

Graph colorings, flows
and
arithmetic Tutte polynomial
(joint work with Luca Moci)

Michele D'Adderio

Max Planck Institut für Mathematik

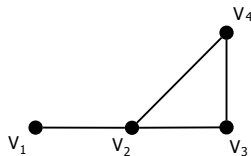
Bordeaux, February 2nd 2012

Journées de Combinatoire de Bordeaux

Graphs

Graphs

Let $\mathcal{G} := (V, E)$, where
 $V := \{v_1, v_2, v_3, v_4\}$ and
 $E := \{\{v_1, v_2\}, \{v_2, v_3\},$
 $\{v_2, v_4\}, \{v_3, v_4\}\}$.



Colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) q -*coloring* of \mathcal{G} is a map

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that if $\{x, y\} \in E$ then $c(x) \neq c(y)$.

The *chromatic polynomial* $\chi_{\mathcal{G}}(q)$ of \mathcal{G} is defined as the number of q -colorings of \mathcal{G} .

Colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) q -*coloring* of \mathcal{G} is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that if $\{x, y\} \in E$ then $c(x) \neq c(y)$.

The *chromatic polynomial* $\chi_{\mathcal{G}}(q)$ of \mathcal{G} is defined as the number of q -colorings of \mathcal{G} .

Colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) q -*coloring* of \mathcal{G} is a map

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that if $\{x, y\} \in E$ then $c(x) \neq c(y)$.

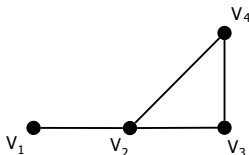
The *chromatic polynomial* $\chi_{\mathcal{G}}(q)$ of \mathcal{G} is defined as the number of q -colorings of \mathcal{G} .

Colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) q -coloring of \mathcal{G} is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that if $\{x, y\} \in E$ then $c(x) \neq c(y)$.

The *chromatic polynomial* $\chi_{\mathcal{G}}(q)$ of \mathcal{G} is defined as the number of q -colorings of \mathcal{G} .

Let $\mathcal{G} := (V, E)$, where
 $V := \{v_1, v_2, v_3, v_4\}$ and
 $E := \{\{v_1, v_2\}, \{v_2, v_3\},$
 $\{v_2, v_4\}, \{v_3, v_4\}\}$.



For $q = 4$, $c(v_1) = c(v_3) = \bar{0}$, $c(v_2) = \bar{1}$ and $c(v_4) = \bar{3}$ is a 4-coloring of \mathcal{G} .

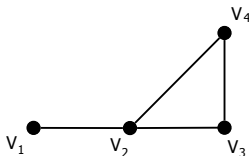
While $\chi_{\mathcal{G}}(q) = q(q-1)^2(q-2) = q^4 - 4q^3 + 5q^2 - 2q$.

Colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) q -coloring of \mathcal{G} is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that if $\{x, y\} \in E$ then $c(x) \neq c(y)$.

The *chromatic polynomial* $\chi_{\mathcal{G}}(q)$ of \mathcal{G} is defined as the number of q -colorings of \mathcal{G} .

Let $\mathcal{G} := (V, E)$, where
 $V := \{v_1, v_2, v_3, v_4\}$ and
 $E := \{\{v_1, v_2\}, \{v_2, v_3\},$
 $\{v_2, v_4\}, \{v_3, v_4\}\}.$



For $q = 4$, $c(v_1) = c(v_3) = \bar{0}$, $c(v_2) = \bar{1}$ and $c(v_4) = \bar{3}$ is a 4-coloring of \mathcal{G} .

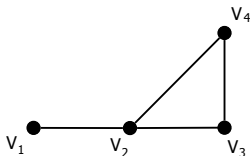
While $\chi_{\mathcal{G}}(q) = q(q-1)^2(q-2) = q^4 - 4q^3 + 5q^2 - 2q$.

Colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) q -*coloring* of \mathcal{G} is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that if $\{x, y\} \in E$ then $c(x) \neq c(y)$.

The *chromatic polynomial* $\chi_{\mathcal{G}}(q)$ of \mathcal{G} is defined as the number of q -colorings of \mathcal{G} .

Let $\mathcal{G} := (V, E)$, where
 $V := \{v_1, v_2, v_3, v_4\}$ and
 $E := \{\{v_1, v_2\}, \{v_2, v_3\},$
 $\{v_2, v_4\}, \{v_3, v_4\}\}$.



For $q = 4$, $c(v_1) = c(v_3) = \bar{0}$, $c(v_2) = \bar{1}$ and $c(v_4) = \bar{3}$ is a 4-coloring of \mathcal{G} .

While $\chi_{\mathcal{G}}(q) = q(q-1)^2(q-2) = q^4 - 4q^3 + 5q^2 - 2q$.

Flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) q -flow on an oriented graph $\mathcal{G}_\theta = (V, E_\theta)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z}) \setminus \{\bar{0}\}$ such that $\forall v \in V$,
$$\sum_{e^+=v; e \in E_\theta} w(e) - \sum_{e^-=v; e \in E_\theta} w(e) = \bar{0}.$$
The *flow polynomial* $\chi_{\mathcal{G}}^*(q)$ of \mathcal{G} is defined as the number of q -flows of \mathcal{G}_θ (it doesn't depend on θ).

Flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) q -flow on an oriented graph $\mathcal{G}_\theta = (V, E_\theta)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z}) \setminus \{\bar{0}\}$ such that $\forall v \in V$,
$$\sum_{e^+=v; e \in E_\theta} w(e) - \sum_{e^-=v; e \in E_\theta} w(e) = \bar{0}.$$

The *flow polynomial* $\chi_{\mathcal{G}}^*(q)$ of \mathcal{G} is defined as the number of q -flows of \mathcal{G}_θ (it doesn't depend on θ).

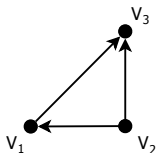
Flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) q -flow on an oriented graph $\mathcal{G}_\theta = (V, E_\theta)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z}) \setminus \{\bar{0}\}$ such that $\forall v \in V$,
$$\sum_{e^+=v; e \in E_\theta} w(e) - \sum_{e^-=v; e \in E_\theta} w(e) = \bar{0}.$$
 The *flow polynomial* $\chi_{\mathcal{G}}^*(q)$ of \mathcal{G} is defined as the number of q -flows of \mathcal{G}_θ (it doesn't depend on θ).

Flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) q -flow on an oriented graph $\mathcal{G}_\theta = (V, E_\theta)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z}) \setminus \{\bar{0}\}$ such that $\forall v \in V$, $\sum_{e^+=v; e \in E_\theta} w(e) - \sum_{e^-=v; e \in E_\theta} w(e) = \bar{0}$.
 The *flow polynomial* $\chi_{\mathcal{G}}^*(q)$ of \mathcal{G} is defined as the number of q -flows of \mathcal{G}_θ (it doesn't depend on θ).

Let $\mathcal{G}_\theta := (V, E_\theta)$, where
 $V := \{v_1, v_2, v_3\}$ and
 $E_\theta := \{(v_1, v_2), (v_3, v_2), (v_3, v_1)\}$.

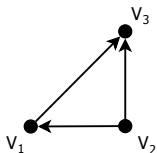


For $q = 4$, $w((v_1, v_2)) = w((v_3, v_1)) = \bar{1}$ and $w((v_3, v_2)) = \bar{3}$ is a 4-flow of \mathcal{G} . While $\chi_{\mathcal{G}}^*(q) = q - 1$.

Flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) q -flow on an oriented graph $\mathcal{G}_\theta = (V, E_\theta)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z}) \setminus \{\bar{0}\}$ such that $\forall v \in V$, $\sum_{e^+=v; e \in E_\theta} w(e) - \sum_{e^-=v; e \in E_\theta} w(e) = \bar{0}$.
 The *flow polynomial* $\chi_{\mathcal{G}}^*(q)$ of \mathcal{G} is defined as the number of q -flows of \mathcal{G}_θ (it doesn't depend on θ).

Let $\mathcal{G}_\theta := (V, E_\theta)$, where
 $V := \{v_1, v_2, v_3\}$ and
 $E_\theta := \{(v_1, v_2), (v_3, v_2), (v_3, v_1)\}$.

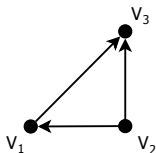


For $q = 4$, $w((v_1, v_2)) = w((v_3, v_1)) = \bar{1}$ and $w((v_3, v_2)) = \bar{3}$ is a 4-flow of \mathcal{G} . While $\chi_{\mathcal{G}}^*(q) = q - 1$.

Flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) q -flow on an oriented graph $\mathcal{G}_\theta = (V, E_\theta)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z}) \setminus \{\bar{0}\}$ such that $\forall v \in V$, $\sum_{e^+=v; e \in E_\theta} w(e) - \sum_{e^-=v; e \in E_\theta} w(e) = \bar{0}$.
 The *flow polynomial* $\chi_{\mathcal{G}}^*(q)$ of \mathcal{G} is defined as the number of q -flows of \mathcal{G}_θ (it doesn't depend on θ).

Let $\mathcal{G}_\theta := (V, E_\theta)$, where
 $V := \{v_1, v_2, v_3\}$ and
 $E_\theta := \{(v_1, v_2), (v_3, v_2), (v_3, v_1)\}$.



For $q = 4$, $w((v_1, v_2)) = w((v_3, v_1)) = \bar{1}$ and $w((v_3, v_2)) = \bar{3}$ is a 4-flow of \mathcal{G} . While $\chi_{\mathcal{G}}^*(q) = q - 1$.

Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \rightarrow \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- 2 if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
- 3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.
 $rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:
 $rk(A) = |\text{maximal independent sublist of } A|$.

Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \rightarrow \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- 2 if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
- 3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.
 $rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:

$rk(A) = |\text{maximal independent sublist of } A|$.

Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \rightarrow \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- 2 if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
- 3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.
 $rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:

$rk(A) = |\text{maximal independent sublist of } A|$.

Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \rightarrow \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- 2 if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
- 3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.
 $rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:

$rk(A) = |\text{maximal independent sublist of } A|$.

Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \rightarrow \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- 2 if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
- 3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.
 $rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:

$rk(A) = |\text{maximal independent sublist of } A|$.

Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \rightarrow \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- 2 if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
- 3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.
 $rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:

$rk(A) = |\text{maximal independent sublist of } A|$.

Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \rightarrow \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- 2 if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
- 3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.
 $rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:

$rk(A) = |\text{maximal independent sublist of } A|$.

Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \rightarrow \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- 2 if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
- 3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.
 $rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:
 $rk(A) = |\text{maximal independent sublist of } A|$.

Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \rightarrow \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- 2 if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
- 3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.
 $rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:
 $rk(A) = |\text{maximal independent sublist of } A|$.

Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \rightarrow \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- 2 if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
- 3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.
 $rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:
 $rk(A) = |\text{maximal independent sublist of } A|$.

Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \rightarrow \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- 2 if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
- 3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.
 $rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:

$rk(A) = |\text{maximal independent sublist of } A|$.

Definition of Matroid

We use the word *list* for *multiset* (repetitions allowed).

A *matroid* $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of *vectors* X with a *rank function* $rk : \mathbb{P}(X) \rightarrow \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- 2 if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;
- 3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is *independent* $\Leftrightarrow rk(A) = |A|$.

An independent sublist of maximal rank $rk(X)$ is called a *basis*.
 $rk(X)$ is called the *rank* of the matroid.

The independent sublists determine the matroid structure:
 $rk(A) = |\text{maximal independent sublist of } A|$.

Examples

- 1 X is a finite list of vectors of a vector space (e.g. \mathbb{R}^n);
 $rk(A) = \dim(\text{span}(A))$;
independent = linearly independent;
- 2 X a finite list of edges of a graph \mathcal{G} ;
 $rk(A) = |\text{maximal subforest of } A|$;
independent = cycle-free (forests).

Examples

- 1 X is a finite list of vectors of a vector space (e.g. \mathbb{R}^n);
 $rk(A) = \dim(\text{span}(A))$;
independent = linearly independent;
- 2 X a finite list of edges of a graph \mathcal{G} ;
 $rk(A) = |\text{maximal subforest of } A|$;
independent = cycle-free (forests).

