

The scaling limit of the MST of a complete graph

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The minimum spanning tree

Definition.

$G = (V, E)$ a connected graph

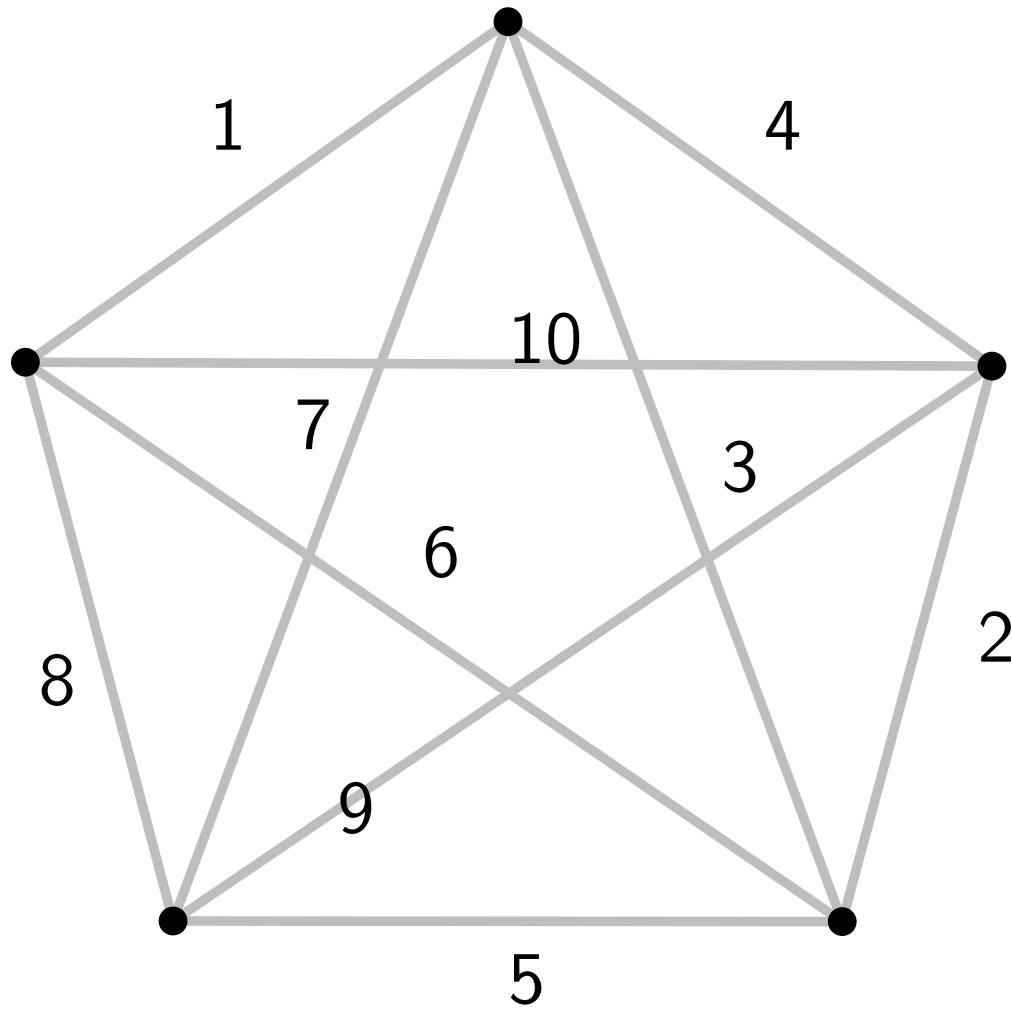
$w_e \geq 0, e \in E$ weights

MST = lightest connected subgraph of G

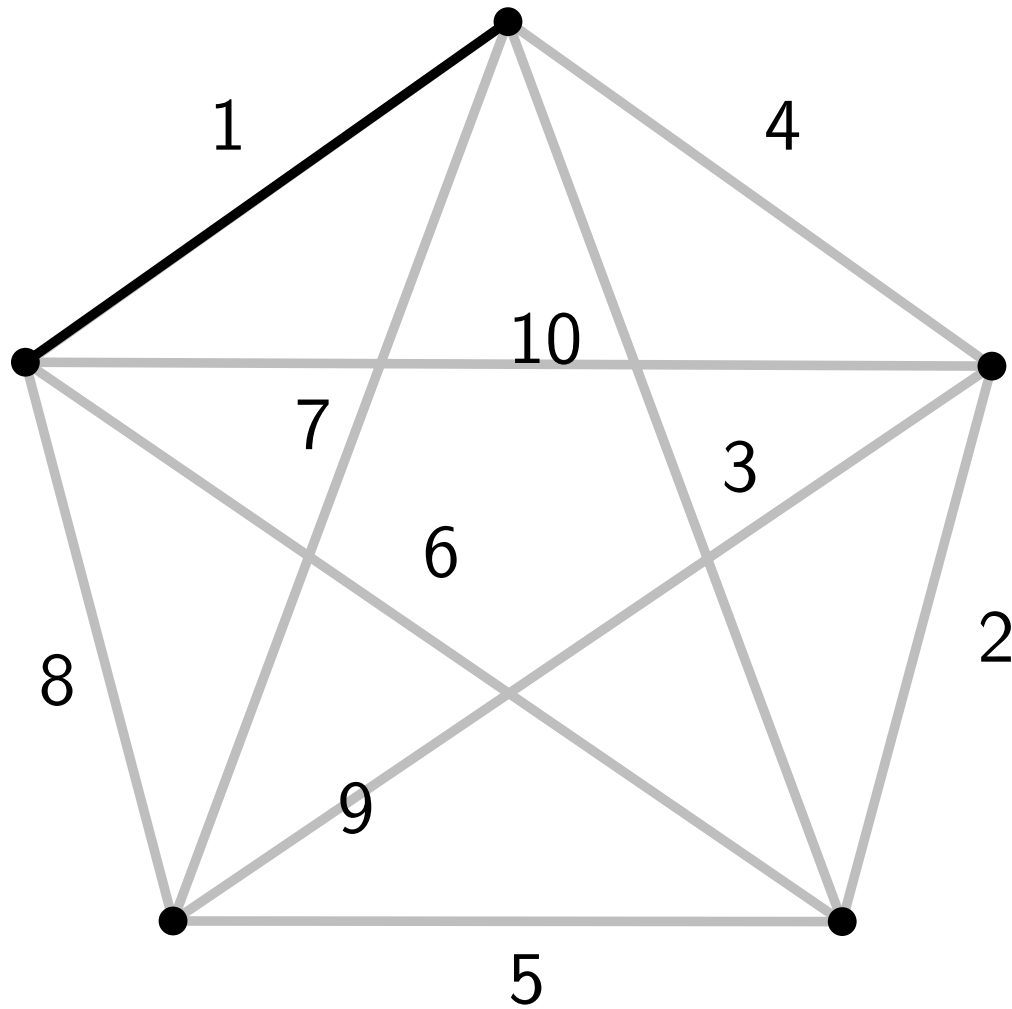
Kruskal's algorithm.

1. sort the edges by increasing weight, $e_i, 1 \leq i \leq |E|$
2. Initially set $T_0 = (V, \emptyset)$
3. Set $T_{i+1} = T_i \cup \{e_i\}$ iff it does not create a cycle

