

Asymptotic of geometrical navigation on a random set of points of the plane

Nicolas Bonichon

Jean-François Marckert

CNRS, LaBRI, Université de Bordeaux

351 cours de la Libération

33405 Talence cedex, France

email: name@labri.fr

Abstract

A navigation on a set of points is a rule for choosing which point to move to from the present point in order to progress toward a specified target. In the present paper we study some “geometrical based” navigations in the two dimensional plane, that is, navigations where the point to move to is chosen according to some rules of geometrical nature. In particular, we are interested in asymptotic results, when the number of points goes to $+\infty$, and are chosen according to a (non-uniform) probability distribution with a bounded support. We obtain asymptotic results concerning the asymptotic geometry of the navigations paths, their asymptotic lengths, the number of stages of the traveller, and the behaviour of various cost functions. These results hold for every source and target simultaneously. This leads to asymptotic results on the stretch factors of random Yao-graphs and random θ -graphs.

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1 Introduction

1.1 Navigations

In a lot of situations, a traveller who can be a human being, a migratory animal, a letter, a radio message, a message in a wireless ad hoc network, ... wanting to reach a point t starting from a point s has to stop along the route where, according to the case, he can sleep, eat, be sorted, be amplified, or routed. Generally, the traveller can not stop everywhere: only some special places offer what is needed (a hostel, a river, a post-office, a radio relay station, a router, ...). Also, in many situations, the traveller can not compute the optimal route from its initial position: it has to choose the next point to move to using only some local information. This paper deals with this problem: a random set S of possible stops given, what happens generally when a traveller in the plane desiring to reach a point t , stops in the first point S that is in the right direction up to an angle θ ? How many steps are done? What is the total distance done? What happens if it has to follow some pre-existing road (some applications appear in computer science in relation with Yao’s graphs)? In this paper we answer these questions in the asymptotic case, when the number of points in S goes to $+\infty$.

Formally, consider a traveller on \mathbb{R}^2 beginning its travel at the *starting position* s and wanting to reach the *target* t using as *set of possible stopping places* S , a finite subset of \mathbb{R}^2 . We call navigation¹ with set of stopping places S , a function \mathbf{X} from $\mathbb{R}^2 \times \mathbb{R}^2$ onto \mathbb{R}^2 such that for any (s, t) , $\mathbf{X}(s, t)$ belongs to $S \cup \{t\}$, and satisfies moreover $\mathbf{X}(s, s) = s$ for any $s \in \mathbb{R}^2$. The position $\mathbf{X}(s, t)$ corresponds to the first stop of the traveller in its travel from s to t . Hence,

$$\mathbf{X}(s, t, j) := \mathbf{X}(\mathbf{X}(s, t, j - 1), t), \quad j \geq 1$$

are the successive stops of the traveller, where by convention $\mathbf{X}(s, t, j) = s$ for $j = 0$. When no confusion is possible on s, t and \mathbf{X} , we will write s_j instead of $\mathbf{X}(s, t, j)$. The quantity

$$\Delta^{\mathbf{X}}(s, t, j) := \mathbf{X}(s, t, j) - \mathbf{X}(s, t, j - 1)$$

is called the *j*th stage. For a general navigation algorithm \mathbf{X} , if $\#S < +\infty$, either $\mathbf{X}(s, t, j) = t$ for j large enough, or $\mathbf{X}(s, t, j) \neq t$ for all j . In the first case, the global navigation from s to t succeeds, when in the second case, it fails. In case of success, the (global) path from s to t is

$$\text{Path}^{\mathbf{X}}(s, t) := \left(s_0, \dots, s_{\text{Nb}^{\mathbf{X}}(s, t)} \right) \tag{1}$$

where $\text{Nb}^{\mathbf{X}}(s, t) := \min \{j, s_j = t\}$ is the number of stages needed to go from s to t .

We are interested in some “geometrical based” navigations in \mathbb{R}^2 , that is, navigations where the point to move to is chosen according to some rules of geometrical nature: we consider two classes of so-called *compass navigations*; these navigations select the next stopping place to move to as the “nearest” point of s in the set $S \cup \{t\}$, which is in the “direction” of t (see Section 1.2).

All along the paper, \mathcal{D} refers to a bounded and simply connected open domain in \mathbb{R}^2 . The sets of stopping places S we consider are finite random subsets of \mathcal{D} taken according to two models: S will be either the set $\{p_1, \dots, p_n\}$ where the points p_i ’s are picked independently according to a distribution having a regular density f (with respect to the Lebesgue measure) in \mathcal{D} , or S will be a Poisson point process with intensity nf , for some $n > 0$ (see Section 1.3).

The main goal of this paper is to study the global asymptotic behaviour of the paths of the traveller. Global means all the possible trajectories corresponding to all starting points s and targets t of \mathcal{D} are considered all together. Several quantities are then studied, describing the “deviation” of the paths of the traveller (or functionals of the path, as the length) to a deterministic limit (see Section 1.4). The asymptotic is made on the number of points of S , which will go to $+\infty$ (that is $n \rightarrow +\infty$ in one of the models).

Convention Throughout the paper, the two dimensional real plane \mathbb{R}^2 is identified with the set of complex numbers \mathbb{C} and according to what appears simpler, the complex notation or the real one is used without warning. The real part, the imaginary part, the modulus of z are respectively written $\Re(z)$, $\Im(z)$, and $|z|$; the argument $\arg(z)$ of any real number $z \neq 0$ is the unique real number

¹also called in the literature *memoryless routing algorithm*

θ in $[0, 2\pi)$ such that $z = \rho e^{i\theta}$, for some $\rho > 0$ (we set $\arg(0) = 0$). The characteristic function of the set A is denoted by $\mathbf{1}_A$. Notation $\llbracket x, y \rrbracket$ refers to the set of integers included in $[x, y]$ and $B(x, r) = \{y \in \mathbb{C}, |x - y| < r\}$ to the open ball with centre x and radius r in \mathbb{C} , the closed ball is $\overline{B}(x, r)$. For $x \in \mathbb{C}$, $A \subset \mathbb{C}$, $d(x, A) = \inf\{|x - y|, y \in A\}$.

1.2 Two types of navigations

The two types of navigation introduced below, namely *cross navigation* and *straight navigation*, may appear very similar, but their asymptotic behaviours as well as their analysis are quite different.

For any $\beta \in (0, 2\pi)$, as illustrated on Figure 1, let

$$\begin{aligned} \text{Sect}(\beta) &:= \{\rho e^{i\nu}, \rho > 0, |\nu| \leq \beta/2\}, \\ \text{Cam}(\beta)(h) &:= \text{Sect}(\beta) \cap \overline{B}(0, h), \\ \text{Tri}(\beta)(h) &:= \{x + iy, x \in (0, h], y \in \mathbb{R}, |y| \leq x \tan(\beta/2)\}. \end{aligned}$$

Notation Cam and Tri are short version for ‘‘Camembert portion’’ and ‘‘triangle’’.

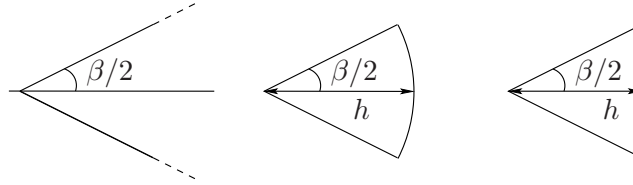


Figure 1: Representation of $\text{Sect}(\beta)$, $\text{Tri}(\beta)(h)$ and $\text{Cam}(\beta)(h)$. Notice that each peak point of the figures is excluded from the corresponding set.

1.2.1 First type of navigations: Cross navigations

Cross navigations are parameterized by a parameter θ satisfying $\theta = 2\pi/p_\theta$ for some $p_\theta \in \{3, 4, \dots\}$. For any $x \in \mathbb{C}$, for κ in $\llbracket 0, p_\theta - 1 \rrbracket$ the κ th *angular sector* around x is

$$\text{Sect}[\kappa, x] := x + e^{i\kappa\theta} \text{Sect}(\theta).$$

The two half-lines $\text{HL}_\kappa(x)$ and $\text{HL}_{\kappa+1}(x)$ defined by

$$\text{HL}_j(x) := x + \left\{ \rho e^{i\theta(j-1/2)}, \rho > 0 \right\}, \quad j \in \llbracket 0, p_\theta \rrbracket, \quad (2)$$

are called the *first* and *last border* of $\text{Sect}[\kappa, x]$. As illustrated in Figure 2, for $h > 0$, the κ th triangle and κ th Camembert section around z with height h are respectively:

$$\text{Tri}[\kappa, x](h) := x + e^{i\kappa\theta} \text{Tri}(\theta)(h), \quad \text{Cam}[\kappa, x](h) := x + e^{i\kappa\theta} \text{Cam}(\theta)(h),$$

where for any $z_1, z_2 \in \mathbb{C}$, and any $A \subset \mathbb{C}$, $z_1 + z_2A$ is the set $\{z_1 + z_2y, y \in A\}$.

p_θ Yao navigation and p_θ **CT** navigation, noted **CY** and **CT** are defined given a set of stopping places S , a finite subset of \mathbb{R}^2 .

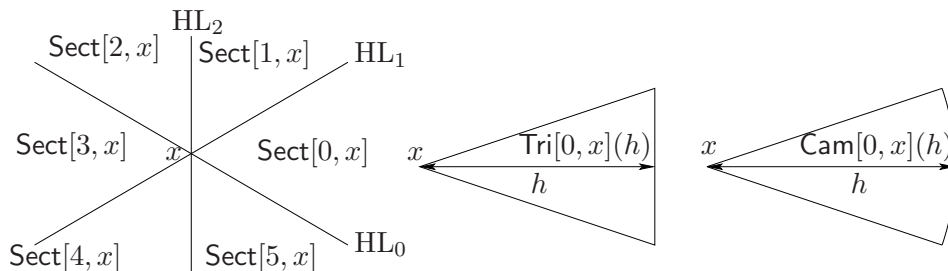


Figure 2: Representation of the 6 angular sectors around a point x when $p_\theta = 6$; on the second picture, representation of $Tri[0, x](h)$, and on the third $Cam[0, x](h)$.

Definition of p_θ Yao navigation

For $(s, t) \in \mathbb{R}^2 \times \mathbb{R}^2$, consider the smallest integer κ in $\llbracket 0, p_\theta - 1 \rrbracket$ such that t lies in $Sect[\kappa, s]$. Consider the smallest r such that $Cam[\kappa, s](r) \cap (S \cup \{t\})$ is not empty. We set $\mathbf{CY}(s, t) = z$, the element of $Cam[\kappa, s](r) \cap (S \cup \{t\})$ that is the closest of the first border of $Sect[\kappa, s]$ (see Figure 3).

This navigation has some applications in computer science (see the discussion in Section 2.5) where it appears to be the canonical navigation in Yao's graphs [Yao82].

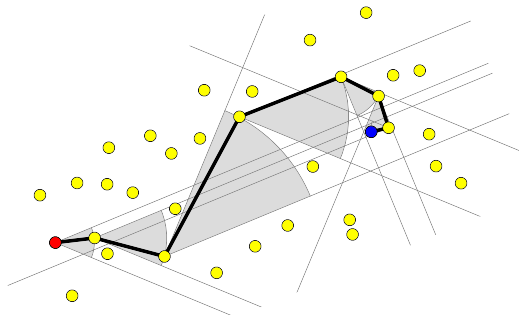


Figure 3: An example of Yao navigation with $p_\theta = 8$ the starting point is on the left. Are drawn the decision domains for each stage, and the sectors that contain the destination.

Definition of p_θ CT navigation

p_θ CT navigation CT is a simple variation of p_θ Yao navigation. It is defined as p_θ Yao navigation, except that $Tri[\kappa, s](r)$ replaces $Cam[\kappa, s](r)$.

Comments 1 When t belongs to $HL_\kappa(s)$ then t lies in $Sect[\kappa, s]$ and in $Sect[\kappa - 1 \bmod p_\theta, s]$; in the other cases, t belongs to a unique sector $Sect[\kappa, s]$. In the sequel, the domains $Cam[\kappa, s](r)$ and $Tri[\kappa, s](r)$ which are, depending on the algorithm considered, the minimum domains allowing to decide to which point to move to, are called the decision domains.

Now the terminology “cross navigation” comes from the fact that, as one can see on Figure 2, the half-lines $(HL_\kappa(t), \kappa \in \llbracket 1, p_\theta \rrbracket)$, forms a cross around t . We set $\mathbf{Cross}(t) := \bigcup_{\kappa=1}^{p_\theta} HL_\kappa(t)$.

Definition of Navigating Graph

Given a set of stopping places S and a navigation \mathbf{X} , the *navigating graph* $G_S^{\mathbf{X}}$ is the graph with set of vertices S , and whose edges are the pairs $\{s, t\}$, with $(s, t) \in S^2$, such that there exists a point $t' \in \mathcal{D}$ satisfying $\mathbf{X}(s, t') = t$. Yao $_{p_\theta}$ -graph (resp. Θ_{p_θ} -Graph) denotes the graph associated with p_θ Yao navigation (resp. p_θ **CT** navigation).

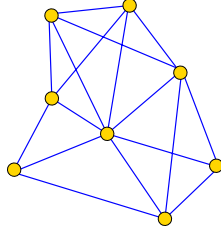


Figure 4: Navigating graph for **CT**, in the case $p_\theta = 6$.

1.2.2 Second type of navigations: Straight navigation

Straight navigations are natural in real life where often a traveller tries to stop not too far from the segment leading to the target. We now introduce two straight navigations, parameterized by a real number $\theta \in (0, 2\pi)$. The difference between the following objects and those before defined is that the sectors are oriented such that their bisecting lines are the straight line (s, t) :

$$\begin{aligned} \overrightarrow{\text{Sect}}(s, t) &:= s + \exp(i \arg(t - s)) \text{Sect}(\theta), \\ \overrightarrow{\text{Cam}}(s, t)(h) &:= s + \exp(i \arg(t - s)) \text{Cam}(\theta)(h), \\ \overrightarrow{\text{Tri}}(s, t)(h) &:= s + \exp(i \arg(t - s)) \text{Tri}(\theta)(h). \end{aligned}$$

θ Straight Yao navigation and θ Straight T navigation will be denoted **SY** and **ST**.

θ Straight Yao navigation

For $(s, t) \in S^2$, consider the smallest r such that the set $\overrightarrow{\text{Cam}}(s, t)(r) \cap (S \cup \{t\})$ is not empty. Set **SY** $(s, t) = z$, where z is the closest of the first border of $\overrightarrow{\text{Cam}}(s, t)$ (see Figure 5).

θ Straight T navigation

The θ Straight T navigation **ST** is similarly defined except that $\overrightarrow{\text{Tri}}(s, t)(r)$ replaces $\overrightarrow{\text{Cam}}(s, t)(r)$.

1.3 The model of random stopping places set: a Poisson point process

Denote by $\text{Lip}^*(\mathcal{D})$ the set of Lipschitzian functions $f : \mathcal{D} \rightarrow \mathbb{R}^+$ with a positive infimum on \mathcal{D} : a function f is in $\text{Lip}^*(\mathcal{D})$, if there exists $\alpha_f > 0$, such that $|f(x) - f(y)| \leq \alpha_f |x - y|$ for all $x, y \in \mathcal{D}$, and $m_f := \inf\{f(x), x \in \mathcal{D}\} > 0$. Since \mathcal{D} is bounded, $M_f := \max\{f(x), x \in \mathcal{D}\} < +\infty$.

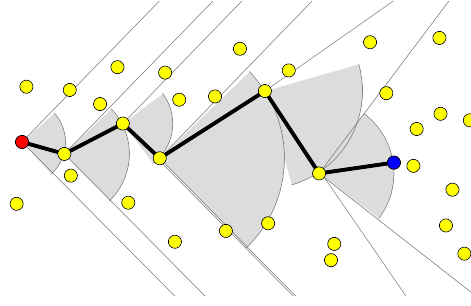


Figure 5: An example of Yao straight navigation with $\theta = \pi/2$ the starting point is on the left. Are drawn the decision domains for each stage, and the sectors that contain the destination.

Here for technical reasons – a differential equation will appear – we consider Poisson point processes \mathbf{K}_f with intensity a finite measure μ_f on \mathcal{D} having a density $f \in \text{Lip}^*(\mathcal{D})$ with respect to the Lebesgue measure on \mathcal{D} (for short, we say that f is the intensity of the PPP instead of μ_f); its support is denoted by $\mathbf{S}(f)$. The distribution of $\mathbf{S}(f)$ is denoted by \mathbb{P}_f . Notice that a.s. \mathbf{K}_f has no two atoms at the same position and then $\#\mathbf{S}(f) \sim \text{Poisson}(\int_{\mathcal{D}} f(u) du)$. For any disjoint Borelian subsets B_1, \dots, B_k of \mathbb{R}^2 , $(\mathbf{S}(f) \cap B_1), \dots, (\mathbf{S}(f) \cap B_k)$ are independent and $\#(\mathbf{S}(f) \cap B_j) \sim \text{Poisson}(\int_{B_j} f(u) du)$ (for more explanations on PPP, see Chap. 10 in [Kal97]). As usual, \mathbf{K}_f can be represented as $\mathbf{K}_f = \sum_{i=1}^n \delta_{x_i}$ where δ denotes the Dirac measure, $\mathbf{n}, x_1, x_2, \dots$, is a family of independent random variables, $\mathbf{n} \sim \text{Poisson}(\int_{\mathcal{D}} f)$ and each x_i has density $f / \int_{\mathcal{D}} f$.

In the paper, we assume \mathbf{K}_f to be defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and is considered as a random variable taking its values in \mathbb{N} the set of counting measures in \mathbb{C} , equipped with the σ -field \mathcal{N} generated by the sets $E_{B,k} = \{\mu, \mu(B) = k\}$ for compacts sets $B \subset \mathbb{C}$, k integer.

The model consisting of n i.i.d. points chosen according to a density f is considered in Section 2.3.3.

Comments 2 *The navigations \mathbf{X} we study, as well as the function $\text{Path}^{\mathbf{X}}, \text{Pos}^{\mathbf{X}}$, and various cost functions are all defined given a stopping places set S , and then, there are some functions of S (one should write $\mathbf{X}(S)$, etc. but we choose to delete this S to lighten the notation). On $(\Omega, \mathcal{A}, \mathbb{P})$, these functions are random variables with values in some functional spaces.*

1.4 Quantities of interest

In (1) is defined the global path $\text{Path}^{\mathbf{X}}(s, t)$ of the traveller between two points s and t . In many applications, a quantity of interest is the comparison between the Euclidean distance $|s - t|$ and the *path length*, the total distance done by the traveller in case of success:

$$|\text{Path}^{\mathbf{X}}(s, t)| := \sum_{j=1}^{\text{Nb}^{\mathbf{X}}(s, t)} |s_j - s_{j-1}|.$$

The associated *trajectory* is the compact subset of \mathbb{R}^2 formed by the union of the segments $[s_j, s_{j+1}]$:

$$[\text{Path}^{\mathbf{X}}(s, t)] := \bigcup_{j=1}^{\text{Nb}^{\mathbf{X}}(s, t)} [s_{j-1}, s_j].$$

One of the results of the paper is the comparison between $[\text{Path}^{\mathbf{X}}(s, t)]$ and a limiting object.

