?

**Une continuité discrète**  
 Problèmes ouverts

$$\sigma \in \mathcal{S}_n(k) \setminus \mathcal{S}_n(k-1) \xrightarrow{h_k} \sigma \cdot \overbrace{\langle \sigma^{-1}(k-1), n \rangle}^{d_s} \in \mathcal{S}_n(k-1)$$





## Conséquence

- $\text{Card}(S_n(k)) = (n-1)!$  pour  $1 \leq k \leq n$
- Pour tout  $k \in \{1, 2, \dots, n\}$ , on a:

$$n! = (n-k+1) \cdot \sum_{i=0}^{k-1} \binom{k-1}{i} (n-k+i)! (k-i-1)!$$

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Cardinalité de  $S_{n+1}(k) = n!$

## Conséquence

- $\text{Card}(S_n(k)) = (n-1)!$  pour  $1 \leq k \leq n$
- Pour tout  $k \in \{1, 2, \dots, n\}$ , on a:

$$n! = (n-k+1) \cdot \sum_{i=0}^{k-1} \binom{k-1}{i} (n-k+i)! (k-i-1)!$$

$$\sigma \in S_{n+1}(k) \iff \begin{cases} \sigma(n+1) = c_1 \geq k \\ [k, n+1] \subseteq c = \langle n+1, c_1, c_2, c_3, \dots \rangle \end{cases}$$

Il y a  $(n-k+1)$  choix pour  $c_1 = \sigma(n+1) \in [k, n]$

## Conséquence

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- Pour tout  $k \in \{1, 2, \dots, n\}$ , on a:

$$n! = (n-k+1) \cdot \sum_{i=0}^{k-1} \binom{k-1}{i} (n-k+i)! (k-i-1)!$$

$$\sigma \in S_{n+1}(k) \iff \begin{cases} \sigma(n+1) = c_1 \geq k \\ [k, n+1] \subseteq c = \langle n+1, c_1, c_2, c_3, \dots \rangle \end{cases}$$

Le cycle  $c$ , est constitué de  $i$ ,  $0 \leq i \leq k-1$ , valeurs dans  $\{1, 2, \dots, k-1\}$ , et des valeurs de  $[k, n+1]$ .

## Conséquence

- $\text{Card}(S_n(k)) = (n-1)!$  pour  $1 \leq k \leq n$
- Pour tout  $k \in \{1, 2, \dots, n\}$ , on a:

$$n! = (n-k+1) \cdot \sum_{i=0}^{k-1} \binom{k-1}{i} (n-k+i)! (k-i-1)!$$

$$\sigma \in S_{n+1}(k) \iff \begin{cases} \sigma(n+1) = c_1 \geq k \\ [k, n+1] \subseteq c = \langle n+1, c_1, c_2, c_3, \dots \rangle \end{cases}$$

Il reste alors  $(k-i-1)!$  choix pour les valeurs restantes.

## Théorème (B. 2012)

Pour  $k \in \{2, \dots, n\}$ , la bijection  $h_k : S_n(k) \rightarrow S_n(k-1)$  satisfait pour tout  $\sigma \in S_n(k)$ ,

$$WE(\sigma) \cap \{1, 2, \dots, n-1\} = WE(h_k(\sigma)).$$

## Corollaire

Il existe une bijection  $h$  de  $S_n$  vers  $C_{n+1}$  qui préserve l'ensemble des excédences faibles.

| $S_4(1)$ | $S_4(2)$ | $S_4(3)$ | $\sigma \in S_4(4) = S_3 \cdot 4$ | $WE(\sigma) \cap \{1, 2, 3\}$ |
|----------|----------|----------|-----------------------------------|-------------------------------|
| 4123     | 4123     | 4123     | 3124                              | {1}                           |
| 3142     | 3142     | 2143     | 2134                              | {1, 3}                        |
| 4312     | 4312     | 4213     | 3214                              | {1, 2}                        |
| 2413     | 2413     | 2413     | 2314                              | {1, 2}                        |
| 2341     | 1342     | 1243     | 1234                              | {1, 2, 3}                     |
| 3421     | 1423     | 1423     | 1324                              | {1, 2}                        |

Table: Les ensembles  $S_4(k)$  pour  $k = 1, 2, 3, 4$ .

| $S_4(1)$ | $S_4(2)$ | $S_4(3)$ | $\sigma \in S_4(4) = S_3 \cdot 4$ | $WE(\sigma) \cap \{1, 2, 3\}$ |
|----------|----------|----------|-----------------------------------|-------------------------------|
| 4123     | 4123     | 4123     | 3124                              | {1}                           |
| 3142     | 3142     | 2143     | 2134                              | {1, 3}                        |
| 4312     | 4312     | 4213     | 3214                              | {1, 2}                        |
| 2413     | 2413     | 2413     | 2314                              | {1, 2}                        |
| 2341     | 1342     | 1243     | 1234                              | {1, 2, 3}                     |
| 3421     | 1423     | 1423     | 1324                              | {1, 2}                        |