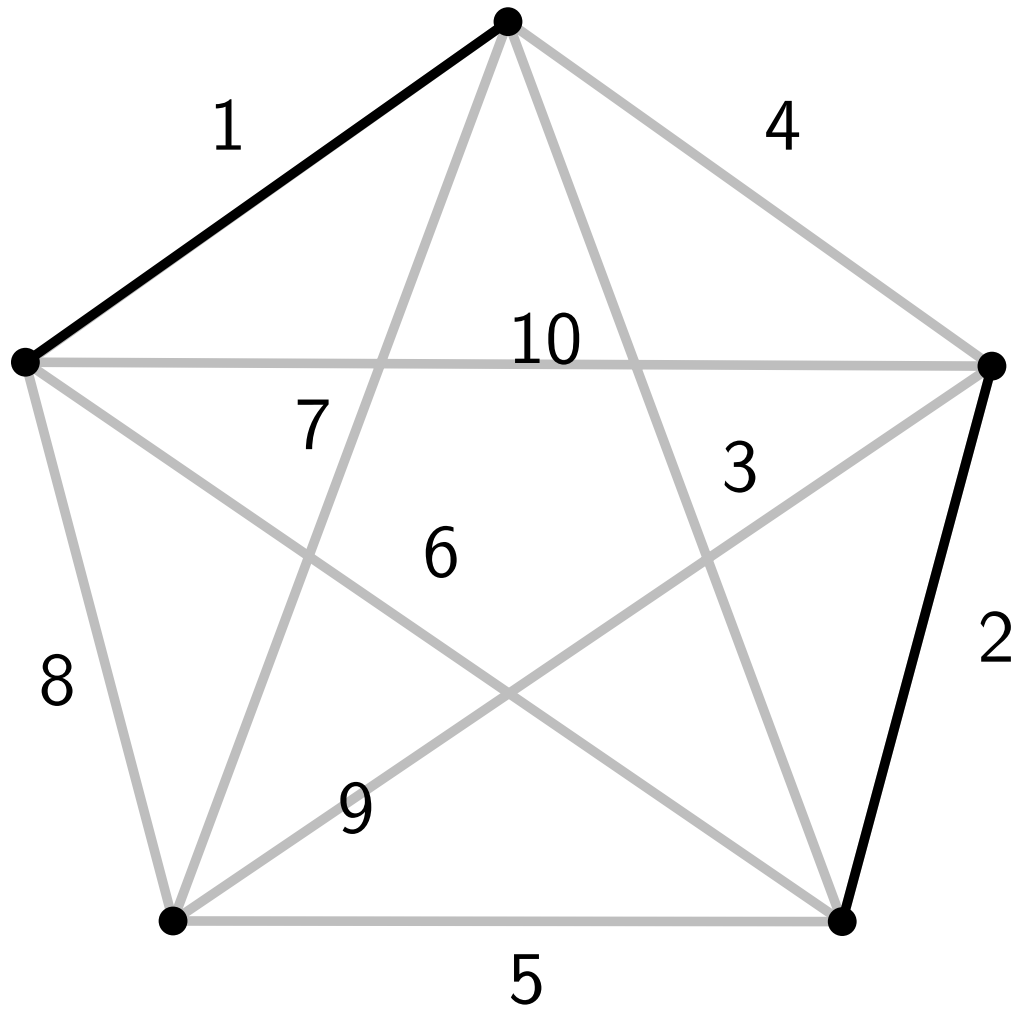
Kruskal – Example



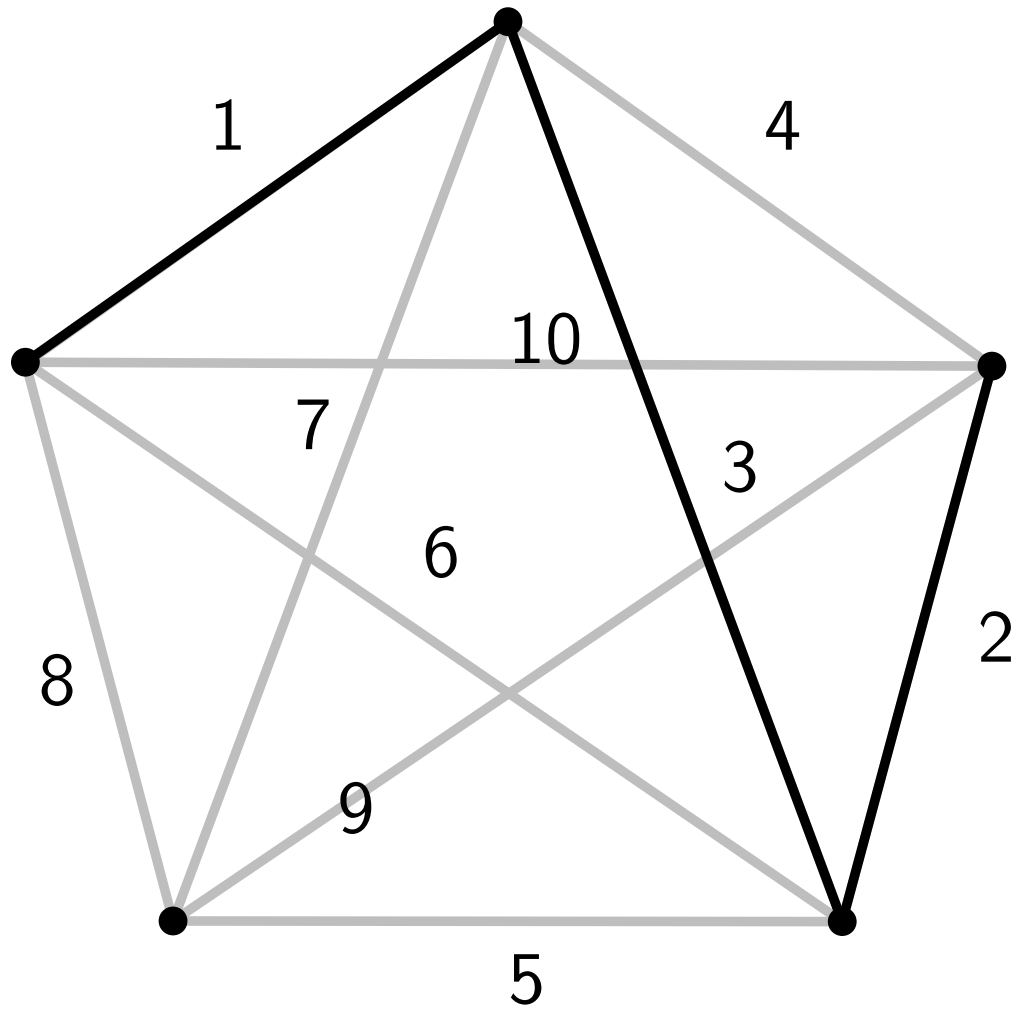
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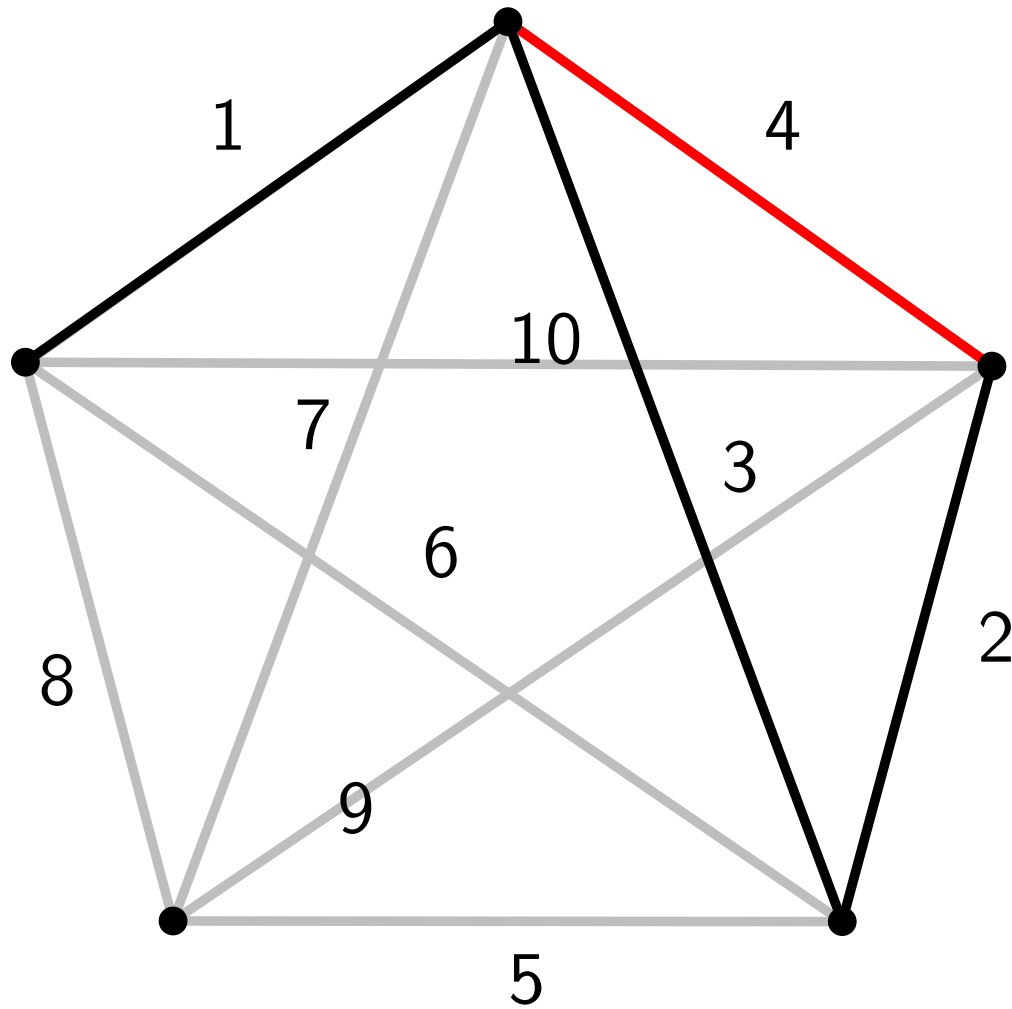
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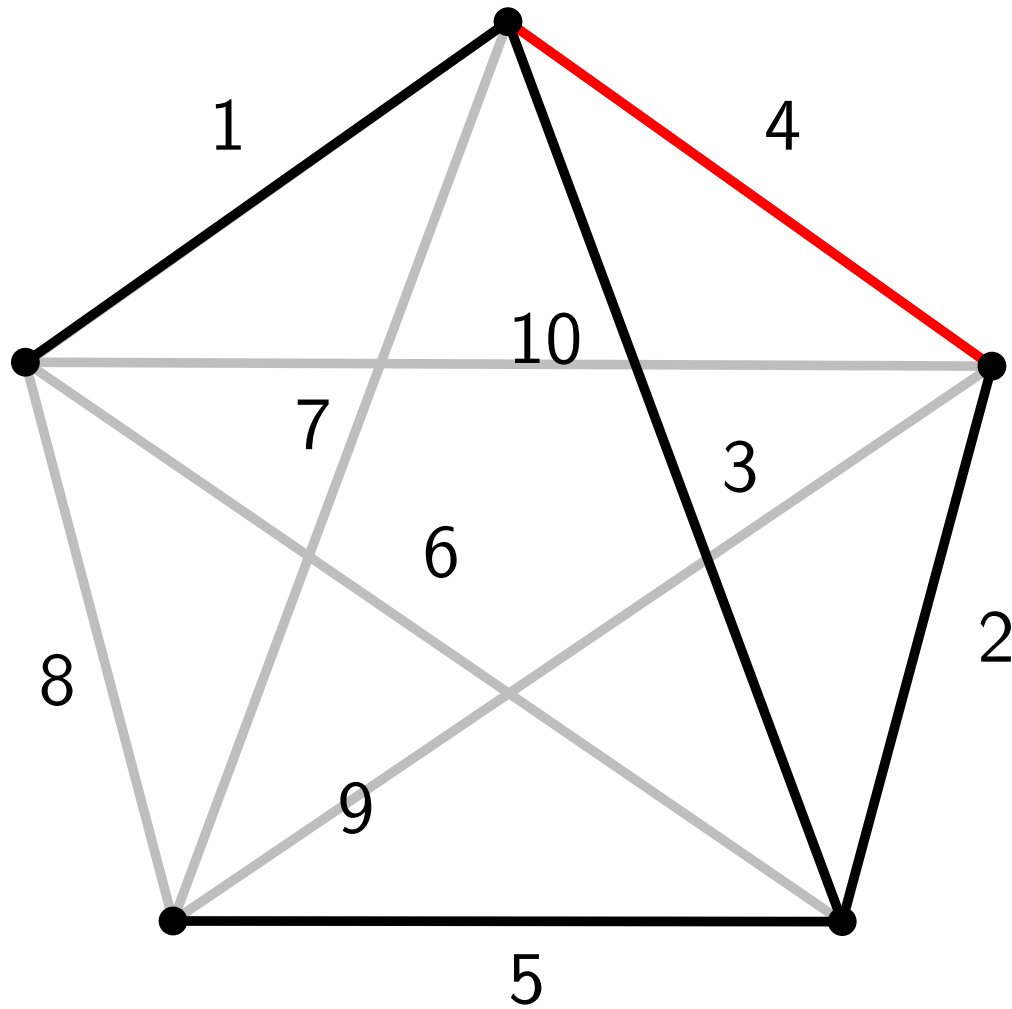
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Random Model

”Mean-field” model

graph: complete graph K_n

weights: iid uniform

A little history.

Frieze ('85): total weight converges to $\zeta(3)$

Janson ('95): CLT

Aldous: degree of the node 1

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But... all these informations are local

What is the global *metric* structure?

The continuum spanning tree

The rescaled minimum spanning tree

- T_n the minimum spanning tree of K_n
- $n^{-1/3}d_n$, for d_n the graph distance
- μ_n mass n^{-1} on each vertex

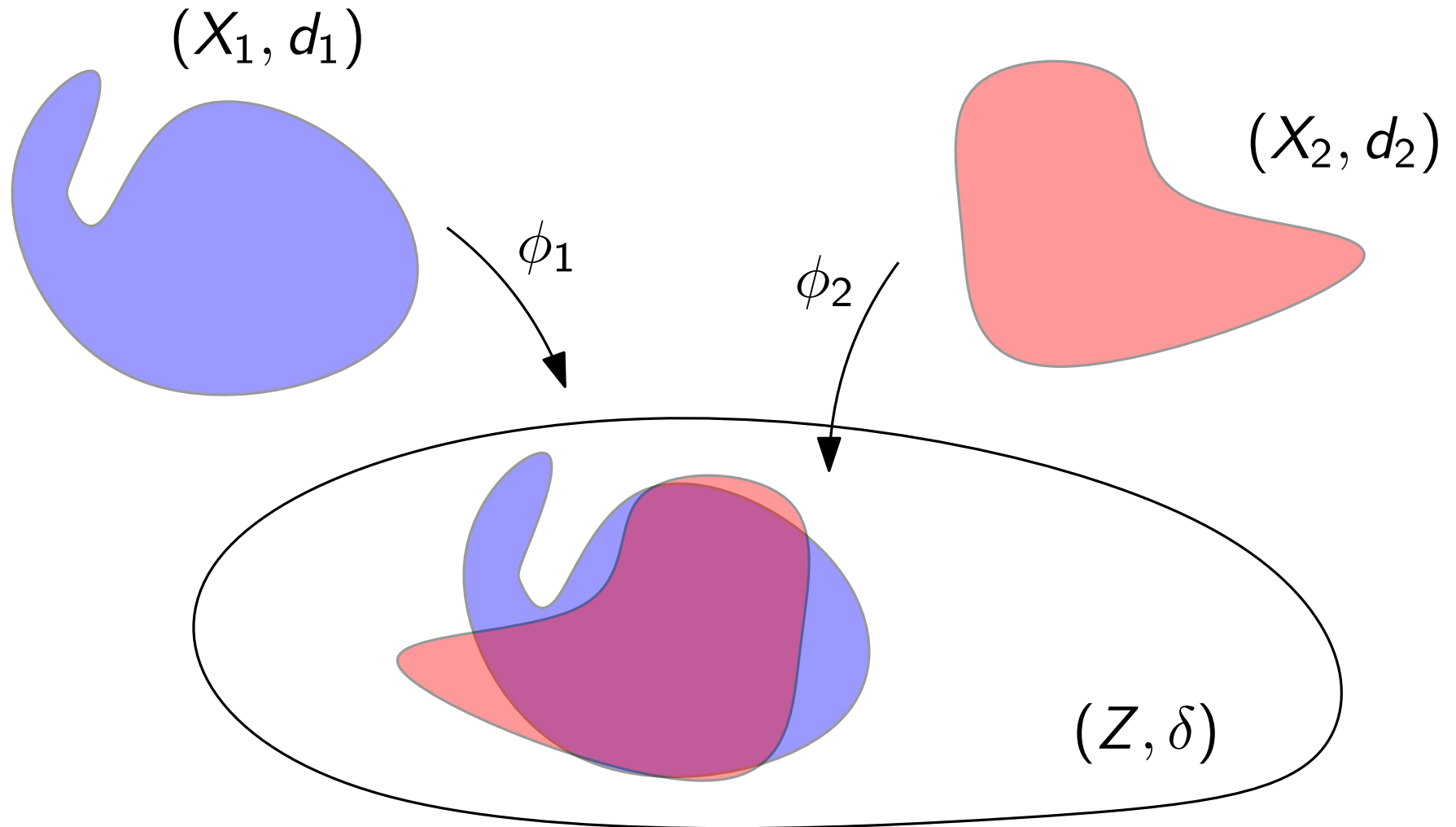
Theorem (ABGM '13)

There exists a random compact metric space \mathcal{M} s.t.