When the point process is not homogeneous (case where f is no constant), in order to evaluate the traveller trajectory, a precise study of the “speed” of the traveller is needed. We introduce the function $\text{Pos}_{s,t}^{\mathbf{X}} : [0, \text{Nb}^{\mathbf{X}}(s, t)] \rightarrow \mathbb{R}^2$ whose values coincides with those of $\mathbf{X}(s, t, \cdot)$ at integer points, and interpolated in between. Hence, $\text{Pos}_{s,t}^{\mathbf{X}}$ gives the position of the traveller according to the number of stages; for any $s, t \in \mathcal{D}$, $[\text{Path}^{\mathbf{X}}(s, t)]$ equals the range of $\text{Pos}_{s,t}^{\mathbf{X}}$. From our point of view, the asymptotic results obtained for $\text{Pos}_{s,t}$ are among the main contributions of this paper.

A consequence is the control of the stretch of the considered navigation algorithms. A geometric graph $G(S)$ has a *stretch* (α, β) – we can say also that $G(S)$ is a (α, β) -*spanner* – if $\forall s, t \in D, d_G(s, t) \leq \alpha|s - t| + \beta$. If $\beta = 0$ the stretch is said to be *multiplicative* and if $\alpha = 0$ the stretch is said *additive*. Here, similarly, we consider that a navigation \mathbf{X} has stretch (α, β) if $\forall s, t \in S, |\text{Path}^{\mathbf{X}}(s, t)| \leq \alpha|t - s| + \beta$. Some theoretical bound of the stretch are known; they correspond to the worst locations for the points of S .

In some applications (see Section 2.5), the sum of a function of the stage lengths appears to be the relevant quantity instead of the length. Formally a “unitary cost function” $H : \mathbb{C} \rightarrow \mathbb{R}$ is considered. The total cost associated with H corresponding to $\text{Path}^{\mathbf{X}}(s, j) = (s_0, \dots, s_{\text{Nb}^{\mathbf{X}}(s, t)})$ is

$$\text{Cost}_H^{\mathbf{X}}(s, t) := \sum_{j=1}^{\text{Nb}^{\mathbf{X}}(s, t)} H(s_j - s_{j-1}). \quad (3)$$

Important elementary cost functions are $H : x \rightarrow |x|$ which gives $\text{Cost}_H(s, t) = |\text{Path}(s, t)|$, the function $H : x \rightarrow 1$ which gives $\text{Cost}_H^{\mathbf{X}}(s, t) = \text{Nb}^{\mathbf{X}}(s, t)$, and functions of the type $H(x) = |x|^\beta$ which corresponds, at least for some β , to some real applications (see Section 2.5).

2 Statements of the main results

In the rest of the paper, f is a fixed element of $\text{Lip}^*(\mathcal{D})$ and a a fixed positive number. The set $\mathcal{D}[a] = \{x \in \mathcal{D}, d(x, \mathcal{C}\mathcal{D}) \geq a\}$ is the set of points in \mathcal{D} at distance at least a to $\mathcal{C}\mathcal{D}$.

The asymptotic behaviour of $\text{Path}^{\mathbf{X}}(s, t)$ (when the set of stopping places S increases) is difficult to predict if \mathcal{D} is not regular or if s or t are close to $\mathcal{C}\mathcal{D}$. The cases where in some sense paths may be not regular have to be excluded: this leads to some restrictions on the pairs (s, t) in the theorems. We also add restrictions on θ to avoid situations where the navigation can fail.

All the results are stated under \mathbb{P}_{nf} ; nevertheless, in Section 2.3.3, the case where the point process is constituted by n i.i.d. points chosen in \mathcal{D} is treated, and in some sense reduced to \mathbb{P}_{nf} .

Before giving the results, we define some constants, computed in Sections 3.3.1 and 3.3.3, and related to the speed of the traveller along its trajectory, and to some ratio “mean length of a stage” divided by “length of the projection of a stage” with respect to some ad hoc directions.

$\mathbf{C}_{\text{bis}}^{\text{ST}} = \mathbf{C}_{\text{bis}}^{\text{CT}} = \frac{1}{2} \sqrt{\frac{\pi}{\tan(\theta/2)}}$	$\mathbf{C}_{\text{bis}}^{\text{SY}} = \mathbf{C}_{\text{bis}}^{\text{CY}} = \frac{\sqrt{2\pi} \sin(\theta/2)}{\theta^{3/2}}$	(4)
$\mathbf{C}_{\text{bor}}^{\text{CT}} = \sqrt{\frac{\pi \cos^3(\theta/2)}{4 \sin(\theta/2)}}$	$\mathbf{C}_{\text{bor}}^{\text{CY}} = \frac{\sqrt{\pi/2} \sin(\theta)}{\theta^{3/2}}$	
$\mathbf{Q}_{\text{bis}}^{\text{ST}} = \mathbf{Q}_{\text{bis}}^{\text{CT}} = \frac{1}{2} \left(\frac{1}{\cos(\theta/2)} + \frac{\text{arcsinh}(\tan(\theta/2))}{\tan(\theta/2)} \right)$	$\mathbf{Q}_{\text{bis}}^{\text{SY}} = \mathbf{Q}_{\text{bis}}^{\text{CY}} = \frac{\theta/2}{\sin(\theta/2)}$	
$\mathbf{Q}_{\text{bor}}^{\text{CT}} = \frac{\theta}{\sin(\theta)}$	$\mathbf{Q}_{\text{bor}}^{\text{CY}} = \frac{1}{2} \left(\frac{1}{\cos^2(\theta/2)} + \frac{\text{arcsinh}(\tan(\theta/2))}{\sin(\theta/2)} \right)$	

For any $r > 0$ and any subset A of \mathbb{C} , set $A^r := \bigcup_{x \in A} B(x, r)$. The Hausdorff distance on \mathbb{C} is the distance on the set of compact subsets of \mathbb{C} defined by

$$d_H(K_1, K_2) := \max \{ \inf \{ r > 0, K_2 \subset K_1^r \}, \inf \{ r > 0, K_1 \subset K_2^r \} \}.$$

2.1 Limit theorems for the straight navigations

The first theorem expresses that for straight navigations, the traveller paths are asymptotically segments. The path length $|\text{Path}^{\mathbf{X}}(s, t)|$ is uniformly compared with a multiple of $|s - t|$.

Theorem 3 *Let $\mathbf{X} = \mathbf{SY}$ and $\theta \leq 2\pi/3$, or $\mathbf{X} = \mathbf{ST}$ and $\theta \leq \pi/2$. For any $\alpha \in (0, 1/8)$, any $\beta > 0$, for n large enough*

$$\mathbb{P}_{nf} \left(\sup_{(s,t) \in \mathcal{D}, [s,t] \subset \mathcal{D}[a]} |\text{Path}^{\mathbf{X}}(s, t) - \mathbf{Q}_{\text{bis}}^{\mathbf{X}} \times |s - t|| \geq n^{-\alpha} \right) \leq n^{-\beta}$$

and the trajectory converges to the straight line for the Hausdorff metric:

$$\mathbb{P}_{nf} \left(\sup_{(s,t) \in \mathcal{D}, [s,t] \subset \mathcal{D}[a]} d_H([\text{Path}^{\mathbf{X}}(s, t)], [s, t]) \geq n^{-\alpha} \right) \leq n^{-\beta}.$$

The terms $\mathbf{Q}_{\text{bis}}^{\text{ST}}$ and $\mathbf{Q}_{\text{bis}}^{\text{SY}}$ appear to be some quotients measuring some local stretch of the traveller relatively to the direction of the bisecting lines of the decision sectors used.

This theorem implies the following result concerning the stretch for straight navigations.

Corollary 4 *Under the hypothesis of Theorem 3, for any $\varepsilon > 0$ and $\nu > -1/8$*

$$\mathbb{P}_{nf} \left(\sup_{(s,t) \in \mathcal{D}, [s,t] \subset \mathcal{D}[a], |t-s| \geq n^\nu} \left| \frac{|\text{Path}^{\mathbf{X}}(s, t)|}{|s - t|} - \mathbf{Q}_{\text{bis}}^{\mathbf{X}} \right| \geq \varepsilon \right) \xrightarrow{n} 0. \quad (5)$$

Notice that in the corollary are excluded the pairs (s, t) where s and t are too close; for those pairs of points, large number type compensations, used or far away points, may fail.

We now describe the asymptotic of the position function $\text{Pos}_{s,t}$. This question is of primary importance: we first study the position and then deduce the distance done by the traveller by integrating a cost function along the traveller's path. In the case of straight navigation, the limiting position function $\text{Pos}^{\infty,c}$ is deterministic, and depends on a real parameter c , function of θ and of the navigation algorithm \mathbf{X} used. This function is the solution of a differential equation.

We introduce a family of differential equations indexed by $(\lambda, \nu) \in (0, +\infty) \times [0, 2\pi]$. For any such (λ, ν) , let $F_{\lambda,\nu}$ be the function from \mathcal{D} into \mathbb{C} defined by

$$F_{\lambda,\nu}(z) := \lambda e^{i\nu} / \sqrt{f(z)}. \quad (6)$$

For $s_0 \in \mathcal{D}$ given, consider the following ordinary differential equation

$$\text{ODE}(\lambda, \nu, s_0) := \begin{cases} \rho(0) & = s_0, \\ \frac{\partial \rho(x)}{\partial x} & = F_{\lambda,\nu}(\rho(x)). \end{cases} \quad (7)$$

$\text{ODE}(\lambda, \nu, s_0)$ admits a unique solution $x \mapsto \text{SOL}_{s_0}^{\lambda,\nu}(x)$ valid for $x \geq 0$ and while $\text{SOL}_{s_0}^{\lambda,\nu}(x)$ stays inside \mathcal{D} , thanks to the Cauchy-Lipschitz theorem; indeed since f is in $\text{Lip}^*(\mathcal{D})$, so do $F_{\lambda,\nu}$.

Let us discuss a bit the behaviour of $\text{SOL}_{s_0}^{\lambda,\nu}$. Since f takes its values in \mathbb{R}^+ , $\arg(F_{\lambda,\nu}) = \nu$. Hence, $\text{SOL}_{s_0}^{\lambda,\nu}$ appears to be a monotone parameterization of the segment $\{s_0 + x e^{i\nu}, x > 0\} \cap \mathcal{D}$ (more precisely, of the component containing s_0 in this intersection). The speed along this segment at position z is given by $F_{\lambda,\nu}(z)$. Now, choose two points (s, t) in \mathcal{D} such that $[s, t] \subset \mathcal{D}$ and consider $\text{SOL}_s^{\lambda, \arg(t-s)}$ for some fixed $\lambda > 0$. In this case, the range of $\text{SOL}_s^{\lambda, \arg(t-s)}$ contains $[s, t]$ and by continuity there exists a time $\text{Time}_{s,t}^\lambda$ such that

$$\text{SOL}_s^{\lambda, \arg(t-s)} \left(\text{Time}_{s,t}^\lambda \right) = t.$$

We then define the function $\text{Pos}_{s,t}^{\infty,\lambda}$ as the function $\text{SOL}_s^{\lambda, \arg(t-s)}$ stopped at t .

Theorem 5 *Let $\mathbf{X} = \mathbf{SY}$ and $\theta \leq 2\pi/3$, or $\mathbf{X} = \mathbf{ST}$ and $\theta \leq \pi/2$. For any $\alpha \in (0, 1/8)$, any $\beta > 0$ for n large enough*

$$\mathbb{P}_{nf} \left(\sup_{(s,t) \in \mathcal{D}, [s,t] \subset \mathcal{D}[a]} \sup_{x \geq 0} \left| \text{Pos}_{s,t}^{\mathbf{X}}(x\sqrt{n}) - \text{Pos}_{s,t}^{\infty, \mathbf{C}_{\text{bis}}^{\mathbf{X}}}(x) \right| \geq n^{-\alpha} \right) \leq n^{-\beta}.$$

The asymptotic behaviour of different cost functions are studied in Section 4.2.

2.2 Limit theorems for cross navigation

Let s and t in \mathcal{D} , and let κ be such that $t \in \text{Sect}[\kappa, s]$. Consider $\mathbf{B}_\kappa(s) = \{s + r e^{i\kappa\theta}, r \geq 0\}$ the bisecting half-line of $\text{Sect}[\kappa, s]$. Consider the lines parallel to $\text{HL}_\kappa(s)$ and $\text{HL}_{\kappa+1}(s)$ passing via t .

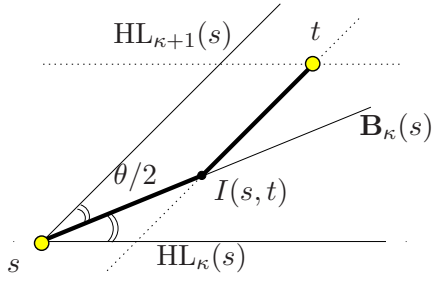


Figure 6: Definition of $I(s, t)$. On the picture, $\mathbf{B}_\kappa(s)$ is the bisecting line of $\text{Sect}[\kappa, s]$. $\Gamma(s, t)$ is represented by the heavy segments.

These two lines intersect the half-line $\mathbf{B}_\kappa(s)$ in two points. The point which is the closest from s is called $I(s, t)$, as represented on Figure 6. The compact $\Gamma(s, t)$ is:

$$\Gamma(s, t) := [s, I(s, t)] \cup [I(s, t), t]. \quad (8)$$

For any $c_1 > 0, c_2 > 0$, and any $s, t \in \mathbb{C}$, let

$$\mathbf{D}_{c_1, c_2}(s, t) = c_1 \times |I(s, t) - s| + c_2 \times |t - I(s, t)| \quad (9)$$

be a kind of weighted length of $\Gamma(s, t)$.

The “cross navigations” analogous of Theorem 3 is Theorem 6 below (see Figure 7).

Theorem 6 *Let $\mathbf{X} \in \{\mathbf{CT}, \mathbf{CY}\}$ and $\theta \leq \pi/3$. For any $\alpha \in (0, 1/8)$, any $\beta > 0$, for n large enough*

$$\mathbb{P}_{nf} \left(\sup_{(s,t) \in \mathcal{D}, \Gamma(s,t) \subset \mathcal{D}[a]} \left| |\text{Path}^{\mathbf{X}}(s, t)| - \mathbf{D}_{\mathbf{Q}_{\text{bis}}^{\mathbf{X}}, \mathbf{Q}_{\text{bor}}^{\mathbf{X}}}(s, t) \right| \geq n^{-\alpha} \right) \leq n^{-\beta}$$

and the trajectories converge to $\Gamma(s, t)$ for the Hausdorff metric:

$$\mathbb{P}_{nf} \left(\sup_{(s,t) \in \mathcal{D}, \Gamma(s,t) \subset \mathcal{D}[a]} d_H([\text{Path}^{\mathbf{X}}(s, t)], \Gamma(s, t)) \geq n^{-\alpha} \right) \leq n^{-\beta}.$$

The terms $\mathbf{Q}_{\text{bor}}^{\text{ST}}$ and $\mathbf{Q}_{\text{bor}}^{\text{SY}}$ appear to measure some local stretch relatively to the direction of the border of the used decision sectors on the second part of its journey.

A first corollary is that for s fixed, for $\mathbf{X} \in \{\mathbf{CT}, \mathbf{CY}\}$, the set $\{t \in \mathcal{D}, |\text{Path}^{\mathbf{X}}(s, t)| \leq r\}$ of targets at “traveller distance” smaller than r , does not converge to the Euclidean ball for the Hausdorff distance as in the case of straight navigations but converges in probability to a regular p_θ -gone (provided that this p_θ -gone is included in \mathcal{D}). Another corollary that can be easily deduced concerning the stretch:

Corollary 7 *Let $\mathbf{X} \in \{\mathbf{CT}, \mathbf{CY}\}$. For any $\theta \geq \pi/3$, any $\varepsilon > 0$, and $\nu > -1/8$*

$$\mathbb{P}_{nf} \left(\left| \mathbf{Q}_{\text{bor}}^{\mathbf{X}} - \sup_{(s,t) \in \mathcal{D}, \Gamma(s,t) \subset \mathcal{D}[a], |t-s| \geq n^\nu} \frac{|\text{Path}^{\mathbf{X}}(s, t)|}{|s-t|} \right| \geq \varepsilon \right) \xrightarrow[n]{} 0. \quad (10)$$

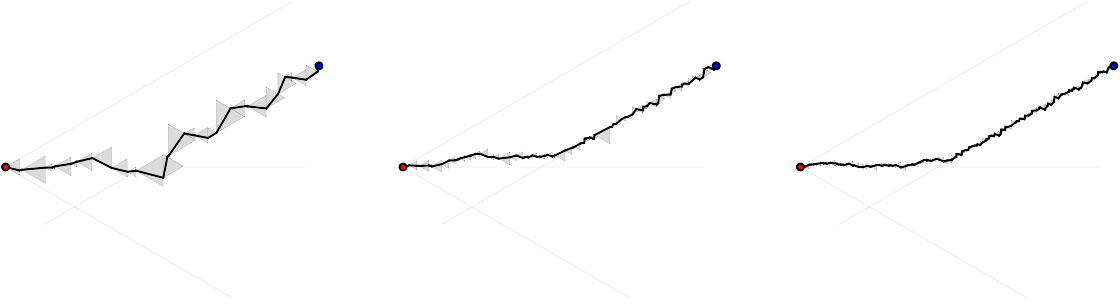


Figure 7: Examples of p_θ CT navigation (for $p_\theta = 6$) with 500 nodes, 5000 nodes and 20000 nodes uniformly distributed in the unit square.

This is much different that in the straight case: the supremum is inside the absolute value, when it was outside in the cross case. Here the limiting traveller distance \mathbf{D}_{c_1, c_2} is not invariant by rotation: the largest quotient $|\text{Path}^{\mathbf{X}}(s, t)|/|s - t|$ for far away points s, t occurs when $t \in \text{Cross}(s)$.