Table: Les ensembles  $S_4(k)$  pour  $k = 1, 2, 3, 4$ .



| $S_4(1)$ | $S_4(2)$ | $S_4(3)$ | $\sigma \in S_4(4) = S_3 \cdot 4$ | $WE(\sigma) \cap \{1, 2, 3\}$ |
|----------|----------|----------|-----------------------------------|-------------------------------|
| 4123     | 4123     | 4123     | 3124                              | {1}                           |
| 3142     | 3142     | 2143     | 2134                              | {1, 3}                        |
| 4312     | 4312     | 4213     | 3214                              | {1, 2}                        |
| 2413     | 2413     | 2413     | 2314                              | {1, 2}                        |
| 2341     | 1342     | 1243     | 1234                              | {1, 2, 3}                     |
| 3421     | 1423     | 1423     | 1324                              | {1, 2}                        |

Table: Les ensembles  $S_4(k)$  pour  $k = 1, 2, 3, 4$ .

| $S_4(1)$ | $S_4(2)$ | $S_4(3)$ | $\sigma \in S_4(4) = S_3 \cdot 4$ | $WE(\sigma) \cap \{1, 2, 3\}$ |
|----------|----------|----------|-----------------------------------|-------------------------------|
| 4123     | 4123     | 4123     | 3124                              | {1}                           |
| 3142     | 3142     | 2143     | 2134                              | {1, 3}                        |
| 4312     | 4312     | 4213     | 3214                              | {1, 2}                        |
| 2413     | 2413     | 2413     | 2314                              | {1, 2}                        |
| 2341     | 1342     | 1243     | 1234                              | {1, 2, 3}                     |
| 3421     | 1423     | 1423     | 1324                              | {1, 2}                        |

Table: Les ensembles  $S_4(k)$  pour  $k = 1, 2, 3, 4$ .

## Problèmes ouverts

### Question 1 : Dérangements et descentes

Existe-t-il une bijection de  $D_n$  vers  $C_{n+1}^q$  qui préserve l'ensemble des descentes sur les positions  $\{1, 2, \dots, n-1\}$ ?

| $D_4$       | $C_5^q$      | $D(\sigma) \cap [3]$ |
|-------------|--------------|----------------------|
| <b>4123</b> | <b>51234</b> | $\{1\}$              |
| <b>3412</b> | <b>45123</b> | $\{2\}$              |
| <b>2413</b> | <b>45231</b> | $\{2\}$              |
| <b>2341</b> | <b>34512</b> | $\{3\}$              |
| <b>4312</b> | <b>54132</b> | $\{1, 2\}$           |
| <b>3142</b> | <b>41532</b> | $\{1, 3\}$           |
| <b>2143</b> | <b>31524</b> | $\{1, 3\}$           |
| <b>3421</b> | <b>35214</b> | $\{2, 3\}$           |
| <b>4321</b> | <b>54213</b> | $\{1, 2, 3\}$        |

Pour  $1 \leq k \leq n$ , soit l'ensemble  $S'_n(k)$  des permutations  $\sigma \in S_n$  telles que le cycle de  $\sigma$  qui contient  $n$  est de taille  $k$ .

### Question 2 : Continuité discrète et descente

Pour  $k \in \{2, \dots, n\}$ , existe-t-il une bijection  $h_k$  de  $S'_n(k)$  vers  $S'_n(k-1)$  qui préserve l'ensemble des descentes sur les positions  $\{1, 2, \dots, n-2\}$ , *i.e.*,

$$D(\sigma) \cap \{1, 2, \dots, n-2\} = D(h'_k(\sigma)) \cap \{1, 2, \dots, n-2\}?$$

| $S'_4(1) = S_3 \cdot 4$ | $S'_4(2)$   | $S'_4(3)$   | $\sigma \in S'_4(4) = C_4$ | $D(\sigma) \cap \{1, 2\}$ |
|-------------------------|-------------|-------------|----------------------------|---------------------------|
| 1234                    | 1243        | 1342        | 2341                       | $\{\}$                    |
| <b>3124</b>             | <b>4231</b> | <b>4132</b> | <b>4123</b>                | $\{1\}$                   |
| <b>2134</b>             | <b>2143</b> | <b>3241</b> | <b>3142</b>                | $\{1\}$                   |
| <b>1324</b>             | <b>1432</b> | <b>2431</b> | <b>2413</b>                | $\{2\}$                   |
| <b>2314</b>             | <b>3412</b> | <b>1423</b> | <b>3421</b>                | $\{2\}$                   |
| <b>3214</b>             | <b>4321</b> | <b>4213</b> | <b>4312</b>                | $\{1, 2\}$                |

Merci