$$T_n \xrightarrow[GHP]{d} \mathcal{M}$$

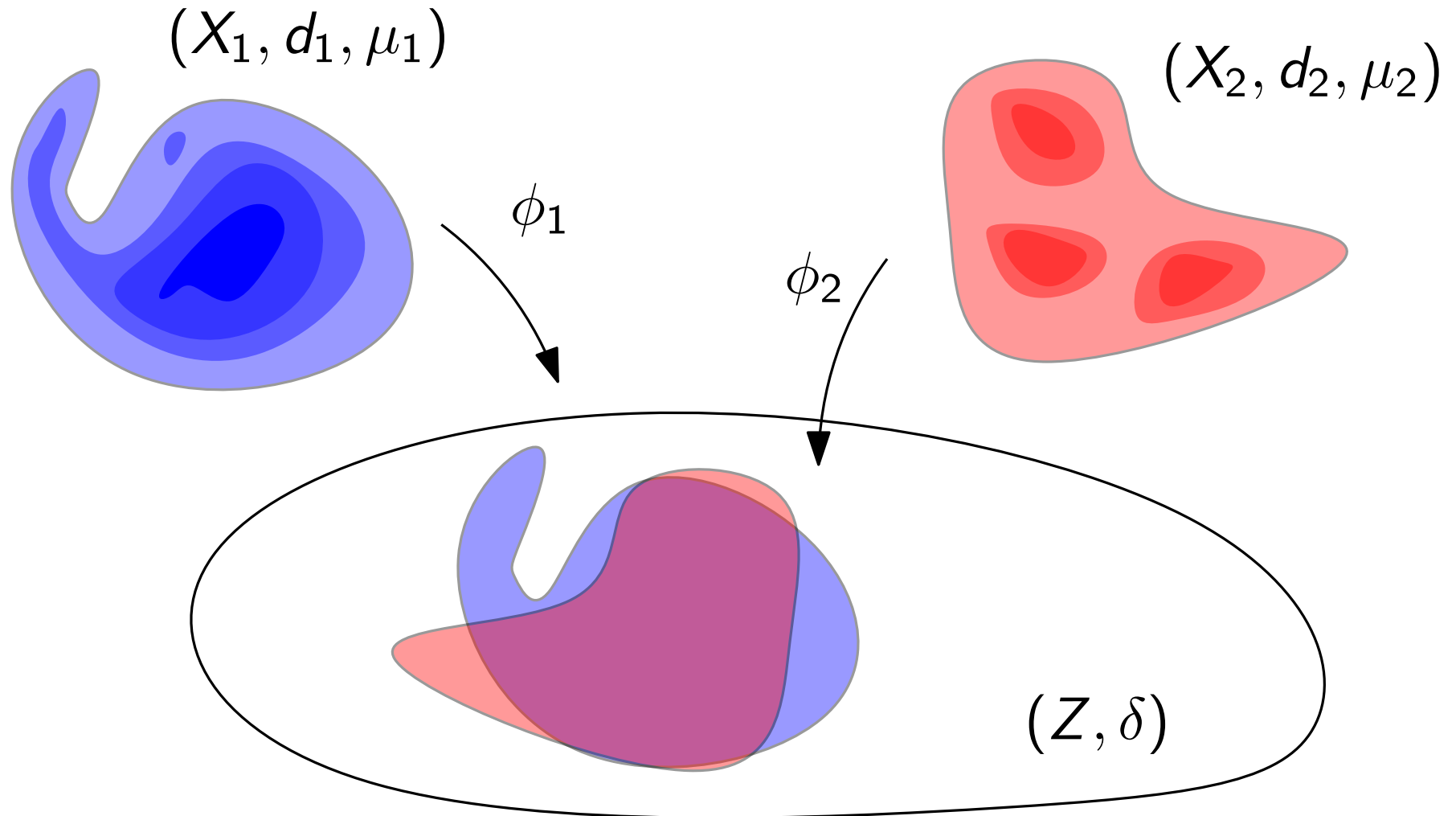
Comparing metric spaces

Gromov-Hausdorff topology.



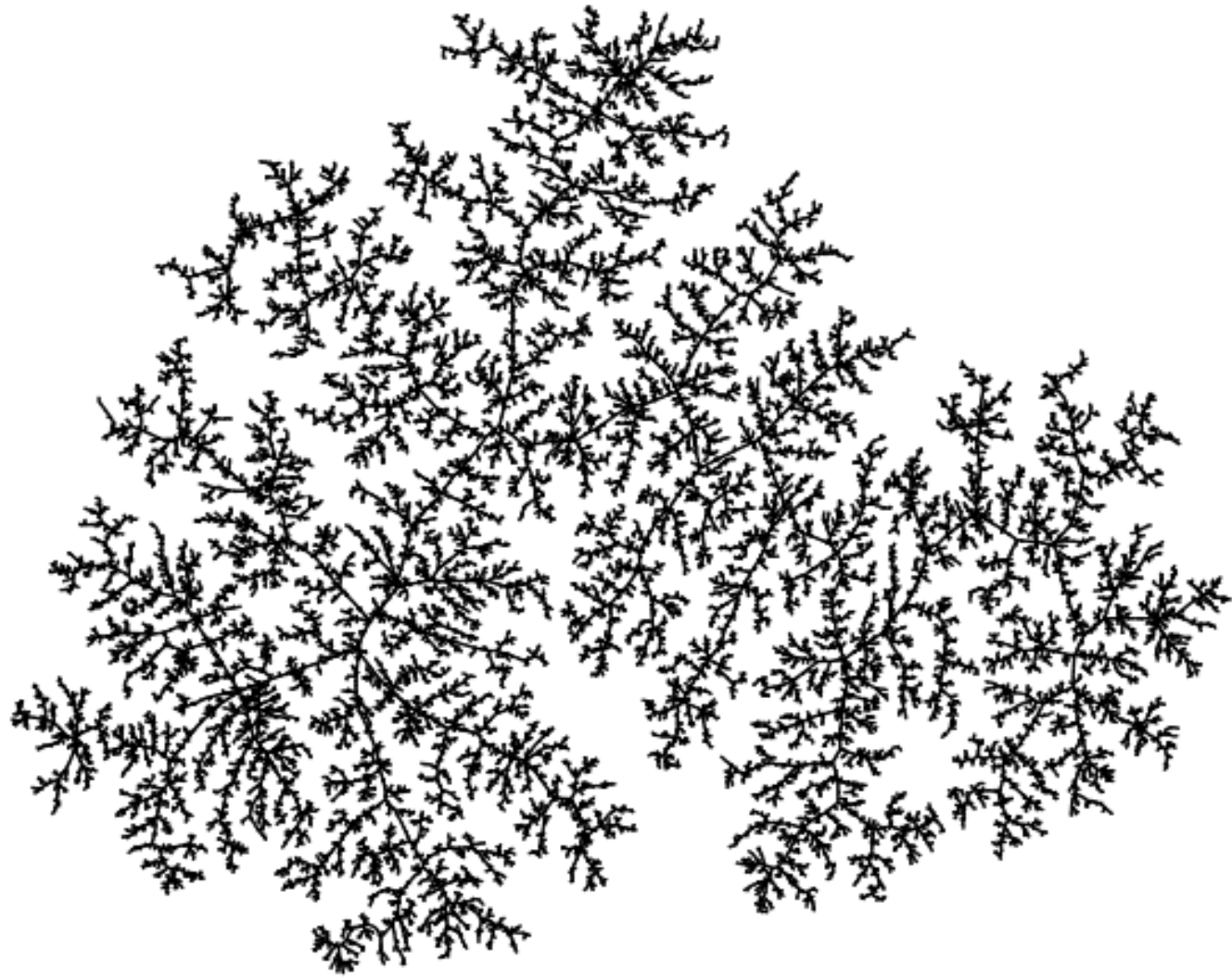
Comparing measured metric spaces

Gromov-Hausdorff-Prokhorov topology.



What does it look like?

M



A few properties of \mathcal{M}

Proposition.

1. \mathcal{M} is a tree-like metric space
2. \mathcal{M} has maximum degree 3
3. for μ -almost every x , $\deg(x) = 1$

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Proposition.

\mathcal{M} is not Aldous' Continuum Random Tree (CRT)

Elements of proof

Random graphs

Phase transition

Scaling limit of large trees / CRT

Structure of critical random graphs

Minimum spanning tree

Erdős–Rényi random graphs

Definition. Random graph $G(n, p)$

graph on $\{1, 2, \dots, n\}$

independently, take edges with probability p

C_i^n the connected components in decreasing order of size

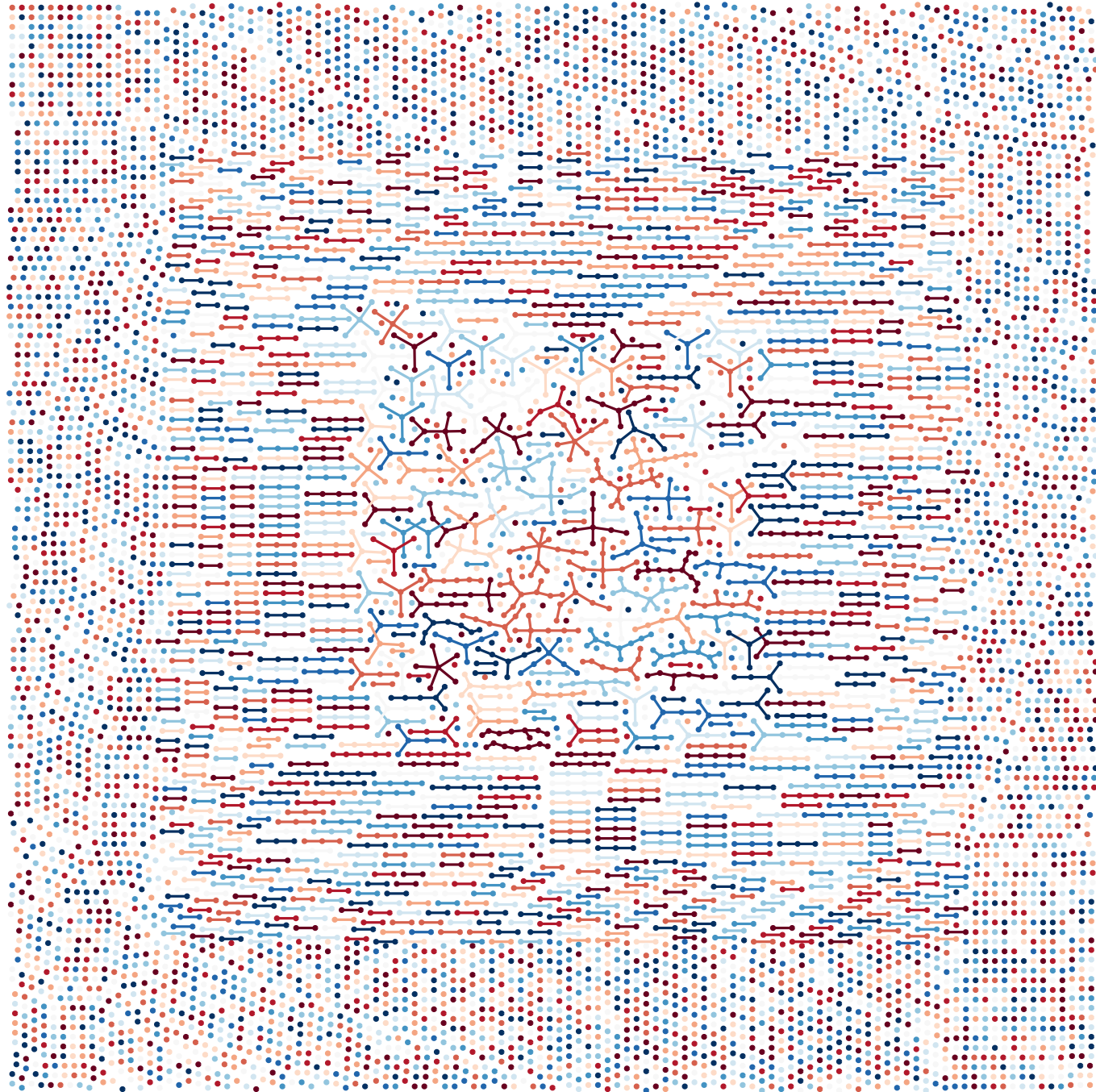
Phase transition: $G(n, c/n)$

$$c < 1: \quad |C_1^n| = O(\log n)$$

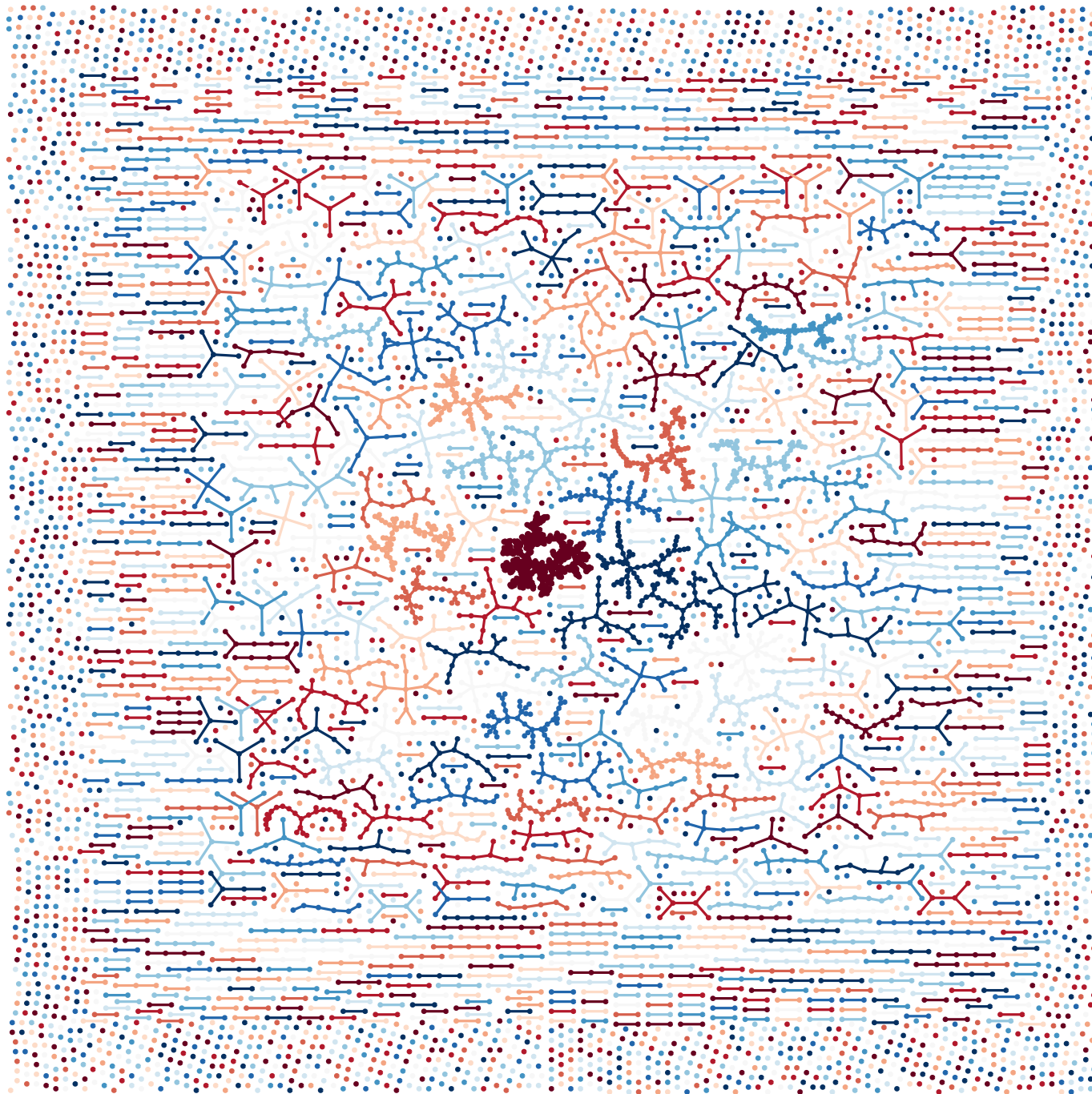
$$c = 1: \quad |C_1^n|, |C_2^n|, \dots, |C_k^n| \approx n^{2/3}$$