Examples

- 1 X is a finite list of vectors of a vector space (e.g. \mathbb{R}^n);
 $rk(A) = \dim(\text{span}(A))$;
independent = linearly independent;
- 2 X a finite list of edges of a graph \mathcal{G} ;
 $rk(A) = |\text{maximal subforest of } A|$;
independent = cycle-free (forests).

Examples

- 1 X is a finite list of vectors of a vector space (e.g. \mathbb{R}^n);
 $rk(A) = \dim(\text{span}(A))$;
independent = linearly independent;
- 2 X a finite list of edges of a graph \mathcal{G} ;
 $rk(A) = |\text{maximal subforest of } A|$;
independent = cycle-free (forests).

Examples

- 1 X is a finite list of vectors of a vector space (e.g. \mathbb{R}^n);
 $rk(A) = \dim(\text{span}(A))$;
independent = linearly independent;
- 2 X a finite list of edges of a graph \mathcal{G} ;
 $rk(A) = |\text{maximal subforest of } A|$;
independent = cycle-free (forests).

Examples

- 1 X is a finite list of vectors of a vector space (e.g. \mathbb{R}^n);
 $rk(A) = \dim(\text{span}(A))$;
independent = linearly independent;
- 2 X a finite list of edges of a graph \mathcal{G} ;
 $rk(A) = |\text{maximal subforest of } A|$;
independent = cycle-free (forests).

Examples

- 1 X is a finite list of vectors of a vector space (e.g. \mathbb{R}^n);
 $rk(A) = \dim(\text{span}(A))$;
independent = linearly independent;
- 2 X a finite list of edges of a graph \mathcal{G} ;
 $rk(A) = |\text{maximal subforest of } A|$;
independent = cycle-free (forests).

Tutte Polynomial

The *Tutte polynomial* of the matroid $\mathfrak{M} = (X, rk)$ is defined as

$$T_X(x, y) := \sum_{A \subseteq X} (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}.$$

Theorem (Tutte 1954)

Let $\mathcal{G} = (V, E)$ with k connected components, and let $T_{\mathcal{G}}(x, y)$ the Tutte polynomial of \mathcal{G} . We have

- 1 $\chi_{\mathcal{G}}(q) = (-1)^{|V| - k} q^k T_{\mathcal{G}}(1 - q, 0);$
- 2 $\chi_{\mathcal{G}}^*(q) = (-1)^{|E| - |V| + k} T_{\mathcal{G}}(0, 1 - q).$

Tutte Polynomial

The *Tutte polynomial* of the matroid $\mathfrak{M} = (X, rk)$ is defined as

$$T_X(x, y) := \sum_{A \subseteq X} (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}.$$

Theorem (Tutte 1954)

Let $\mathcal{G} = (V, E)$ with k connected components, and let $T_{\mathcal{G}}(x, y)$ the Tutte polynomial of \mathcal{G} . We have

- 1 $\chi_{\mathcal{G}}(q) = (-1)^{|V| - k} q^k T_{\mathcal{G}}(1 - q, 0);$
- 2 $\chi_{\mathcal{G}}^*(q) = (-1)^{|E| - |V| + k} T_{\mathcal{G}}(0, 1 - q).$

Tutte Polynomial

The *Tutte polynomial* of the matroid $\mathfrak{M} = (X, rk)$ is defined as

$$T_X(x, y) := \sum_{A \subseteq X} (x - 1)^{rk(X) - rk(A)} (y - 1)^{|A| - rk(A)}.$$

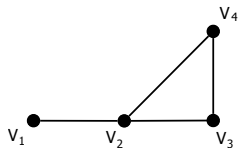
Theorem (Tutte 1954)

Let $\mathcal{G} = (V, E)$ with k connected components, and let $T_{\mathcal{G}}(x, y)$ the Tutte polynomial of \mathcal{G} . We have

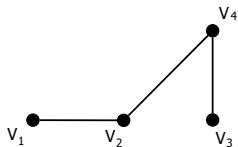
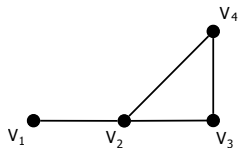
- 1 $\chi_{\mathcal{G}}(q) = (-1)^{|V| - k} q^k T_{\mathcal{G}}(1 - q, 0);$
- 2 $\chi_{\mathcal{G}}^*(q) = (-1)^{|E| - |V| + k} T_{\mathcal{G}}(0, 1 - q).$

Deletion Contraction I

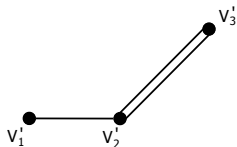
Deletion Contraction I



Deletion Contraction I



Classical deletion of
 $\{v_2, v_3\}$.



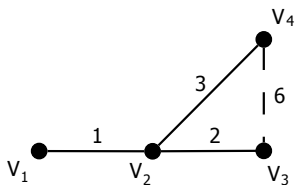
Classical contraction of
 $\{v_2, v_3\}$.

Labelled graphs

Let (\mathcal{G}, ℓ) , where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$,
 $R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$ the *regular edges*,
 $D := \{\{v_3, v_4\}\}$ the *dotted edges*, so that
 $E = R \cup D = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}$;
let $\ell(\{v_1, v_2\}) = 1$, $\ell(\{v_2, v_3\}) = 2$, $\ell(\{v_2, v_4\}) = 3$,
 $\ell(\{v_3, v_4\}) = 6$ be the *labels*.

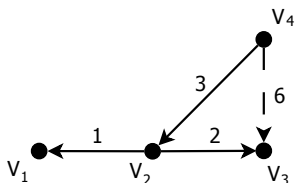
Labelled graphs

Let (\mathcal{G}, ℓ) , where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$,
 $R := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$ the *regular edges*,
 $D := \{\{v_3, v_4\}\}$ the *dotted edges*, so that
 $E = R \cup D = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}$;
 let $\ell(\{v_1, v_2\}) = 1$, $\ell(\{v_2, v_3\}) = 2$, $\ell(\{v_2, v_4\}) = 3$,
 $\ell(\{v_3, v_4\}) = 6$ be the *labels*.



Oriented labelled graphs

Let (\mathcal{G}, ℓ) , where $\mathcal{G} := (V, E)$, $V := \{v_1, v_2, v_3, v_4\}$,
 $R := \{(v_1, v_2), (v_3, v_2), (v_2, v_4)\}$ the *regular edges*,
 $D := \{(v_3, v_4)\}$ the *dotted edges*, so that
 $E = R \cup D = \{(v_1, v_2), (v_3, v_2), (v_2, v_4), (v_3, v_4)\}$;
 let $\ell((v_1, v_2)) = 1$, $\ell((v_3, v_2)) = 2$, $\ell((v_2, v_4)) = 3$,
 $\ell((v_3, v_4)) = 6$ be the *labels*.



Arithmetic colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) *arithmetic q -coloring* of a labelled graph (\mathcal{G}, ℓ) is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

- (1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
- (2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The *arithmetic chromatic polynomial* $\chi_{\mathcal{G}, \ell}(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -colorings of (\mathcal{G}, ℓ) .

Arithmetic colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) *arithmetic q -coloring* of a labelled graph (\mathcal{G}, ℓ) is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

- (1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
- (2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The *arithmetic chromatic polynomial* $\chi_{\mathcal{G}, \ell}(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -colorings of (\mathcal{G}, ℓ) .

Arithmetic colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) *arithmetic q -coloring* of a labelled graph (\mathcal{G}, ℓ) is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

- (1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
- (2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The *arithmetic chromatic polynomial* $\chi_{\mathcal{G}, \ell}(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -colorings of (\mathcal{G}, ℓ) .

Arithmetic colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) *arithmetic q -coloring* of a labelled graph (\mathcal{G}, ℓ) is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

- (1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
- (2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The *arithmetic chromatic polynomial* $\chi_{\mathcal{G}, \ell}(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -colorings of (\mathcal{G}, ℓ) .

Arithmetic colorings

For $q \in \mathbb{N} \setminus \{0\}$, a *(proper) arithmetic q -coloring* of a labelled graph (\mathcal{G}, ℓ) is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

- (1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
- (2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The *arithmetic chromatic polynomial* $\chi_{\mathcal{G}, \ell}(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -colorings of (\mathcal{G}, ℓ) .

Arithmetic colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) *arithmetic q -coloring* of a labelled graph (\mathcal{G}, ℓ) is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

- (1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
- (2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The *arithmetic chromatic polynomial* $\chi_{\mathcal{G}, \ell}(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -colorings of (\mathcal{G}, ℓ) .

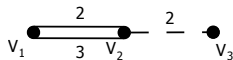
Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$$V = \{v_1, v_2, v_3\},$$

$$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\},$$

$$D := \{e_3 := \{v_2, v_3\}\} \text{ so } E = R \cup D,$$

$$\ell(e_1) = 2, \ell(e_2) = 3 \text{ and } \ell(e_3) = 2.$$



For $q = 6$, $c(v_1) = \bar{0}$ and $c(v_2) = c(v_3) = \bar{1}$ is an arithmetic 4-coloring of \mathcal{G} . For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q - 4) = 2q^2 - 8q$.

Arithmetic colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) *arithmetic q -coloring* of a labelled graph (\mathcal{G}, ℓ) is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

- (1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
- (2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The *arithmetic chromatic polynomial* $\chi_{\mathcal{G}, \ell}(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -colorings of (\mathcal{G}, ℓ) .

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$$V = \{v_1, v_2, v_3\},$$

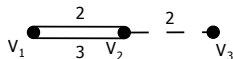
$$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\},$$

$$D := \{e_3 := \{v_2, v_3\}\} \text{ so } E = R \cup D,$$

$$\ell(e_1) = 2, \ell(e_2) = 3 \text{ and } \ell(e_3) = 2.$$

For $q = 6$, $c(v_1) = \bar{0}$ and $c(v_2) = c(v_3) = \bar{1}$ is an arithmetic

4-coloring of \mathcal{G} . For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q - 4) = 2q^2 - 8q$.



Arithmetic colorings

For $q \in \mathbb{N} \setminus \{0\}$, a (*proper*) *arithmetic q -coloring* of a labelled graph (\mathcal{G}, ℓ) is a map $c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

- (1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;
- (2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

The *arithmetic chromatic polynomial* $\chi_{\mathcal{G}, \ell}(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -colorings of (\mathcal{G}, ℓ) .

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

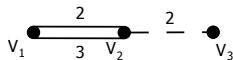
$$V = \{v_1, v_2, v_3\},$$

$$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\},$$

$$D := \{e_3 := \{v_2, v_3\}\} \text{ so } E = R \cup D,$$

$$\ell(e_1) = 2, \ell(e_2) = 3 \text{ and } \ell(e_3) = 2.$$

For $q = 6$, $c(v_1) = \bar{0}$ and $c(v_2) = c(v_3) = \bar{1}$ is an arithmetic 4-coloring of \mathcal{G} . For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q - 4) = 2q^2 - 8q$.



Arithmetic colorings

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$V = \{v_1, v_2, v_3\}$,

$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\}$,

$D := \{e_3 := \{v_2, v_3\}\}$ so $E = R \cup D$,

$\ell(e_1) = 2$, $\ell(e_2) = 3$ and $\ell(e_3) = 2$.

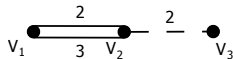
For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q - 4) = 2q^2 - 8q$.

For $c(v_2)$: q choices.

$2c(v_3) = 2c(v_2) \Rightarrow c(v_3) = c(v_2)$ or $c(v_2) + \overline{q/2}$: 2 choices.

$2c(v_1) \neq 2c(v_2)$ and $3c(v_1) \neq 3c(v_2) \Rightarrow c(v_1) \neq c(v_2)$,

$c(v_2) + \overline{q/2}$, $c(v_2) + \overline{q/3}$, $c(v_2) + \overline{2q/3}$: $q - 4$ choices.



Arithmetic colorings

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$V = \{v_1, v_2, v_3\}$,

$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\}$,

$D := \{e_3 := \{v_2, v_3\}\}$ so $E = R \cup D$,

$\ell(e_1) = 2$, $\ell(e_2) = 3$ and $\ell(e_3) = 2$.

