The height and width of simple trees

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The limit law of the couple height-width for simple trees can be seen as a consequence of deep results of Aldous, Drmota and Gittenberger, and Jeulin. We give here an elementary proof in the case of binary trees.

1 Introduction

Let $Z_i(t)$ denote the number of nodes at distance i from the root of a rooted tree t. The *profile* of the tree t is the sequence $(Z_i(t))_{i\geq 0}$. The *width* $w(t)$ and *height* $h(t)$ of the tree t are defined by:

$$\begin{aligned} w(t) &= \max_i \{Z_i(t)\}, \\ h(t) &= \max\{i | Z_i(t) > 0\}. \end{aligned}$$

Let $T_B^{(n)}$ denote the set of *binary trees* with n leaves ($2n-1$ nodes), endowed with the uniform probability, and let $H_B^{(n)}$ (resp. $W_B^{(n)}$) be the restriction of h (resp. w) to $T_B^{(n)}$. One can also see $H_B^{(n)}$ and $W_B^{(n)}$ as the height and width of a Galton-Watson tree with offspring distribution 0 or 2 with probability 1/2, conditioned to have total progeny $2n-1$ (see [1, pp. 27-28]). Then, the limit law of the height [15, 23] and of the width [7, 13, 25] are given by:

$$\frac{H_B^{(n)}}{\sqrt{2n}} \xrightarrow[n\to+\infty]{law} 2V, \tag{1.1}$$

$$\frac{W_B^{(n)}}{\sqrt{2n}} \xrightarrow[n\to+\infty]{law} V, \tag{1.2}$$

where:

$$\Pr(V \leq x) = \sum_{-\infty<k<+\infty} (1-4k^2x^2)\exp\left(-2k^2x^2\right). \tag{1.3}$$

Connections between the distribution of V on one hand, the Brownian motion and Jacobi's Theta function on the other hand, are discussed in [5, 9, 20]. For instance, let $\big(e(s)\big)_{0\leq s\leq 1}$ denote a standard normalized Brownian excursion (see Subsection 3.1). Then, the random variables

$$H = \int_0^1 \frac{ds}{e(s)}$$

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and

$$W = \max_{0 \le s \le 1} e(s)$$

satisfy

$$V \overset{law}{=} W \overset{law}{=} \frac{H}{2}. \tag{1.4}$$

The first identity is due to Chung [9], the second was first stressed in [4, p. 69]. The aim of this paper is to give a simple proof of the following theorem:

Theorem 1.1

$$\left(\frac{H_B^{(n)}}{\sqrt{2n}}, \frac{W_B^{(n)}}{\sqrt{2n}} \right) \xrightarrow[n \to +\infty]{law} (H, W).$$

Note that the obvious negative correlation between height and width of a tree with given size n, is reflected in the dependence between $\int_0^1 \frac{ds}{e(s)}$ and $\max_{0 \le s \le 1} e(s)$. Previous results [15, 23] about height and width of simple trees belongs to the foundations of computer science. Surprisingly, Theorem 1.1 does not seem to be stated anywhere, though it can be deduced easily from deep results of Aldous on one hand (about the continuum random tree [1, 2]) and on the other hand of Drmota & Gittenberger [12], using a clever idea due to Aldous [3, Th. 3] again. We felt that this consequence of [3, Th. 3] deserved to be pointed out, and that the reader would welcome an 'elementary' and direct proof.

Let $\Phi(\alpha, \gamma, z)$ denote the confluent hypergeometric function, defined, for $|z| < +\infty, \gamma \neq 0, -1, -2, \cdots$, by:

$$\Phi(\alpha, \gamma, z) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k \, z^k}{(\gamma)_k \, k!}$$

where $(\lambda)_k = (\lambda)(\lambda+1) \ldots (\lambda+k-1)$. The joint law of (H, W) has been investigated recently by Catherine Donati-Martin [11]. With the help of the agreement formula for the Itô measure (see [4, 22] and [26]), she obtains the following results:

Theorem 1.2 *For $\lambda \ge 0, \alpha \ge 0$,*

$$E\Big(W \exp\Big(-\frac{\lambda^2}{2W^2} - \frac{\alpha^2 H}{2W} \Big) \Big) = \sqrt{\frac{\pi}{2}} \frac{\exp(2\lambda)}{\Phi^2(1 + \alpha^2/(2\lambda), 2, 2\lambda)}.$$