$$c > 1: \quad |C_1^n| = \Omega(n), \quad |C_2^n| = O(\log n)$$

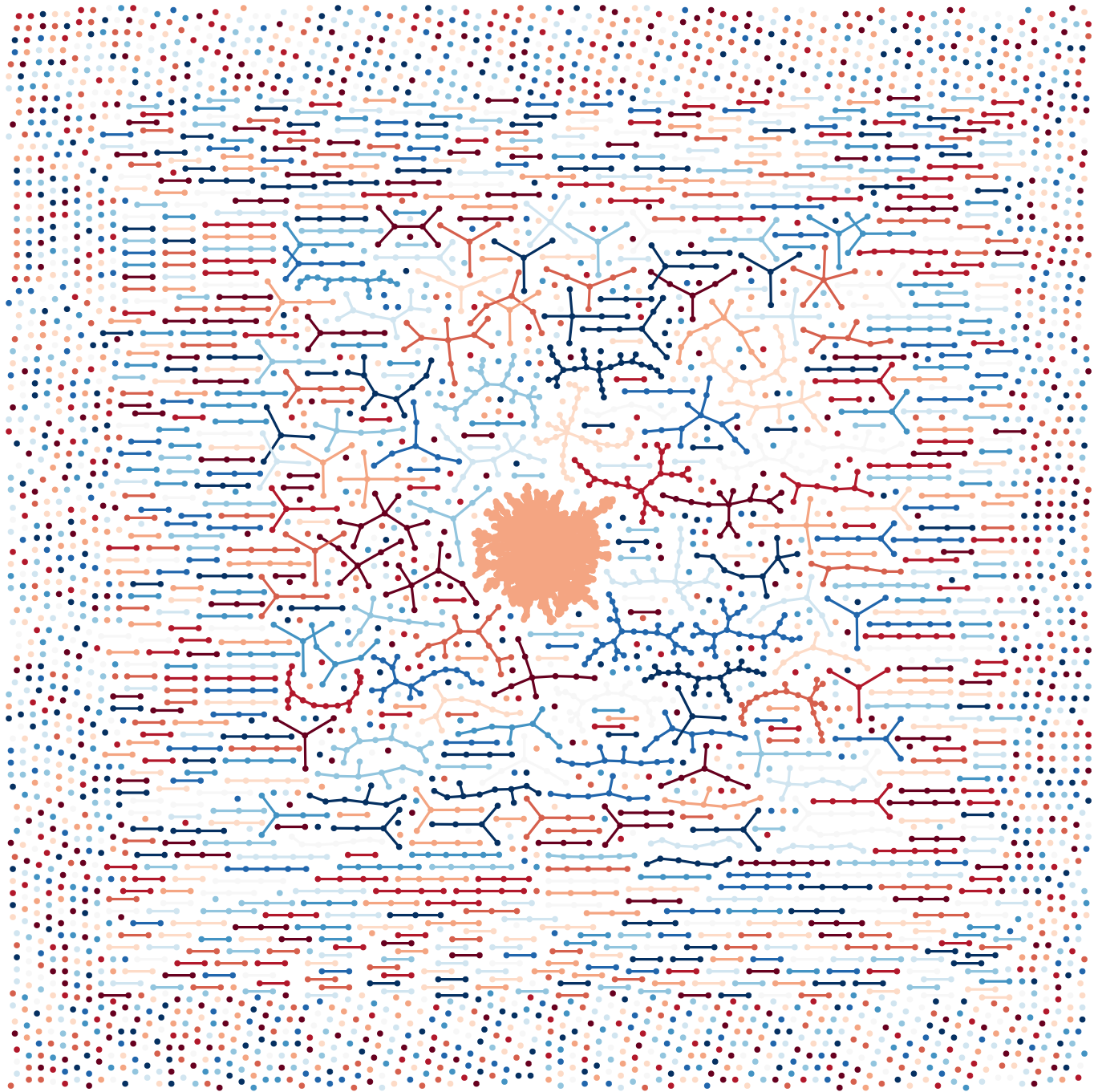
The phase transition in pictures



The phase transition in pictures



The phase transition in pictures



When is the metric structure built?

$T(n, p)$ portion of the MST that is in $G(n, p)$

$$T(n, p) = (T_1(n, p), T_2(n, p), \dots)$$

Evolution of distances:

- for all $p < (1 - \epsilon)/n$

$$d_{\text{GH}}(T(n, p); \text{“empty graph”}) = O(\log n)$$

- for all $p > (1 + \epsilon)/n$

$$d_{\text{GH}}(T_1(n, p); \text{MST}) = O(\log^{10} n)$$

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Look around the *critical phase*

$$p^* = 1/n + \lambda n^{-4/3} \quad \lambda \in \mathbb{R} \text{ large}$$

The phase transition

Theorem. (Aldous '97) For $np = 1 + \lambda n^{-1/3}$ $\lambda \in \mathbb{R}$
 $(n^{-2/3}|C_i^n|, s(C_i^n))_{i \geq 1} \rightarrow (|\gamma_i|, s(\gamma_i))_{i \geq 1}$

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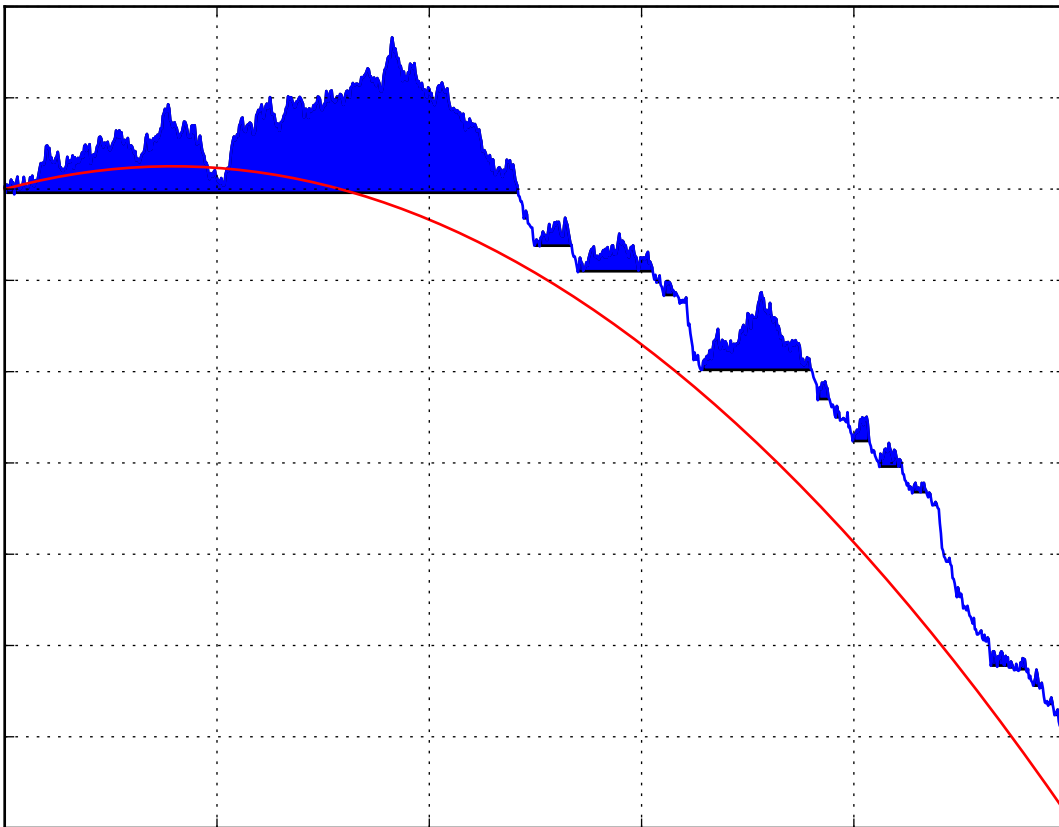
W Brownien