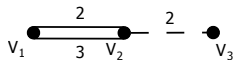
For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q - 4) = 2q^2 - 8q$.

For $c(v_2)$: q choices.

$2c(v_3) = 2c(v_2) \Rightarrow c(v_3) = c(v_2)$ or $c(v_2) + \overline{q/2}$: 2 choices.

$2c(v_1) \neq 2c(v_2)$ and $3c(v_1) \neq 3c(v_2) \Rightarrow c(v_1) \neq c(v_2)$,

$c(v_2) + \overline{q/2}$, $c(v_2) + \overline{q/3}$, $c(v_2) + \overline{2q/3}$: $q - 4$ choices.



Arithmetic colorings

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$V = \{v_1, v_2, v_3\}$,

$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\}$,

$D := \{e_3 := \{v_2, v_3\}\}$ so $E = R \cup D$,

$\ell(e_1) = 2$, $\ell(e_2) = 3$ and $\ell(e_3) = 2$.

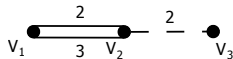
For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q - 4) = 2q^2 - 8q$.

For $c(v_2)$: q choices.

$2c(v_3) = 2c(v_2) \Rightarrow c(v_3) = c(v_2)$ or $c(v_2) + \overline{q/2}$: 2 choices.

$2c(v_1) \neq 2c(v_2)$ and $3c(v_1) \neq 3c(v_2) \Rightarrow c(v_1) \neq c(v_2)$,

$c(v_2) + \overline{q/2}$, $c(v_2) + \overline{q/3}$, $c(v_2) + \overline{2q/3}$: $q - 4$ choices.



Arithmetic colorings

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$V = \{v_1, v_2, v_3\}$,

$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\}$,

$D := \{e_3 := \{v_2, v_3\}\}$ so $E = R \cup D$,

$\ell(e_1) = 2$, $\ell(e_2) = 3$ and $\ell(e_3) = 2$.

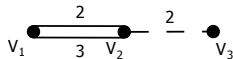
For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q-4) = 2q^2 - 8q$.

For $c(v_2)$: q choices.

$2c(v_3) = 2c(v_2) \Rightarrow c(v_3) = c(v_2)$ or $c(v_2) + \overline{q/2}$: 2 choices.

$2c(v_1) \neq 2c(v_2)$ and $3c(v_1) \neq 3c(v_2) \Rightarrow c(v_1) \neq c(v_2)$,

$c(v_2) + \overline{q/2}$, $c(v_2) + \overline{q/3}$, $c(v_2) + \overline{2q/3}$: $q-4$ choices.



Arithmetic colorings

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$V = \{v_1, v_2, v_3\}$,

$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\}$,

$D := \{e_3 := \{v_2, v_3\}\}$ so $E = R \cup D$,

$\ell(e_1) = 2$, $\ell(e_2) = 3$ and $\ell(e_3) = 2$.

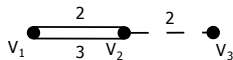
For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q - 4) = 2q^2 - 8q$.

For $c(v_2)$: q choices.

$2c(v_3) = 2c(v_2) \Rightarrow c(v_3) = c(v_2)$ or $c(v_2) + \overline{q/2}$: 2 choices.

$2c(v_1) \neq 2c(v_2)$ and $3c(v_1) \neq 3c(v_2) \Rightarrow c(v_1) \neq c(v_2)$,

$c(v_2) + \overline{q/2}$, $c(v_2) + \overline{q/3}$, $c(v_2) + \overline{2q/3}$: $q - 4$ choices.



Arithmetic colorings

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$V = \{v_1, v_2, v_3\}$,

$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\}$,

$D := \{e_3 := \{v_2, v_3\}\}$ so $E = R \cup D$,

$\ell(e_1) = 2$, $\ell(e_2) = 3$ and $\ell(e_3) = 2$.

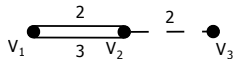
For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q-4) = 2q^2 - 8q$.

For $c(v_2)$: q choices.

$2c(v_3) = 2c(v_2) \Rightarrow c(v_3) = c(v_2)$ or $c(v_2) + \overline{q/2}$: 2 choices.

$2c(v_1) \neq 2c(v_2)$ and $3c(v_1) \neq 3c(v_2) \Rightarrow c(v_1) \neq c(v_2)$,

$c(v_2) + \overline{q/2}$, $c(v_2) + \overline{q/3}$, $c(v_2) + \overline{2q/3}$: $q-4$ choices.



Arithmetic colorings

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$V = \{v_1, v_2, v_3\}$,

$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\}$,

$D := \{e_3 := \{v_2, v_3\}\}$ so $E = R \cup D$,

$\ell(e_1) = 2$, $\ell(e_2) = 3$ and $\ell(e_3) = 2$.

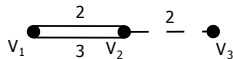
For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q - 4) = 2q^2 - 8q$.

For $c(v_2)$: q choices.

$2c(v_3) = 2c(v_2) \Rightarrow c(v_3) = c(v_2)$ or $c(v_2) + \overline{q/2}$: 2 choices.

$2c(v_1) \neq 2c(v_2)$ and $3c(v_1) \neq 3c(v_2) \Rightarrow c(v_1) \neq c(v_2)$,

$c(v_2) + \overline{q/2}$, $c(v_2) + \overline{q/3}$, $c(v_2) + \overline{2q/3}$: $q - 4$ choices.



Arithmetic colorings

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$V = \{v_1, v_2, v_3\}$,

$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\}$,

$D := \{e_3 := \{v_2, v_3\}\}$ so $E = R \cup D$,

$\ell(e_1) = 2$, $\ell(e_2) = 3$ and $\ell(e_3) = 2$.

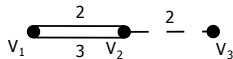
For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q - 4) = 2q^2 - 8q$.

For $c(v_2)$: q choices.

$2c(v_3) = 2c(v_2) \Rightarrow c(v_3) = c(v_2)$ or $c(v_2) + \overline{q/2}$: 2 choices.

$2c(v_1) \neq 2c(v_2)$ and $3c(v_1) \neq 3c(v_2) \Rightarrow c(v_1) \neq c(v_2)$,

$c(v_2) + \overline{q/2}$, $c(v_2) + \overline{q/3}$, $c(v_2) + \overline{2q/3}$: $q - 4$ choices.



Arithmetic colorings

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$V = \{v_1, v_2, v_3\}$,

$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\}$,

$D := \{e_3 := \{v_2, v_3\}\}$ so $E = R \cup D$,

$\ell(e_1) = 2$, $\ell(e_2) = 3$ and $\ell(e_3) = 2$.

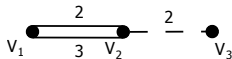
For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q-4) = 2q^2 - 8q$.

For $c(v_2)$: q choices.

$2c(v_3) = 2c(v_2) \Rightarrow c(v_3) = c(v_2)$ or $c(v_2) + \overline{q/2}$: 2 choices.

$2c(v_1) \neq 2c(v_2)$ and $3c(v_1) \neq 3c(v_2) \Rightarrow c(v_1) \neq c(v_2)$,

$c(v_2) + \overline{q/2}$, $c(v_2) + \overline{q/3}$, $c(v_2) + \overline{2q/3}$: $q-4$ choices.



Arithmetic colorings

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$V = \{v_1, v_2, v_3\}$,

$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\}$,

$D := \{e_3 := \{v_2, v_3\}\}$ so $E = R \cup D$,

$\ell(e_1) = 2$, $\ell(e_2) = 3$ and $\ell(e_3) = 2$.

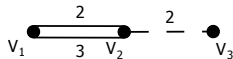
For $q = 6h$, $\chi_{\mathcal{G}, \ell}(q) = 2q(q - 4) = 2q^2 - 8q$.

For $c(v_2)$: q choices.

$2c(v_3) = 2c(v_2) \Rightarrow c(v_3) = c(v_2)$ or $c(v_2) + \overline{q/2}$: 2 choices.

$2c(v_1) \neq 2c(v_2)$ and $3c(v_1) \neq 3c(v_2) \Rightarrow c(v_1) \neq c(v_2)$,

$c(v_2) + \overline{q/2}$, $c(v_2) + \overline{q/3}$, $c(v_2) + \overline{2q/3}$: $q - 4$ choices.



Arithmetic colorings

$c : V \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that:

(1) if $e := \{u, v\} \in R$, then $\ell(e) \cdot c(u) \neq \ell(e) \cdot c(v)$;

(2) if $e := \{u, v\} \in D$, then $\ell(e) \cdot c(u) = \ell(e) \cdot c(v)$.

Consider (\mathcal{G}, ℓ) with $\mathcal{G} := (V, E)$,

$$V = \{v_1, v_2, v_3\},$$

$$R := \{e_1 := \{v_1, v_2\}, e_2 := \{v_1, v_2\}\},$$

$$D := \{e_3 := \{v_2, v_3\}\} \text{ so } E = R \cup D,$$

$$\ell(e_1) = 2, \ell(e_2) = 3 \text{ and } \ell(e_3) = 2.$$

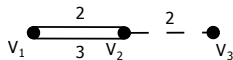
$$\text{For } q = 6h, \chi_{\mathcal{G}, \ell}(q) = 2q(q - 4) = 2q^2 - 8q.$$

For $c(v_2)$: q choices.

$$2c(v_3) = 2c(v_2) \Rightarrow c(v_3) = c(v_2) \text{ or } c(v_2) + \overline{q/2}: \text{ 2 choices.}$$

$$2c(v_1) \neq 2c(v_2) \text{ and } 3c(v_1) \neq 3c(v_2) \Rightarrow c(v_1) \neq c(v_2),$$

$$c(v_2) + \overline{q/2}, c(v_2) + \overline{q/3}, c(v_2) + \overline{2q/3}: \text{ } q - 4 \text{ choices.}$$



Arithmetic flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) *arithmetic q -flow* on an oriented labelled graph $(\mathcal{G}_\theta, \ell)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

(1) for all $v \in V$,

$$\sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0} \in \mathbb{Z}/q\mathbb{Z};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi_{\mathcal{G}, \ell}^*(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -flows of $(\mathcal{G}_\theta, \ell)$ (it doesn't depend on θ).

Arithmetic flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) *arithmetic q -flow* on an oriented labelled graph $(\mathcal{G}_\theta, \ell)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

(1) for all $v \in V$,

$$\sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0} \in \mathbb{Z}/q\mathbb{Z};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi_{\mathcal{G}, \ell}^*(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -flows of $(\mathcal{G}_\theta, \ell)$ (it doesn't depend on θ).

Arithmetic flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) *arithmetic q -flow* on an oriented labelled graph $(\mathcal{G}_\theta, \ell)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

(1) for all $v \in V$,

$$\sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0} \in \mathbb{Z}/q\mathbb{Z};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi_{\mathcal{G}, \ell}^*(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -flows of $(\mathcal{G}_\theta, \ell)$ (it doesn't depend on θ).

Arithmetic flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) *arithmetic q -flow* on an oriented labelled graph $(\mathcal{G}_\theta, \ell)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

(1) for all $v \in V$,

$$\sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0} \in \mathbb{Z}/q\mathbb{Z};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi_{\mathcal{G}, \ell}^*(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -flows of $(\mathcal{G}_\theta, \ell)$ (it doesn't depend on θ).

Arithmetic flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) *arithmetic q -flow* on an oriented labelled graph $(\mathcal{G}_\theta, \ell)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

(1) for all $v \in V$,

$$\sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0} \in \mathbb{Z}/q\mathbb{Z};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi_{\mathcal{G}, \ell}^*(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -flows of $(\mathcal{G}_\theta, \ell)$ (it doesn't depend on θ).

Arithmetic flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) *arithmetic q -flow* on an oriented labelled graph $(\mathcal{G}_\theta, \ell)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

(1) for all $v \in V$,

$$\sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0} \in \mathbb{Z}/q\mathbb{Z};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi_{\mathcal{G}, \ell}^*(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -flows of $(\mathcal{G}_\theta, \ell)$ (it doesn't depend on θ).