As a consequence, for $Re(s) > 1, Re(t) < 0$ and $Re(s+t) > 1$:

$$E(W^s H^t) = \sqrt{\frac{\pi}{2}} \frac{2^{\frac{5+t-s}{2}}}{\Gamma(-t)\Gamma(\frac{s+t-1}{2})} \int_0^{+\infty} \int_0^{+\infty} \frac{\lambda^{s+t-2} \alpha^{-(1+2t)} \exp(2\lambda)}{\Phi^2(1 + \alpha^2/(2\lambda), 2, 2\lambda)} \, d\alpha \, d\lambda.$$

2 First proof of Theorem 1.1

Aldous [3, Th. 3] proves that, suitably rescaled, the depth-first walk and the profile of a random rooted labeled tree with n nodes converges jointly to $(2e, l/2)$, where l is the *local time* of the normalized Brownian excursion e, defined by:

$$\int_0^a l(x)\, dx = \int_0^1 I_{[0,a]}\big(e(s)\big) ds.$$

Let $H_L^{(n)}$ (resp. $W_L^{(n)}$) denote the restriction of h (resp. w) to the set of rooted labeled trees with n nodes, endowed with the uniform probability. Invariance principle yields at once that:

$$\frac{1}{\sqrt{n}}\big(H_L^{(n)}, W_L^{(n)}\big) \xrightarrow{law} \Big(2 \max_{0\le t\le 1} e(t), \frac{1}{2} \max_{x\ge 0} l(x)\Big).$$

For a general class of simple trees with n leaves, the proof of [3, Th. 3] is still valid (see [2, Th. 23], and for binary trees, [17]), the limit being now $\left(\frac{2e}{\sigma}, \frac{\sigma l}{2}\right)$. Here σ^2 denotes the variance of the offspring distribution of the corresponding critical Galton-Watson tree (see [1, p. 28, formula (8)] for the meaning of σ in term of simple trees). In the special case of binary trees with $n-1$ internal nodes and n leaves, it yields:

$$\frac{1}{\sqrt{2n}}\big(H_B^{(n)}, W_B^{(n)}\big) \xrightarrow{law} \Big(2 \max_{0\le t\le 1} e(t), \frac{1}{2} \max_{x\ge 0} l(x)\Big). \tag{2.5}$$

Theorem 1.1 is deduced from (2.5) through Jeulin's description of the local time of Brownian excursion. Let $\big(e(s)\big)_{0\le s\le 1}$ be a normalized Brownian excursion with local time $\big(l(x)\big)_{x\ge 0}$. Define

$$L(y) = \int_0^y l(x)\, dx$$

and

$$\psi(t) = L^{-1}(t) = \sup\Big\{y\Big| \int_0^y l(x) dx < t\Big\}.$$

Jeulin [18] proved that the process $\big(\tilde{e}(s)\big)_{0\le s\le 1}$ defined by:

$$\tilde{e}(s) = \frac{1}{2} l(\psi(s)) \tag{2.6}$$

is itself a normalized Brownian excursion (see also [4, p. 70] and interesting heuristic arguments [1, pp. 47-48]). Taking the derivative in $\psi(t) = L^{-1}(t)$, we obtain $\psi' = \frac{1}{L'\circ\psi} = \frac{1}{2\,\tilde{e}}$ and

$$\psi(t) = \int_0^t \frac{du}{2\,\tilde{e}(u)},$$

so Jeulin's representation can be rewritten:

$$l\Big(\int_0^s \frac{du}{2\,\tilde{e}(u)}\Big) \;=\; 2\,\tilde{e}(s). \tag{2.7}$$

A direct consequence is the identity:

$$\Big(2\max_{0\le t\le 1} e(t), \frac{1}{2}\max_{x\ge 0} l(x)\Big) = \Big(\int_0^1 \frac{ds}{\tilde{e}(s)}, \max_{0\le s\le 1} \tilde{e}(s)\Big). \tag{2.8}$$

The equality between first components of (2.8) follows from (2.7) because

$$\int_0^1 \frac{ds}{2\,\tilde{e}(s)} = \psi(1) = \max_{0\le s\le 1} e(s)\,, \tag{2.9}$$

while the equality between second components follows by taking the maximum on each side of (2.7). Thus, (2.5) is equivalent to Theorem 1.1. □

Thus Theorem 1.1 is a direct consequence of Jeulin's representation [18], and of [3, Theorem 3] which relies itself on two deep, but technical, papers [2, 12]. The line of the second proof of Theorem 1.1 is close to that of [7, 25]: the profile of the tree is seen as the breadth-first search random walk, changed of time, giving a discrete converse of Jeulin's representation. That the change of time has precisely the form given by Jeulin, follows, in the discrete case, from a counting principle due to Odlyzko [8, 21].