Again, for $\mathbf{X} \in \{\mathbf{CT}, \mathbf{CY}\}$, the function $x \mapsto \text{Pos}_{s,t}^{\mathbf{X}}(\cdot, \sqrt{n})$ admits a deterministic limit $\text{Pos}_{s,t}^{\infty, a, b}$ depending on two parameters a and b (which depend on \mathbf{X} and θ) related to the speed of the traveller along the two branches of $\Gamma(s, t)$. Set

$$\text{Pos}_{s,t}^{\infty, \lambda, \lambda'}(x) = \begin{cases} \text{SOL}_s^{\lambda, \arg(I(s,t)-s)}(x) & \text{for } x \in [0, \text{Time}_{s,I(s,t)}^\lambda] \\ \text{SOL}_{I(s,t)}^{\lambda', \arg(t-I(s,t))}(x - \text{Time}_{s,I(s,t)}^{\lambda(1)}) & \text{if } x \in \text{Time}_{s,I(s,t)}^\lambda + [0, \text{Time}_{I(s,t),t}^{\lambda'}] \\ t & \text{if } x \geq \text{Time}_{s,I(s,t)}^\lambda + \text{Time}_{I(s,t),t}^{\lambda'} \end{cases} \quad (11)$$

Theorem 8 *Let $\mathbf{X} \in \{\mathbf{CY}, \mathbf{CT}\}$ and $\theta \leq \pi/3$, for any $\alpha \in (0, 1/8)$, any $\beta > 0$, any n large enough*

$$\mathbb{P}_{nf} \left(\sup_{(s,t) \in \mathcal{D}} \sup_{[s,t] \subset \mathcal{D}[a]} \sup_{x \geq 0} \left| \text{Pos}_{s,t}^{\mathbf{X}}(x\sqrt{n}) - \text{Pos}_{s,t}^{\infty, \mathbf{C}_{\text{bis}}^{\mathbf{X}}, \mathbf{C}_{\text{bor}}^{\mathbf{X}}}(x) \right| \geq n^{-\alpha} \right) \leq n^{-\beta}.$$

Again, other results concerning other cost functions are studied in Section 4.2.

2.3 Extensions

The following extensions can be treated with the material available in the present paper. We just provide the main lines of their analysis.

2.3.1 Random north navigation

This is a version of the cross navigation where each point $s \in S$ has its own (random) north $n(s)$ used to compute $\text{Cross}(s)$. The random variables $(n(s), s \in S)$ are assumed to be i.i.d. and takes their values in $[0, 2\pi]$. This can be used to model some imprecisions in the Yao's construction,

where the north is not exactly known by the points of S . The corresponding navigation algorithm is defined as follows: A traveller at s , choose the smallest integer κ in $\llbracket 0, p_\theta - 1 \rrbracket$ such that t lies in $s + e^{in(s)+ik\theta}\text{Sect}(\theta)$ and consider the smallest r such that $s + e^{in(s)+ik\theta}\text{Tri}(\theta)(r) \cap (S \cup \{t\})$ is not empty. Then set $\mathbf{RNT}(s, t) = z$, the element of this set with smallest argument.

Using the Camembert sections instead leads to the random north Yao's navigation, \mathbf{RNY} .

Each distribution of $n(s)$ leads to a different behaviour for the traveller. If the value of $n(s)$ is 0 a.s., then this models coincides with the standard cross navigation. If the distribution of $n(s)$ owns several atoms, then the speed of the traveller may be different along numerous directions. Assume that $n(s)$ is uniform on $[0, 2\pi]$ (or on $[0, k\theta]$ for some $k > 0$). In this case, this model is not much different to the straight navigation, the limiting path being straight, and the limiting path length being a multiple of the Euclidean distance. In this case, using the same arguments than those given in the present paper, one can prove that the distance done by the traveller satisfies, for any $\theta \leq \pi/3$, for any $\alpha \in (0, 1/8)$ and $\beta > 0$, if n is large enough

$$\mathbb{P}_{n,f} \left(\sup_{(s,t) \in \mathcal{D}, [s,t] \subset \mathcal{D}[a]} |\text{Path}^{\mathbf{X}}(s, t) - \mathbf{Q}^{\mathbf{X}}|_{s-t}| \geq n^{-\alpha} \right) \leq n^{-\beta}$$

where

$$\mathbf{Q}^{\mathbf{RNT}} = \frac{\theta/2}{2 \sin(\theta/2)} \left(\frac{1}{\cos(\theta/2)} + \frac{\text{arcsinh}(\tan(\theta/2))}{\tan(\theta/2)} \right), \quad \mathbf{Q}^{\mathbf{RNY}} = \frac{\theta^2}{2 - 2 \cos(\theta)}.$$

The computation of these constants are done in Section 3.3.4; the proof of the globalization of the bounds do not present any problem in this case.

2.3.2 Half- Θ_6 -graph

We are here interested in a sub-graph of Θ_6 -graph (defined in Section 1.2.1) called *half- Θ_6 -graph*: this graph is the sub-graph of Θ_6 -graph composed of edges defined in even sectors (see Figure 8)². Observe that a point u is in an even sector of a point v if and only if v is in an odd

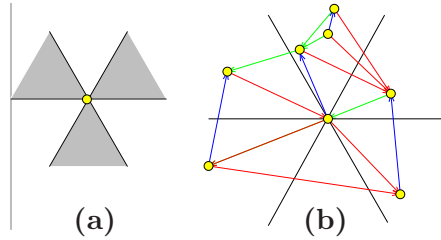


Figure 8: (a) Edges in grey sectors are ignored in half- Θ_6 -graphs.(b) An example of a directed half- Θ_6 -graphs (the colour of the edges correspond to the number of the sector), and the arrows figure out the construction of the edge, which are considered as non oriented.

²Note that the x -axis is the right border of the first sector and not its bisector as in the rest of the paper. This is the convention used in the literature.

sector of u . As observed in [BGHI10], the half- Θ_6 -graph of S appears to be exactly the *TD-Delaunay triangulation* of S , where TD comes from *Triangular Distance*, introduced by Chew in [Che89] which is a generalisation of the Delaunay graph. TD-Delaunay triangulations, which are planar, appear to be (2,0)-spanners [Che89].

In order to bound the stretch of random half- Θ_6 -graph built with a PPP under \mathbb{P}_{nf} , let us compute the length of a special path between two points s and t ; this path is not a traveller path but rather corresponds to “two paths”, for two travellers starting from s and from t , till they meet at a point $M(s, t)$.

First, for $i \in \{0, 2, 4\}$, let $\text{Path}^i(s)$ be the directed path starting at s and composed of edges of the sector i . In words, for a fixed i , $\text{Path}^i(s) = (s_0 = s, s_1, s_2, \dots)$ where s_j is in the sector i of s_{j-1} , and $\mathbf{CT}(s_{j-1}, z) = s_j$ for some z ; therefore s_j does not depend on t . Build in parallel the three paths, $\text{Path}^i(t), i \in \{0, 2, 4\}$. Assume without loss of generality that s is in the sector 0 of t (the other cases can be deduced by symmetry or by interchanging the roles of s and t). Therefore, $\text{Path}^0(s)$ crosses either $\text{Path}^2(t)$ or $\text{Path}^4(t)$ at a point $M(s, t)$ which is necessarily a point of S , since the graph is planar. We call the *rendezvous path*, denoted $\text{Path}^{\text{RdV}}(s, t)$ the part of the path $\text{Path}^0(s)$ between s and $M(s, t)$ union the part of the path $\text{Path}^2(t)$ or $\text{Path}^4(t)$ between t and $M(s, t)$ (depending on which path $M(s, t)$ is, as represented on Figure 9). The sum of the lengths of the stages in $\text{Path}^{\text{RdV}}(s, t)$, denoted by $|\text{Path}^{\text{RdV}}(s, t)|$, is clearly a bound of the distance in the half- Θ_6 -graph between s and t .

In order to define the limiting path, notice that the bisecting line of the sector 0 of s crosses the bisecting line of sector 2 or that of sector 4 of t at a point; denote it $M^\infty(s, t)$. From the study of directed navigation of Section 3.3, it is straightforward to get the following result:

Theorem 9 *For any $\alpha \in (0, 1/8)$, any $\beta > 0$, for n large enough*

$$\mathbb{P}_{nf} \left(\sup |\text{Path}^{\text{RdV}}(s, t)| - \mathbf{Q}_{\text{bis}}^{\mathbf{T}} \times (|s - M^\infty(s, t)| + |M^\infty(s, t) - t|) \geq n^{-\alpha} \right) \leq n^{-\beta}$$

and the considered paths converge:

$$\mathbb{P}_{nf} \left(\sup d_H([\text{Path}^{\text{RdV}}(s, t)], [s, M^\infty(s, t)] \cup [M^\infty(s, t), t]) \geq n^{-\alpha} \right) \leq n^{-\beta}$$

$$\text{where } \mathbf{Q}_{\text{bis}}^{\text{ST}} = \frac{1}{2} \left(\frac{1}{\cos(\pi/6)} + \frac{\text{arcsinh}(\tan(\pi/6))}{\tan(\pi/6)} \right) \quad (12)$$

and where the supremum is taken in these formulas on the set $\{(s, t)^2 \in \mathcal{D}, [s, M^\infty(s, t)] \cup [M^\infty(s, t), t] \subset \mathcal{D}[a]\}$.

Under the assumptions of the previous theorem, since $|s - M^\infty(s, t)| + |M^\infty(s, t) - t| \leq \frac{|t-s|}{\cos\pi/6}$, the TD-Delaunay of \mathbf{S} is a $(1.216, n^{-\alpha})$ -spanner with probability $1 - n^{-\beta}$.

2.3.3 Point process made with n i.i.d. positions

Consider g a function in $\text{Lip}^*(\mathcal{D})$ which integrates to 1. It is the density (with respect to the Lebesgue measure) of a probability measure μ_g with support the closure of \mathcal{D} . The point process

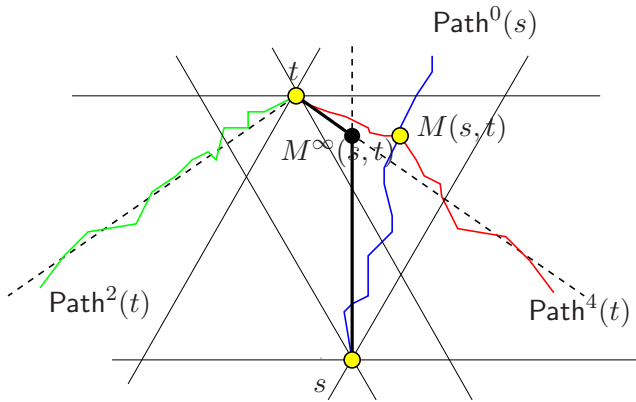


Figure 9: Rendezvous path between s and t and the limit path $[s, M^\infty(s, t)] \cup [M^\infty(s, t), t]$.

considered here is $\mathbf{K}_g^n := \sum_{j=1}^n \delta_{p_j}$ where p_1, \dots, p_n are n i.i.d. random points picked according to μ_g . We denote by \mathbf{P}_g^n the distribution of $\mathbf{S}_g^n := \{p_1, \dots, p_n\}$, the support of \mathbf{K}_g^n .

The analysis of the navigation algorithms under \mathbb{P}_{ng} is simpler than under \mathbf{P}_g^n since under \mathbb{P}_{ng} Markovian properties can be used; the derivation of the results under \mathbf{P}_g^n will appear to be simple consequences of those on \mathbb{P}_{ng} . Indeed, the two models \mathbf{P}_g^n and \mathbb{P}_{ng} are related via the classical fact

$$\mathbf{P}_g^n(\cdot) = \mathbb{P}_{ng}(\cdot \mid \#\mathbf{S} = n); \quad (13)$$

in other words \mathbf{K}_{ng} conditioned by $\#\mathbf{S}(ng) = n$ has the same distribution as \mathbf{K}_g^n .

For any measurable event A (element of \mathcal{N} as defined in Section 1.3) and any $c > 0$,

$$\mathbf{P}_g^n(A) = \mathbb{P}_{ng}(A \mid \#\mathbf{S} = n) \leq \frac{\mathbb{P}_{ng}(A)}{\mathbb{P}_{ng}(\#\mathbf{S} = n)}.$$

Since $\#\mathbf{S}(ng)$ is Poisson(n) distributed, one sees that $\mathbb{P}_{ng}(\#\mathbf{S} = n) \geq c_1 n^{-1/2}$, for n large enough, for a constant $c_1 > 0$ (this is an application of the Stirling formula) and therefore

$$\mathbf{P}_g^n(A) \leq (1/c_1) n^{1/2} \mathbb{P}_{ng}(A),$$

for n large enough. All the results of the present paper under the form $\mathbb{P}_{ng}(A_n) = o(x_n)$ for some decreasing function x_n can be transferred under \mathbf{P}_g^n to the form $\mathbf{P}_g^n(A_n) = O(n^{1/2} \mathbb{P}_{ng}(A_n)) = o(n^{1/2} x_n)$, that is a factor $n^{1/2}$ on the bound must be added. Therefore, the main theorems of the present paper can be transferred to \mathbf{P}_g^n without any problems since all results (or intermediate results) have the form $\mathbb{P}_{ng}(A_n) = o(x_n)$ for $x_n = o(n^{-1/2})$.

2.4 Compared performance of the different navigations

Are presented below some comparisons of the constants arising in the Theorem of the paper, for some special values of θ : constants $\mathbf{Q}_{\text{bis}}^{\text{ST}}$ and $\mathbf{Q}_{\text{bis}}^{\text{SY}}$ gives the limit stretches for the straight algorithms, \mathbf{Q}^{RNT} and \mathbf{Q}^{RNY} for the random north models, $\mathbf{Q}_{\text{bor}}^{\text{CY}}$ and $\mathbf{Q}_{\text{bor}}^{\text{CT}}$ for the cross models.

They can be compared to $\mathbf{Q}^{\text{cross}}$, the worst case analysis of the multiplicative stretch of cross navigations [Yao82, Cla87, Kei88, RS91]. Only the 3 first digits are given.

$\theta \setminus \mathbf{Q}$	$\mathbf{Q}_{\text{bis}}^{\text{ST}}$	$\mathbf{Q}_{\text{bis}}^{\text{SY}}$	\mathbf{Q}^{RNT}	\mathbf{Q}^{RNY}	$\mathbf{Q}_{\text{bor}}^{\text{CY}}$	$\mathbf{Q}_{\text{bor}}^{\text{CT}}$	$\mathbf{Q}^{\text{cross}}$
$\pi/3$	1.053	1.047	1.103	1.097	1.209	1.216	∞
$\pi/4$	1.028	1.026	1.054	1.053	1.111	1.113	4.262
$\pi/6$	1.012	1.012	1.023	1.023	1.047	1.048	2.618
$\pi/9$	1.005	1.006	1.011	1.008	1.020	1.020	1.532
θ	$1 + \frac{\theta^2}{24} + \frac{31\theta^4}{5760} + o(\theta^5)$	$1 + \frac{\theta^2}{24} + \frac{7\theta^4}{5760} + o(\theta^5)$	$1 + \frac{\theta^2}{12} + \frac{\theta^4}{120} + o(\theta^5)$	$1 + \frac{\theta^2}{12} + \frac{\theta^4}{240} + o(\theta^5)$	$1 + \frac{\theta^2}{6} + \frac{7\theta^4}{360} + o(\theta^5)$	$1 + \frac{\theta^2}{6} + \frac{17\theta^4}{720} + o(\theta^5)$	$1 + 4\theta + 16\theta^2 + \frac{184\theta^3}{3} + o(\theta^3)$

2.5 Related works and comments

As presented at the very beginning of [GS06], a generic question arising in the construction of geometric graphs is the following one: “Given a set S of n points (in \mathcal{R}^d), how to construct a *good* network connecting these points?”

Depending on the context (wireless networks mobile or not, sensor networks, road networks, ...) the word “good” may have several meanings. A network can be considered as “good” if: there exists a short route between any pair of points, a short route between any pair of points can easily be found, the number of edges (or the total length of edges) is not too big, the length of the edges is bounded, the network remains connected even if some edges are deleted, the network can be easily built... Clearly some of these criteria are conflicting and trade-off have to be made.

Stretch factor In this paper we are focused on the length of routes that can be defined greedily. For p_θ Yao navigation [Yao82] and p_θ CT navigation [Cla87, Kei88, RS91], a worst case study have been made showing that the stretch factor of these navigations is at most $\frac{1}{1-2\sin\pi/p_\theta}$, for every $p_\theta > 6$. For $p_\theta \leq 6$ the stretch factor of these navigations can be unbounded. However, it has been proved that Θ_6 is a (2,0)-spanner [BGHI10], Yao₆ is a (c , 0)-spanner (for some c) [O’R10] and Yao₄ is a $(8(29 + 23\sqrt{2}), 0)$ -spanner [BDD⁺10].

Another graph family intensively studied is the Delaunay triangulation. The current best bounds on the multiplicative stretch factor st of Delaunay triangulation are $1.584 < st < 2.419$, proved respectively by [BDL⁺09] and [KG92]. For TD-Delaunay triangulations [Che89] the multiplicative stretch factor is known to be 2, and this is one of the rare families for which the stretch factor is known.

On the average/standard behaviour of these navigations or these navigating graphs very little has been made. The main contribution in this domain are the two following ones. The first one [BTZ00] states that the expected length of a natural greedy routing algorithm on a random Delaunay triangulation is $4/\pi$ the Euclidean distance between these two points. The second one [DGM09], the authors show that the expected maximum degree of Yao-graph is $\Theta(\log n / \log \log n)$ in probability for n points drawn uniformly at random in the unit square. Many other results on random instances

are obtained from simulations (see for example [WL06, GGH⁺01]). In Section 2.3.2 we obtained new bounds on the stretch factor for random TD-Delaunay triangulations.

Other costs In the context of wireless Ad Hoc network other costs of paths are considered. In [RM99], the cost of an edge (u, v) is $|u - v|^\beta + c$ for some $\beta > 2$ and some $c \geq 0$. In [LWW01] they show that the power stretch factor of a Yao-graph is at most $\frac{1}{1 - (2 \sin(\theta))^\beta}$ when $c = 0$. In the context where the edge length is bounded (Unit Disk Graph), another cost for a path is sometime defined: the number of hops (or number of stages). Some stretch factors have been proposed for some graph families [WL02, CCV10], but not for greedy routing algorithm. Once again, no result have been obtained on random instances.

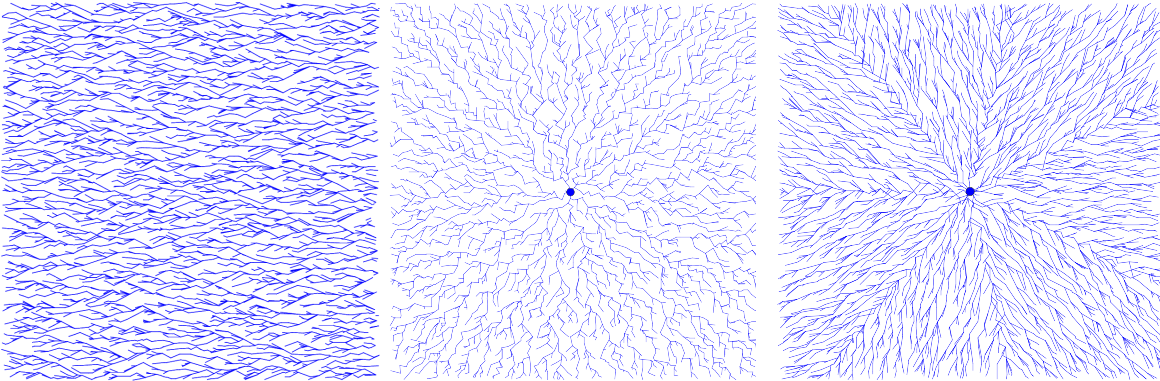


Figure 10: Union of all paths (toward in a unique target) on 5000 uniform random points for: (a) a DT navigation with $\theta = 2\pi/7$. (b) a SY navigation with $\theta = 2\pi/3$. (c) a CT navigation with $p_\theta = 7$.