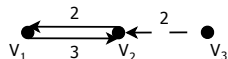
Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$$V = \{v_1, v_2, v_3\}, R_\theta := \{e_1 := (v_1, v_2),$$

$$e_2 := (v_2, v_1)\}, D_\theta := \{e_3 := (v_2, v_3)\}$$

$$\text{so } E_\theta = R_\theta \cup D_\theta, \ell(e_1) = 2, \ell(e_2) = 3$$

$$\text{and } \ell(e_3) = 2.$$



For $q = 6$, $w(e_1) = \bar{3}$, $w(e_2) = \bar{2}$ and $w(e_3) = \bar{0}$ is an arithmetic

6-flow of \mathcal{G} . For $q = 6h$, $\chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8$.

Arithmetic flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) *arithmetic q -flow* on an oriented labelled graph $(\mathcal{G}_\theta, \ell)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

(1) for all $v \in V$,

$$\sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0} \in \mathbb{Z}/q\mathbb{Z};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi_{\mathcal{G}, \ell}^*(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -flows of $(\mathcal{G}_\theta, \ell)$ (it doesn't depend on θ).

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$$V = \{v_1, v_2, v_3\}, R_\theta := \{e_1 := (v_1, v_2),$$

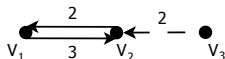
$$e_2 := (v_2, v_1)\}, D_\theta := \{e_3 := (v_2, v_3)\}$$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

For $q = 6$, $w(e_1) = \bar{3}$, $w(e_2) = \bar{2}$ and $w(e_3) = \bar{0}$ is an arithmetic

6-flow of \mathcal{G} . For $q = 6h$, $\chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8$.



Arithmetic flows

For $q \in \mathbb{N} \setminus \{0\}$, a (*nowhere zero*) *arithmetic q -flow* on an oriented labelled graph $(\mathcal{G}_\theta, \ell)$ is a map $w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

(1) for all $v \in V$,

$$\sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0} \in \mathbb{Z}/q\mathbb{Z};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

The *arithmetic flow polynomial* $\chi_{\mathcal{G}, \ell}^*(q)$ of (\mathcal{G}, ℓ) is defined as the number of arithmetic q -flows of $(\mathcal{G}_\theta, \ell)$ (it doesn't depend on θ).

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$$V = \{v_1, v_2, v_3\}, R_\theta := \{e_1 := (v_1, v_2),$$

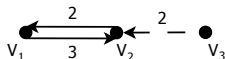
$$e_2 := (v_2, v_1)\}, D_\theta := \{e_3 := (v_2, v_3)\}$$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

For $q = 6$, $w(e_1) = \bar{3}$, $w(e_2) = \bar{2}$ and $w(e_3) = \bar{0}$ is an arithmetic

6-flow of \mathcal{G} . For $q = 6h$, $\chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8$.



Arithmetic flows

$w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$V = \{v_1, v_2, v_3\}$, $R_\theta := \{e_1 := (v_1, v_2)$,

$e_2 := (v_2, v_1)\}$, $D_\theta := \{e_3 := (v_2, v_3)\}$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

For $q = 6h$, $\chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8$.

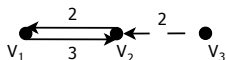
$-2w(e_3) = \bar{0} \Rightarrow w(e_3) = \bar{0}$ or $\overline{q/2}$: 2 choices.

$2w(e_1) - 3w(e_2) = \bar{0} : w(e_2) = 2\bar{a}$ ($q/2$ choices) $\Rightarrow w(e_1) = 3\bar{a}$

or $3\bar{a} + \overline{q/2}$ (2 choices), but $w(e_1) \neq \bar{0} \neq w(e_2) \Rightarrow$

$(w(e_1), w(e_2)) \neq (\bar{0}, \bar{0}), (\bar{0}, \overline{q/3}), (\bar{0}, \overline{2q/3}), (\overline{q/2}, \bar{0})$: $q - 4$

choices.



Arithmetic flows

$w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$V = \{v_1, v_2, v_3\}$, $R_\theta := \{e_1 := (v_1, v_2)$,

$e_2 := (v_2, v_1)\}$, $D_\theta := \{e_3 := (v_2, v_3)\}$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

For $q = 6h$, $\chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8$.

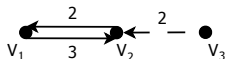
$-2w(e_3) = \bar{0} \Rightarrow w(e_3) = \bar{0}$ or $\overline{q/2}$: 2 choices.

$2w(e_1) - 3w(e_2) = \bar{0} : w(e_2) = 2\bar{a}$ ($q/2$ choices) $\Rightarrow w(e_1) = 3\bar{a}$

or $3\bar{a} + \overline{q/2}$ (2 choices), but $w(e_1) \neq \bar{0} \neq w(e_2) \Rightarrow$

$(w(e_1), w(e_2)) \neq (\bar{0}, \bar{0}), (\bar{0}, \overline{q/3}), (\bar{0}, \overline{2q/3}), (\overline{q/2}, \bar{0})$: $q - 4$

choices.



Arithmetic flows

$w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{e^+=v} \ell(e) \cdot w(e) - \sum_{e^-=v} \ell(e) \cdot w(e) = \bar{0};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$V = \{v_1, v_2, v_3\}$, $R_\theta := \{e_1 := (v_1, v_2)$,

$e_2 := (v_2, v_1)\}$, $D_\theta := \{e_3 := (v_2, v_3)\}$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

For $q = 6h$, $\chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8$.

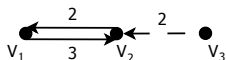
$-2w(e_3) = \bar{0} \Rightarrow w(e_3) = \bar{0}$ or $\overline{q/2}$: 2 choices.

$2w(e_1) - 3w(e_2) = \bar{0} : w(e_2) = 2\bar{a}$ ($q/2$ choices) $\Rightarrow w(e_1) = 3\bar{a}$

or $3\bar{a} + \overline{q/2}$ (2 choices), but $w(e_1) \neq \bar{0} \neq w(e_2) \Rightarrow$

$(w(e_1), w(e_2)) \neq (\bar{0}, \bar{0}), (\bar{0}, \overline{q/3}), (\bar{0}, \overline{2q/3}), (\overline{q/2}, \bar{0})$: $q - 4$

choices.



Arithmetic flows

$w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{e^+=ve \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=ve \in E_\theta} \ell(e) \cdot w(e) = \bar{0};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$V = \{v_1, v_2, v_3\}$, $R_\theta := \{e_1 := (v_1, v_2)$,

$e_2 := (v_2, v_1)\}$, $D_\theta := \{e_3 := (v_2, v_3)\}$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

For $q = 6h$, $\chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8$.

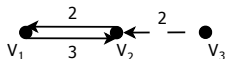
$-2w(e_3) = \bar{0} \Rightarrow w(e_3) = \bar{0}$ or $\overline{q/2}$: 2 choices.

$2w(e_1) - 3w(e_2) = \bar{0} : w(e_2) = 2\bar{a}$ ($q/2$ choices) $\Rightarrow w(e_1) = 3\bar{a}$

or $3\bar{a} + \overline{q/2}$ (2 choices), but $w(e_1) \neq \bar{0} \neq w(e_2) \Rightarrow$

$(w(e_1), w(e_2)) \neq (\bar{0}, \bar{0}), (\bar{0}, \overline{q/3}), (\bar{0}, \overline{2q/3}), (\overline{q/2}, \bar{0})$: $q - 4$

choices.



Arithmetic flows

$w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0};$$

$$(2) \text{ for all } e \in R_\theta, w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}.$$

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$$V = \{v_1, v_2, v_3\}, R_\theta := \{e_1 := (v_1, v_2),$$

$$e_2 := (v_2, v_1)\}, D_\theta := \{e_3 := (v_2, v_3)\}$$

$$\text{so } E_\theta = R_\theta \cup D_\theta, \ell(e_1) = 2, \ell(e_2) = 3$$

$$\text{and } \ell(e_3) = 2.$$

$$\text{For } q = 6h, \chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8.$$

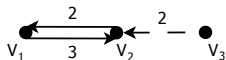
$$-2w(e_3) = \bar{0} \Rightarrow w(e_3) = \bar{0} \text{ or } \overline{q/2}: \text{ 2 choices.}$$

$$2w(e_1) - 3w(e_2) = \bar{0} \text{ and } 2w(e_3) + 3w(e_2) - 2w(e_1) = \bar{0} :$$

$$w(e_2) = 2\bar{a} \text{ (} q/2 \text{ choices)} \Rightarrow w(e_1) = 3\bar{a} \text{ or } 3\bar{a} + \overline{q/2} \text{ (2 choices),}$$

$$\text{but } w(e_1) \neq \bar{0} \neq w(e_2) \Rightarrow (w(e_1), w(e_2)) \neq (\bar{0}, \bar{0}), (\bar{0}, \overline{q/3}),$$

$$(\bar{0}, \overline{2q/3}), (\overline{q/2}, \bar{0}): q - 4 \text{ choices.}$$



Arithmetic flows

$w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$V = \{v_1, v_2, v_3\}$, $R_\theta := \{e_1 := (v_1, v_2)$,

$e_2 := (v_2, v_1)\}$, $D_\theta := \{e_3 := (v_2, v_3)\}$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

For $q = 6h$, $\chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8$.

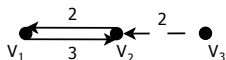
$-2w(e_3) = \bar{0} \Rightarrow w(e_3) = \bar{0}$ or $\overline{q/2}$: 2 choices.

$2w(e_1) - 3w(e_2) = \bar{0} : w(e_2) = 2\bar{a}$ ($q/2$ choices) $\Rightarrow w(e_1) = 3\bar{a}$

or $3\bar{a} + \overline{q/2}$ (2 choices), but $w(e_1) \neq \bar{0} \neq w(e_2) \Rightarrow$

$(w(e_1), w(e_2)) \neq (\bar{0}, \bar{0}), (\bar{0}, \overline{q/3}), (\bar{0}, \overline{2q/3}), (\overline{q/2}, \bar{0})$: $q - 4$

choices.



Arithmetic flows

$w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{e^+=ve \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=ve \in E_\theta} \ell(e) \cdot w(e) = \bar{0};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$V = \{v_1, v_2, v_3\}$, $R_\theta := \{e_1 := (v_1, v_2)$,

$e_2 := (v_2, v_1)\}$, $D_\theta := \{e_3 := (v_2, v_3)\}$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

For $q = 6h$, $\chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8$.

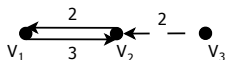
$-2w(e_3) = \bar{0} \Rightarrow w(e_3) = \bar{0}$ or $\overline{q/2}$: 2 choices.

$2w(e_1) - 3w(e_2) = \bar{0} : w(e_2) = 2\bar{a}$ ($q/2$ choices) $\Rightarrow w(e_1) = 3\bar{a}$

or $3\bar{a} + \overline{q/2}$ (2 choices), but $w(e_1) \neq \bar{0} \neq w(e_2) \Rightarrow$

$(w(e_1), w(e_2)) \neq (\bar{0}, \bar{0}), (\bar{0}, \overline{q/3}), (\bar{0}, \overline{2q/3}), (\overline{q/2}, \bar{0})$: $q - 4$

choices.



Arithmetic flows

$w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$V = \{v_1, v_2, v_3\}$, $R_\theta := \{e_1 := (v_1, v_2)$,

$e_2 := (v_2, v_1)\}$, $D_\theta := \{e_3 := (v_2, v_3)\}$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

For $q = 6h$, $\chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8$.

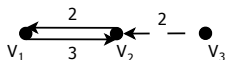
$-2w(e_3) = \bar{0} \Rightarrow w(e_3) = \bar{0}$ or $\overline{q/2}$: 2 choices.

$2w(e_1) - 3w(e_2) = \bar{0} : w(e_2) = 2\bar{a}$ ($q/2$ choices) $\Rightarrow w(e_1) = 3\bar{a}$

or $3\bar{a} + \overline{q/2}$ (2 choices), but $w(e_1) \neq \bar{0} \neq w(e_2) \Rightarrow$

$(w(e_1), w(e_2)) \neq (\bar{0}, \bar{0}), (\bar{0}, \overline{q/3}), (\bar{0}, \overline{2q/3}), (\overline{q/2}, \bar{0})$: $q - 4$

choices.