3 Second proof of Theorem 1.1

3.1 Brownian excursion and Bernoulli excursion

Let us call *Bernoulli excursion* of size $2n$, any $2n$-steps random walk $\omega = \big(S_k(\omega)\big)_{k=0,\cdots,2n}$ that satisfy:

$$S_0(\omega) = 0\,,\quad S_{2n}(\omega) = 0,\quad S_{k+1}(\omega) = S_k(\omega) \pm 1$$

and

$$S_k(\omega) > 0 \text{ for } k \in \{1, \cdots, 2n-1\}.$$

Let $Es(2n)$ denote the set of Bernoulli excursions of size $2n$, endowed with the uniform probability. It is well known that

$$\#Es(2n) = \#T_B^{(n)} = \binom{2n-1}{n-1}\frac{1}{2n-1},$$

is the $n-1^{th}$ Catalan number: C_{n-1} (see [24, pp.220-221, and 256-257]). Note that there is an obvious one-to-one correspondence between Bernoulli excursions and Dyck paths.

Any Bernoulli excursion ω defines a random element

$$e_n(t) = \frac{S_{\lfloor 2nt\rfloor}}{\sqrt{2n}},\quad 0 \le t \le 1,$$

of the set $D([0,1])$ of right continuous left limit functions, endowed with the Skorohod topology. The weak limit of e_n is called the *normalized Brownian excursion* (see [16]). The normalized Brownian excursion e is usually defined by the following path transformation of the standard linear Brownian motion $\mathcal{B} = (\mathcal{B}_t)_{t\geq 0}$: let g (resp. d) be the last zero of $\mathcal{B}$ before 1 (resp. after 1), and set

$$e(t) = \frac{\mathcal{B}_{g+t(d-g)}}{\sqrt{d-g}}, \qquad 0 \leq t \leq 1.$$

3.2 Breadth-first search correspondence

Let S_k be the height of the queue at the k^{th} step of the breadth-first search of a rooted binary tree $t \in T_B^{(n)}$ (see [10, Section 23.2], and Figure (3.1) for an example). Then $\omega = \left(S_k\right)_{k=0,\cdots,2n}$ belongs to $Es(2n)$, and this is a one-to-one correspondence (for instance, one can adapt [24, p. 256, 6.19.d]). We explain below how to obtain an expression of $(h(t), w(t))$ in term of functionals of the corresponding Bernoulli excursion ω.

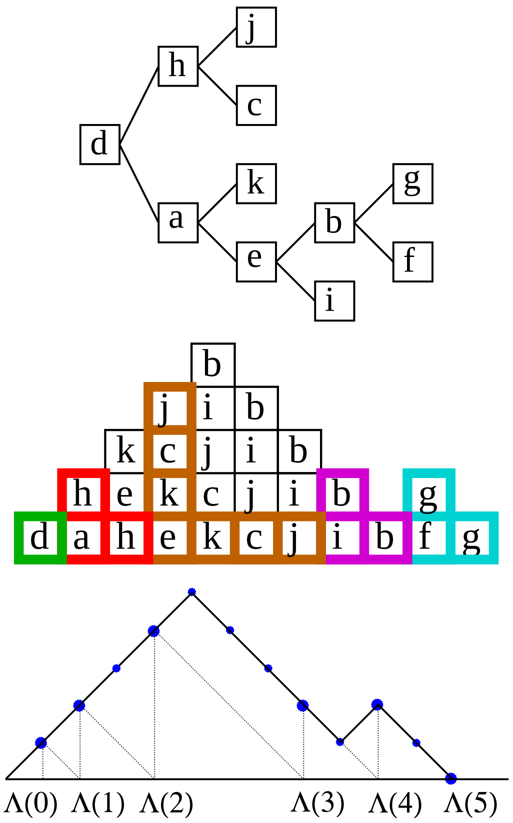

Figure 3.1 : Excursion - Binary tree

The width

As already noted in [7, 19, 25], the profile of t can be read on ω: assuming $S_k(\omega) = 0$ for $k \geq 2n+1$, we have

$$\begin{array}{rcl} Z_0(t) &=& S_1(\omega) = 1 \\ Z_1(t) &=& S_{1+Z_0(t)}(\omega) \\ Z_2(t) &=& S_{1+Z_0(t)+Z_1(t)}(\omega) \\ & \cdots & \\ Z_{k+1}(t) &=& S_{1+Z_0(t)+\cdots+Z_k(t)}(\omega). \end{array}$$