Link with other random objects In [BB07], the authors investigate a navigation \mathbf{N} on a homogeneous PPP \mathbf{S} on \mathbb{C} , where the traveller wants to reach the origin 0 of the plane: for a given s , $\mathbf{N}(s)$ is the closest element of s in $\mathbf{S} \cup \{0\}$, which has the additional property to be closer to 0 than s . Adding an edge between each $s \in \mathbf{S}$ and $\mathbf{N}(s)$ one gets a tree, called the “radial spanning tree” of the PPP.³ The authors provides numerous results concerning this tree; some local properties concerning the degree of the vertices, the length of edges, and some more global properties, as the behaviour of the subset of elements s of \mathbf{S} such that $\mathbf{N}(s, k) = 0$. In [Bor08], the author goes on this study. Among other, functional along a path are studied (the tree has infinite number of ends, paths going to infinity).

Another object that may be related to directed navigation is the Brownian web. In [FFW05], the authors construct a binary tree using as set of vertices the points of a homogeneous PPP in the plane. The parent-children relation is then induced by a kind of navigation, where, instead of Camembert section, a rectangle section is used; this is deeply similar to what is represented in picture 1 of Figure

³On the second picture in Figure 10 straight p_θ Yao navigation with $\theta = 2\pi/3$; this picture is quite similar to the Figure 1 in [BB07] even if the navigation is less homogeneous in that paper

10. They then show that in a certain sense, their object converges to the Brownian web (see the references therein for definitions and criteria of convergences to the Brownian web). What is done in the present paper concerning directed navigation let us conjecture that what is done in [FFW05] could be extended to Yao’s graphs or Θ graphs associated to directed navigations, and this may be also valid when the intensity of the PPP owns a non-zero Lipschitz density. The main structural difference here is that the infinite tree constructed by directed navigation has an unbounded degree, when it was binary in [FFW05].

In a sequence of papers, Aldous and coauthors [AC09, AK07, AS10, Ald09]... investigates numerous questions related to navigation (or traveller salesman type problem) in a PPP.

3 Toward the proofs, first considerations

3.1 Presentation of the analysis

The analysis rests on some key ideas that we would like to present first, before entering into the details of the proofs.

- Under \mathbb{P}_{nf} , a ball of radius r included in \mathcal{D} contains $O(nr^2)$ point of $\mathbf{S}(nf)$ in average, and with huge probability less than $n^{1+\varepsilon}r^2$, uniformly on all balls (Lemma 18). Moreover, when the traveller enters in the ball $B(t, r)$ around the target t , it is in somehow trapped: we restrict ourselves to the case where the navigation has the property to force the traveller to come closer and closer to t at each stage. Hence using a good $r = r_n$, the uniform control of the number of points in a ball (Lemma 18) allows one to show that the contribution of these last stages is negligible, for a well chosen r_n . See Section 3.5.
- To study the behaviour of the traveller far from its target, a local argument is used: under \mathbb{P}_{nf} for a non constant function f , the stages $(s_j - s_{j-1}, j \geq 1)$ are not identically distributed, neither independent since the value of s_{j-1} affects the distribution of $s_j - s_{j-1}$. But, if one considers only a_n successive stages, with $a_n \rightarrow +\infty$ and $a_n n^\varepsilon / \sqrt{n} \rightarrow 0$, these stages stay in a small window around s_0 with a huge probability; these stages appear moreover to roughly behave as i.i.d. random variables under the homogeneous PPP $\mathbb{P}_{nf(s_0)}$; the behaviour of these a_n stages are seen to depend at the first order, only of $f(s_0)$. Moreover, since $a_n \rightarrow +\infty$ some regularizations of the type “law of large numbers” occur. See Section 3.3.
- This local theorem is one of the cornerstones of the study: in some sense, the global trajectory between any two points is a concatenation of these parts of length a_n ; the successive local values $(f(s_{k \times a_n}), k \geq 0)$ appearing successively as ruling the local behaviour, yields directly to a differential equation. See section 3.4.
- The speed of convergence as stated in the theorems is roughly given for each part with size a_n . The distance between the trajectory to the limiting solution of a ODE are then controlled: it is shown that some deviations are exponential rare. Hence, they resist by the union bounds to polynomial union of events. This allows us to get a first globalisation theorem: we extend the

bounds we have to all trajectories starting and ending on the two dimensional grid $n^{-\mathbf{w}}\mathbb{Z}^2 \cap \mathcal{D}$, for some $\mathbf{w} > 0$. The final globalisation consists in the comparison between the paths between any two points s and t of \mathcal{D} , and some well chosen points of the grid $n^{-\mathbf{w}}\mathbb{Z}^2 \cap \mathcal{D}$. This is possible when \mathbf{w} is large enough. See Section 4.3.

3.2 About the termination of the navigations

The main of this section is to state and prove the following proposition.

Proposition 10 *Let $\mathbf{X} = \mathbf{SY}$ and $\theta \leq 2\pi/3$, or $\mathbf{X} = \mathbf{ST}$ and $\theta \leq \pi/2$, or $\mathbf{X} \in \{\mathbf{CT}, \mathbf{CY}\}$ and $\theta \leq 2\pi/3$. Under \mathbb{P}_{nf} , a.s. for all $(s, t) \in \mathcal{D}[a]^2$*

1. *the navigation from s to t succeeds (i.e. $\exists k$ such that $\mathbf{X}(s, t, k) = t$);*
2. *the traveller comes closer to the target at each step, $\text{Path}^{\mathbf{X}}(s, t) \subset B(t, |s - t|)$;*
3. *$|\text{Path}^{\mathbf{X}}(s, t)| \leq 2|s - t| \times \#S \cap B(t, |s - t|)$;*
4. *if $|s_1 - s| \leq |t - s|/2$ then $|t - s| - |t - s_1| \geq (2 - \sqrt{3})|s_1 - s|$.*

Proof. Let κ be the integer such that $t \in \text{Sect}[\kappa, s]$ and let t_κ be the orthogonal projection of t on the bisecting line of the sector κ . Let Area be $\text{Cam}[\kappa, s](|s - t|)$ (resp. $\text{Tri}[\kappa, s](|s - t_\kappa|)$, $\overrightarrow{\text{Cam}}(s, t)(|s - t|)$ and $\overrightarrow{\text{Tri}}(s, t)(|s - t|)$) if \mathbf{X} is \mathbf{CY} (resp. \mathbf{CT} , \mathbf{SY} and \mathbf{ST}). For the values of θ considered, $\text{Area} \subset B(t, |s - t|)$. In each case, $s_1 \in \text{Area}$, and so $|s_1 - t| \leq |s - t|$, which implies the second item of the proposition. The previous inequality is strict except in one particular case for each navigation considered and only for the maximal values of θ considered: when Area contains only the two points t and s_1 , and s_1 is on the first border of Area such that $|s_1 - t| = |s - t|$. Observe that in that case, $\arg(s_1 - t) - \arg(s - t)$ equals $\pi/2$ if $X = \mathbf{ST}$ and $\pi/3$ for the other navigations (still for the maximal values of θ considered). Hence the navigation fails only if there exists $(s, t) \in \mathcal{D}^2$ such that $S \cup \{s\}$ contains 4 points (resp. 6 points) forming an empty square (resp. an empty hexagon) centred at t with no other points of S inside this polygon. Under \mathbb{P}_{nf} , almost surely S doesn't contain such a configuration. This implies the first item of the proposition.

The third item comes directly from the two first ones: the length of each stage of $\text{Path}^{\mathbf{X}}(s, t)$ is at most $2|s - t|$ and $\text{Path}^{\mathbf{X}}(s, t)$ is composed of at most $\#S \cap B(t, |s - t|)$ stages.

Now let us prove the last item. Let $\alpha := \arg(t - s) - \arg(s_1 - s)$. For the navigation considered, $\alpha \in [0, \pi/3]$. By the cosine law (also called Al-Kashi's theorem), we have:

$$|t - s_1|^2 = |t - s|^2 + |s_1 - s|^2 - 2|t - s| \cdot |s_1 - s| \cos \alpha. \quad (14)$$

Let $x \geq 2$ such that $|t - s| = x|s_1 - s|$. By (14), $|t - s_1| = |s_1 - s|\sqrt{x^2 + 1 - 2x \cos \alpha}$ and then $|t - s| - |t - s_1| = |s_1 - s|(x - \sqrt{x^2 + 1 - 2x \cos \alpha})$. For $x \geq 2$, using that $\alpha \mapsto \cos \alpha$ is decreasing on $[0, \pi/3]$,

$$x - (x^2 + 1 - x)^{1/2} \leq x - \sqrt{x^2 + 1 - 2x \cos \alpha} \leq x - (x - 1)$$

Therefore $(x - \sqrt{x^2 + 1 - 2x \cos \alpha}) \in [2 - \sqrt{3}, 1]$, we get the last item. \square

3.3 A notion of directed navigation

In the straight navigation under \mathbb{P}_{nf} , when the traveller is far from its target, the fluctuations of $\arg(t - s_i)$ stay small for the first values of i (for large n). It is then intuitive that the trajectory of the traveller would not be much changed if he'd use the constant direction given by $\arg(t - s_0)$ instead of $\arg(t - s_i)$; similarly, in the cross navigation, when the traveller is far from $\text{Cross}(t)$ it is easily seen that along its first stages, the bisecting lines of its decision sectors are parallel to the first one, in other words, he follows the direction of the bisecting line of the sector around s_0 containing t . In order to make clear these phenomena, the directed navigation **DT** is defined below.

Again θ is a fixed parameter chosen in $(0, 2\pi)$.

Definition 11 **DT** is a map from \mathbb{C} onto S defined as follows. Let $s \in \mathbb{C}$. Consider the smallest r such that $(s + \text{Tri}(\theta)(r)) \cap S$ is not empty. Then $\mathbf{DT}(s)$ is defined to be the element of this intersection, that is the closest of the first border of $\overrightarrow{\text{Sect}}(s, +\infty)$.

For any $s \in \mathbb{C}$, the successive stopping places $(\mathbf{DT}(s, j), j \geq 0)$ of the traveller satisfy $\mathbf{DT}(s, 0) = s$, $\mathbf{DT}(s, j) = \mathbf{DT}(\mathbf{DT}(s, j - 1))$. Informally, **DT** coincides with **ST** when the target is $t = +\infty$. When the directed navigation is done on a homogeneous PPP on \mathbb{C} , the stages

$$\Delta^{\mathbf{DT}}(s, j) := \mathbf{DT}(s, j) - \mathbf{DT}(s, j - 1), \quad j \geq 1$$

are i.i.d.. Under \mathbb{P}_f , for a non constant f , the process $(\mathbf{DT}(s, j), j \geq 0)$ is Markovian but the stages are not i.i.d.. (see the asymptotic behaviour of the directed navigation on Figure 10(a)).

3.3.1 Directed navigation on a PPP with constant intensity

We consider now a directed navigation process on a PPP with constant intensity $f(z) = c$ on all \mathbb{C} . In the sequel, the index c will refer to the intensity, that is we write $\Delta_c^{\mathbf{DT}}(j)$ instead of $\Delta^{\mathbf{DT}}(s, j)$ under \mathbb{P}_c (the distribution of $\Delta^{\mathbf{DT}}(s, j)$ does not depend on s). For any $j \geq 1$, write

$$\Delta_c^{\mathbf{DT}}(j) = x_c^{\mathbf{DT}}(j) + i y_c^{\mathbf{DT}}(j) \quad \text{for } x_c^{\mathbf{DT}}(j), y_c^{\mathbf{DT}}(j) \in \mathbb{R}, \quad (15)$$

and set $l_c^{\mathbf{DT}}(j) = |\Delta_c^{\mathbf{DT}}(j)|$, the length of this stage. We remove now the superscript **DT** from everywhere. The stages $(\Delta_c(j), j \geq 1)$ are i.i.d., as well as the lengths $(l_c(j), j \geq 1)$ and the pairs $((x_c(j), y_c(j)), j \geq 1)$. Thanks to a clear space rescaling argument, for any $j \geq 1$,

$$\Delta_c(j) \stackrel{(d)}{=} \frac{1}{\sqrt{c}} \Delta_1(j), \quad (x_c(j), y_c(j)) \stackrel{(d)}{=} \frac{1}{\sqrt{c}} (x_1(j), y_1(j)), \quad l_c(j) \stackrel{(d)}{=} \frac{1}{\sqrt{c}} l_1(j) \quad (16)$$

where $\stackrel{(d)}{=}$ means distribution equality. Note x_1 instead of $x_1(1)$, y_1 instead of $y_1(1)$, etc. For any $r > 0$,

$$\mathbb{P}(x_1 > r) = \mathbb{P}_1(\#\text{Tri}[\kappa, 0](r) \cap \mathbf{S} = 0) = \exp(-r^2 \tan(\theta/2)), \quad (17)$$

since a Poisson r.v. with parameter λ takes the value 0 with probability $e^{-\lambda}$. Hence,

$$\mathbb{E}(x_1) = \int_0^{+\infty} \exp(-r^2 \tan(\theta/2)) dr = \frac{1}{2} \sqrt{\frac{\pi}{\tan(\theta/2)}}.$$

Notice that

$$\mathbb{E}(x_1^{\text{DT}}) = \mathbf{C}_{\text{bis}}^{\text{ST}} = \mathbf{C}_{\text{bis}}^{\text{CT}}.$$

Now, since $y_1 = x_1 U$ where U is independent of x_1 and uniform on $[-\tan(\theta/2), \tan(\theta/2)]$,

$$l_1 = |\Delta_1| = \sqrt{x_1^2 + y_1^2} = x_1 (1 + \tan^2(\theta/2)V^2)^{1/2}$$

for V uniform in $[0, 1]$, independent of x_1 . A trite computation leads to $\mathbb{E}\left(\sqrt{1 + \tan^2(\theta/2)V^2}\right) = \mathbf{Q}_{\text{bis}}^{\text{ST}}$, as given in the beginning of Section 2, and then

$$\mathbb{E}(l_1^{\text{DT}}) = \mathbb{E}(x_1^{\text{DT}}) \mathbf{Q}_{\text{bis}}^{\text{ST}} = \mathbf{Q}_{\text{bis}}^{\text{ST}} \mathbf{C}_{\text{bis}}^{\text{ST}}.$$

We consider another quantity that will play a special role during the analysis of cross navigation. Consider the coordinates (ξ_1, ξ'_1) of a stage Δ_1 in the coordinate system $(0, e^{-i\theta/2}, e^{-i\theta/2+i\pi/2})$:

$$\Delta_1 = \xi_1 e^{-i\theta/2} + \xi'_1 e^{-i\theta/2+i\pi/2}.$$

As seen on Figure 11, ξ_1 is the length of the projection of Δ_1 on $\text{HL}_0(s)$. Recall that $y_1 = x_1 U$

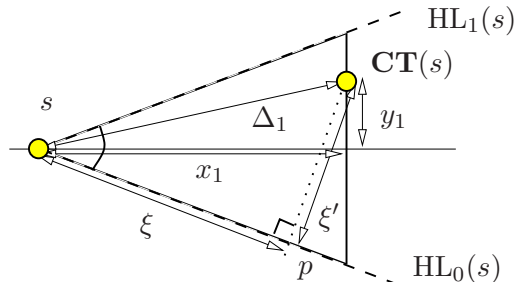


Figure 11: Representation of ξ and ξ' .

where U is independent of x_1 and uniform in $[-\tan(\theta/2), \tan(\theta/2)]$. Therefore, by symmetry, the middle of the projections of (x_1, y_1) and $(x_1, -y_1)$ coincides with the projection of $(x_1, 0)$, and then

$$\mathbb{E}(\xi_1^{\text{DT}}) = \mathbb{E}(x_1^{\text{DT}}) \cos(\theta/2) = \mathbf{C}_{\text{bor}}^{\text{CT}}. \quad (18)$$

Consider now the distance and the position of the traveller after k stops

$$X_1^{\text{DT}}(k) = \sum_{j=1}^k x_1^{\text{DT}}(j), \quad Y_1^{\text{DT}}(k) = \sum_{j=1}^k y_1^{\text{DT}}(j), \quad L_1^{\text{DT}}(k) = \sum_{j=1}^k l_1^{\text{DT}}(j), \quad \Xi_1^{\text{DT}}(k) = \sum_{j=1}^k \xi_1^{\text{DT}}(j).$$

By the law of large numbers, we have immediately

Proposition 12 *In the DT case, the following 4-tuple converges a.s.*

$$\left(\frac{X_1^{\text{DT}}(k)}{k}, \frac{Y_1^{\text{DT}}(k)}{k}, \frac{L_1^{\text{DT}}(k)}{k}, \frac{\Xi_1^{\text{DT}}(k)}{k} \right) \xrightarrow[k]{a.s.} (\mathbf{C}_{\text{bis}}^{\text{CT}}, 0, \mathbb{E}(l_1^{\text{DT}}), \mathbf{C}_{\text{bor}}^{\text{CT}})$$

and then, $\frac{L_1^{\text{DT}}(k)}{X_1^{\text{DT}}(k)} \xrightarrow[k]{a.s.} \frac{\mathbb{E}(l_1^{\text{DT}})}{\mathbb{E}(x_1^{\text{DT}})} = \mathbf{Q}_{\text{bis}}^{\text{ST}}$, and $\frac{\Xi_1^{\text{DT}}(k)}{X_1^{\text{DT}}(k)} \xrightarrow[k]{a.s.} \frac{\mathbb{E}(l_1^{\text{DT}})}{\mathbb{E}(\xi_1^{\text{DT}})} = \mathbf{Q}_{\text{bor}}^{\text{CT}}$.

These numbers are then the crucial coefficients appearing in Theorems 3 and 6.

3.3.2 Control of the fluctuations of $X_1^{\text{DT}}(k)$, $Y_1^{\text{DT}}(k)$ and $L_1^{\text{DT}}(k)$

First, it is easily checked that the random variables l_1 , x_1 , y_1 , and ξ_1 have exponential moments. Indeed, by (17), $\mathbb{E}(e^{t|x_1|}) < +\infty$ for any $t > 0$, and the other variables are controlled as follows: there exist some constants β, β', β'' (that depends only on θ) such that

$$0 \leq |y_1| \leq \beta x_1, 0 \leq l_1 \leq \beta' x_1; 0 \leq \xi_1 \leq \beta'' x_1.$$

Therefore, the variables $|x_1 - \mathbb{E}(x_1)|$, $|y_1 - \mathbb{E}(y_1)|$ and $|l_1 - \mathbb{E}(l_1)|$, $|\xi_1 - \mathbb{E}(\xi_1)|$ own also exponential moments, which permits to control the fluctuations of $X_1(k)$, $Y_1(k)$, $L_1(k)$, $\Xi_1(k)$ around their means by Petrov's [Pet75] Theorem 15 p.52, and Lemma 5 p.54 that we adapt a bit; if $Z_k = z_1 + \dots + z_k$, and the z_i are centred, have exponential moments ($\mathbb{E}(e^{t|z_1|}) < +\infty$, for some $t > 0$), then there exist constants $\alpha > 0$ and $c > 0$ such that

$$\begin{aligned} \mathbb{P}(|Z_m| \geq x_m \sqrt{m}) &\leq 2 \exp(-c x_m^2) && \text{if } x_m \leq \alpha \sqrt{m} \\ \mathbb{P}(|Z_m| \geq x_m \sqrt{m}) &\leq 2 \exp(-c x_m \sqrt{m}) && \text{if } x_m \geq \alpha \sqrt{m}; \end{aligned} \quad (19)$$

constants c, α depend only on the distribution of z_1 , but not on m (the same constant c can be chosen for both bounds; we do this). A consequence is the following proposition.