Arithmetic flows

$w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{e^+=v, e \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=v, e \in E_\theta} \ell(e) \cdot w(e) = \bar{0};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$V = \{v_1, v_2, v_3\}$, $R_\theta := \{e_1 := (v_1, v_2)$,

$e_2 := (v_2, v_1)\}$, $D_\theta := \{e_3 := (v_2, v_3)\}$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

For $q = 6h$, $\chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8$.

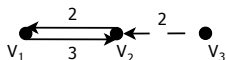
$-2w(e_3) = \bar{0} \Rightarrow w(e_3) = \bar{0}$ or $\overline{q/2}$: 2 choices.

$2w(e_1) - 3w(e_2) = \bar{0} : w(e_2) = 2\bar{a}$ ($q/2$ choices) $\Rightarrow w(e_1) = 3\bar{a}$

or $3\bar{a} + \overline{q/2}$ (2 choices), but $w(e_1) \neq \bar{0} \neq w(e_2) \Rightarrow$

$(w(e_1), w(e_2)) \neq (\bar{0}, \bar{0}), (\bar{0}, \overline{q/3}), (\bar{0}, \overline{2q/3}), (\overline{q/2}, \bar{0})$: $q - 4$

choices.



Arithmetic flows

$w : E_\theta \rightarrow (\mathbb{Z}/q\mathbb{Z})$ such that:

$$(1) \forall v \in V, \sum_{e^+=ve \in E_\theta} \ell(e) \cdot w(e) - \sum_{e^-=ve \in E_\theta} \ell(e) \cdot w(e) = \bar{0};$$

(2) for all $e \in R_\theta$, $w(e) \neq \bar{0} \in \mathbb{Z}/q\mathbb{Z}$.

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$V = \{v_1, v_2, v_3\}$, $R_\theta := \{e_1 := (v_1, v_2)$,

$e_2 := (v_2, v_1)\}$, $D_\theta := \{e_3 := (v_2, v_3)\}$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

For $q = 6h$, $\chi_{\mathcal{G}, \ell}^*(q) = 2(q - 4) = 2q - 8$.

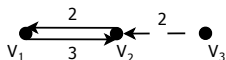
$-2w(e_3) = \bar{0} \Rightarrow w(e_3) = \bar{0}$ or $\overline{q/2}$: 2 choices.

$2w(e_1) - 3w(e_2) = \bar{0} : w(e_2) = 2\bar{a}$ ($q/2$ choices) $\Rightarrow w(e_1) = 3\bar{a}$

or $3\bar{a} + \overline{q/2}$ (2 choices), but $w(e_1) \neq \bar{0} \neq w(e_2) \Rightarrow$

$(w(e_1), w(e_2)) \neq (\bar{0}, \bar{0}), (\bar{0}, \overline{q/3}), (\bar{0}, \overline{2q/3}), (\overline{q/2}, \bar{0})$: $q - 4$

choices.



Definition of Arithmetic Matroid

An *arithmetic matroid* is a pair (\mathfrak{M}_X, m) , where \mathfrak{M}_X is a matroid on a list of vectors X , and m is a *multiplicity function*, i.e.

$m : \mathbb{P}(X) \rightarrow \mathbb{N} \setminus \{0\}$ has the following properties:

- 1 if $A \subseteq X$ and $v \in X$ is dependent on A , then $m(A \cup \{v\})$ divides $m(A)$;
- 2 if $A \subseteq X$ and $v \in X$ is independent on A , then $m(A)$ divides $m(A \cup \{v\})$;
- 3 if $A \subseteq B \subseteq X$ and B is a disjoint union $B = A \cup F \cup T$ such that for all $A \subseteq C \subseteq B$ we have $rk(C) = rk(A) + |C \cap F|$, then $m(A) \cdot m(B) = m(A \cup F) \cdot m(A \cup T)$.
- 4 if $A \subseteq B \subseteq X$ and $rk(A) = rk(B)$, then $\mu_B(A) := \sum_{A \subseteq T \subseteq B} (-1)^{|T|-|A|} m(T) \geq 0$;
- 5 if $A \subseteq B \subseteq X$ and $rk^*(A) = rk^*(B)$, then $\mu_B^*(A) := \sum_{A \subseteq T \subseteq B} (-1)^{|T|-|A|} m(X \setminus T) \geq 0$.

Definition of Arithmetic Matroid

An *arithmetic matroid* is a pair (\mathfrak{M}_X, m) , where \mathfrak{M}_X is a matroid on a list of vectors X , and m is a *multiplicity function*, i.e.

$m : \mathbb{P}(X) \rightarrow \mathbb{N} \setminus \{0\}$ has the following properties:

- 1 if $A \subseteq X$ and $v \in X$ is dependent on A , then $m(A \cup \{v\})$ divides $m(A)$;
- 2 if $A \subseteq X$ and $v \in X$ is independent on A , then $m(A)$ divides $m(A \cup \{v\})$;
- 3 if $A \subseteq B \subseteq X$ and B is a disjoint union $B = A \cup F \cup T$ such that for all $A \subseteq C \subseteq B$ we have $rk(C) = rk(A) + |C \cap F|$, then $m(A) \cdot m(B) = m(A \cup F) \cdot m(A \cup T)$.
- 4 if $A \subseteq B \subseteq X$ and $rk(A) = rk(B)$, then $\mu_B(A) := \sum_{A \subseteq T \subseteq B} (-1)^{|T| - |A|} m(T) \geq 0$;
- 5 if $A \subseteq B \subseteq X$ and $rk^*(A) = rk^*(B)$, then $\mu_B^*(A) := \sum_{A \subseteq T \subseteq B} (-1)^{|T| - |A|} m(X \setminus T) \geq 0$.

Definition of Arithmetic Matroid

An *arithmetic matroid* is a pair (\mathfrak{M}_X, m) , where \mathfrak{M}_X is a matroid on a list of vectors X , and m is a *multiplicity function*, i.e.

$m : \mathbb{P}(X) \rightarrow \mathbb{N} \setminus \{0\}$ has the following properties:

- 1 if $A \subseteq X$ and $v \in X$ is dependent on A , then $m(A \cup \{v\})$ divides $m(A)$;
- 2 if $A \subseteq X$ and $v \in X$ is independent on A , then $m(A)$ divides $m(A \cup \{v\})$;
- 3 if $A \subseteq B \subseteq X$ and B is a disjoint union $B = A \cup F \cup T$ such that for all $A \subseteq C \subseteq B$ we have $rk(C) = rk(A) + |C \cap F|$, then $m(A) \cdot m(B) = m(A \cup F) \cdot m(A \cup T)$.
- 4 if $A \subseteq B \subseteq X$ and $rk(A) = rk(B)$, then $\mu_B(A) := \sum_{A \subseteq T \subseteq B} (-1)^{|T|-|A|} m(T) \geq 0$;
- 5 if $A \subseteq B \subseteq X$ and $rk^*(A) = rk^*(B)$, then $\mu_B^*(A) := \sum_{A \subseteq T \subseteq B} (-1)^{|T|-|A|} m(X \setminus T) \geq 0$.



Definition of Arithmetic Matroid

Setting $m(A) = 1$ for all $A \subseteq X$ we get a *trivial* multiplicity function, which essentially does not add anything to the matroid structure.

So any matroid is trivially an arithmetic matroid.

In this sense the notion of an arithmetic matroid is a generalization of the one of a matroid.

But of course there are more interesting examples.

Definition of Arithmetic Matroid

Setting $m(A) = 1$ for all $A \subseteq X$ we get a *trivial* multiplicity function, which essentially does not add anything to the matroid structure.

So any matroid is trivially an arithmetic matroid.

In this sense the notion of an arithmetic matroid is a generalization of the one of a matroid.

But of course there are more interesting examples.

Definition of Arithmetic Matroid

Setting $m(A) = 1$ for all $A \subseteq X$ we get a *trivial* multiplicity function, which essentially does not add anything to the matroid structure.

So any matroid is trivially an arithmetic matroid.

In this sense the notion of an arithmetic matroid is a generalization of the one of a matroid.

But of course there are more interesting examples.

Definition of Arithmetic Matroid

Setting $m(A) = 1$ for all $A \subseteq X$ we get a *trivial* multiplicity function, which essentially does not add anything to the matroid structure.

So any matroid is trivially an arithmetic matroid.

In this sense the notion of an arithmetic matroid is a generalization of the one of a matroid.

But of course there are more interesting examples.

Definition of Arithmetic Matroid

Setting $m(A) = 1$ for all $A \subseteq X$ we get a *trivial* multiplicity function, which essentially does not add anything to the matroid structure.

So any matroid is trivially an arithmetic matroid.

In this sense the notion of an arithmetic matroid is a generalization of the one of a matroid.

But of course there are more interesting examples.

The main example

Let X be a finite list of elements of a finitely generated abelian group $G \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_s\mathbb{Z}$.

For $A \subseteq X$ we set

$rk(A) :=$ maximal rank of a free abelian subgroup of $\langle A \rangle$;

$m(A) := |G_A : \langle A \rangle|$, where G_A is the maximal subgroup of G such that $\langle A \rangle \leq G_A$ and $|G_A : \langle A \rangle| < \infty$.

Theorem (D.-Moci)

If we set $\mathfrak{M}_X := (X, rk)$, then (\mathfrak{M}_X, m) is an arithmetic matroid.

The main example

Let X be a finite list of elements of a finitely generated abelian group $G \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_s\mathbb{Z}$.

For $A \subseteq X$ we set

$rk(A) :=$ maximal rank of a free abelian subgroup of $\langle A \rangle$;

$m(A) := |G_A : \langle A \rangle|$, where G_A is the maximal subgroup of G such that $\langle A \rangle \leq G_A$ and $|G_A : \langle A \rangle| < \infty$.

Theorem (D.-Moci)

If we set $\mathfrak{M}_X := (X, rk)$, then (\mathfrak{M}_X, m) is an arithmetic matroid.

The main example

Let X be a finite list of elements of a finitely generated abelian group $G \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_s\mathbb{Z}$.

For $A \subseteq X$ we set

$rk(A) :=$ maximal rank of a free abelian subgroup of $\langle A \rangle$;

$m(A) := |G_A : \langle A \rangle|$, where G_A is the maximal subgroup of G such that $\langle A \rangle \leq G_A$ and $|G_A : \langle A \rangle| < \infty$.

Theorem (D.-Moci)

If we set $\mathfrak{M}_X := (X, rk)$, then (\mathfrak{M}_X, m) is an arithmetic matroid.

The main example

Let X be a finite list of elements of a finitely generated abelian group $G \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_s\mathbb{Z}$.

For $A \subseteq X$ we set

$rk(A) :=$ maximal rank of a free abelian subgroup of $\langle A \rangle$;

$m(A) := |G_A : \langle A \rangle|$, where G_A is the maximal subgroup of G such that $\langle A \rangle \leq G_A$ and $|G_A : \langle A \rangle| < \infty$.

Theorem (D.-Moci)

If we set $\mathfrak{M}_X := (X, rk)$, then (\mathfrak{M}_X, m) is an arithmetic matroid.

The main example

Let X be a finite list of elements of a finitely generated abelian group $G \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_s\mathbb{Z}$.

For $A \subseteq X$ we set

$rk(A) :=$ maximal rank of a free abelian subgroup of $\langle A \rangle$;

$m(A) := |G_A : \langle A \rangle|$, where G_A is the maximal subgroup of G such that $\langle A \rangle \leq G_A$ and $|G_A : \langle A \rangle| < \infty$.

Theorem (D.-Moci)

If we set $\mathfrak{M}_X := (X, rk)$, then (\mathfrak{M}_X, m) is an arithmetic matroid.

A concrete example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Remark

The multiplicity of $A \subseteq X$ is the *GCD* of the minors of maximal rank in the submatrix corresponding to A .

So $m(\emptyset) = 1$, $m(\{v_2\}) = 2$, $m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_2, v_3\}) = 1$,
 $m(\{v_1, v_2\}) = 6$, $m(\{v_1\}) = m(\{v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

A concrete example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Remark

The multiplicity of $A \subseteq X$ is the *GCD* of the minors of maximal rank in the submatrix corresponding to A .