$$W_t^\lambda = \lambda t - t^2/2 + W_t$$

$$B_t^\lambda = W_t^\lambda - \inf_{s \leq t} W_s^\lambda$$

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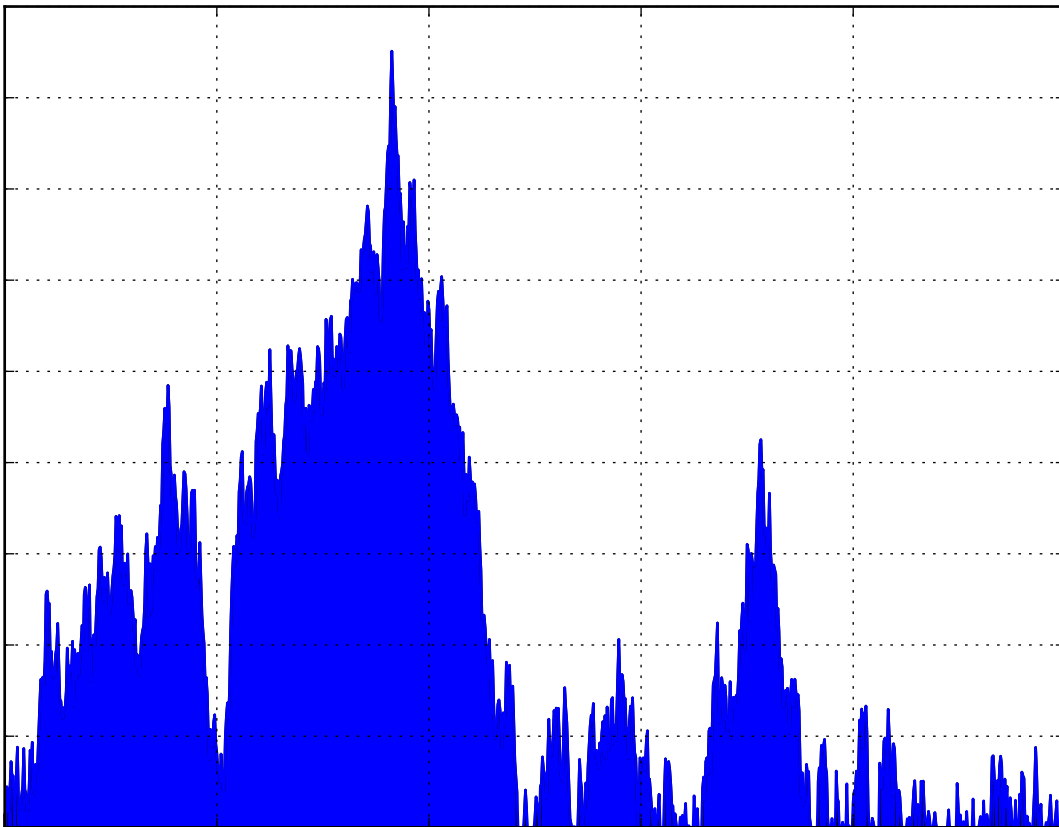
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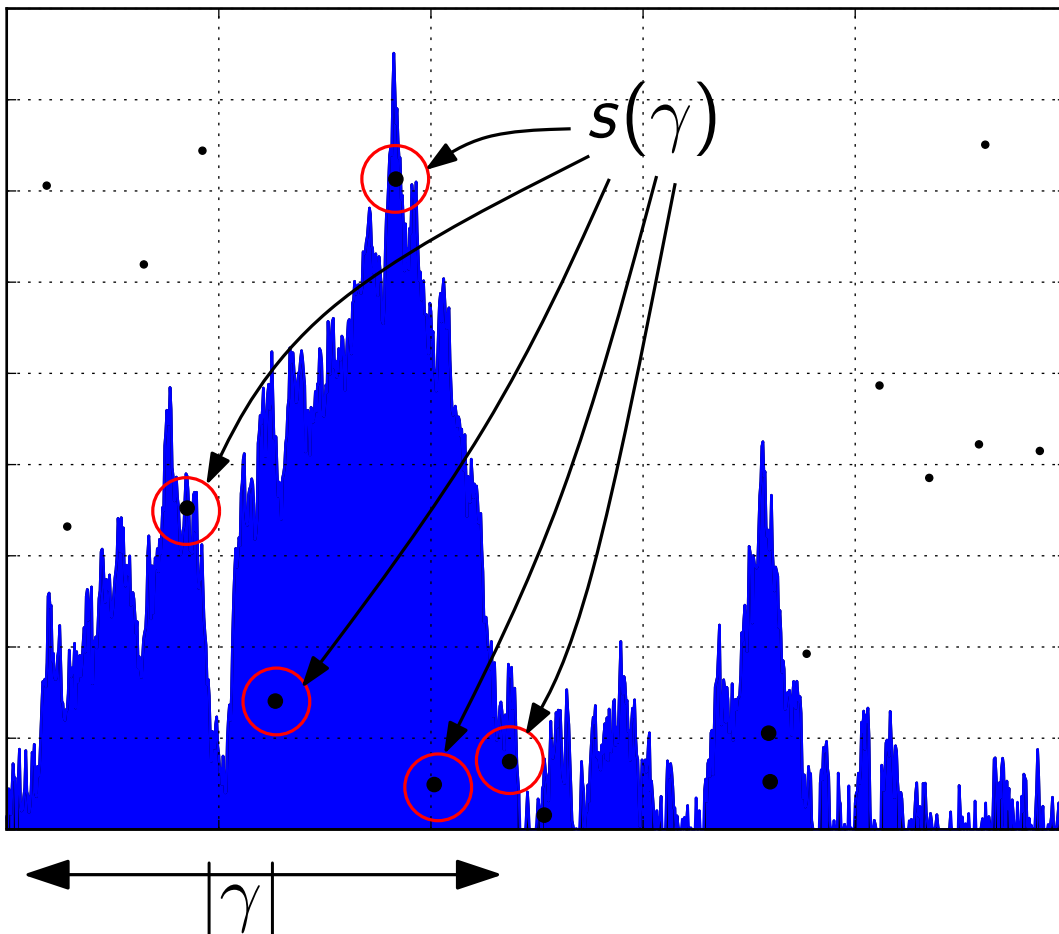
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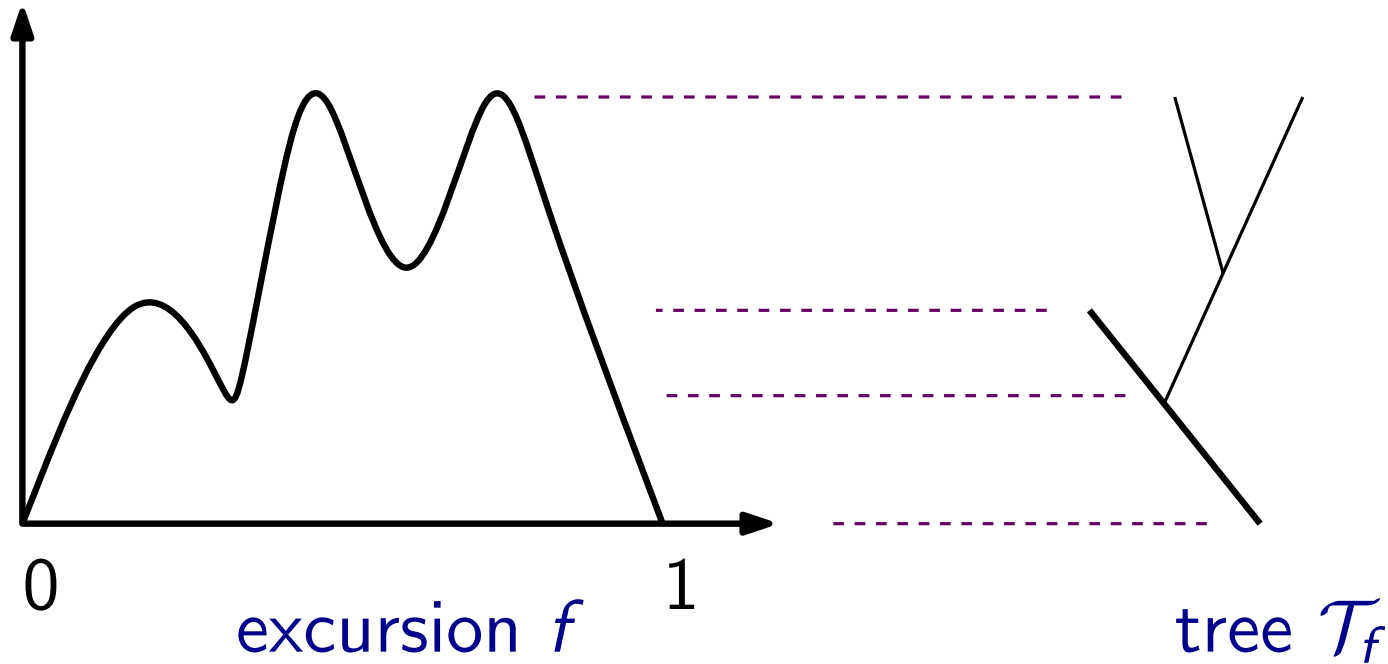
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Poisson rate 1 on \mathbb{R}_+^2

The tree encoded by an excursion



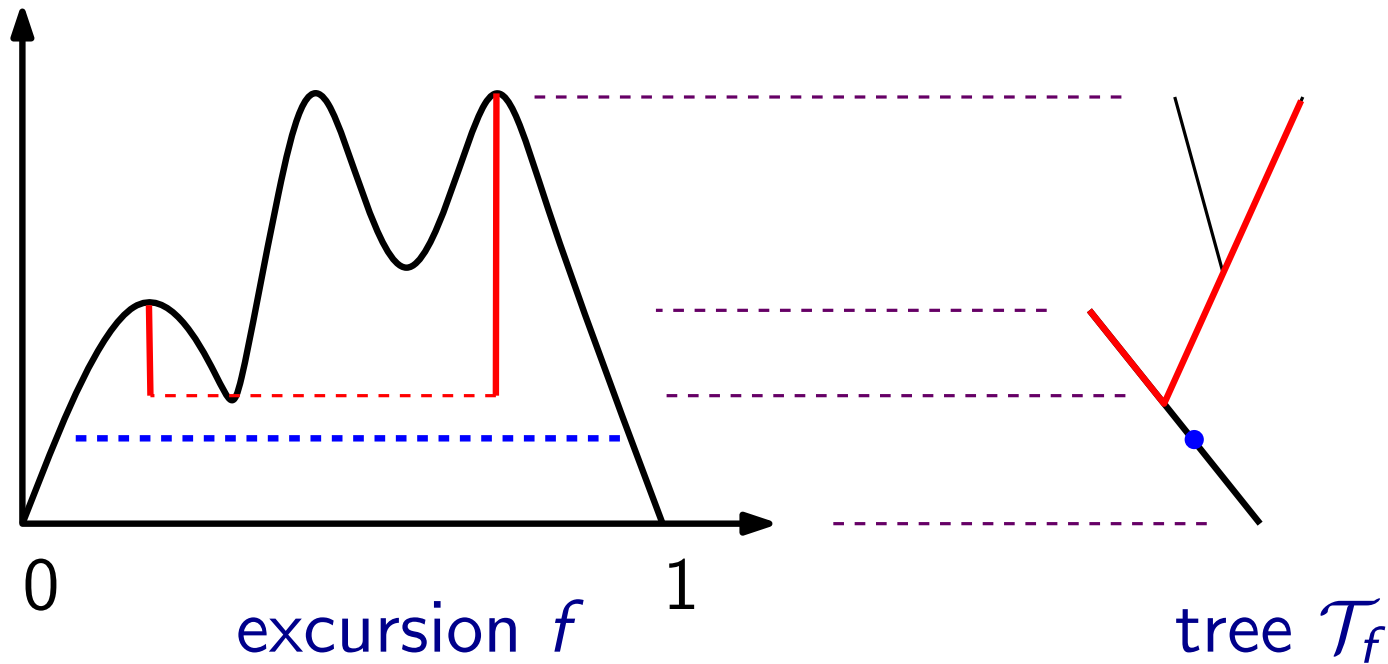
Definition: For a continuous excursion f

$$d_f(x, y) = f(x) + f(y) - 2 \inf_{x \wedge y \leq t \leq x \vee y} f(t)$$

$x \sim_f y$ if $d_f(x, y) = 0$

$([0, 1]/\sim_f, d_f)$ is a tree-like metric space

The tree encoded by an excursion



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Aldous' Continuum Random Tree (CRT)

Theorem. (Aldous '91)

T_n a uniformly random tree on $\{1, 2, \dots, n\}$

$$n^{-1/2} T_n \xrightarrow{d} \mathcal{T}_{2e}$$

Aldous' Continuum Random Tree (CRT)

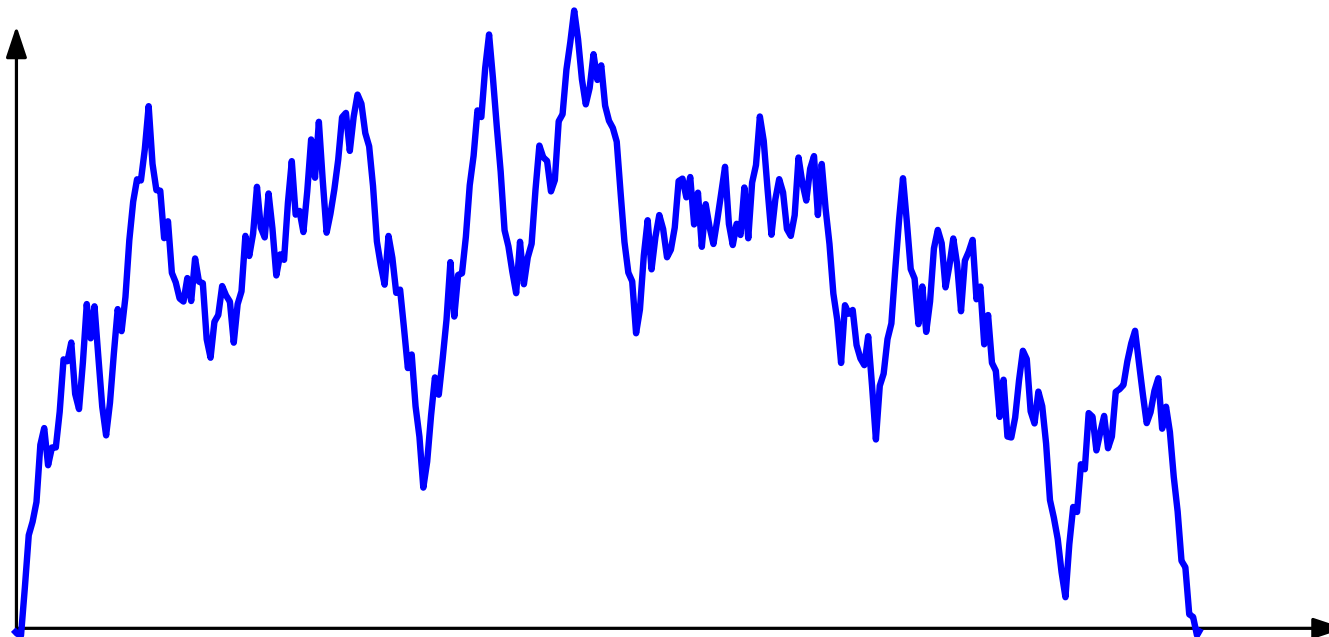
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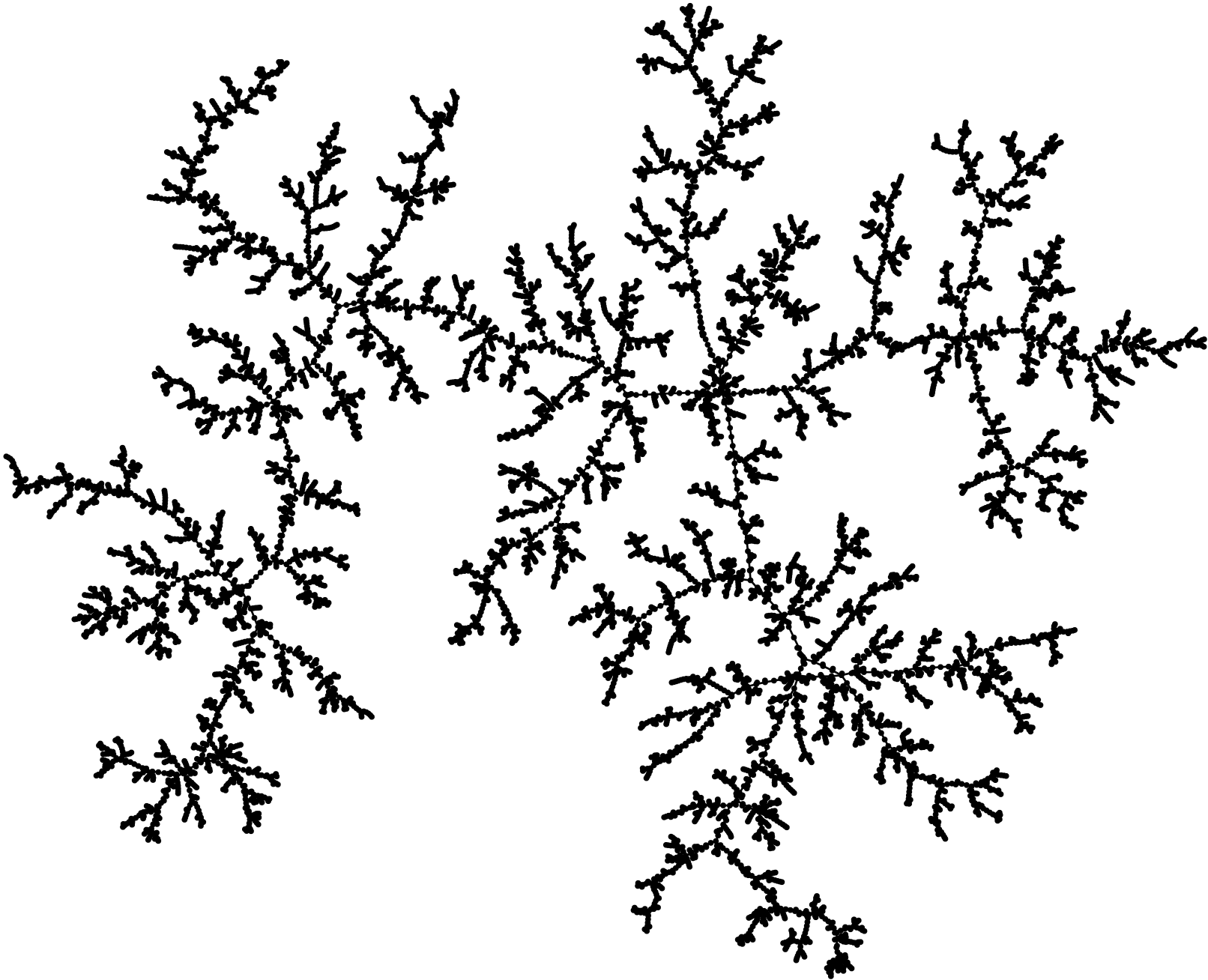
e standard Brownian excursion