Proposition 13 *Let (x_n) be a sequence of real numbers such that $x_n \rightarrow +\infty$ and $x_n = O(\sqrt{n})$. There exists a constant $\gamma > 0$, such that for any m large enough*

$$\mathbb{P} \left(\sup_{k \in [1, m]} \left| \frac{X_1^{\text{DT}}(k) - k\mathbb{E}(x_1^{\text{DT}})}{\sqrt{m}} \right| \geq x_m \right) \leq 2m \exp(-\gamma x_m);$$

The same results hold for $Y_1^{\text{DT}}(k) - k\mathbb{E}(y_1^{\text{DT}})$, $L_1^{\text{DT}}(k) - k\mathbb{E}(l_1^{\text{DT}})$ and $\Xi_1^{\text{DT}}(k) - k\mathbb{E}(\xi_1^{\text{DT}})$ and the same γ can be chosen for all the cases.

Proof. The proofs for $X_1(k)$, $Y_1(k)$, $L_1(k)$ and $\Xi_1(k)$ are identical. For $X_1(k)$, write

$$\mathbb{P} \left(\sup_{k \in [1, m]} \left| \frac{X_1(k) - k\mathbb{E}(x_1)}{\sqrt{m}} \right| \geq x_m \right) \leq \sum_{k=1}^m \mathbb{P} \left(\left| \frac{X_1(k) - k\mathbb{E}(x_1)}{\sqrt{k}} \right| \geq \frac{m^{1/2}}{\sqrt{k}} x_m \right) \quad (20)$$

then use the first or second bounds of (19) according to whether $\frac{m^{1/2} x_m}{\sqrt{k}} \leq \alpha \sqrt{k}$ or not. \square

Now, let (x_n) be a sequence of real numbers, such that $x_n \rightarrow +\infty$ and $x_n = O(\sqrt{n})$. There exists $\gamma > 0$, such that for any m large enough

$$\mathbb{P} \left(\sup_{k \in \llbracket 1, m \rrbracket} \left| \frac{X_1^{\mathbf{DT}}(k) + iY_1^{\mathbf{DT}}(k) - k\mathbb{E}(x_1^{\mathbf{DT}})}{\sqrt{m}} \right| \geq x_m \right) \leq 4m \exp(-\gamma x_m). \quad (21)$$

To see this, use Proposition 13 and that for u and v in \mathbb{R} , if $|u + iv| \geq a$ then $|u| \geq a/\sqrt{2}$ or $|v| \geq a/\sqrt{2}$.

When the intensity of the PPP is constant and equal to $c > 0$, for any $k \geq 1$

$$X_c(k) \stackrel{d}{=} \frac{1}{\sqrt{c}} X_1(k), \quad Y_c(k) \stackrel{d}{=} \frac{1}{\sqrt{c}} Y_1(k), \quad L_c(k) \stackrel{d}{=} \frac{1}{\sqrt{c}} L_1(k) \quad \Xi_c(k) \stackrel{d}{=} \frac{1}{\sqrt{c}} \Xi_1(k). \quad (22)$$

This allows one to transfer results obtained under \mathbb{P}_1 to \mathbb{P}_c , since for any Borelian A in \mathbb{R}^k ,

$$\mathbb{P}((X_1(1), \dots, X_1(k)) \in A) = \mathbb{P}((X_c(1), \dots, X_c(k)) \in \sqrt{c}A). \quad (23)$$

3.3.3 Computations for directed p_θ Yao navigation

We use the same notation as in the previous part; this time, for $r > 0$,

$$\mathbb{P}(l_1^{\mathbf{DY}} > r) = \mathbb{P}_1(\#\text{Cam}[k, 0](r) \cap \mathbf{S} = 0) = \exp(-r^2\theta/2), \quad (24)$$

and $x_1^{\mathbf{DY}} = \cos(\alpha)l_1^{\mathbf{DY}}$ and $y_1^{\mathbf{DY}} = \sin(\alpha)l_1^{\mathbf{DY}}$ for a variable α uniform on $[-\theta/2, \theta/2]$, independent of $l_1^{\mathbf{DY}}$. Again, $l_1^{\mathbf{DY}}$, $x_1^{\mathbf{DY}}$ and $y_1^{\mathbf{DY}}$ own exponential moments, and one gets $\mathbb{E}(y_1^{\mathbf{DY}}) = 0$ and

$$\mathbb{E}(l_1^{\mathbf{DY}}) = \sqrt{\frac{\pi}{2\theta}}, \quad \mathbb{E}(x_1^{\mathbf{DY}}) = \mathbb{E}(l_1^{\mathbf{DY}}) \frac{\sin(\theta/2)}{\theta/2} = \mathbf{C}_{\text{bis}}^{\mathbf{SY}} = \mathbf{C}_{\text{bis}}^{\mathbf{CY}}.$$

Moreover since $\xi_1^{\mathbf{DY}} = l_1^{\mathbf{DY}} \cos(\theta/2 + V)$ where V is uniform on $[-\theta/2, \theta/2]$ and independent of $l_1^{\mathbf{DY}}$,

$$\mathbb{E}(\xi_1^{\mathbf{DY}}) = \mathbb{E}(l_1^{\mathbf{DY}}) \frac{\int_{-\theta/2}^{\theta/2} \cos(\alpha + \theta/2) d\alpha}{\theta} = \mathbb{E}(l_1^{\mathbf{DY}}) \frac{\sin(\theta)}{\theta} = \mathbf{C}_{\text{bor}}^{\mathbf{CY}}$$

and again, notice that $\mathbf{Q}_{\text{bor}}^{\mathbf{CY}} = \mathbb{E}(l_1^{\mathbf{DY}})/\mathbb{E}(\xi_1^{\mathbf{DY}})$, $\mathbf{Q}_{\text{bis}}^{\mathbf{CY}} = \mathbb{E}(l_1^{\mathbf{DY}})/\mathbb{E}(x_1^{\mathbf{DY}})$. This leads to

Proposition 14 *In the DY case, we have*

$$\left(\frac{L_1^{\mathbf{DY}}(k)}{k}, \frac{X_1^{\mathbf{DY}}(k)}{k}, \frac{Y_1^{\mathbf{DY}}(k)}{k}, \frac{\Xi_1^{\mathbf{DY}}(k)}{k} \right) \xrightarrow[k]{a.s.} (\mathbb{E}(l_1^{\mathbf{DY}}), \mathbf{C}_{\text{bis}}^{\mathbf{CY}}, 0, \mathbf{C}_{\text{bor}}^{\mathbf{CY}}), \quad (25)$$

and then $\frac{L_1^{\mathbf{DY}}(k)}{X_1^{\mathbf{DY}}(k)} \xrightarrow[k]{a.s.} \mathbf{Q}_{\text{bis}}^{\mathbf{CY}}$, and $\frac{L_1^{\mathbf{DY}}(k)}{\Xi_1^{\mathbf{DY}}(k)} \xrightarrow[k]{a.s.} \mathbf{Q}_{\text{bor}}^{\mathbf{CY}}$.

3.3.4 Computations for the random north model

We here consider the case where the $n(s)$ are i.i.d. uniform on $[0, 2\pi]$. Let $\Delta^{\mathbf{RNT}}$ be the traveller stage (when going from s to t), and let $x^{\mathbf{RNT}}$ be the orthogonal projection of $\Delta^{\mathbf{RNT}}$ on the line (s, t) and $y^{\mathbf{RNT}}$ that on the orthogonal of (s, t) . A simple computation (and a picture as Figure 12) gives $\mathbb{E}(y_1^{\mathbf{RNT}}) = \mathbb{E}(y_1^{\mathbf{RNY}}) = 0$ and

$$\mathbb{E}(x_1^{\mathbf{RNT}}) = \mathbb{E}(x_1^{\mathbf{DT}}) \int_0^{\theta/2} \frac{\cos(\nu)}{\theta/2} d\nu = \mathbb{E}(x_1^{\mathbf{DT}}) \frac{\sin(\theta/2)}{\theta/2} \quad (26)$$

$$\mathbb{E}(x_1^{\mathbf{RNY}}) = \mathbb{E}(l_1^{\mathbf{DY}}) \int_0^\theta \int_0^\theta \frac{\cos(\nu - \gamma)}{\theta^2} d\nu d\gamma = \mathbb{E}(l_1^{\mathbf{DY}}) \frac{2 - 2\cos(\theta)}{\theta^2}. \quad (27)$$

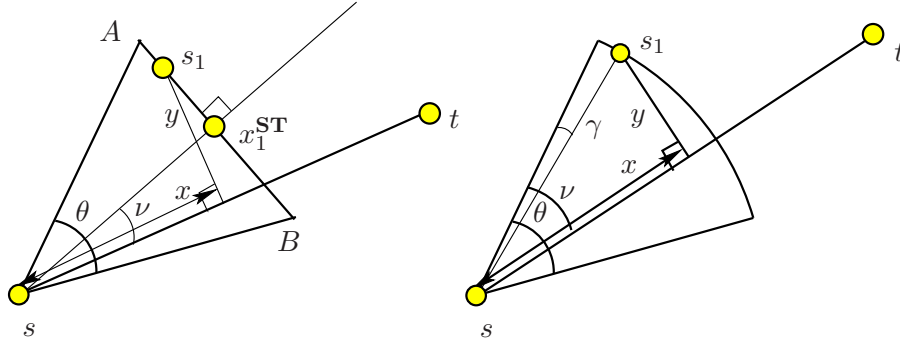


Figure 12: Computation of the constants in the random north models. The first picture deals with **RNT**. If $n(s)$ is uniform on $[0, 2\pi]$, the angle ν is clearly uniform on $[-\pi/p_\theta, \pi/p_\theta]$. The expectation of x knowing ν and l_1 , is equal to the projection of the middle of the segment $[A, B]$, that is $\mathbb{E}(x_1^{\mathbf{ST}})$, on the straight line (s, t) . In the second one is illustrated the **RNY**. The angle ν is uniform on $[0, 2\pi/p_\theta]$ as well as γ when $|s_1 - s|$ is given. Besides $|s_1 - s|$ is distributed as $l_1^{\mathbf{DY}}$.

The limiting quotients $\mathbf{Q}^{\mathbf{RNT}}$ and $\mathbf{Q}^{\mathbf{RNY}}$ are $\mathbf{Q}^{\mathbf{RNT}} = \frac{\mathbb{E}(l_1^{\mathbf{DT}})}{\mathbb{E}(x_1^{\mathbf{RNT}})}$, and $\mathbf{Q}^{\mathbf{RNY}} = \frac{\mathbb{E}(l_1^{\mathbf{DY}})}{\mathbb{E}(x_1^{\mathbf{RNY}})}$.

3.4 From local to global or why differential equations come into play

In Proposition 13 and in (21) a control of the difference between $X_c^{\mathbf{DT}}(k)$ and $Y_c^{\mathbf{DT}}(k)$ and their mean $k\mathbb{E}(x_1^{\mathbf{DT}})/\sqrt{c}$ and 0 is given. When the intensity is nf , the “local intensity” around s is $nf(s)$. Roughly, the first stages of the traveller starting from s are close in distribution under \mathbb{P}_{nf} and under $\mathbb{P}_{nf(s)}$. Hence, at the first order, $X_{nf}^{\mathbf{DT}}(k)$ should be close to $k\mathbb{E}(x_1^{\mathbf{DT}})/\sqrt{nf(s)}$: the speed of the traveller depends on the position, and therefore a differential equation appears.

Since $X_{nf}^{\mathbf{DT}}(k)$ is not deterministic but random, to approximate $\text{Pos}^{\mathbf{DT}}$ by the solution at a differential equation, we will split the traveller’s trajectory into some windows corresponding to some sequence of consecutive stages; the windows have to be small enough to keep the approximation of the local intensity nf by $nf(s)$ to be relevant, and large enough to let the approximation $X_{nf}^{\mathbf{DT}}$ by $k\mathbb{E}(x_1^{\mathbf{DT}})/\sqrt{nf(s)}$ to be relevant too, that is large enough to let large number type compensations occur.

In this section, we present the tools related to the approximation of the traveller position by differential equations. Their proofs are postponed at the Appendix of the paper.

We start with some deterministic considerations. Consider D an open subset of \mathbb{R}^d and let $G : D \rightarrow \mathbb{R}^d$ a Lipschitz function. Denote by $\text{Eq}(G, z)$ the following ordinary differential equation

$$\text{Eq}(G, z) := \begin{cases} y(0) &= z \in D, \\ y'(x) &= G(y(x)), \quad \text{for } x \geq 0. \end{cases} \quad (28)$$

This class of equations contains the family of equations ODE defined in (7). By the Cauchy-Lipschitz theorem (also called Picard-Lindelöf theorem), $\text{Eq}(G, z)$ admits a unique solution $y_{\text{sol}(G, z)}$, or more simply y_{sol} when no confusion on G and z is possible. This solution is defined on a maximal interval $[0, \lambda(G, z))$, where $\lambda(G, z)$ is the hitting time of the border of D by y_{sol} .

Before giving a convergence criterion for random trajectories, here is a deterministic criterion very close to the so-called explicit Euler scheme convergence theorem.

Lemma 15 *Let $G : D \rightarrow \mathbb{R}^d$ be a Lipschitz function, $z \in D$ fixed, and $\lambda \in [0, \lambda(G, z))$ given.*

Let (a_n) and (c_n) two sequences of positive real numbers going to 0, and (y_n) a sequence of continuous functions from $[0, \lambda + a_n]$ onto D satisfying the following conditions:

- a) $y_n(0) = z$ for all $n \geq 1$,*
- b) for all $n \geq 1$, y_n is linear between the points $(ja_n, j \in \llbracket 0, \lfloor \lambda/a_n \rfloor \rrbracket)$, and the slope of y_n on these windows of size a_n is well approximated by $G \circ y_n$ in the following sense:*

$$\sup_{j=0, \dots, \lfloor \lambda/a_n \rfloor} \left| \frac{y_n((j+1)a_n) - y_n(ja_n)}{a_n} - G(y_n(ja_n)) \right| \leq c_n$$

where we denote with an absolute value a norm in \mathbb{R}^d . Under these hypothesis, there exists $C_\lambda > 0$, such that for n large enough

$$\sup_{x \in [0, \lambda]} |y_n(x) - y_{\text{sol}(G, z)}(x)| \leq C_\lambda \max\{a_n, c_n\}.$$

Moreover, the constants C_λ can be chosen in such a way that the function $\lambda \mapsto C_\lambda$ is bounded on all compact subsets of $[0, \lambda(G, z))$ and does not depend on the initial condition $z \in D$.

Note that we need y_n to be defined on a slightly larger interval than $[0, \lambda]$ because of border effects.

We now extend this lemma to the convergence of a sequence of stochastic processes (\mathbf{Z}_n) .

Corollary 16 *Let $G : D \rightarrow \mathbb{R}^n$ be a Lipschitz function, $z \in D$ fixed, and $\lambda \in [0, \lambda(G, z))$ given.*

Let (a_n) , (b_n) , (c_n) , (c'_n) and (d_n) be five sequences of positive real numbers going to 0; let (\mathbf{Z}_n) be a sequence of continuous stochastic processes from $[0, \lambda + a_n]$ onto D satisfying the following conditions:

- a) a.s. $\mathbf{Z}_n(0) = z$,*
- b) the slope of \mathbf{Z}_n on windows of size a_n is well approximated by $G \circ \mathbf{Z}_n$, with a large probability:*

$$\sup_{j \in \llbracket 0, \lfloor \lambda/a_n \rfloor \rrbracket} \mathbb{P} \left(\left| \frac{\mathbf{Z}_n((j+1)a_n) - \mathbf{Z}_n(ja_n)}{a_n} - G(\mathbf{Z}_n(ja_n)) \right| \geq c_n \right) \leq a_n d_n; \quad (29)$$

c) inside the windows, the fluctuations of \mathbf{Z}_n are small:

$$\sup_{j \in \llbracket 0, \lambda/a_n \rrbracket} \mathbb{P} \left(\sup_{x \in [ja_n, (j+1)a_n]} |\mathbf{Z}_n(x) - \mathbf{Z}_n(ja_n) - (x - ja_n)G(\mathbf{Z}_n(ja_n))| \geq c'_n \right) \leq a_n b_n. \quad (30)$$

If these three conditions are satisfied, then

$$\mathbb{P} \left(\sup_{x \in [0, \lambda]} |y_{sol(G, z)}(x) - \mathbf{Z}_n(x)| \leq C_\lambda \max\{a_n, c_n, c'_n\} \right) \geq 1 - (\lambda + 1)(d_n + b_n),$$

for a function $\lambda \mapsto C_\lambda$ independent of z , bounded on every compact subsets of $[0, \lambda(G, z))$.

Notice that condition (c) contains (b) when $c'_n \leq c_n a_n$ which will be the case in the applications we have, but the present presentation allows one to better understand the underlying phenomenon.

3.5 On the largest stage and the maximum number of points in a ball

The two following lemmas will be repeatedly used in the paper. The first one deals with the maximal length of a stage, the second one, with the maximum number of points of a PPP in a ball.

The random variable

$$\text{Navmax}[\theta](S) := \sup_{s \in \mathcal{D}[a], \eta \in [0, 2\pi]} \inf \{h : \#(S \cap (s + e^{i\eta} \text{Cam}(\theta)(h))) \geq 1\}.$$

is a bound on the largest stages length under θ Straight Yao navigation, starting from any point, with any target, when the set of possible stops is S . It is also a bound for all stages length, all starting points ($s \in \mathcal{D}[a]$) and targets ($t \in \mathcal{D}$) and all navigations considered in this paper.