Set

$$\begin{array}{rcl} \Lambda(k) &=& 1 + Z_0(t) + \cdots + Z_{k-1}(t), \\ M_{2n}(\omega) &=& \max_k S_k(\omega). \end{array}$$

The triplet (S, Z, Λ) can be seen as the discrete version of $(\tilde{e}, l, L)$ appearing in Jeulin's representation. Since

$$W_B^{(n)}(t) = \max_k S_{\Lambda(k)}(\omega),$$

we obtain:

$$M_{2n}(\omega) \geq W_B^{(n)}(\omega),$$

but, actually, moderate variation of S_k (see Lemma 3.3) yields that:

Lemma 3.1

$$E\left[|W_B^{(n)}(t) - M_{2n}(\omega)|\right] = O(n^{1/4}\sqrt{\log n}).$$

The height

Set

$$\begin{array}{rcl} \Psi(k) &=& \displaystyle\sum_{j=1}^{2k-1} \frac{1}{S_j(\omega)}, \\ \tilde{\Psi}(k) &=& -1 + \inf\{j \mid \Lambda(j) = 2k\}. \end{array}$$

We see easily that

$$H_B^{(n)}(t) = \tilde{\Psi}(n). \tag{3.10}$$

The following Lemma can be seen as the discrete version of (2.9):

Lemma 3.2

$$E\left[|H_B^{(n)}(t) - \Psi(n)|\right] = o(\sqrt{n}).$$

Remark. Obviously, if the speed of a traveller at point y of the line is $s(y)$, then the duration t of the journey from point 0 to point x satisfies:

$$t = \int_0^x \frac{dy}{s(y)}. \tag{3.11}$$

Lemma 3.2 can be seen as a stochastic analog of relation (3.11), as $H_B^{(n)}(t)$ is the time needed to go from point 0 to point $2n$, doing one step (from $\Lambda(k)$ to by $\Lambda(k+1)$) by time unit, so the speed at point $\Lambda(k)$ is $\Lambda(k+1) - \Lambda(k) = S_{\Lambda(k)}$. This counting principle was used in [8, Section 2] and [21] in order to study the average cost of some search algorithms.

3.3 Proofs of Lemmata 3.1 and 3.2

The proofs of Lemmata 3.1 and 3.2 rely on a property of moderate variation of Bernoulli excursions, inherited from the simple symmetric random walk.
Let $\tilde{\Omega}_c(2n)$ denote the set of Bernoulli excursions $\omega = (S_k(\omega))_{k=0,\cdots,2n}$ such that for any l, k in the set $\{0, 1, \cdots, 2n\}$,

$$|S_k(\omega) - S_l(\omega)| \le c\sqrt{|k-l| \log n}.$$

Lemma 3.3 *For every $\beta > 0$ there exist $c > 0$ such that, for n sufficiently large:*

$$\Pr(\tilde{\Omega}_c(2n)) \ge 1 - n^{-\beta}.$$

Proof : The lemma is easily proved for a simple symmetric random walk $\omega = (S_k(\omega))_{k=0,\cdots,2n}$, using Chernoff bounds:

$$\forall x \ge 0,\ \forall k,\ \Pr(|S_k(\omega)| > x) \le 2\exp(-\frac{x^2}{2k})$$

(see for instance [8]). But $\Pr(\tilde{\Omega}_c(2n))$ in Lemma 3.3 is just $\Pr(\omega \in \tilde{\Omega}_c(2n) | \omega \in Es(2n))$, and in the other hand the probability that a simple symmetric random walk ω belongs to $Es(2n))$ is $\Theta(n^{-\frac{3}{2}})$. Finally, choose $A = \complement\tilde{\Omega}_c(2n)$ in:

$$\Pr(\omega \in A \,|\, \omega \in Es(2n)) \le \frac{\Pr(\omega \in A)}{\Pr(\omega \in Es(2n))} \le c_1 n^{\frac{3}{2}} \Pr(\omega \in A). \quad \Box$$