\mathcal{T}_{2e} : **Continuum random tree**



What does it look like?

\mathcal{T}_{2e}



Scaling critical random graphs

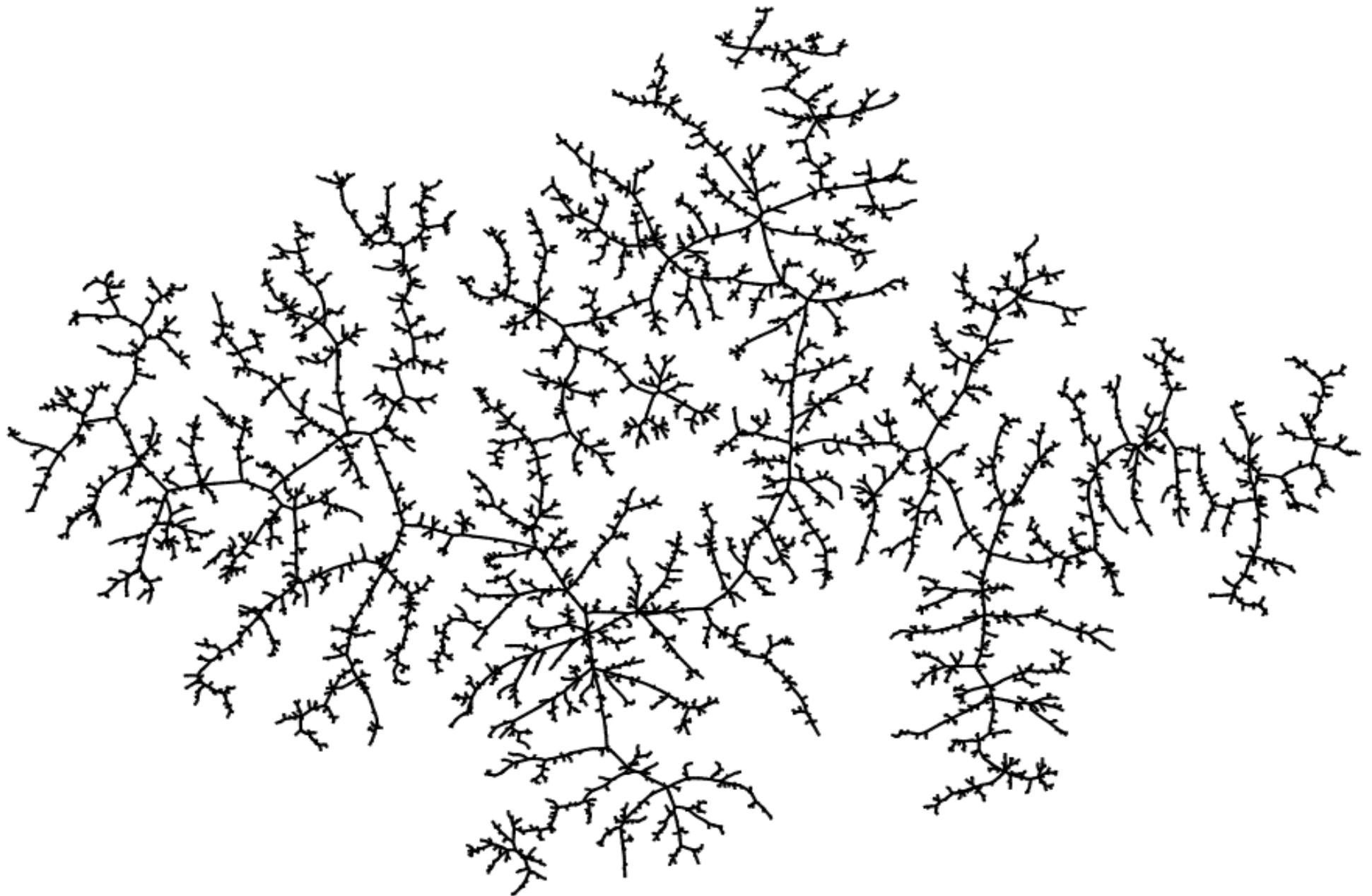
$G(n, p)$ critical window: for $pn = 1 + \lambda n^{-1/3}$, $\lambda \in \mathbb{R}$

- C_i^n the i th largest c.c.
- distances rescaled by $n^{-1/3}$
- mass $n^{-2/3}$ on each vertex

Theorem. (ABG'12) There exists a sequence of random compact measured metric spaces s.t.

$$(C_i^n)_{i \geq 1} \xrightarrow{d} (\mathcal{C}_i)_{i \geq 1} \quad \text{for the GHP distance}$$

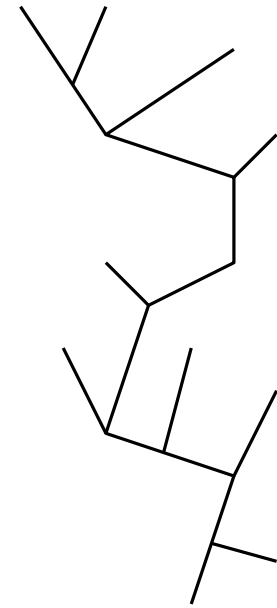
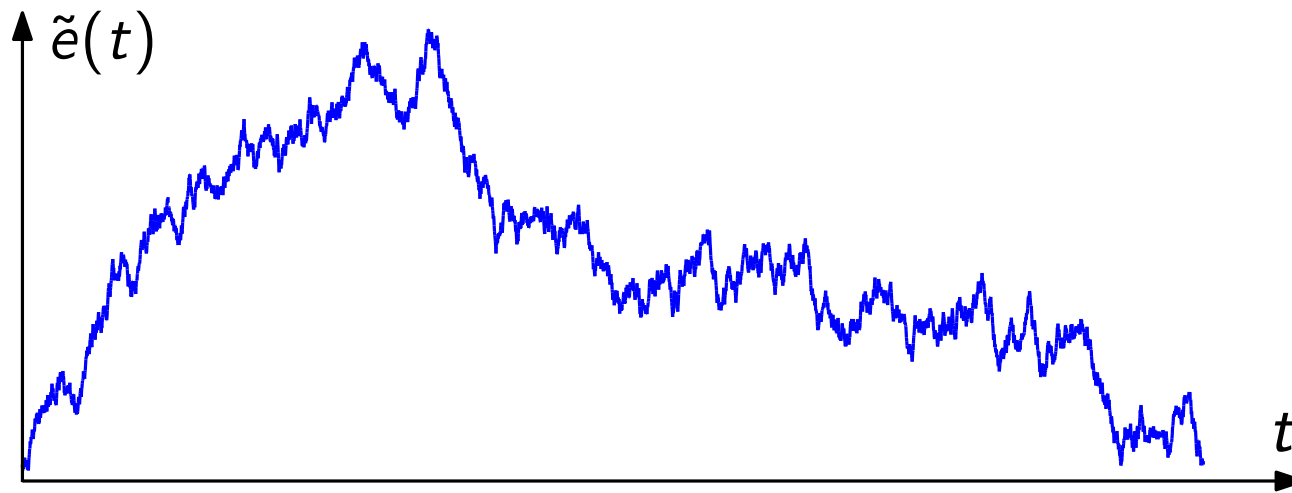
A (limit) random connected component



A limit connected component I

Identifying points in excursions

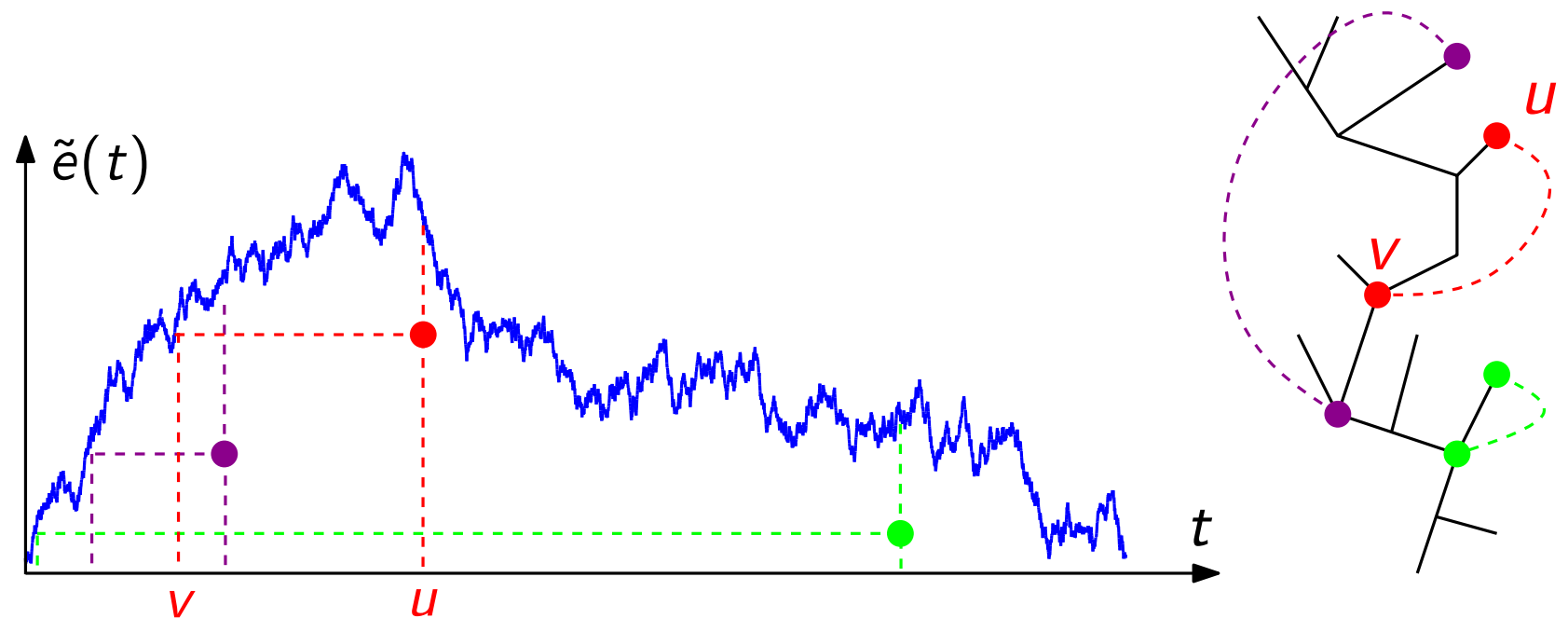
\approx “Random foldings of a random tree”



A limit connected component I

Identifying points in excursions

\approx “Random foldings of a random tree”



- **Poisson process** rate one under \tilde{e}

For each point $\{\bullet, \bullet, \bullet\}$ identify two points of $\mathcal{T}_{2\tilde{e}}$