Lemma 17 For any $\theta \in (0, 2\pi]$, any $\mathbf{C} > 0$, if n is large enough,

$$\mathbb{P}_{nf} \left(\text{Navmax}[\theta] \geq n^{\mathbf{C}-1/2} \right) \leq \exp \left(-n^{\mathbf{C}/2} \right).$$

Proof. Consider a tiling of the plane with squares of size $n^{-1/2+\mathbf{C}/2}$. If n is large enough, each element of the family $(s + e^{i\eta} \text{Cam}(\theta)(n^{-1/2+\mathbf{C}}), s \in \mathbb{C}, \eta \in [0, 2\pi])$ contains in its interior at least a square of the tiling. To conclude, it suffices to show that any square \square of the tiling intersecting $\mathcal{D}[a]$ intersects also $\mathbf{S}(nf)$. Such a square \square contains no point of $\mathbf{S}(nf)$ with probability $\exp(-\int_{\square} nf(z) dz) \leq \exp(-m_f n^{\mathbf{C}})$ since nf is larger than nm_f , and the area of \square is $n^{-1+\mathbf{C}}$. Since there is $O(n^{1-\mathbf{C}})$ squares intersecting the bounded domain $\mathcal{D}[a]$, by the union bound, the probability that there exists at least one square containing no elements of $\mathbf{S}(nf)$ is $O(n^{1-\mathbf{C}} \exp(-m_f n^{\mathbf{C}}))$. \square .

Now we turn our attention to

$$\text{Maxball}[r](S) := \max\{\#(S \cap B(x, r)) \mid x \in \mathcal{D}[a]\},$$

the maximum number of elements of S in a ball with radius r and having its centre in $\mathcal{D}[a]$.

Lemma 18 For any $\mathbf{B} > 0$, $\varepsilon > 0$, if n is large enough

$$\mathbb{P}_{nf} \left(\text{Maxball} \left[n^{-1/2+\mathbf{B}} \right] \geq n^{2\mathbf{B}+\varepsilon} \right) \leq \exp(-n^{2\mathbf{B}+\varepsilon}).$$

Note that for $s \in \mathcal{D}$ the mean number of elements in $B(s, n^{-1/2+\mathbf{B}}) \cap \mathbf{S}(nf)$ is $O(n^{2\mathbf{B}})$.

Proof. Consider a tiling of the plane with squares of size $a_n = n^{\mathbf{B}-1/2}$; denote by Squ_a the subset of those having a distance to $\mathcal{D}[a]$ smaller than a_n . Any disk $B(x, a_n)$ with $x \in \mathcal{D}[a]$ intersects a bounded number d of such squares, with d independent of \mathbf{B} . Hence for any positive sequence (b_n) ,

$$\{\text{Maxball}[a_n](\mathbf{S}(nf)) \geq b_n\} \subset \left\{ \sup_{\square \in \text{Squ}_a} \#(\mathbf{S}(nf) \cap \square) \geq b_n/d \right\}.$$

By the union bound, it suffices to control $\mathbb{P}(\#(\mathbf{S}(nf) \cap \square) > b_n/d)$ for a given square \square of the family, the total number of squares being $O(1/a_n^2)$. Since $\#(\mathbf{S}(nf) \cap \square)$ is Poisson($\int_{\square} nf(u)du$) distributed, one may use the following usual bound (Markov's inequality): if $X \sim \text{Poisson}(\lambda)$,

$$\mathbb{P}(X \geq r) = \mathbb{P}(e^{tX} \geq e^{tr}) \leq e^{-tr} \mathbb{E}(e^{tX}) = e^{-tr+\lambda(e^t-1)} \quad \text{for } t \in (0, +\infty),$$

and then, if λ' is an upper bound on λ , then $\mathbb{P}(X \geq r) \leq \sup\{e^{-tr+\lambda'(e^t-1)}, t > 0\}$; the function $t \mapsto e^{-tr+\lambda'(e^t-1)}$ reaches its minimum $t^* = \ln(r/\lambda')$ (if $r > \lambda'$), its value being then $\exp(-r \ln(r/\lambda') + r - \lambda')$. Here an upper bound on the parameter of the Poisson random variable $\#(\mathbf{S}(nf) \cap \square)$ is $\lambda' = \int_{\square} nM_f du = na_n^2 M_f$. Hence

$$\mathbb{P}_{nf}(\#(\mathbf{S} \cap \square) > b_n/d) \leq \exp\left(-\frac{b_n}{d} \ln\left(\frac{b_n}{dna_n^2 M_f}\right) + \frac{b_n}{d} - na_n^2 M_f\right). \quad (31)$$

Substituting $a_n = n^{-1/2+\mathbf{B}}$ and $b_n = n^{2\mathbf{B}+\varepsilon}$ gives the result. \square

3.6 About the constants in the paper

The aim of this section is to discuss the role of the constants in this paper, and maybe, to help the reader to follow more easily the computations.

- In any case, we will use $n^{\mathbf{C}-1/2}$ as a bound on Navmax (this is a bound with probability $1 - \exp(-n^{\mathbf{C}/2})$ at least, for n large).
- When the traveller is close to its target, the approximation of its position by the solution of the differential equation fails and its behaviour is then treated thanks to the content of Section 3.2. “Far to its target” (or to some particular points in the trajectory) means $|t - s| \geq n^{\mathbf{B}-1/2}$. When the traveller enters in the final ball $B(t, n^{\mathbf{B}-1/2})$, before reaching t he will make at most $\text{Maxball}[n^{\mathbf{B}-1/2}]$ additional stages, that is at most $n^{3\mathbf{B}}$ with probability $1 - \exp(-n^{3\mathbf{B}})$.
- In Corollary 16, when the approximation by the solution of a differential equation is needed, a window with size a_n arises. Here, we will always take

$$a_n = n^{\mathbf{w}-1/2}.$$

Taking into account the space normalisation, this corresponds to consider $n^{\mathbf{w}}$ stages of the traveller.

Some assumptions are made on the relative values of $\mathbf{w}, \mathbf{C}, \mathbf{B}$ (for example $2\mathbf{w} + 3\mathbf{C} < 1/2$) in the statements of this paper. In any case, there is a choice of $(\mathbf{w}, \mathbf{C}, \mathbf{B})$ which fulfils all the requirements of all intermediate results, that is

$$0 < \mathbf{w} < 1/4, \quad 2\mathbf{w} + 3\mathbf{C} < 1/2, \quad \mathbf{C} + \mathbf{w} < \mathbf{B}, \quad 0 < \mathbf{B} < 1/4, \quad \mathbf{C} \in (0, 1/2) \quad (32)$$

as can easily be checked. Another quantity appearing later on is

$$0 < \mathbf{w}' < 1/2, \quad (33)$$

related to the bound n^α , $\alpha < 1/8$ appearing in the paper in most of the important theorems. This α appears to be $\max\{\mathbf{w}(\mathbf{w}' - 1/2), \mathbf{w}(\mathbf{w}' + 1/2) - 1/2, \mathbf{w} - 1/2\}$ where $(\mathbf{w}, \mathbf{C}, \mathbf{B}, \mathbf{w}')$ are subject to (32) and (33). To reach an α close to $(1/8)^-$, take $\mathbf{w} = (1/4)^-$, $\mathbf{C} = 0^+$, $\mathbf{B} = (1/4)^-$, $\mathbf{w}' = 0^+$.

3.7 Navigations in homogeneous / non homogeneous PPP

We first introduce the notion of simple stages, notion related to non-intersecting decision domains (see Figure 13).

Definition 19 *Let (d_1, \dots, d_k) be the k first stages of a traveller going from s to t . We say that these stages are $\mathbf{X}(s, t)$ -simple if the corresponding decision domains do not intersect (this amounts to saying that if the traveller uses as set of possible stops the set $\{s + \sum_{j=1}^i d_j, i \leq k\}$, then the decision sectors are non intersecting when going from s to t). A Borelian subset Θ of \mathbb{C}^k is said to be $\mathbf{X}(s, t)$ -simple if for any sequence $(d_1, \dots, d_k) \in \Theta$, the stages (d_1, \dots, d_k) are $\mathbf{X}(s, t)$ -simple.*

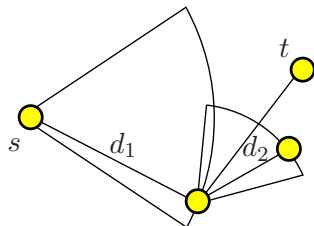


Figure 13: Example of configuration of points in the straight Yao navigation leading to an intersection of the decision domains. For **DT** the decision sectors are simple for $\theta \leq \pi$, and for **DY**, the condition is $\theta \leq \pi/2$. For **ST** and **SY**, for any θ , the decision sectors can intersect.

The following proposition is one of the cornerstones of this paper. It provides around a point s , a bound of the deviations of the traveller stages under the non homogeneous \mathbb{P}_{nf} using the much simpler measure homogeneous one $\mathbb{P}_{nf(s)}$. We were unable to find a coupling argument giving sufficient bounds; our argument relies on the comparison of variables distributions.

Let us introduce the following vectorial notation:

$$\begin{aligned} \mathbf{DT}[s, j] &:= (\mathbf{DT}(s, 1), \dots, \mathbf{DT}(s, j)) \\ \Delta^{\mathbf{DT}}[s, j] &:= (\Delta^{\mathbf{DT}}(s, 1), \dots, \Delta^{\mathbf{DT}}(s, j)); \end{aligned}$$

in the sequel, $\Delta^{\mathbf{X}}[s, t, j]$ and other similar notation will be used for the other navigation processes \mathbf{X} when the target is needed to be specified.

Proposition 20 *Let $\mathbf{X} \in \{\mathbf{CY}, \mathbf{CT}\}$ and $\theta \leq \pi/3$, or $\mathbf{X} = \mathbf{SY}$ and $\theta \leq 2\pi/3$, or $\mathbf{X} = \mathbf{ST}$ and $\theta \leq \pi/2$, or $\mathbf{X} = \mathbf{DT}$ and $\theta \leq \pi$, or $\mathbf{X} = \mathbf{DY}$ and $\theta \leq \pi/2$. Assume that (γ_n) is a sequence of integers such that $\gamma_n \sim n^{\mathbf{w}}$ for some $\mathbf{w} \in (0, 1/4)$. Let $\mathbf{C} \in (0, 1/2)$ such that $2\mathbf{w} + 3\mathbf{C} < 1/2$, and $s, t \in \mathcal{D}[a]$. For any Θ Borelian subset of \mathbb{C}^{γ_n} , $\mathbf{X}(s, t)$ -simple, for n large enough*

$$\mathbb{P}_{n,f}(\Delta^{\mathbf{X}}[s, t, \gamma_n] \in \Theta) \leq \exp(-n^{\mathbf{C}/2}) + 2 \mathbb{P}_{n,f(s)}(\Delta^{\mathbf{X}}[s, t, \gamma_n] \in \Theta),$$

where “ t ” has to be omitted when $\mathbf{X} \in \{\mathbf{ST}, \mathbf{SY}\}$.

Example of application of this proposition: the probability to observe a large value for $\sum_{i=1}^{\gamma_n} \Delta^{\mathbf{DT}}(s, i) - \mathbb{E}(\sum_{i=1}^{\gamma_n} \Delta^{\mathbf{DT}}(s, i))$ is small under $\mathbb{P}_{n,f(s)}$ as says Proposition 13, therefore it is also small under $\mathbb{P}_{n,f}$. Many of the functionals of interest under $\mathbb{P}_{n,f}$ will be controlled like this.

Remark 21 *Notice that the condition of simplicity of the stages is unlikely to arise when s is close to t since the decision domains loose the property to have “a constant” direction, and then can intersect more likely. This is the reason why we will use this proposition only when s is far from t , the last steps of the traveller being treated differently. Notice also that the factor 2 in the RHS can be replaced by any number strictly larger than 1, but 2 will be sufficient for our purpose.*

Proof of Proposition 20. We give the proof in the easiest case, the case \mathbf{DT} (or \mathbf{DY}), where the stages are simple by nature. Nevertheless, we emphasis in the text the place where the condition of non-intersection need to be used for different navigation algorithms.

For $\mathbf{C} > 0$, consider the following event $\Omega(n, \mathbf{C}) := \{\text{Navmax}[\theta] \leq n^{\mathbf{C}-1/2}\}$. By Lemma 17, for n large enough, $\mathbb{P}_{n,f}(\Omega(n, \mathbf{C})) \leq \exp(-n^{\mathbf{C}/2})$. Write

$$\mathbb{P}_{n,f}(\Delta^{\mathbf{DT}}[s, \gamma_n] \in \Theta) \leq \mathbb{P}_{n,f}(\complement\Omega(n, \mathbf{C})) + \mathbb{P}_{n,f}(\Delta^{\mathbf{DT}}[s, \gamma_n] \in \Theta, \Omega(n, \mathbf{C})).$$

We then just have to bound the second term in the RHS. For this, we will show that under the hypothesis of the proposition

$$\begin{aligned} \mathbb{P}_{n,f}(\Delta^{\mathbf{DT}}[s, \gamma_n] \in \Theta, \Omega(n, \mathbf{C})) &\leq 2 \mathbb{P}_{n,f(s)}(\Delta^{\mathbf{DT}}[s, \gamma_n] \in \Theta, \Omega(n, \mathbf{C})) \\ &\leq 2 \mathbb{P}_{n,f(s)}(\Delta^{\mathbf{DT}}[s, \gamma_n] \in \Theta), \end{aligned}$$

only the first inequality deserving to be proved. We will show more; it is easy to see that under $\mathbb{P}_{n,f}$ and under $\mathbb{P}_{n,f(s)}$, the distribution of $\Delta^{\mathbf{DT}}[s, \gamma_n]$ owns a density with respect to the Lebesgue measure on \mathbb{C}^{γ_n} . We will then compare these densities, and bound their quotient on a set of interest. Here, it is then convenient to see the measures not on \mathbb{C}^{γ_n} , but on $\mathbb{R}^{2\gamma_n}$ (see Subsection 3.3.1). The density of $\Delta^{\mathbf{DT}}[s, \gamma_n]$ has to be understood as the density on $\mathbb{R}^{2\gamma_n}$ of the vector

$$V[\gamma_n] := (\Re(\Delta^{\mathbf{DT}}(s, 1)), \Im(\Delta^{\mathbf{DT}}(s, 1)), \dots, \Re(\Delta^{\mathbf{DT}}(s, \gamma_n)), \Im(\Delta^{\mathbf{DT}}(s, \gamma_n))).$$

Write for short \mathbf{v} instead of the generic vector $(x_1, y_1, \dots, x_{\gamma_n}, y_{\gamma_n})$ of $\mathbb{R}^{2\gamma_n}$ where the density is expressed. Since Θ is \mathbf{DT} -simple, by (16), the density of $V[\gamma_n]$ under $\mathbb{P}_{nf(s)}$ is

$$\mathbf{D}_1(\mathbf{v}) = (nf(s))^{\gamma_n} \prod_{m=1}^{\gamma_n} \exp\left(-\frac{nf(s)\tan(\theta/2)}{2} x_m^2\right) \mathbf{1}_{x_m \geq 0} \mathbf{1}_{|y_m| \leq x_m \tan(\theta/2)}. \quad (34)$$

Let us focus on \mathbf{D}_2 , the density of $V[\gamma_n]$ under \mathbb{P}_{nf} . A complication arises since the $\Delta^{\mathbf{DT}}(s, j)$'s are no more independent; by the Markovianity of the sequence $(\mathbf{DT}(s, j), j \geq 0)$ under the non intersecting condition, by successively conditioning on the first stages of the traveller, one has

$$\mathbf{D}_2(\mathbf{v}) = \prod_{m=1}^{\gamma_n} \mathbb{P}_{nf} \left(\Delta^{\mathbf{DT}} \left(s + \sum_{j=1}^{m-1} x_j + iy_j, 1 \right) \in d(x_m + iy_m) \right),$$

where the generic term in the product designates the density at point (x_m, y_m) of the random variable $\Delta^{\mathbf{DT}} \left(s + \sum_{j=1}^{m-1} x_j + iy_j, 1 \right)$ under \mathbb{P}_{nf} , the stage random variables starting from position $s + \sum_{j=1}^{m-1} x_j + iy_j$ (we use the convention $\sum_{i=1}^0 g(i) = 0$ for any g). Recall, that as long as $s + \text{Tri}(\theta)(x) \subset \mathcal{D}$, we have

$$\mathbb{P}_{nf}(\Re(\Delta^{\mathbf{DT}}(s, 1)) \geq x) = \exp\left(-n \int_{s+\text{Tri}(\theta)(x)} f(u) du\right)$$

which implies that the density of $\Re(\Delta^{\mathbf{DT}}(s, 1))$ at a point $x \in \mathbb{R}$ under \mathbb{P}_{nf} is

$$\mathbf{1}_{x \geq 0} n \left(\int_{|y| \leq \tan(\theta/2)x} f(s + x + iy) dy \right) \exp\left(-n \int_{s+\text{Tri}(\theta)(x)} f(u) du\right);$$

conditionally on $\Re(\Delta^{\mathbf{DT}}(s, 1)) = x$ (with $x > 0$) the density of $\Im(\Delta^{\mathbf{DT}}(s, 1))$ at a point y is

$$\mathbf{1}_{|y| \leq \tan(\theta/2)x} \frac{f(s + x + iy)}{\int_{|y| \leq \tan(\theta/2)x} f(s + x + iy) dy}.$$

Denoting by $\lambda_m = \sum_{j \leq m} x_j + iy_j$,

$$\mathbf{D}_2(\mathbf{v}) = \prod_{m=1}^{\gamma_n} nf(s + \lambda_m) \exp\left(-n \int_{s+\lambda_{m-1}+\text{Tri}(\theta)(x_m)} f(u) du\right) \mathbf{1}_{x_m \geq 0} \mathbf{1}_{|y_m| \leq x_m \tan(\theta/2)}. \quad (35)$$

We now show that for n large enough, for any element \mathbf{v} in $\Omega(n, \mathbf{C}) \cap \Theta$, (the non intersecting condition is here again crucial),

$$\mathbf{D}_2(\mathbf{v}) \leq 2\mathbf{D}_1(\mathbf{v}),$$

which is enough to conclude. For this, consider the first factor in the product (35). If $\mathbf{v} \in \Omega(n, \mathbf{C})$, for $m \leq \gamma_n$, since f is Lipschitzian, $f(s + \lambda_m) \leq f(s) + \alpha_f \gamma_n n^{\mathbf{C}-1/2} \leq f(s)(1 + c_1 \gamma_n n^{\mathbf{C}-1/2})$, for a constants $c_1 > 0$ and α_f , since f is bounded.