So $m(\emptyset) = 1$, $m(\{v_2\}) = 2$, $m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_2, v_3\}) = 1$,
 $m(\{v_1, v_2\}) = 6$, $m(\{v_1\}) = m(\{v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

A concrete example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Remark

The multiplicity of $A \subseteq X$ is the *GCD* of the minors of maximal rank in the submatrix corresponding to A .

So $m(\emptyset) = 1$, $m(\{v_2\}) = 2$, $m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_2, v_3\}) = 1$,
 $m(\{v_1, v_2\}) = 6$, $m(\{v_1\}) = m(\{v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

A concrete example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Remark

The multiplicity of $A \subseteq X$ is the *GCD* of the minors of maximal rank in the submatrix corresponding to A .

So $m(\emptyset) = 1$, $m(\{v_2\}) = 2$, $m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_2, v_3\}) = 1$,
 $m(\{v_1, v_2\}) = 6$, $m(\{v_1\}) = m(\{v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

A concrete example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Remark

The multiplicity of $A \subseteq X$ is the *GCD* of the minors of maximal rank in the submatrix corresponding to A .

So $m(\emptyset) = 1$, $m(\{v_2\}) = 2$, $m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_2, v_3\}) = 1$,
 $m(\{v_1, v_2\}) = 6$, $m(\{v_1\}) = m(\{v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

A concrete example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Remark

The multiplicity of $A \subseteq X$ is the *GCD* of the minors of maximal rank in the submatrix corresponding to A .

So $m(\emptyset) = 1$, $m(\{v_2\}) = 2$, $m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_2, v_3\}) = 1$,
 $m(\{v_1, v_2\}) = 6$, $m(\{v_1\}) = m(\{v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

A concrete example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Remark

The multiplicity of $A \subseteq X$ is the *GCD* of the minors of maximal rank in the submatrix corresponding to A .

So $m(\emptyset) = 1$, $m(\{v_2\}) = 2$, $m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_2, v_3\}) = 1$,
 $m(\{v_1, v_2\}) = 6$, $m(\{v_1\}) = m(\{v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

A concrete example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Remark

The multiplicity of $A \subseteq X$ is the *GCD* of the minors of maximal rank in the submatrix corresponding to A .

So $m(\emptyset) = 1$, $m(\{v_2\}) = 2$, $m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_2, v_3\}) = 1$,
 $m(\{v_1, v_2\}) = 6$, $m(\{v_1\}) = m(\{v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

Arithmetic Tutte Polynomial

The *arithmetic Tutte polynomial* of the arithmetic matroid (\mathfrak{M}_X, m) is defined as

$$M_X(x, y) := \sum_{A \subseteq X} m(A) (x-1)^{rk(X)-rk(A)} (y-1)^{|A|-rk(A)}.$$

For the trivial multiplicity function $m(A) = 1$ for all $A \subseteq X$ we get the Tutte polynomial $T_X(x, y)$ of \mathfrak{M}_X .

Arithmetic Tutte Polynomial

The *arithmetic Tutte polynomial* of the arithmetic matroid (\mathfrak{M}_X, m) is defined as

$$M_X(x, y) := \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}.$$

For the trivial multiplicity function $m(A) = 1$ for all $A \subseteq X$ we get the Tutte polynomial $T_X(x, y)$ of \mathfrak{M}_X .

Arithmetic Tutte Polynomial

The *arithmetic Tutte polynomial* of the arithmetic matroid (\mathfrak{M}_X, m) is defined as

$$M_X(x, y) := \sum_{A \subseteq X} m(A) (x-1)^{rk(X)-rk(A)} (y-1)^{|A|-rk(A)}.$$

For the trivial multiplicity function $m(A) = 1$ for all $A \subseteq X$ we get the Tutte polynomial $T_X(x, y)$ of \mathfrak{M}_X .

An example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Then $m(\emptyset) = m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_2\}) = 2$,
 $m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

$$M_X(x, y) = \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} = \\ = (x-1)^2 + (3+2+3)(x-1) + (x-1)(y-1) + (6+9) + 3(y-1) = \\ = x^2 + 5x + 6 + xy + 2y.$$

An example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Then $m(\emptyset) = m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_2\}) = 2$,
 $m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

$$M_X(x, y) = \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} = \\ = (x-1)^2 + (3+2+3)(x-1) + (x-1)(y-1) + (6+9) + 3(y-1) = \\ = x^2 + 5x + 6 + xy + 2y.$$

An example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Then $m(\emptyset) = m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_2\}) = 2$,
 $m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

$$M_X(x, y) = \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} = \\ = (x-1)^2 + (3+2+3)(x-1) + (x-1)(y-1) + (6+9) + 3(y-1) = \\ = x^2 + 5x + 6 + xy + 2y.$$

An example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Then $m(\emptyset) = m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_2\}) = 2$,
 $m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

$$M_X(x, y) = \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} =$$

$$= (x-1)^2 + (3+2+3)(x-1) + (x-1)(y-1) + (6+9) + 3(y-1) =$$

$$x^2 + 5x + 6 + xy + 2y.$$

An example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Then $m(\emptyset) = m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_2\}) = 2$,
 $m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

$$M_X(x, y) = \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} = \\ = (x-1)^2 + (3+2+3)(x-1) + (x-1)(y-1) + (6+9) + 3(y-1) = \\ = x^2 + 5x + 6 + xy + 2y.$$

An example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Then $m(\emptyset) = m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_2\}) = 2$,
 $m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

$$M_X(x, y) = \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} = \\ = (x-1)^2 + (3+2+3)(x-1) + (x-1)(y-1) + (6+9) + 3(y-1) = \\ = x^2 + 5x + 6 + xy + 2y.$$

An example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Then $m(\emptyset) = m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_2\}) = 2$,
 $m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

$$M_X(x, y) = \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} = \\ = (x-1)^2 + (3+2+3)(x-1) + (x-1)(y-1) + (6+9) + 3(y-1) = \\ = x^2 + 5x + 6 + xy + 2y.$$

An example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Then $m(\emptyset) = m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_2\}) = 2$,
 $m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

$$M_X(x, y) = \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} = \\ = (x-1)^2 + (3+2+3)(x-1) + (x-1)(y-1) + (6+9) + 3(y-1) = \\ = x^2 + 5x + 6 + xy + 2y.$$

An example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Then $m(\emptyset) = m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_2\}) = 2$,
 $m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

$$M_X(x, y) = \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} = \\ = (x-1)^2 + (3+2+3)(x-1) + (x-1)(y-1) + (6+9) + 3(y-1) = \\ = x^2 + 5x + 6 + xy + 2y.$$

An example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Then $m(\emptyset) = m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_2\}) = 2$,
 $m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

$$M_X(x, y) = \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} = \\ = (x-1)^2 + (3+2+3)(x-1) + (x-1)(y-1) + (6+9) + 3(y-1) = \\ = x^2 + 5x + 6 + xy + 2y.$$

An example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Then $m(\emptyset) = m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_2\}) = 2$,
 $m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

$$M_X(x, y) = \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} =$$

$$= (x-1)^2 + (3+2+3)(x-1) + (x-1)(y-1) + (6+9) + 3(y-1) =$$

$$x^2 + 5x + 6 + xy + 2y. \text{ Positive coefficients!}$$

An example

Let $X = \{v_1 := (3, 0), v_2 := (2, -2), v_3 := (-3, 3)\} \subseteq G := \mathbb{Z}^2$.

Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 .

Then $m(\emptyset) = m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_2\}) = 2$,
 $m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

$M_X(x, y) = \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} =$
 $= (x-1)^2 + (3+2+3)(x-1) + (x-1)(y-1) + (6+9) + 3(y-1) =$
 $x^2 + 5x + 6 + xy + 2y$. **Positive coefficients!**

Theorem (D.-Moci)

The arithmetic Tutte polynomial has a combinatorial interpretation that generalizes Crapo's one for the classical Tutte polynomial.

From (\mathcal{G}, ℓ) to $\mathfrak{M}_{\mathcal{G}, \ell}$

To $e = (v_i, v_j) \in E_\theta$ associate the vector $x_e \in \mathbb{Z}^{|V|}$ with $\ell(e)$ in the i -th position, $-\ell(e)$ in the j -th position, and 0 elsewhere.

Set $X_D := \{x_e \mid e \in D\}$, $G := \mathbb{Z}^{|V|} / \langle X_D \rangle$,

$X_R := \{\bar{x}_e \mid e \in R\} \subseteq G$.

Take $\mathfrak{M}_{\mathcal{G}, \ell}$ to be the one associated to $X_R \subseteq G$.

From (\mathcal{G}, ℓ) to $\mathfrak{M}_{\mathcal{G}, \ell}$

To $e = (v_i, v_j) \in E_\theta$ associate the vector $x_e \in \mathbb{Z}^{|V|}$ with $\ell(e)$ in the i -th position, $-\ell(e)$ in the j -th position, and 0 elsewhere.

Set $X_D := \{x_e \mid e \in D\}$, $G := \mathbb{Z}^{|V|} / \langle X_D \rangle$,

$X_R := \{\bar{x}_e \mid e \in R\} \subseteq G$.

Take $\mathfrak{M}_{\mathcal{G}, \ell}$ to be the one associated to $X_R \subseteq G$.

From (\mathcal{G}, ℓ) to $\mathfrak{M}_{\mathcal{G}, \ell}$

To $e = (v_i, v_j) \in E_\theta$ associate the vector $x_e \in \mathbb{Z}^{|V|}$ with $\ell(e)$ in the i -th position, $-\ell(e)$ in the j -th position, and 0 elsewhere.

Set $X_D := \{x_e \mid e \in D\}$, $G := \mathbb{Z}^{|V|} / \langle X_D \rangle$,

$X_R := \{\bar{x}_e \mid e \in R\} \subseteq G$.

Take $\mathfrak{M}_{\mathcal{G}, \ell}$ to be the one associated to $X_R \subseteq G$.

From (\mathcal{G}, ℓ) to $\mathfrak{M}_{\mathcal{G}, \ell}$

To $e = (v_i, v_j) \in E_\theta$ associate the vector $x_e \in \mathbb{Z}^{|V|}$ with $\ell(e)$ in the i -th position, $-\ell(e)$ in the j -th position, and 0 elsewhere.

Set $X_D := \{x_e \mid e \in D\}$, $G := \mathbb{Z}^{|V|} / \langle X_D \rangle$,

$X_R := \{\bar{x}_e \mid e \in R\} \subseteq G$.

Take $\mathfrak{M}_{\mathcal{G}, \ell}$ to be the one associated to $X_R \subseteq G$.

From (\mathcal{G}, ℓ) to $\mathfrak{M}_{\mathcal{G}, \ell}$

To $e = (v_i, v_j) \in E_\theta$ associate the vector $x_e \in \mathbb{Z}^{|V|}$ with $\ell(e)$ in the i -th position, $-\ell(e)$ in the j -th position, and 0 elsewhere.

Set $X_D := \{x_e \mid e \in D\}$, $G := \mathbb{Z}^{|V|} / \langle X_D \rangle$,

$X_R := \{\bar{x}_e \mid e \in R\} \subseteq G$.

Take $\mathfrak{M}_{\mathcal{G}, \ell}$ to be the one associated to $X_R \subseteq G$.

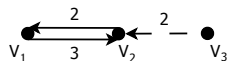
Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$V = \{v_1, v_2, v_3\}$, $R_\theta := \{e_1 := (v_1, v_2)$,

$e_2 := (v_2, v_1)\}$, $D_\theta := \{e_3 := (v_2, v_3)\}$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.



From (\mathcal{G}, ℓ) to $\mathfrak{M}_{\mathcal{G}, \ell}$

To $e = (v_i, v_j) \in E_\theta$ associate the vector $x_e \in \mathbb{Z}^{|V|}$ with $\ell(e)$ in the i -th position, $-\ell(e)$ in the j -th position, and 0 elsewhere.

Set $X_D := \{x_e \mid e \in D\}$, $G := \mathbb{Z}^{|V|} / \langle X_D \rangle$,

$X_R := \{\bar{x}_e \mid e \in R\} \subseteq G$.

Take $\mathfrak{M}_{\mathcal{G}, \ell}$ to be the one associated to $X_R \subseteq G$.