Proof of Lemma 3.1: We have

$$0 \le E\big(M_{2n}(\omega) - W_B^{(n)}(t)\big) = E\big(M_{2n}(\omega) - \max_k S_{\Lambda(k)}(\omega)\big).$$

We consider an index $K(\omega)$ such that $M_{2n}(\omega) = S_{K(\omega)}(\omega)$. There exists an integer $i(\omega)$ such that

$$\Lambda(i) \le K \le \Lambda(i+1).$$

Then,

$$\begin{aligned}
E(M_{2n}-\max_k S_{l(k)}) &\leq E(S_K - S_{\Lambda(i)}) \\
&\leq E\big(\mathbb{I}_{\tilde{\Omega}_c} c\sqrt{(K-\Lambda(i))\log(2n)} + n\,\mathbb{I}_{\complement\tilde{\Omega}_c}\big) \\
&\leq E\Big(c\sqrt{M_{2n}\log(2n)}\Big) + n\Pr(\complement\tilde{\Omega}_c) \\
&= O\Big(n^{1/4}(\log n))^{3/4}\Big),
\end{aligned}$$

for c large enough. □

Proof of Lemma 3.2. For any positive integers l,k, such that $k\leq l\leq 2n-k$, we have

$$\Pr(S_k = l) = \frac{n\,l^2}{k(2n-k)} \frac{\binom{k}{\frac{k-l}{2}}\binom{2n-k}{\frac{2n-k-l}{2}}}{\binom{2n-2}{n-1}} \mathbb{I}_{k\equiv l[2]}, \tag{3.12}$$

since $\frac{l}{k}\binom{k}{\frac{k-l}{2}}$ is the number of positive paths from $(0,0)$ to (k,l), and $\frac{l}{2n-k}\binom{2n-k}{\frac{2n-k-l}{2}}$ is the number of positive paths from (k,l) to $(2n,0)$. We have

$$H_B^{(n)} = \tilde{\Psi}(n) = \sum_{i=0}^{\tilde{\Psi}(n)-1}\sum_{h=\Lambda(i)}^{\Lambda(i+1)-1}\frac{1}{Z(i)}.$$

Let α be a real number in $]0,1/2[$. Then

$$\Big|\sum_{i=0}^{\tilde{\Psi}(n)-1}\sum_{h=\Lambda(i)}^{\Lambda(i+1)-1}\frac{1}{Z(i)} - \sum_{h=1}^{2n-1}\frac{1}{S_h}\Big| \leq 2n^{1/2-\alpha} + A_n + B_n$$

where

$$\begin{aligned}
A_n &\leq \sum_{i=0}^{\tilde{\Psi}(n)-1}\sum_{h=\Lambda(i)}^{\Lambda(i+1)-1}\Big|\frac{1}{Z(i)}-\frac{1}{S_h}\Big|\,\mathbb{I}_{Z(i)\geq(\log^{1+\varepsilon} n)}\;\mathbb{I}_{[n^{1/2-\alpha},2n-n^{1/2-\alpha}]}(h), \\
B_n &\leq \sum_{i=0}^{\tilde{\Psi}(n)-1}\sum_{h=\Lambda(i)}^{\Lambda(i+1)-1}\Big|\frac{1}{Z(i)}-\frac{1}{S_h}\Big|\,\mathbb{I}_{Z(i)\leq(\log n)^{1+\varepsilon}}\;\mathbb{I}_{[n^{1/2-\alpha},2n-n^{1/2-\alpha}]}(h).
\end{aligned}$$

First,

$$\begin{aligned}
E(B_n) &\leq E\big(\#\{h|h\in[n^{1/2-\alpha},2n-n^{1/2-a}], S_h\leq(\log n)^{1+\varepsilon}\}\big) \\
&\leq \sum_{h=n^{1/2-\alpha}}^{2n-n^{1/2-\alpha}}\sum_{l=1}^{[\log^{1+\varepsilon} n]}\Pr(S_h = l) \\
&= O(\log^{3+3\varepsilon} n)
\end{aligned}$$

where the last equality follows from (3.12), Stirling formula and $\binom{k}{k/2} \geq \binom{k}{(k-l)/2}$. Clearly,

$$\mathbb{I}_{\complement\tilde{\Omega}_c(2n)}(\omega)A_n \leq nI_{\complement\tilde{\Omega}_c(2n)}(\omega).$$