Let us bound the second factor in (35):

$$\exp\left(-n \int_{s+\lambda_{m-1}+\text{Tri}(\theta)(x_m)} f(u) du\right) = \exp\left(-n \int_{s+\lambda_{m-1}+\text{Tri}(\theta)(x_m)} f(u) - f(s) du\right) \quad (36)$$

$$\times \exp\left(-n \int_{s+\lambda_{m-1}+\text{Tri}(\theta)(x_m)} f(s) du\right) \quad (37)$$

the second term of this product is simply $\exp\left(-\frac{nf(s)\tan(\theta/2)}{2}x_m^2\right)$; let us bound the first one. Using that the area of $s+\lambda_{m-1}+\text{Tri}(\theta)(x_m)$ is bounded by $c_2n^{2\mathbf{C}-1}$ (since $x_m \leq n^{\mathbf{C}-1/2}$), and that $f(u)-f(s)$ is greater than $-\alpha_f mn^{\mathbf{C}-1/2}$ (Lipschitz, and since we are in $\Omega(n, \mathbf{C})$, $|t-s| \leq mn^{\mathbf{C}-1/2}$), we get that the LHS of (36) is bounded by

$$\exp\left(-n \int_{s+\lambda_{m-1}+\text{Tri}(\theta)(x_m)} f(u) - f(s) du\right) \leq \exp\left(c_2n \times mn^{\mathbf{C}-1/2}n^{2\mathbf{C}-1}\right) = \exp(c_2 m n^{3\mathbf{C}-1/2})$$

for some constant $c_2 > 0$. Finally, putting together the γ_n terms involved,

$$\mathbf{D}_2(v) \leq \mathbf{D}_1(v) \times \left(1 + c_1\gamma_n n^{\mathbf{C}-1/2}\right)^{\gamma_n} \times \prod_{m=1}^{\gamma_n} \exp\left(c_2mn^{3\mathbf{C}-1/2}\right) \quad (38)$$

Recall that $\gamma_n \sim n^{\mathbf{w}} \rightarrow +\infty$. The first term $(1+c_1\gamma_n n^{\mathbf{C}-1/2})^{\gamma_n}$ goes to 1 if $\mathbf{C} < 1/2$. The second term goes to 1 if $\sum_{m=1}^{\gamma_n} c_2mn^{3\mathbf{C}-1/2}$ goes to 0, that is if $2\mathbf{w}+3\mathbf{C}-1/2 < 0$. Finally, taking $\mathbf{C} \in (0, 1/2)$ and $\mathbf{w} > 0$ satisfying $2\mathbf{w}+3\mathbf{C}-1/2 < 0$, we see that $(1+c_1\gamma_n n^{\mathbf{C}-1/2})^{\gamma_n} \times \prod_{m=1}^{\gamma_n} \exp(c_2mn^{3\mathbf{C}-1/2}) \rightarrow 1$, and thus is less than 2 for n large enough. \square

3.8 Limiting behaviour for a traveller using DT or DY

In this section, we deal with **DT**, but **DY** can be studied similarly. The aim is to show that when $n \rightarrow +\infty$, after rescaling, the function $\text{Pos}_s^{\mathbf{DT}}$ under \mathbb{P}_{nf} converges in distribution to $\text{SOL}_s^{\mathbf{CT}_{\text{bis}},0}$ the solution of $\text{ODE}(\mathbf{C}_{\text{bis}}^{\mathbf{CT}}, 0, s)$ as defined in (7).

Theorem 22 *Let $\mathbf{X} = \mathbf{DT}$ and $\theta \leq \pi$ or $\mathbf{X} = \mathbf{DY}$ and $\theta \leq \pi/2$. For any $s \in \mathcal{D}[a]$, any $\alpha \in (0, 1/8)$, for $\lambda \in [0, \lambda(F_{\mathbf{C}_{\text{bis}}^{\mathbf{X}},0}, s))$, there exists a constant $d > 0$ such that for n large enough*

$$\mathbb{P}_{nf} \left(\sup_{x \in [0, \lambda]} \left| \text{Pos}_s^{\mathbf{DT}}(x\sqrt{n}) - \text{SOL}_s^{\mathbf{CT}_{\text{bis}},0}(x) \right| \leq n^{-\alpha} \right) \geq 1 - \exp(-n^d).$$

Notice that $\text{ODE}(\mathbf{C}_{\text{bis}}^{\mathbf{CT}}, 0, s)$ coincides with $\text{Eq}(F_{\mathbf{C}_{\text{bis}}^{\mathbf{CT}},0}, s)$ as introduced in (28) (see also the considerations about $\lambda(G, z)$, the hitting time of \mathcal{CD} by the solution of $\text{Eq}(G, z)$). Hence, this theorem is a consequence of the following lemma

Lemma 23 *Let $\lambda \in [0, \lambda(F_{\mathbf{C}_{\text{bis}}^{\mathbf{CT}},0}, s))$, $\gamma_n = \lfloor n^{\mathbf{w}} \rfloor$ and $a_n = \gamma_n/\sqrt{n} \sim n^{\mathbf{w}-1/2}$ be the size of the “windows”. Let $\mathbf{C}, \mathbf{w}, \mathbf{w}'$ be some positive constants, such that $2\mathbf{w} + 3\mathbf{C} - 1/2 < 0$ and $\mathbf{w}' < 1/2$.*

Then for any $s \in \mathcal{D}[a]$, the sequence $(x \mapsto \text{Pos}_s^{\text{DT}}(x\sqrt{n}))$ satisfies the hypothesis of Corollary 16, for $G = F_{\mathbf{C}_{\text{bis}}^{\text{CT}},0}$, $c_n = n^{\mathbf{w}(\mathbf{w}'-1/2)}/\sqrt{mf}$, $b_n = \exp(-n^{\min(\mathbf{w}\mathbf{w}'/2, \mathbf{C}/3)})$, $c'_n = n^{\mathbf{w}(\mathbf{w}'+1/2)-1/2}/\sqrt{mf}$, $d_n = b_n$. Thus

$$\mathbb{P}_{nf} \left(\sup_{x \in [0, \lambda]} \left| \text{SOL}_s^{\mathbf{C}_{\text{bis}}^{\text{CT}},0}(x) - \text{Pos}_s^{\text{DT}}(x\sqrt{n}) \right| \leq C_\lambda \max\{a_n, c'_n, c_n\} \right) \geq 1 - 2(\lambda + 1)b_n,$$

for $\lambda \rightarrow C_\lambda$ bounded on compact sets.

Here, the minimum value for $\max\{a_n, c_n, c'_n\}$ is $n^{-1/8+}$ as explained in Section 3.6 ($\mathbf{w} = (1/4)^-$, $\mathbf{C} = 0^+$, $\mathbf{w}' = 0^+$).

In the homogeneous PPP \mathbb{P}_{nc} (for some $c > 0$), the right order of the variance of $\text{Pos}_s^{\text{DT}}(x\sqrt{n})$ is $1/\sqrt{n}$, since it is a sum of $\sqrt{n}\lambda$ random variables with variance of order $1/n$ by (16). Then standard deviations have order $n^{1/4}$. Here the constant $1/8$ arising in the results is not so good, but gives exponential bounds needed here, and are valid also for non homogeneous PPP.

Proof of Lemma 23. For short, we write $\mathbf{Z}_s^{(n)}(x)$ instead of $\text{Pos}_s^{\text{DT}}(x\sqrt{n})$. We will use Proposition 13, Formulas (21) and (23) and will establish some bounds valid for the first γ_n stages of a traveller starting from a generic point $s_0 \in \mathbb{C}$ (that is $\mathbf{Z}_s^{(n)}(0) = s_0$). Recall that $\mathbf{C}_{\text{bis}}^{\text{CT}} = \mathbb{E}(x_1^{\text{DT}})$ and consider the following Borelian subset of \mathbb{C}^{γ_n} ,

$$\Theta_n^{(1)} := \left\{ (\lambda_1, \dots, \lambda_{\gamma_n}) \in \mathbb{C}^{\gamma_n}, \sup_{l \in \llbracket 1, \gamma_n \rrbracket} \left| \sum_{j=1}^l \left(\lambda_j - \frac{\mathbf{C}_{\text{bis}}^{\text{CT}}}{\sqrt{nf(s_0)}} \right) \right| \leq y_n \right\} \quad (39)$$

for (y_n) a sequence that will be fixed later on; notice that $\frac{\mathbf{C}_{\text{bis}}^{\text{CT}}}{\sqrt{f(s_0)}} = F_{\mathbf{C}_{\text{bis}}^{\text{CT}},0}(s_0) = F_{\mathbf{C}_{\text{bis}}^{\text{CT}},0}(\mathbf{Z}_s^{(n)}(0))$. In term of events,

$$\left\{ \Delta^{\text{DT}}[s_0, \gamma_n] \in \Theta_n^{(1)} \right\} = \left\{ \sup_{l \in \llbracket 1, \gamma_n \rrbracket} \left| \sum_{j=1}^l \left(\Delta^{\text{DT}}(s_0, j) - \frac{\mathbf{C}_{\text{bis}}^{\text{CT}}}{\sqrt{nf(s_0)}} \right) \right| \leq y_n \right\}.$$

Using $\mathbf{Z}_{s_0}^{(n)}(j/\sqrt{n}) = s_0 + \sum_{l=1}^j \Delta^{\text{DT}}(s_0, l)$,

$$\left\{ \Delta^{\text{DT}}[s_0, \gamma_n] \in \Theta_n^{(1)} \right\} = \left\{ \sup_{l \in \llbracket 1, \gamma_n \rrbracket} \left| \mathbf{Z}_{s_0}^{(n)}(l/\sqrt{n}) - s_0 - l \frac{\mathbf{C}_{\text{bis}}^{\text{CT}}}{\sqrt{nf(s_0)}} \right| \leq y_n \right\}. \quad (40)$$

Therefore, the event $\Theta_n^{(1)}$ contains (for $l = \gamma_n$)

$$\left\{ \left| \frac{\mathbf{Z}_{s_0}^{(n)}(a_n) - s_0}{a_n} - F_{\mathbf{C}_{\text{bis}}^{\text{CT}},0}(\mathbf{Z}_{s_0}^{(n)}(0)) \right| \leq y_n/a_n \right\},$$

and

$$\left\{ \sup_{x \in [0, a_n]} \left| \mathbf{Z}_{s_0}^{(n)}(x) - \mathbf{Z}_{s_0}^{(n)}(0) - (x-0)F_{\mathbf{C}_{\text{bis}}^{\text{CT}},0}(\mathbf{Z}_{s_0}^{(n)}(0)) \right| \leq y_n \right\}.$$

We now take $y_n = \frac{x_{\gamma_n} \sqrt{\gamma_n}}{\sqrt{n m_f}}$ for $x_n \sim n^{\mathbf{w}'}$. Since $x_n = O(\sqrt{n})$, it fulfils the requirement of Proposition 13. Moreover, since $2\mathbf{w} + 3\mathbf{C} - 1/2 < 0$, the comparison provided by Proposition 20 is valid and thus we work for a moment under $\mathbb{P}_{nf(s_0)}$. We get, using also the change of scale (22),

$$\begin{aligned} \mathbb{P}_{nf(s_0)} \left(\Delta^{\mathbf{DT}}[s_0, \gamma_n] \in \mathfrak{C}\Theta_n^{(1)} \right) &= \mathbb{P}_{nf(s_0)} \left(\sup_{l \in \llbracket 1, \gamma_n \rrbracket} \left| \sum_{j=1}^l \left(\Delta^{\mathbf{DT}}(s_0, j) - \frac{\mathbf{C}_{\text{bis}}^{\mathbf{CT}}}{\sqrt{nf(s_0)}} \right) \right| \geq \frac{x_{\gamma_n} \sqrt{\gamma_n}}{\sqrt{n m_f}} \right) \\ &\leq \mathbb{P}_1 \left(\sup_{l \in \llbracket 1, \gamma_n \rrbracket} \left| \sum_{j=1}^l \frac{\Delta^{\mathbf{DT}}(s_0, j) - \mathbf{C}_{\text{bis}}^{\mathbf{CT}}}{\sqrt{\gamma_n}} \right| \geq x_{\gamma_n} \right) \\ &\leq 2\gamma_n \exp(-\gamma' x_{\gamma_n}) \leq \exp(-\gamma'' n^{\mathbf{w}\mathbf{w}'}) \end{aligned}$$

for constants $\gamma' > 0, \gamma'' > 0$, and for n large enough (where it has been used that $\frac{\sqrt{f(s_0)}}{\sqrt{m_f}} \geq 1$). Take $c_n = \frac{x_{\gamma_n}}{\sqrt{\gamma_n m_f}} \sim n^{\mathbf{w}\mathbf{w}' - \mathbf{w}/2} / \sqrt{m_f}$ (this goes to 0) and $c'_n = \frac{x_{\gamma_n} \sqrt{\gamma_n}}{\sqrt{n m_f}} \sim n^{\mathbf{w}\mathbf{w}' + \mathbf{w}/2 - 1/2} / \sqrt{m_f}$ (this goes to 0). Then we have established that Formulas (29) and (30) in Corollary 16 hold true under \mathbb{P}_{nf} , with the left most signs ‘‘sup’’ deleted, for $j = 1$, $G = F_{\mathbf{C}_{\text{bis}}^{\mathbf{CT}}, 0}$, $\mathbf{Z}_n = \mathbf{Z}^{(n)}$, $b_n = d_n \geq (2 \exp(-\gamma'' n^{\mathbf{w}\mathbf{w}'}) + \exp(-n^{\mathbf{C}/2})) / a_n$. For example we can take $b_n = d_n = \exp(-n^{\min(\mathbf{w}\mathbf{w}'/2, \mathbf{C}/3)})$.

Since these bounds are valid for any starting points s_0 , and any j , Formulas (29) and (30) in Corollary 16 hold true in this case with the supremum sign re-established, by Markovianity of the sequence $(\mathbf{Z}^n(j/\sqrt{n}), j \geq 0)$. The assumptions of Corollary 16 are satisfied. The conclusion of this corollary entails those of the present lemma. \square

3.9 Local representation of navigations using directed navigations

The aim is to represent locally around a point s , the first stages of a navigation (cross or straight) under the homogeneous PPP $\mathbb{P}_{nf(s)}$ with the first stages of directed navigation.

Local representation of straight navigation using directed navigation

We are here comparing the stages of **ST** and **DT**. Same results for **SY** holds true also.

Lemma 24 *Let $S \in \Omega(n, \mathbf{C})$, and $(s, t) \in \mathcal{D}^2$ such that $[s, t] \subset \mathcal{D}[a]$ and $|t - s| \geq n^{\mathbf{B}-1/2}$. The **ST**-decision domains of $\gamma_n \sim n^{\mathbf{w}}$ first stages are **ST**(s, t)-simple, if $\mathbf{C} + \mathbf{w} < \mathbf{B}$, for n large enough.*

Proof. If $|t - s| \geq n^{\mathbf{B}-1/2}$, since $S \in \Omega(n, \mathbf{C})$, we have $|t - s_i| \geq n^{\mathbf{B}-1/2} - \gamma_n n^{\mathbf{C}-1/2} \geq n^{\mathbf{B}-1/2}/2$ and $|s - s_i| \leq \gamma_n n^{\mathbf{C}-1/2}$ for $i \leq \gamma_n$ (for n large enough). Therefore, for $i \leq \gamma_n$, $|\arg(t - s_i) - \arg(t - s)|$ is bounded above by $O(n^{\mathbf{C} + \mathbf{w} - \mathbf{B}})$. The angle between the bisecting lines of the decision domains are going to 0 uniformly; then the decision domains are non intersecting. \square

Lemma 25 *Let $(s, t) \in \mathcal{D}^2$ such that $[s, t] \subset \mathcal{D}[a]$ and $|t - s| \geq n^{\mathbf{B}-1/2}$ and let $k \in \llbracket 1, \gamma_n \rrbracket$ with $\gamma_n \sim n^{\mathbf{w}}$. Assume that $\mathbf{C} + \mathbf{w} < \mathbf{B}$. For any $\varepsilon > 0$, for any $x > 0$, under \mathbb{P}_{nx} , the vectors $\Delta^{\mathbf{ST}}[s, t, k] \mathbf{1}_{[0, n^{\mathbf{C}-1/2}]}$ ($\max |\Delta^{\mathbf{ST}}[s, t, k]|$) and*

$$\left(e^{i \arg(t - s_{j-1})} \Delta^{\mathbf{DT}}(s, j), j = 1, \dots, k \right) \mathbf{1}_{[0, n^{\mathbf{C}-1/2}]} \left(\max |\Delta^{\mathbf{DT}}[s, k]| \right),$$

where $s_j = \sum_{i=0}^{j-1} \Delta^{\mathbf{DT}}(s, i) + s$, have same distribution when n is large enough.

Note that under \mathbb{P}_{nx} the stages $\Delta^{\mathbf{DT}}[s, \gamma_n]$ are i.i.d. when the stages $\Delta^{\mathbf{ST}}[s, \gamma_n]$ are not.

Proof. This is a simple consequence that under the hypothesis, the decision domains under **ST** or **DT** are simple; then the distribution of the stages in both cases are given by the same computations (based on areas of triangles). \square

Local representation of cross navigation using directed navigation

We treat here the case **CT** but **CY** can be treated similarly. For any two points $(s, t) \in \mathcal{D}$, let $K_{s,t}$ be an integer κ such that $t \in \text{Sect}[\kappa, s]$:

$$K_{s,t} = \left\lfloor \frac{\arg(t-s)}{\theta} + 1/2 \right\rfloor.$$

Lemma 26 *Let $\theta \leq \pi/3$. Let $S \in \Omega(n, \mathbf{C})$, and $(s, t) \in \mathcal{D}[a]$ such that $|t-s| \geq n^{\mathbf{B}-1/2}$. The **CT**-decision domains for the $\gamma_n \sim n^{\mathbf{w}}$ first stages are **CT**(s, t)-simple, if $\mathbf{C} + \mathbf{w} < \mathbf{B}$, for large n .*

Proof. Two cases have to be considered.

- When t is far from $\text{Cross}(s)$, $K_{s_i,t}$ stays constant for small values of i . Hence the bisecting line of the decision domains of the traveller are parallel, and then the domains do not intersect.
- When t is close to $\text{Cross}(s)$ but $|s-t| \geq n^{\mathbf{B}-1/2}$, then during the first $n^{\mathbf{w}}$ stages, t remains close to $\text{Cross}(s_i)$, but $|t-s_i| \geq |t-s| - n^{\mathbf{w}+\mathbf{C}-1/2} \geq n^{-1/2}(n^{\mathbf{B}} - n^{\mathbf{C}+\mathbf{w}})$. Therefore $K_{s_i,t}$ can take two values $K_{s,t}$ and $K_{s,t} \pm 1 \pmod{p_\theta}$ depending on the position of t with respect to $\text{Cross}(s)$. Therefore the bisecting lines of the decision domains have two possible directions. The angle between these possible directions being $\theta \leq \pi/3$, hence the decision domains are non intersecting. \square

We then give a representation of the increments of the cross navigation, immediate since as for Lemma 25, it just relies on the fact that some triangles have the same area (and on Lemma 26).

Lemma 27 *Let $(s, t) \in \mathcal{D}[a]$ such that $|t-s| \geq n^{\mathbf{B}-1/2}$. Let $\gamma_n \sim n^{\mathbf{w}}$ and $\mathbf{C} + \mathbf{w} < \mathbf{B}$. For any $k \in \llbracket 1, \gamma_n \rrbracket$, for any $x > 0$, under \mathbb{P}_{nx} , the variables*

$$\left(e^{i\theta\kappa_j} \Delta^{\mathbf{DT}}(s, j), j = 1, \dots, k \right) \mathbf{1}_{[0, n^{\mathbf{C}-1/2}]} \left(\max |\Delta^{\mathbf{DT}}[s, k]| \right)$$

and $\Delta^{\mathbf{CT}}[s, t, k] \mathbf{1}_{[0, n^{\mathbf{C}-1/2}]} \left(\max |\Delta^{\mathbf{CT}}[s, t, k]| \right)$ where $\kappa_j = K_{s+\sum_{l=1}^{j-1} \Delta^{\mathbf{DT}}(s,l), t}$ have same distribution.