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$V = \{v_1, v_2, v_3\}$, $R_\theta := \{e_1 := (v_1, v_2)$,

$e_2 := (v_2, v_1)\}$, $D_\theta := \{e_3 := (v_2, v_3)\}$

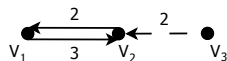
so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

$e_1 \rightsquigarrow x_{e_1} = (2, -2, 0) \in \mathbb{Z}^3$,

$e_2 \rightsquigarrow x_{e_2} = (-3, 3, 0) \in \mathbb{Z}^3$,

$e_3 \rightsquigarrow x_{e_3} = (0, 2, -2) \in \mathbb{Z}^3$



From (\mathcal{G}, ℓ) to $\mathfrak{M}_{\mathcal{G}, \ell}$

To $e = (v_i, v_j) \in E_\theta$ associate the vector $x_e \in \mathbb{Z}^{|V|}$ with $\ell(e)$ in the i -th position, $-\ell(e)$ in the j -th position, and 0 elsewhere.

Set $X_D := \{x_e \mid e \in D\}$, $G := \mathbb{Z}^{|V|} / \langle X_D \rangle$,

$X_R := \{\bar{x}_e \mid e \in R\} \subseteq G$.

Take $\mathfrak{M}_{\mathcal{G}, \ell}$ to be the one associated to $X_R \subseteq G$.

Consider $(\mathcal{G}_\theta, \ell)$ with $\mathcal{G}_\theta := (V, E_\theta)$,

$V = \{v_1, v_2, v_3\}$, $R_\theta := \{e_1 := (v_1, v_2)$,

$e_2 := (v_2, v_1)\}$, $D_\theta := \{e_3 := (v_2, v_3)\}$

so $E_\theta = R_\theta \cup D_\theta$, $\ell(e_1) = 2$, $\ell(e_2) = 3$

and $\ell(e_3) = 2$.

$e_1 \rightsquigarrow x_{e_1} = (2, -2, 0) \in \mathbb{Z}^3$,

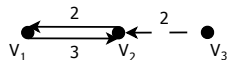
$e_2 \rightsquigarrow x_{e_2} = (-3, 3, 0) \in \mathbb{Z}^3$,

$e_3 \rightsquigarrow x_{e_3} = (0, 2, -2) \in \mathbb{Z}^3$

$G := \mathbb{Z}^{|V|} / \langle X_D \rangle = \mathbb{Z}^3 / \langle x_{e_3} \rangle$,

$X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G$

$X_R \subseteq G \rightsquigarrow \mathfrak{M}_{\mathcal{G}, \ell}$.



Main results

Set $\bar{\mathcal{G}} :=$ the classical contraction of all dotted edges of \mathcal{G} .

We say that q is **admissible** for (\mathcal{G}, ℓ) if $\ell(e)$ divides q for all $e \in E$.

Theorem (D.-Moci)

For q admissible we have

$$1 \quad \chi_{\mathcal{G}, \ell}(q) = (-1)^{|\bar{V}|-k} q^k M_{\bar{\mathcal{G}}, \ell}(1 - q, 0).$$

$$2 \quad \chi_{\mathcal{G}, \ell}^*(q) = (-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{\bar{\mathcal{G}}, \ell}(0, 1 - q).$$

Theorem (Tutte 1954)

Let $\mathcal{G} = (V, E)$ with k connected components, and let $T_{\mathcal{G}}(x, y)$ the Tutte polynomial of \mathcal{G} . We have

$$1 \quad \chi_{\mathcal{G}}(q) = (-1)^{|V|-k} q^k T_{\mathcal{G}}(1 - q, 0);$$

$$2 \quad \chi_{\mathcal{G}}^*(q) = (-1)^{|E|-|V|+k} T_{\mathcal{G}}(0, 1 - q).$$

Main results

Set $\bar{\mathcal{G}} :=$ the classical contraction of all dotted edges of \mathcal{G} .

We say that q is *admissible* for (\mathcal{G}, ℓ) if $\ell(e)$ divides q for all $e \in E$.

Theorem (D.-Moci)

For q admissible we have

- 1 $\chi_{\mathcal{G}, \ell}(q) = (-1)^{|\bar{V}|-k} q^k M_{\bar{\mathcal{G}}, \ell}(1 - q, 0).$
- 2 $\chi_{\mathcal{G}, \ell}^*(q) = (-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{\bar{\mathcal{G}}, \ell}(0, 1 - q).$

Theorem (Tutte 1954)

Let $\mathcal{G} = (V, E)$ with k connected components, and let $T_{\mathcal{G}}(x, y)$ the Tutte polynomial of \mathcal{G} . We have

- 1 $\chi_{\mathcal{G}}(q) = (-1)^{|V|-k} q^k T_{\mathcal{G}}(1 - q, 0);$
- 2 $\chi_{\mathcal{G}}^*(q) = (-1)^{|E|-|V|+k} T_{\mathcal{G}}(0, 1 - q).$

Main results

Set $\bar{\mathcal{G}}$:= the classical contraction of all dotted edges of \mathcal{G} .

We say that q is **admissible** for (\mathcal{G}, ℓ) if $\ell(e)$ divides q for all $e \in E$.

Theorem (D.-Moci)

For q admissible we have

$$1 \quad \chi_{\mathcal{G}, \ell}(q) = (-1)^{|\bar{V}|-k} q^k M_{\bar{\mathcal{G}}, \ell}(1 - q, 0).$$

$$2 \quad \chi_{\mathcal{G}, \ell}^*(q) = (-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{\bar{\mathcal{G}}, \ell}(0, 1 - q).$$

Theorem (Tutte 1954)

Let $\mathcal{G} = (V, E)$ with k connected components, and let $T_{\mathcal{G}}(x, y)$ the Tutte polynomial of \mathcal{G} . We have

$$1 \quad \chi_{\mathcal{G}}(q) = (-1)^{|V|-k} q^k T_{\mathcal{G}}(1 - q, 0);$$

$$2 \quad \chi_{\mathcal{G}}^*(q) = (-1)^{|E|-|V|+k} T_{\mathcal{G}}(0, 1 - q).$$

Main results

Set $\bar{\mathcal{G}} :=$ the classical contraction of all dotted edges of \mathcal{G} .

We say that q is **admissible** for (\mathcal{G}, ℓ) if $\ell(e)$ divides q for all $e \in E$.

Theorem (D.-Moci)

For q admissible we have

$$1 \quad \chi_{\mathcal{G}, \ell}(q) = (-1)^{|\bar{V}|-k} q^k M_{\bar{\mathcal{G}}, \ell}(1 - q, 0).$$

$$2 \quad \chi_{\mathcal{G}, \ell}^*(q) = (-1)^{|R| - |\bar{V}| + k} q^{|D| - |V| + |\bar{V}|} M_{\bar{\mathcal{G}}, \ell}(0, 1 - q).$$

Theorem (Tutte 1954)

Let $\mathcal{G} = (V, E)$ with k connected components, and let $T_{\mathcal{G}}(x, y)$ the Tutte polynomial of \mathcal{G} . We have

$$1 \quad \chi_{\mathcal{G}}(q) = (-1)^{|V|-k} q^k T_{\mathcal{G}}(1 - q, 0);$$

$$2 \quad \chi_{\mathcal{G}}^*(q) = (-1)^{|E|-|V|+k} T_{\mathcal{G}}(0, 1 - q).$$

Main results

Set $\bar{\mathcal{G}}$:= the classical contraction of all dotted edges of \mathcal{G} .

We say that q is **admissible** for (\mathcal{G}, ℓ) if $\ell(e)$ divides q for all $e \in E$.

Theorem (D.-Moci)

For q admissible we have

$$1 \quad \chi_{\mathcal{G}, \ell}(q) = (-1)^{|\bar{V}| - k} q^k M_{\mathcal{G}, \ell}(1 - q, 0).$$

$$2 \quad \chi_{\mathcal{G}, \ell}^*(q) = (-1)^{|R| - |\bar{V}| + k} q^{|D| - |V| + |\bar{V}|} M_{\mathcal{G}, \ell}(0, 1 - q).$$

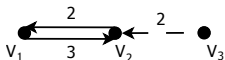
Theorem (Tutte 1954)

Let $\mathcal{G} = (V, E)$ with k connected components, and let $T_{\mathcal{G}}(x, y)$ the Tutte polynomial of \mathcal{G} . We have

$$1 \quad \chi_{\mathcal{G}}(q) = (-1)^{|V| - k} q^k T_{\mathcal{G}}(1 - q, 0);$$

$$2 \quad \chi_{\mathcal{G}}^*(q) = (-1)^{|E| - |V| + k} T_{\mathcal{G}}(0, 1 - q).$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

$m(\emptyset) = 2$, $m(\{x_{e_1}\}) = 4$, $m(\{x_{e_2}\}) = 6$, $m(\{x_{e_1}, x_{e_2}\}) = 2$.

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A) (x-1)^{rk(X)-rk(A)} (y-1)^{|A|-rk(A)} =$$

$$2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

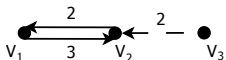
$$(-1)^{|\bar{V}|-k} q^k M_{G,\ell}(1-q, 0) = (-1)q(2(1-q) + 6) =$$

$$= 2q^2 - 8q = \chi_{G,\ell}(q)!!$$

$$(-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{G,\ell}(0, 1-q) =$$

$$= (-1)(6 + 2(1-q)) = 2q - 8 = \chi_{G,\ell}^*(q)!!$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

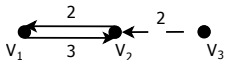
$$m(\emptyset) = 2, m(\{x_{e_1}\}) = 4, m(\{x_{e_2}\}) = 6, m(\{x_{e_1}, x_{e_2}\}) = 2.$$

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A) (x-1)^{rk(X)-rk(A)} (y-1)^{|A|-rk(A)} = 2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

$$\begin{aligned} (-1)^{|\bar{V}|-k} q^k M_{G,\ell}(1-q, 0) &= (-1)q(2(1-q) + 6) = \\ &= 2q^2 - 8q = \chi_{G,\ell}(q)!! \end{aligned}$$

$$\begin{aligned} (-1)^{|R|-|\bar{V}|-k} q^{|D|-|V|+|\bar{V}|} M_{G,\ell}(0, 1-q) &= \\ &= (-1)(6 + 2(1-q)) = 2q - 8 = \chi_{G,\ell}^*(q)!! \end{aligned}$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

Consider the matrix $\begin{pmatrix} 2 & -3 & 0 \\ -2 & 3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$ with columns $x_{e_1}, x_{e_2}, x_{e_3}$.

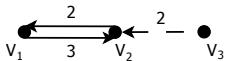
$$m(\emptyset) = 2, \quad m(\{x_{e_1}\}) = 4, \quad m(\{x_{e_2}\}) = 6, \quad m(\{x_{e_1}, x_{e_2}\}) = 2.$$

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A) (x-1)^{\text{rk}(X) - \text{rk}(A)} (y-1)^{|A| - \text{rk}(A)} = 2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

$$\begin{aligned} (-1)^{|\bar{V}| - k} q^k M_{G,\ell}(1-q, 0) &= (-1)q(2(1-q) + 6) = \\ &= 2q^2 - 8q = \chi_{G,\ell}(q)!! \end{aligned}$$

$$(-1)^{|R| - |\bar{V}| + k} q^{|D| - |\bar{V}| + |\bar{V}|} M_{G,\ell}(0, 1-q) =$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

Consider the matrix $\begin{pmatrix} 2 & -3 & 0 \\ -2 & 3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$ with columns $x_{e_1}, x_{e_2}, x_{e_3}$.

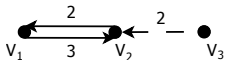
$$m(\emptyset) = 2, m(\{x_{e_1}\}) = 4, m(\{x_{e_2}\}) = 6, m(\{x_{e_1}, x_{e_2}\}) = 2.$$

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A) (x-1)^{\text{rk}(X) - \text{rk}(A)} (y-1)^{|A| - \text{rk}(A)} = 2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

$$\begin{aligned} (-1)^{|\bar{V}| - k} q^k M_{G,\ell}(1-q, 0) &= (-1)q(2(1-q) + 6) = \\ &= 2q^2 - 8q = \chi_{G,\ell}(q)!! \end{aligned}$$

$$(-1)^{|R| - |\bar{V}| + k} q^{|D| - |\bar{V}| + |\bar{V}|} M_{G,\ell}(0, 1-q) =$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

Consider the matrix $\begin{pmatrix} 2 & -3 & 0 \\ -2 & 3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$ with columns $x_{e_1}, x_{e_2}, x_{e_3}$.