Finally, using the moderate variation property to bound

$$\left|\frac{1}{Z(i)} - \frac{1}{S_h}\right|,$$

we obtain:

$$\mathbb{I}_{\tilde{\Omega}_c(2n)}(\omega)A_n \leq (\log^{-\varepsilon/2} n)\ \Psi(n).$$

Lemma 3.2 follows, for c large enough. □

3.4 Convergence of $\big(\Psi(n), M_{2n}\big)$

Lemmata 3.1 and 3.2 together yields that:

$$\frac{1}{\sqrt{n}}\left\|\big(H_B^{(n)}, W_B^{(n)}\big) - \big(\Psi(n), M_{2n}\big)\right\|_1 = o(1).$$

Thus, the proof of Theorem 1.1 reduces to the proof of

Proposition 3.4

$$\frac{1}{\sqrt{2n}}\left(\Psi(n), M_{2n}\right) \stackrel{law}{\longrightarrow} \Big(\int_0^1 \frac{1}{e(s)}\ ds, \max_{0\leq s\leq 1} e(s)\Big).$$

We use the following Lemma [6, Th.4.2 p.25]:

Lemma 3.5 *Let $(X_n)_n$ and $(X_n^{(a)})_{n,a}$ be two families of $\mathbb{R}^2$ valued r.v., defined on the same probability space, such that:*

$$(X_n^{(a)})_n \underset{n\longrightarrow+\infty}{\stackrel{law}{\longrightarrow}} X^{(a)}$$

and

$$X^{(a)} \underset{a\longrightarrow 0}{\stackrel{law}{\longrightarrow}} X.$$

Assume that

$$\lim_{a\longrightarrow 0}\Big[\limsup_{n\longrightarrow+\infty} P(\|X_n - X_n^{(a)}\|_1 \geq \epsilon)\Big] = 0$$

for each positive ϵ. Then

$$X_n \stackrel{law}{\longrightarrow} X.$$

Proof of Proposition 3.4 : We have

$$M_{2n}/\sqrt{2n} = \max_{0\le t\le 1} e_n(t)$$

and

$$\frac{\Psi(n)}{\sqrt{2n}} = \frac{1}{2n}\sum_{k=1}^{2n-1}\frac{1}{e_n(k/2n)}.$$

Define $\Psi^{(a)}(n)$ by

$$\Psi^{(a)}(n) = \frac{1}{2n}\sum_{k=1}^{2n-1}\frac{\mathbb{I}_{\{e_n\ge a\}}}{e_n(k/2n)}.$$

Set

$$\begin{array}{rcl} X_n & = & \frac{1}{\sqrt{2n}}\big(\Psi(n), M_{2n}\big), \\ X_n^{(a)} & = & \frac{1}{\sqrt{2n}}\big(\Psi^{(a)}(n), M_{2n}\big), \\ X & = & \Big(\int_0^1 \frac{1}{e(s)}\, ds, \max_{0\le s\le 1} e(s)\Big), \\ X^{(a)} & = & \Big(\int_0^1 \frac{\mathbb{I}_{\{e(s)\ge a\}}}{e(s)}\, ds, \max_{0\le s\le 1} e(s)\Big). \end{array}$$

Proposition 3.4 is equivalent to

$$X_n \xrightarrow{law} X.$$

The convergence of $X_n^{(a)}$ to $X^{(a)}$ when n goes to ∞ results from the continuity of the functional. To conclude, it suffices to prove the two following lemmas:

Lemma 3.6 *There exists a positive constant C_1 such that, for any $a > 0$,*

$$\|X_n - X_n^{(a)}\|_1 = \frac{1}{\sqrt{2n}} E\Big(\sum_{k=1}^{2n-1}\frac{I_{S_k\le a\sqrt{2n}}}{S_k}\Big) \le C_1\, a.$$

Lemma 3.7 *There exists a positive constant C_2 such that, for any $a > 0$,*

$$\|X - X^{(a)}\|_1 = E\Big(\int_0^1 \frac{\mathbb{I}_{e(s)\le a}}{e(s)}\, ds\Big) \le C_2\, a.$$