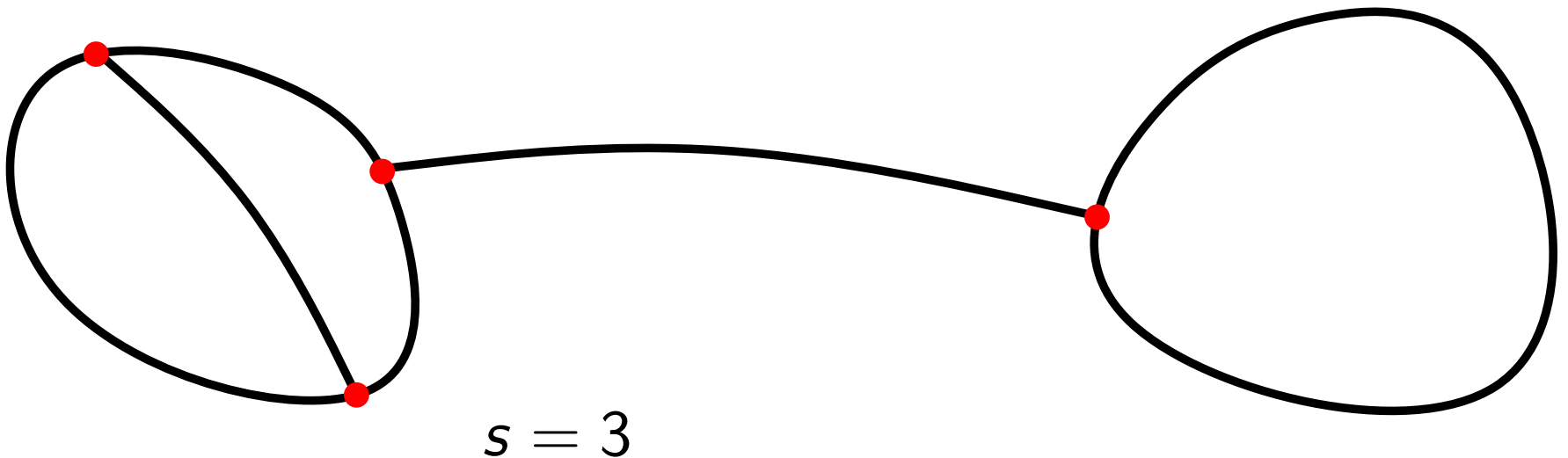
A limit connected component II

Structural approach:

1. Sample a connected 3-regular multigraph with $2(s - 1)$ vertices and $3(s - 1)$ edges
2. respective masses of the bits (“=edges”):

$$(X_1, \dots, X_{3(s-1)}) \sim \text{Dirichlet}(\frac{1}{2}, \dots, \frac{1}{2})$$

3. sample $3(s - 1)$ independent CRT with 2 distinguished points each



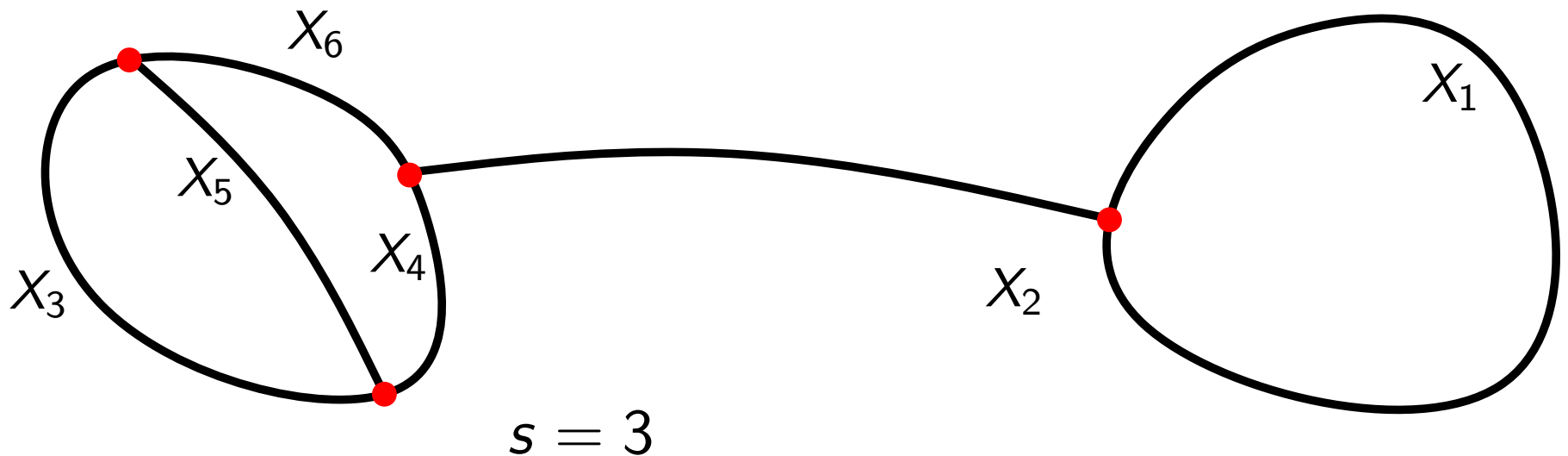
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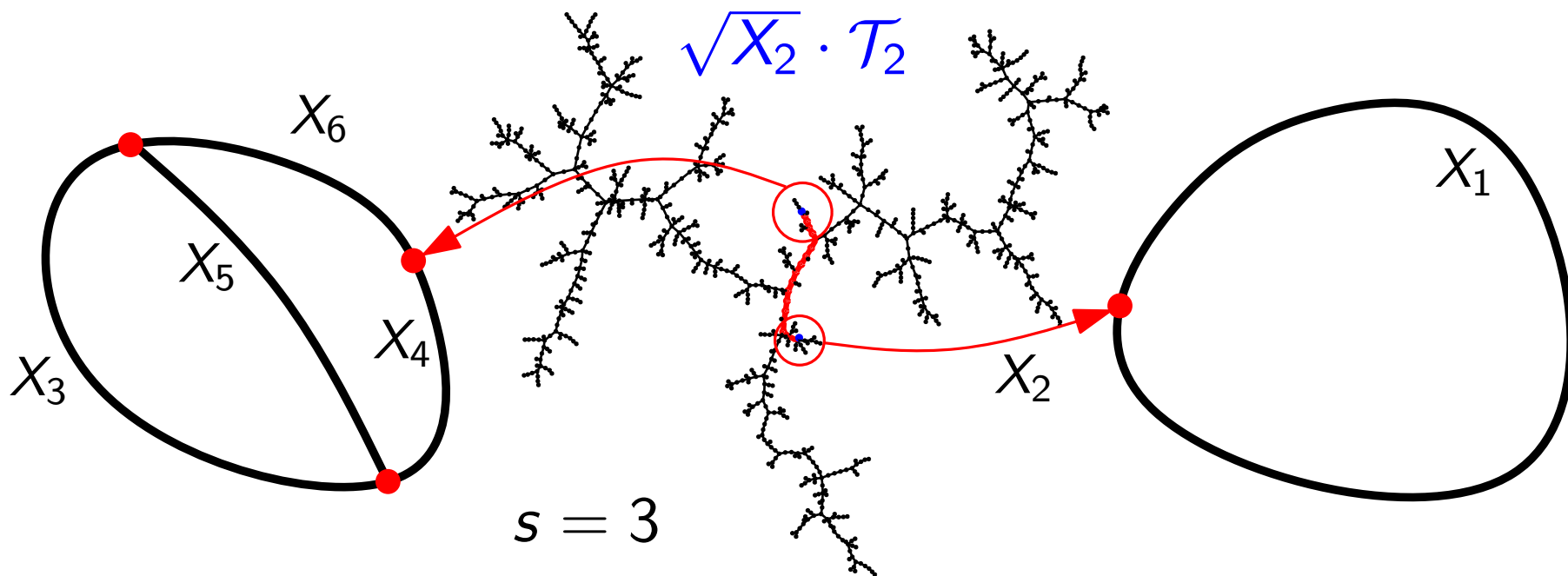
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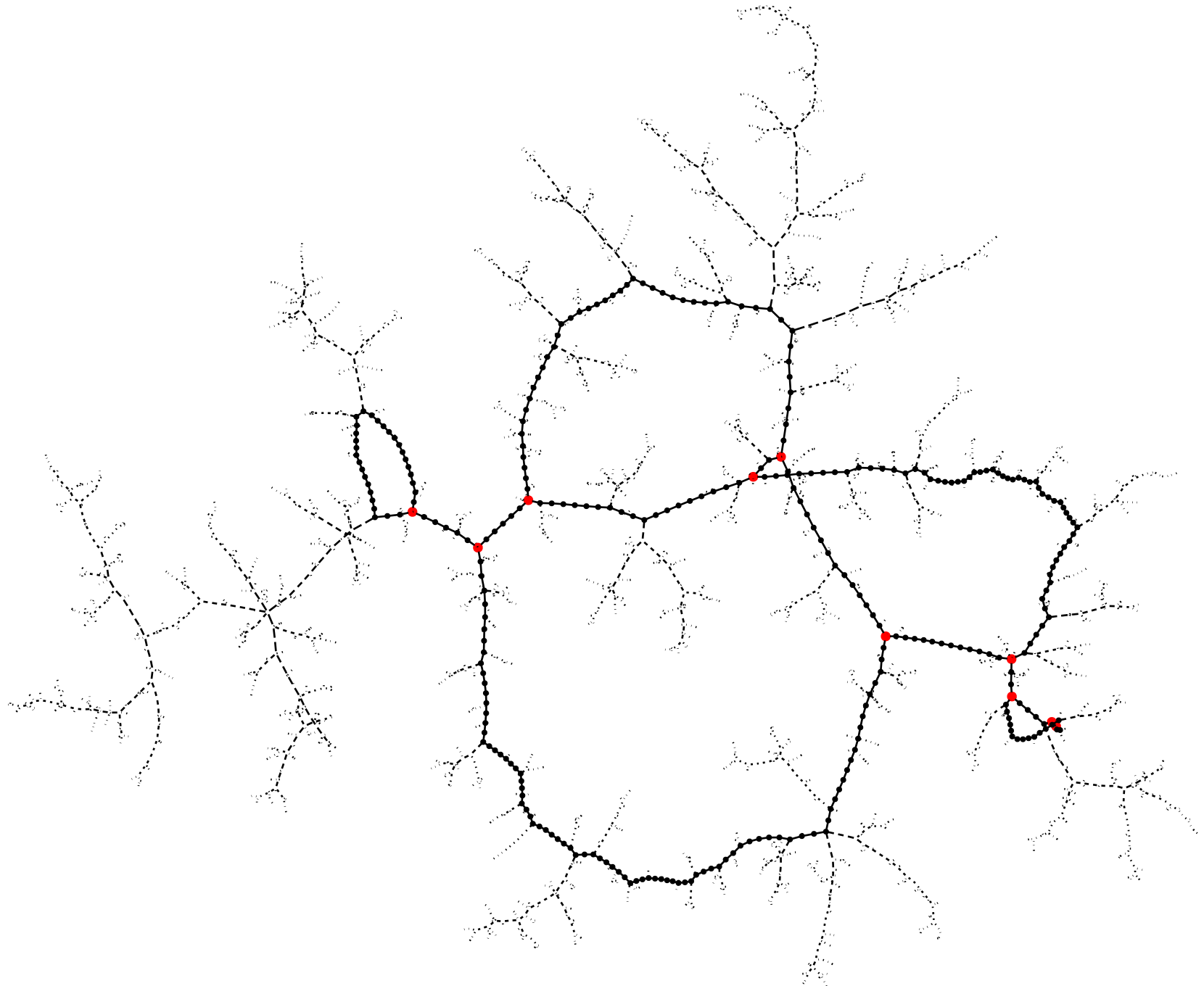
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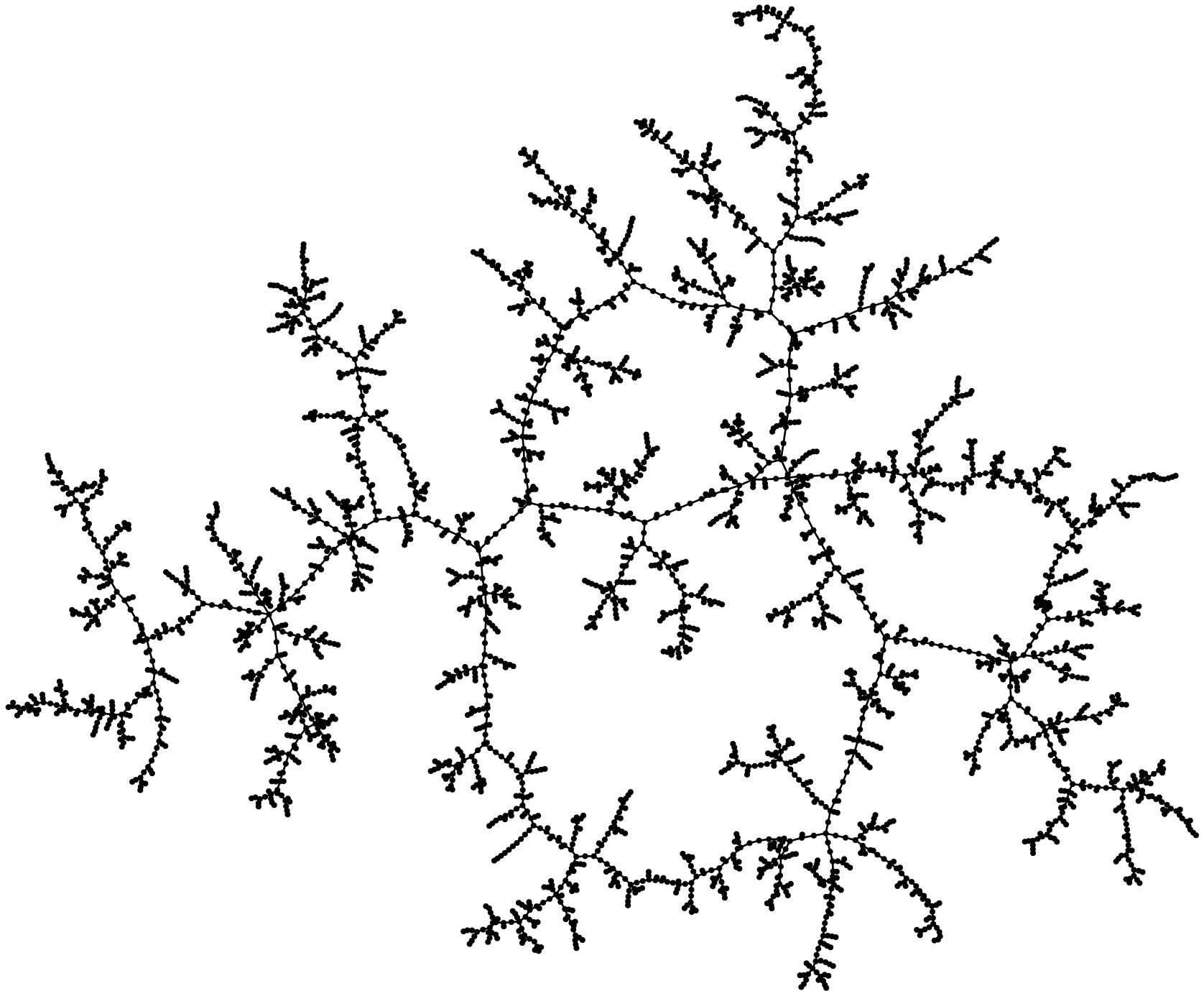
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A large connected graph