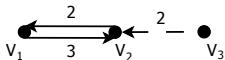
$$m(\emptyset) = 2, m(\{x_{e_1}\}) = 4, m(\{x_{e_2}\}) = 6, m(\{x_{e_1}, x_{e_2}\}) = 2.$$

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A) (x-1)^{\text{rk}(X) - \text{rk}(A)} (y-1)^{|A| - \text{rk}(A)} = 2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

$$\begin{aligned} (-1)^{|\bar{V}| - k} q^k M_{G,\ell}(1-q, 0) &= (-1)q(2(1-q) + 6) = \\ &= 2q^2 - 8q = \chi_{G,\ell}(q)!! \end{aligned}$$

$$(-1)^{|R| - |\bar{V}| + k} q^{|D| - |\bar{V}| + |\bar{V}|} M_{G,\ell}(0, 1-q) =$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

Consider the matrix $\begin{pmatrix} 2 & -3 & 0 \\ -2 & 3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$ with columns $x_{e_1}, x_{e_2}, x_{e_3}$.

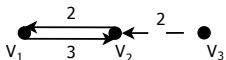
$$m(\emptyset) = 2, m(\{x_{e_1}\}) = 4, m(\{x_{e_2}\}) = 6, m(\{x_{e_1}, x_{e_2}\}) = 2.$$

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A) (x-1)^{\text{rk}(X) - \text{rk}(A)} (y-1)^{|A| - \text{rk}(A)} = 2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

$$\begin{aligned} (-1)^{|\bar{V}| - k} q^k M_{G,\ell}(1-q, 0) &= (-1)q(2(1-q) + 6) = \\ &= 2q^2 - 8q = \chi_{G,\ell}(q)!! \end{aligned}$$

$$(-1)^{|R| - |\bar{V}| + k} q^{|D| - |\bar{V}| + |\bar{V}|} M_{G,\ell}(0, 1-q) =$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

$m(\emptyset) = 2$, $m(\{x_{e_1}\}) = 4$, $m(\{x_{e_2}\}) = 6$, $m(\{x_{e_1}, x_{e_2}\}) = 2$.

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A) (x-1)^{rk(X)-rk(A)} (y-1)^{|A|-rk(A)} =$$

$$2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

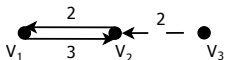
$$(-1)^{|\bar{V}|-k} q^k M_{G,\ell}(1-q, 0) = (-1)q(2(1-q) + 6) =$$

$$= 2q^2 - 8q = \chi_{G,\ell}(q)!!$$

$$(-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{G,\ell}(0, 1-q) =$$

$$= (-1)(6 + 2(1-q)) = 2q - 8 = \chi_{G,\ell}^*(q)!!$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

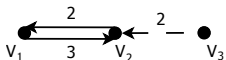
$m(\emptyset) = 2$, $m(\{x_{e_1}\}) = 4$, $m(\{x_{e_2}\}) = 6$, $m(\{x_{e_1}, x_{e_2}\}) = 2$.

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} = 2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

$$(-1)^{|\bar{V}|-k} q^k M_{G,\ell}(1-q, 0) = (-1)q(2(1-q) + 6) = 2q^2 - 8q = \chi_{G,\ell}(q)!!$$

$$(-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{G,\ell}(0, 1-q) = (-1)(6 + 2(1-q)) = 2q - 8 = \chi_{G,\ell}^*(q)!!$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

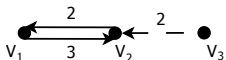
$m(\emptyset) = 2$, $m(\{x_{e_1}\}) = 4$, $m(\{x_{e_2}\}) = 6$, $m(\{x_{e_1}, x_{e_2}\}) = 2$.

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} = 2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

$$(-1)^{|\bar{V}|-k} q^k M_{G,\ell}(1-q, 0) = (-1)q(2(1-q) + 6) = 2q^2 - 8q = \chi_{G,\ell}(q)!!$$

$$(-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{G,\ell}(0, 1-q) = (-1)(6 + 2(1-q)) = 2q - 8 = \chi_{G,\ell}^*(q)!!$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

$m(\emptyset) = 2$, $m(\{x_{e_1}\}) = 4$, $m(\{x_{e_2}\}) = 6$, $m(\{x_{e_1}, x_{e_2}\}) = 2$.

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A) (x-1)^{rk(X)-rk(A)} (y-1)^{|A|-rk(A)} =$$

$$2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

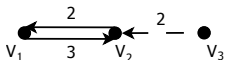
$$(-1)^{|\bar{V}|-k} q^k M_{G,\ell}(1-q, 0) = (-1)q(2(1-q) + 6) =$$

$$= 2q^2 - 8q = \chi_{G,\ell}(q)!!$$

$$(-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{G,\ell}(0, 1-q) =$$

$$= (-1)(6 + 2(1-q)) = 2q - 8 = \chi_{G,\ell}^*(q)!!$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

$m(\emptyset) = 2$, $m(\{x_{e_1}\}) = 4$, $m(\{x_{e_2}\}) = 6$, $m(\{x_{e_1}, x_{e_2}\}) = 2$.

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} =$$

$$2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

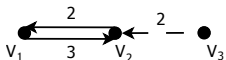
$$(-1)^{|\bar{V}|-k} q^k M_{G,\ell}(1-q, 0) = (-1)q(2(1-q) + 6) =$$

$$= 2q^2 - 8q = \chi_{G,\ell}(q)!!$$

$$(-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{G,\ell}(0, 1-q) =$$

$$= (-1)(6 + 2(1-q)) = 2q - 8 = \chi_{G,\ell}^*(q)!!$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

$m(\emptyset) = 2$, $m(\{x_{e_1}\}) = 4$, $m(\{x_{e_2}\}) = 6$, $m(\{x_{e_1}, x_{e_2}\}) = 2$.

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} =$$

$$2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

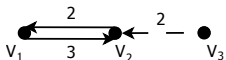
$$(-1)^{|\bar{V}|-k} q^k M_{G,\ell}(1-q, 0) = (-1)q(2(1-q) + 6) =$$

$$= 2q^2 - 8q = \chi_{G,\ell}(q)!!$$

$$(-1)^{|R| - |\bar{V}| + k} q^{|D| - |V| + |\bar{V}|} M_{G,\ell}(0, 1-q) =$$

$$= (-1)(6 + 2(1-q)) = 2q - 8 = \chi_{G,\ell}^*(q)!!$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

$m(\emptyset) = 2$, $m(\{x_{e_1}\}) = 4$, $m(\{x_{e_2}\}) = 6$, $m(\{x_{e_1}, x_{e_2}\}) = 2$.

$$M_{\mathcal{G}, \ell}(x, y) = \sum_{A \subseteq X_R} m(A) (x-1)^{rk(X)-rk(A)} (y-1)^{|A|-rk(A)} =$$

$$2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

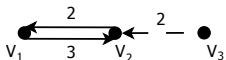
$$(-1)^{|\bar{V}|-k} q^k M_{\mathcal{G}, \ell}(1-q, 0) = (-1)q(2(1-q) + 6) =$$

$$= 2q^2 - 8q = \chi_{\mathcal{G}, \ell}(q)!!$$

$$(-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{\mathcal{G}, \ell}(0, 1-q) =$$

$$= (-1)(6 + 2(1-q)) = 2q - 8 = \chi_{\mathcal{G}, \ell}^*(q)!!$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

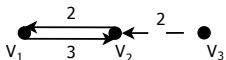
$m(\emptyset) = 2$, $m(\{x_{e_1}\}) = 4$, $m(\{x_{e_2}\}) = 6$, $m(\{x_{e_1}, x_{e_2}\}) = 2$.

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)} = 2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

$$(-1)^{|\bar{V}|-k} q^k M_{G,\ell}(1-q, 0) = (-1)q(2(1-q) + 6) = 2q^2 - 8q = \chi_{G,\ell}(q)!!$$

$$(-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{G,\ell}(0, 1-q) = (-1)(6 + 2(1-q)) = 2q - 8 = \chi_{G,\ell}^*(q)!!$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

$m(\emptyset) = 2$, $m(\{x_{e_1}\}) = 4$, $m(\{x_{e_2}\}) = 6$, $m(\{x_{e_1}, x_{e_2}\}) = 2$.

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A) (x-1)^{rk(X)-rk(A)} (y-1)^{|A|-rk(A)} =$$

$$2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

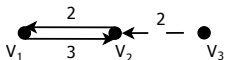
$$(-1)^{|\bar{V}|-k} q^k M_{G,\ell}(1-q, 0) = (-1)q(2(1-q) + 6) =$$

$$= 2q^2 - 8q = \chi_{G,\ell}(q)!!$$

$$(-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{G,\ell}(0, 1-q) =$$

$$= (-1)(6 + 2(1-q)) = 2q - 8 = \chi_{G,\ell}^*(q)!!$$

An example



We have $x_{e_1} = (2, -2, 0)$, $x_{e_2} = (-3, 3, 0)$, $x_{e_3} = (0, 2, -2)$, and $X_R = \{\bar{x}_{e_1}, \bar{x}_{e_2}\} \subseteq G := \mathbb{Z}^3 / \langle x_{e_3} \rangle$.

$m(\emptyset) = 2$, $m(\{x_{e_1}\}) = 4$, $m(\{x_{e_2}\}) = 6$, $m(\{x_{e_1}, x_{e_2}\}) = 2$.

$$M_{G,\ell}(x, y) = \sum_{A \subseteq X_R} m(A) (x-1)^{rk(X)-rk(A)} (y-1)^{|A|-rk(A)} =$$

$$2(x-1) + (4+6) + 2(y-1) = 2x + 6 + 2y.$$

$$(-1)^{|\bar{V}|-k} q^k M_{G,\ell}(1-q, 0) = (-1)q(2(1-q) + 6) =$$

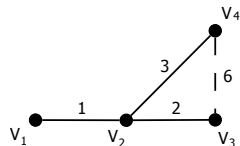
$$= 2q^2 - 8q = \chi_{G,\ell}(q)!!$$

$$(-1)^{|R|-|\bar{V}|+k} q^{|D|-|V|+|\bar{V}|} M_{G,\ell}(0, 1-q) =$$

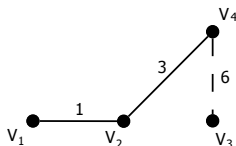
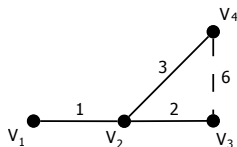
$$= (-1)(6 + 2(1-q)) = 2q - 8 = \chi_{G,\ell}^*(q)!!$$

Deletion Contraction II

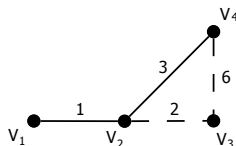
Deletion Contraction II



Deletion Contraction II



Deletion of $\{v_2, v_3\}$.



Contraction of $\{v_2, v_3\}$.

THE END

References

- 1 M. D'Adderio, L. Moci, *Graph colorings, flows and arithmetic Tutte polynomial*, preprint
- 2 M. D'Adderio, L. Moci, *Arithmetic matroids, Tutte polynomial and toric arrangements*, preprint
- 3 C. De Concini, C. Procesi, *Topics in hyperplane arrangements, polytopes and box-splines*, Springer 2010.
- 4 L. Moci, *A Tutte polynomial for toric arrangements*, to appear on Trans. Am. Math. Soc.

THE END

THANKS!

References

- 1 M. D'Adderio, L. Moci, *Graph colorings, flows and arithmetic Tutte polynomial*, preprint
- 2 M. D'Adderio, L. Moci, *Arithmetic matroids, Tutte polynomial and toric arrangements*, preprint
- 3 C. De Concini, C. Procesi, *Topics in hyperplane arrangements, polytopes and box-splines*, Springer 2010.
- 4 L. Moci, *A Tutte polynomial for toric arrangements*, to appear on Trans. Am. Math. Soc.