Proof of Lemma 3.6 : Using Formula (3.12), we have

$$
\begin{aligned}
\frac{1}{\sqrt{2n}} E\Big(\sum_{k=1}^{2n-1} \frac{I_{S_k \le a\sqrt{2n}}}{S_k}\Big) &= \frac{1}{\sqrt{2n}} \sum_{k=1}^{2n-1} \sum_{l=1}^{a\sqrt{2n}} \frac{\tilde{\mathbb{P}}(S_k = l)}{l} \\
&\le c_1 . \frac{n}{\sqrt{n}\binom{2n-2}{n-1}} \sum_{k=1}^{n} \sum_{l=1}^{a\sqrt{2n}} \frac{l\binom{k}{\frac{k-l}{2}}\binom{2n-k}{\frac{2n-k}{2}}}{k(2n-k)} \mathbb{I}_{k\equiv l[2]} \\
&\le c_2 \frac{1}{\sqrt{n}} \sum_{l=1}^{a\sqrt{2n}} \sum_{k=1}^{n} \frac{l\binom{k}{\frac{k-l}{2}}}{2^k k} \mathbb{I}_{k\equiv l[2]}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{l\binom{k}{\frac{k-l}{2}}}{2^k k} \mathbb{I}_{k\equiv l[2]} &= \mathbb{P}(S_k = l)\frac{l}{k} \\
\mathbb{P}(S_k = l)\frac{l}{k} &= \frac{l}{k}\frac{1}{2\pi}\int_0^{2\pi} \cos^k(t)\cos(lt)dt \\
&= \frac{2}{\pi}\int_0^{\pi/2} \sin(lt)\cos^{k-1}(t)\sin t \; dt \text{ for } k \ge 1.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{1}{\sqrt{2n}} E\Big(\sum_{k=1}^{2n-1} \frac{I_{S_k \le a\sqrt{2n}}}{S_k}\Big) &\le \frac{c_3}{\sqrt{n}} \sum_{l=1}^{a\sqrt{2n}} \int_0^{\pi/2} \sin(lt)\sin t \; \frac{1-\cos^n t}{1-\cos t} dt \\
&= \frac{c_3}{\sqrt{n}} \sum_{l=1}^{a\sqrt{2n}} \int_0^{\pi/2} \sin(lt)\frac{(1+\cos t)(1-\cos^n t)}{\sin t} dt
\end{aligned} \tag{3.13}
$$

Let us expand this sum and bound its terms. Set

$$
\begin{aligned}
I_l &= \int_0^{\pi/2} \frac{\sin(lt)\cos t}{\sin t} dt, \\
J_l &= \int_0^{\pi/2} \frac{\sin(lt)}{\sin t} dt.
\end{aligned}
$$

We notice that

$$
J_l = I_{l-1} + \frac{\sin((l-1)\frac{\pi}{2})}{l-1} \text{ and } I_l = J_{l-1} + \frac{\sin(\frac{\pi l}{2})}{l}.
$$

So J_l and I_l are uniformly bounded. We have

$$
\begin{aligned}
\Big|\int_0^{\pi/2} \frac{\sin(lt)}{\sin t}\cos^n t \; dt\Big| &\le \int_0^{\pi/2} \Big|\frac{\sin(lt)}{\sin t}\Big| \cos^n t \; dt \\
&\le l\int_0^{\pi/2} \cos^n t \; dt \\
&= \frac{\sqrt{\pi}}{2}\frac{l\,\Gamma\big(\frac{n+1}{2}\big)}{\Gamma(n/2+1)}
\end{aligned}
$$

Due to Stirling formula, this last term is uniformly bounded for $l \in \{1, \cdots, a\sqrt{2n}\}$, so the terms of the sum in (3.13) are uniformly bounded and the proof is complete. $\square$

Proof of Lemma 3.7 : According to [14, Prop. 3.4],

$$E\Big(\int_0^1 \mathbb{I}_{e(s)\leq a}\, ds\Big) = 1 - 2e^{-2a^2},$$

we have:

$$E\Big(\int_0^1 \frac{\mathbb{I}_{e(s)\leq a}}{e(s)}\, ds\Big) = \int_0^a 8e^{-2a^2}\, da \leq 8\,a. \ \square$$

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