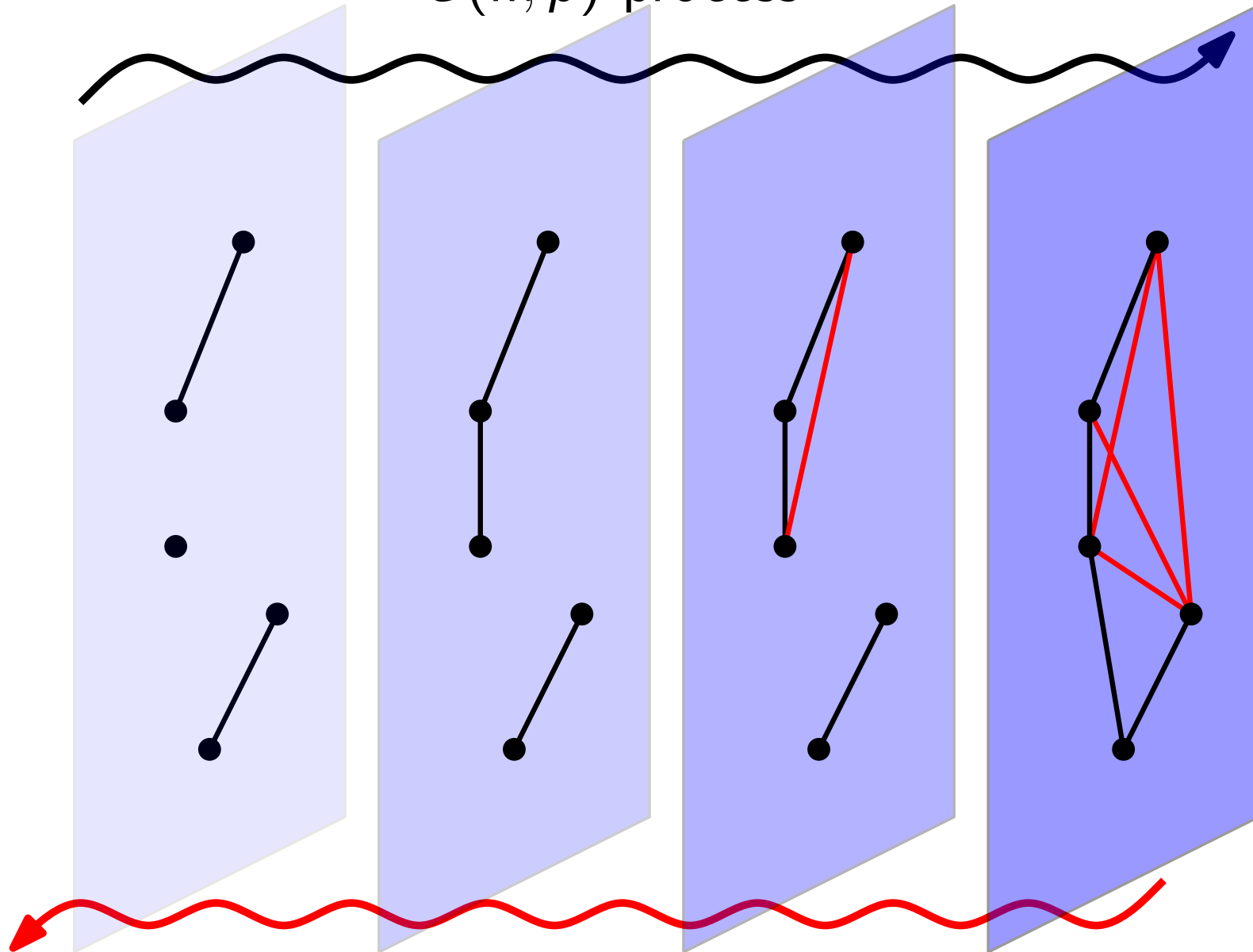


A large connected graph



Use the coupling with $G(n, p)$

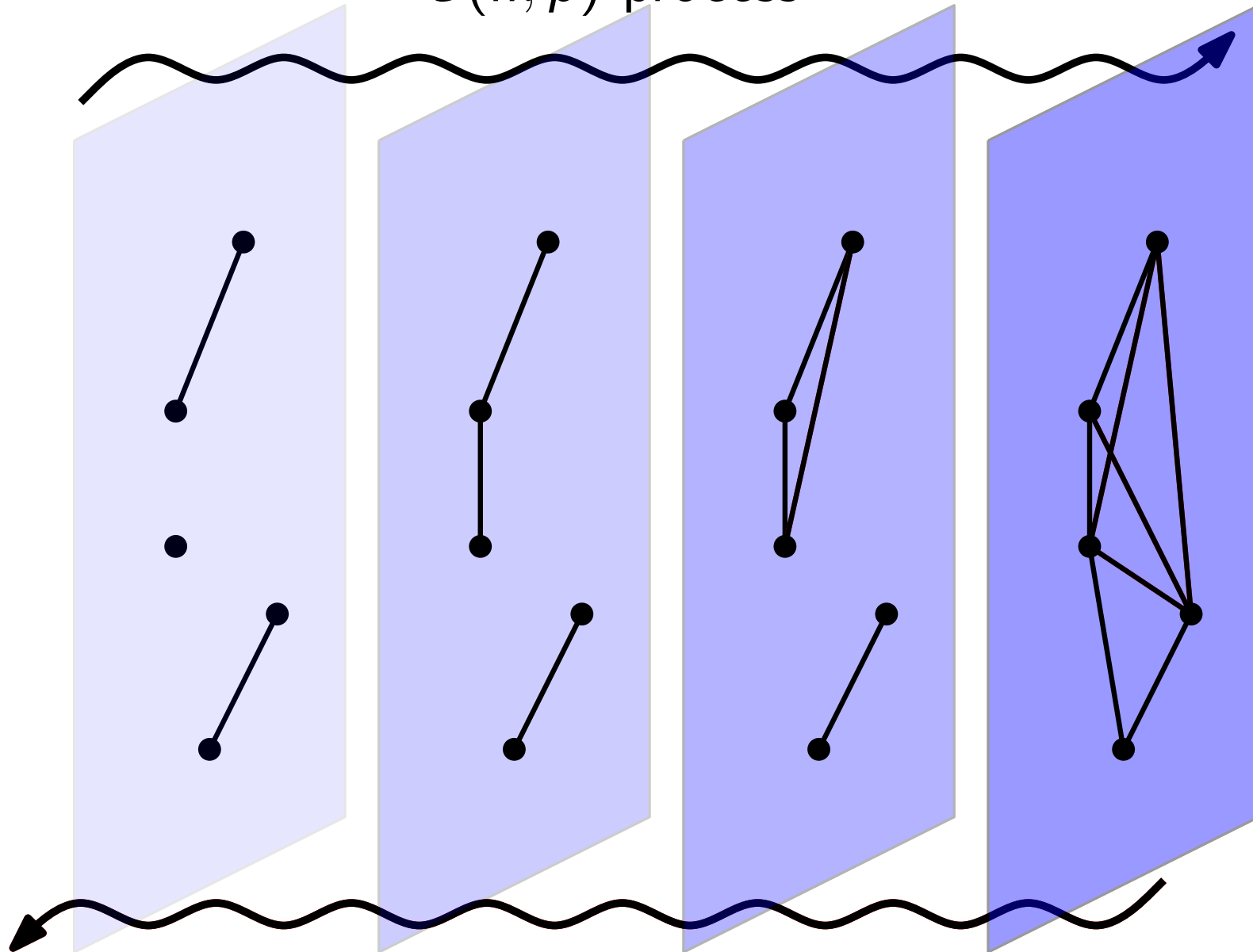
$G(n, p)$ process



Removing non-MST edges

Use the coupling with $G(n, p)$

$G(n, p)$ process



Removing non-MST edges ??

Forward-Backward approach

Strategy.

1. Build $G(n, p)$: Add all edges until some weight p^*
2. Remove the edges that should not have been put

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2'. Conditional on $G(n, p) = G$,
construct a tree **distributed** as $\text{MST}(G)$

Forward-Backward approach

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Cycle breaking: $(e_i)_{i \geq 1}$, i.i.d. uniformly random edges

While “not a tree”

Remove e_i unless it disconnects the graph

Forward-Backward approach – the limit

Strategy.

1. Build $G(n, p)$: Add all edges until some weight p^*
2. Remove the edges that should not have been put

$$G(n, p) \xrightarrow[n \rightarrow \infty]{} (\mathcal{C}_1, \mathcal{C}_2, \dots)$$

Cycle breaking for metric spaces:

$(x_i)_{i \geq 1}$ i.i.d. random points on the cycle structure

While “not a tree”

Remove x_i unless it disconnects the metric space

Construction of the limit

$$G(n, p) \xrightarrow{n \rightarrow \infty} (\mathcal{C}_1^\lambda, \mathcal{C}_2^\lambda, \dots)$$



cycle breaking



$$T(n, p) \xrightarrow{n \rightarrow \infty} (\mathcal{T}_1^\lambda, \mathcal{T}_2^\lambda, \dots)$$

$$\downarrow \lambda \rightarrow \infty$$

$$(T_n, 0, 0, \dots)$$

$$\xrightarrow{n \rightarrow \infty}$$

$$\downarrow \lambda \rightarrow \infty$$

$$(\mathcal{M}, 0, 0, \dots)$$

Fractal dimension

(X, d) a compact metric space

$N(X, r) = \min$ number of balls of radius r to cover X

$$\underline{\dim}(X) = \liminf_{r \rightarrow 0} \frac{\log N(X, r)}{\log(1/r)}$$

$$\overline{\dim}(X) = \limsup_{r \rightarrow 0} \frac{\log N(X, r)}{\log(1/r)}$$

box-counting dimension

$\dim(X)$ is the common value, if they are equal

Example:

$$\dim([0, 1]) = 1$$

$$N([0, 1], r) \approx 1/r$$

$$\dim([0, 1]^2) = 2$$

$$N([0, 1]^2, r) \approx 1/r^2$$

Dimensions of continuum random trees

Theorem. (ABGM 2013)

$$\dim(\mathcal{M}) = 3 \quad \text{with probability one}$$

while

Theorem.

$$\dim(CRT) = 2 \text{ with probability one}$$

Thank you!

Estimating the box-counting dimension

For $p = 1/n + \lambda n^{-4/3}$, λ large

1. mass of the largest component $\sim 2\lambda$
2. surplus of the largest component $\sim 2\lambda^3/3$
3. Each "tree" has mass $\sim \lambda^{-2}$
4. Each tree has diameter $\sim \sqrt{\lambda^{-2}} = \lambda^{-1}$

$$N(\mathcal{C}_1^\lambda, \lambda^{-1}) \asymp \lambda^3$$