Let us examine the consequence of this lemma. As in the proof of Lemma 26 two cases occur.

3.9.1 When t is far from $\text{Cross}(s)$

Assume that we are in $\Omega(n, \mathbf{C})$. And assume that $d(t, \text{Cross}(s)) \geq n^{\mathbf{w}+\mathbf{C}-1/2}$, in words, the distance between the point t and the set $\text{Cross}(s)$ is greater than $n^{\mathbf{w}+\mathbf{C}-1/2}$. In this case, $K_{s_i,t}$ is equal to $K_{s,t}$ for $i \leq n^{\mathbf{w}}$. Therefore the property of the preceding lemma rewrites

$$\Delta^{\mathbf{CT}}[s, t, k] \mathbf{1}_{[0, n^{\mathbf{C}-1/2}]} \left(\max |\Delta^{\mathbf{CT}}[s, t, k]| \right) = e^{i\theta K_{s,t}} \Delta^{\mathbf{DT}}[s, k] \mathbf{1}_{[0, n^{\mathbf{C}-1/2}]} \left(\max |\Delta^{\mathbf{DT}}[s, k]| \right), \quad (41)$$

and then, under these conditions, **CT** coincides with the **DT** with direction $\theta K_{s,t}$.

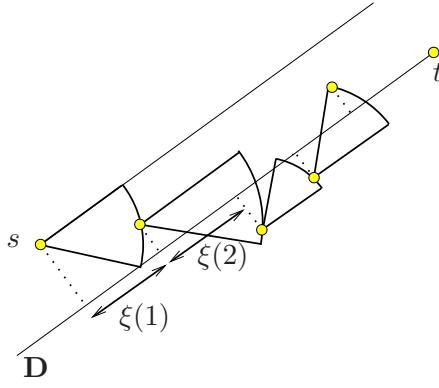


Figure 14: Progression of the traveller near the line \mathbf{D} . Note that the decision sectors are symmetric in distribution with respect to \mathbf{D} , and then two successive progressions (as long as the other branches of $\text{Cross}(t)$ do not interfere)

3.9.2 When t is close to $\text{Cross}(s)$

Assume now that $d(t, \text{Cross}(s)) \leq n^{\mathbf{w}+\mathbf{C}-1/2}$ but $|t - s| \geq n^{\mathbf{B}-1/2}$. There exists a unique k such that $d(t, \text{HL}_k(s)) \leq n^{\mathbf{w}+\mathbf{C}-1/2}$ for n large enough (since $\mathbf{B} > \mathbf{w} + \mathbf{C}$). Consider \mathbf{D} the parallel to $\text{HL}_k(s)$ passing via t (\mathbf{D} is included in $\text{Cross}(t)$ when p_θ is even). We then have $d(\mathbf{D}, s) \leq n^{\mathbf{w}+\mathbf{C}-1/2}$.

Two things are needed to be stated:

- for any $i \leq n^{\mathbf{w}}$, $d(\mathbf{D}, s_i) \leq n^{\mathbf{w}+\mathbf{C}-1/2}$. In words, if the traveller is close to \mathbf{D} at some time, it stays close to it afterward. The reason is simple and comes from the second point: the decision domains of the traveller has a border parallel to \mathbf{D} (see Figure 14).
- the decision domains of the traveller for these $n^{\mathbf{w}}$ steps have a border parallel to \mathbf{D} , and the other border, of course presents an angle θ with \mathbf{D} . Therefore, the orthogonal projection of the stages on \mathbf{D} of all of these stages, have the same distribution (see Figure 14).

4 Proofs of the theorems

The proofs of our theorems are decomposed in several parts.

- (a) First, we prove that for any (s, t) fixed, for any navigation \mathbf{X} , the function $\text{Pos}_{s,t}^{\mathbf{X}}$ admits a limit (specified in the different theorems of the paper). The different costs associated with the path starting by its length – which indeed appears as a cost, and which can not be handled without knowing the position of the traveller – is treated afterward.
- (b) The result for “one trajectory” are then extended to all trajectories in several steps: first, the results from (a) arrive with some probability bounds that allows to handle in once a polynomial number of trajectories (for starting and ending points in a grid). Then the paths between other points are treated by comparison with these trajectories (this is Section 4.3).

4.1 Result for one trajectory

4.1.1 Straight navigation

For any function g and any set L , the hitting time of L by g is

$$\tau[g](L) := \inf\{x, g(x) \in L\},$$

We introduce a uniform big O notation: let (g_n) be a sequence of functions $g_n : A \rightarrow \mathbb{R}$, and (c_n) a sequence of real numbers. Notation $g_n = O_A(c_n)$ means that $\sup_{y \in A} g_n(y) = O(c_n)$.

Theorem 28 *Let $\mathbf{X} = \mathbf{SY}$ and $\theta \leq 2\pi/3$, or $\mathbf{X} = \mathbf{ST}$ and $\theta \leq \pi/2$. Let $\mathcal{D}'[a] = \{(s, t) \in \mathcal{D}^2, [s, t] \subset \mathcal{D}[a]\}$. For any $\alpha \in (0, 1/8)$, there exists $c = c(\alpha)$ such that*

$$\mathbb{P}_{nf} \left(\sup_{x \in [0, \text{Time}_{s,t}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}}] } \left| \text{Pos}_{s,t}^{\mathbf{X}}(x\sqrt{n}) - \text{Pos}_{s,t}^{\infty, \mathbf{C}_{\text{bis}}^{\mathbf{X}}}(x) \right| \geq n^{-\alpha} \right) = O_{\mathcal{D}'[a]}(\exp(-n^c)). \quad (42)$$

Notice that $\tau \left[\text{Pos}_{s,t}^{\infty, \mathbf{C}_{\text{bis}}^{\mathbf{X}}} \right] (\{t\}) = \text{Time}_{s,t}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}}$. Moreover there exists $d > 0$ such that for n large enough

$$\mathbb{P}_{nf} \left(\left| \text{Nb}^{\mathbf{X}}(s, t)/\sqrt{n} - \text{Time}_{s,t}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}} \right| \geq dn^{-\alpha} \right) = O_{\mathcal{D}'[a]}(\exp(-n^c)).$$

Notice that the second assertion of this theorem is not a direct consequence of the first one since when $\|g_n - g\|_{\infty} \rightarrow 0$ it may happen that $\tau[g_n](A) \not\rightarrow \tau[g](A)$ for some set A . Notice also for $\mathbf{X} \in \{\mathbf{ST}, \mathbf{SY}\}$, $\text{Time}_{s,t}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}}$ is implicitly known by

$$\text{Time}_{s,t}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}} = \inf \left\{ x, \int_0^x \frac{e^{i \arg(t-s)} \mathbf{C}_{\text{bis}}^{\mathbf{X}}}{\sqrt{f(s + e^{i \arg(t-s)u})}} du = t - s \right\}. \quad (43)$$

When $f = c$ is constant this simplifies and we get

$$\text{Time}_{s,t}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}} = \frac{|t-s|\sqrt{c}}{\mathbf{C}_{\text{bis}}^{\mathbf{X}}}. \quad (44)$$

Proof of Theorem 28. We give the proof in the case $\mathbf{X} = \mathbf{ST}$ the other case $\mathbf{X} = \mathbf{SY}$ is similar. For short, write $\overline{\mathbf{Z}}_s^{(n)}(x)$ instead of $\text{Pos}_{s,t}^{\mathbf{ST}}(x\sqrt{n})$. For $\mathbf{B} \in (0, 1/4)$, and $\varepsilon > 0$ consider

$$\Omega_{n,\varepsilon} = \left\{ \text{Maxball} \left[n^{\mathbf{B}-1/2} \right] \leq n^{(2+\varepsilon)\mathbf{B}}, \text{Navmax}[\theta] < n^{\mathbf{C}-1/2} \right\}. \quad (45)$$

By Lemmas 17 and 18, $\mathbb{P}_{nf}(\mathbb{C}\Omega_{n,\varepsilon}) \leq \exp(-n^{c_1})$ for some $c_1 > 0$, for n large enough. Let us assume that $S \in \Omega_{n,\varepsilon}$. Consider $T_{s,t}^n$ the hitting time of $B(t, n^{\mathbf{B}-1/2})$ by $\overline{\mathbf{Z}}_s^{(n)}$:

$$T_{s,t}^n = \inf \left\{ x, \left| \overline{\mathbf{Z}}_s^{(n)}(x) - t \right| < n^{\mathbf{B}-1/2} \right\}.$$

By Proposition 10, $T_{s,t}^n < +\infty$ almost surely, and for any $x \geq T_{s,t}^n$,

$$\left| \overline{\mathbf{Z}}_s^{(n)}(x) - t \right| \leq n^{\mathbf{B}-1/2}. \quad (46)$$

For $S \in \Omega_{n,\varepsilon}$, the total number of stops inside $B(t, n^{\mathbf{B}-1/2})$ is at most $n^{(2+\varepsilon)\mathbf{B}}$, which corresponds on the process $\overline{\mathbf{Z}}^{(n)}$ to a negligible time interval $n^{(2+\varepsilon)\mathbf{B}-1/2} = o(1)$ if $(2+\varepsilon)\mathbf{B} < 1/2$. The space fluctuations of these last stages are at most of order $n^{\mathbf{B}-1/2}$, and then, they are negligible when:

$$\mathbf{B} - 1/2 < -\alpha.$$

The control of the position of the traveller along the rest of the trajectory will be done thanks to Corollary 16, using together the elements of the proof of Theorem 28, and Lemma 25.

Assume that $|s - t| > n^{\mathbf{B}-1/2}$, $\mathbf{w} > 0$, $\gamma_n = n^{\mathbf{w}}$ and $y_n = \frac{x\gamma_n\sqrt{\gamma_n}}{\sqrt{nm_f}}$ for $x_n \sim n^{\mathbf{w}'}$, $\mathbf{w}' \in (0, 1/2)$ as in the proof of Theorem 22.

By Proposition 20, we know that for a $\mathbf{ST}(s,t)$ -simple set Θ ,

$$\mathbb{P}_{nf}(\Delta^{\mathbf{ST}}[s, t, \gamma_n] \in \Theta) \leq \exp(-n^{\mathbf{C}/2}) + 2\mathbb{P}_{nf(s)}(\Delta^{\mathbf{ST}}[s, t, \gamma_n] \in \Theta). \quad (47)$$

We then work under $\mathbb{P}_{nf(s)}$ from now on. Recall Lemma 25 which gives a representation of the \mathbf{ST} stages using the \mathbf{DT} stages (under $\mathbb{P}_{nf(s)}$) and set, for (s, t) fixed,

$$\begin{aligned} \Psi : \quad \mathbb{C}^{\gamma_n} &\longrightarrow \mathbb{C}^{\gamma_n} \\ \lambda := (\lambda_1, \dots, \lambda_{\gamma_n}) &\longmapsto (e^{i \arg(t - s_{k-1}(\lambda))} \lambda_k, k = 1, \dots, \gamma_n) \mathbf{1}_{[0, n^{\mathbf{C}-1/2}]}(\max_{l \in \llbracket 1, \gamma_n \rrbracket} |\lambda_l|) \end{aligned}$$

where $s_{k-1}(\lambda) = s + \sum_{i=1}^{k-1} \lambda_i$. Hence, Lemma 25 says that under $\mathbb{P}_{nf(s)}$,

$$\Psi(\Delta^{\mathbf{DT}}[s, \gamma_n]) \mathbf{1}_{[0, n^{\mathbf{C}-1/2}]}(\sup \Delta^{\mathbf{DT}}[s, \gamma_n]) \stackrel{d}{=} \Delta^{\mathbf{ST}}[s, t, \gamma_n] \mathbf{1}_{[0, n^{\mathbf{C}-1/2}]}(\max |\Delta^{\mathbf{ST}}[s, t, \gamma_n]|). \quad (48)$$

Define

$$\Theta_n^{(2)} = \Theta_n^{(1)} \cap \left\{ (\lambda_1, \dots, \lambda_{\gamma_n}) \in \mathbb{C}^{\gamma_n}, \max_{l \in \llbracket 1, \gamma_n \rrbracket} |\lambda_l| < n^{\mathbf{C}-1/2} \right\},$$

where $\Theta_n^{(1)}$ has been defined in the proof of Lemma 23 By Lemmas 23 and 17, for some γ'' and for n large enough.

$$\mathbb{P}_{nf(s)}(\Delta^{\mathbf{DT}}[s, \gamma_n] \in \Theta_n^{(2)}) \geq 1 - \exp(-\gamma'' n^{\mathbf{w}\mathbf{w}'}) - \exp(-n^{\mathbf{C}/2}). \quad (49)$$

When $\Delta^{\mathbf{DT}}[s, \gamma_n] \in \Theta_n^{(2)}$ then it is also in $\Theta_n^{(1)}$ and then most of the equalities or set inclusions of the proof of Lemma 23 can be recycled here, starting from $\left\{ \Delta^{\mathbf{DT}}[s_0, \gamma_n] \in \Theta_n^{(2)} \right\}$

$$= \left\{ \max_{l \in \llbracket 1, \gamma_n \rrbracket} \left| \sum_{j=1}^l \left(\Delta^{\mathbf{DT}}(s_0, j) - \frac{\mathbf{C}_{\text{bis}}^{\mathbf{CT}}}{\sqrt{nf(s_0)}} \right) \right| \leq y_n, \max \Delta^{\mathbf{DT}}[s_0, \gamma_n] < n^{\mathbf{C}-1/2} \right\}.$$

For $\Delta^{\mathbf{DT}}[s_0, \gamma_n] \in \Theta_n^{(2)}$, for $\mathbf{Z}^{(n)}$ defined as in Lemma 25, since $e^{i \arg(t-s)}$ has modulus one,

$$\left\{ \left| e^{i \arg(t-s)} \frac{\mathbf{Z}_{s_0}^{(n)}(a_n) - s_0}{a_n} - e^{i \arg(t-s)} F_{\mathbf{C}_{\text{bis}}^{\mathbf{CT}}, 0}(\mathbf{Z}_{s_0}^{(n)}(0)) \right| \leq y_n/a_n \right\}, \quad (50)$$

and

$$\left\{ \sup_{x \in [0, a_n]} \left| e^{i \arg(t-s)} (\mathbf{Z}_{s_0}^{(n)}(x) - \mathbf{Z}_{s_0}^{(n)}(0)) - (x-0) e^{i \arg(t-s)} F_{\mathbf{C}_{\text{bis}}^{\text{CT}}, 0}(\mathbf{Z}_{s_0}^{(n)}(0)) \right| \leq y_n \right\}. \quad (51)$$

Consider now the increments defined by $\widehat{\Delta}[s, t, \gamma_n] := \Psi(\Delta^{\text{DT}}[s, \gamma_n])$. Using (49) and (48), for any Borelian set Θ ,

$$|\mathbb{P}_{nf(s)}(\widehat{\Delta}[s, t, \gamma_n] \in \Theta) - \mathbb{P}_{nf(s)}(\Delta^{\text{ST}}[s, t, \gamma_n] \in \Theta)| \leq \exp(-\gamma'' n^{\mathbf{w}\mathbf{w}'}) + \exp(-n^{\mathbf{C}/2}).$$

Up to this exponentially small probability, we may work with $\widehat{\Delta}[s, t, \gamma_n]$ instead of $\Delta^{\text{ST}}[s, t, \gamma_n]$. For $\Delta^{\text{DT}}[s, \gamma_n] \in \Theta_n^{(2)}$ and $k \leq \gamma_n$, let us bound

$$d := \left| \sum_{j=1}^k (\widehat{\Delta}(s, t, j) - e^{i \arg(t-s)} \Delta^{\text{DT}}(j)) \right|.$$

Writing $S_j = \sum_{m=1}^j \Delta^{\text{DT}}(j)$,

$$\begin{aligned} d &= \left| \sum_{j=1}^k e^{i \arg(t-S_{j-1})} \Delta^{\text{DT}}(j) - e^{i \arg(t-s)} \Delta^{\text{DT}}(j) \right| \\ &\leq \max\{|e^{i \arg(t-S_{j-1})} - e^{i \arg(t-s)}|, j \in \llbracket 0, k \rrbracket\} \times \sum_{j=1}^k |\Delta^{\text{DT}}(j)|. \end{aligned} \quad (52)$$

Since $\Delta^{\text{DT}}[s, \gamma_n] \in \Theta_n^{(2)}$, the second term in the RHS of (52) is bounded by $k \mathbf{C}_{\text{bis}}^{\text{CT}} / \sqrt{nf(s)} + y_n$ which is smaller than $O_{\mathcal{D}'[a]}(a_n + y_n) = O_{\mathcal{D}'[a]}(n^{\mathbf{w}-1/2})$ (since $y_n = o(a_n)$). Now, to control the maximum, we compare $\arg(t - S_j)$ with $\arg(t - s)$. For $\Delta^{\text{DT}}[s, \gamma_n] \in \Theta_n^{(2)}$, for $j \in \llbracket 0, \gamma_n \rrbracket$,

$$|S_j - s| \leq \frac{\mathbf{C}_{\text{bis}}^{\text{CT}}}{m_f} \gamma_n n^{-1/2} + y_n = O_{\mathcal{D}'[a]}(n^{\mathbf{w}-1/2} + y_n) = O_{\mathcal{D}'[a]}(n^{\mathbf{w}-1/2}) \quad (53)$$

(since $y_n = o(a_n)$) and then

$$|t - S_j| \geq n^{\mathbf{B}-1/2} - |s - S_j| \geq n^{\mathbf{B}-1/2}/2 \quad (54)$$

for n large enough since $\mathbf{w} < \mathbf{B}$ (uniformly in $s \in \mathcal{D}(a)$). Using that $|e^{ia} - 1| = O(|a|)$ we get

$$\max_{j \in \llbracket 0, \gamma_n \rrbracket} \left| e^{i(\arg(t-S_{j-1}(\lambda)) - \arg(t-s))} - 1 \right| = O_{\mathcal{D}'[a]}(y_n / n^{\mathbf{B}-1/2}).$$

(The tangent of the angle $\widehat{S_{j-1}, t, s}$ is $O(y_n / |t - S_j|)$). Then,

$$d = O_{\mathcal{D}'[a]}(n^{\mathbf{w}\mathbf{w}' + \mathbf{w}/2 - \mathbf{B} + \mathbf{w} - 1/2}) = O_{\mathcal{D}'[a]}(n^{\mathbf{w}\mathbf{w}' + 3\mathbf{w}/2 - \mathbf{B} - 1/2}).$$

Set $\widehat{\mathbf{Z}}_s^{(n)}(j/\sqrt{n}) = s + \sum_{l=1}^j \widehat{\Delta}(s, t, l)$. Again, for any Borelian set Θ ,

$$|\mathbb{P}_{nf(s)}(\widehat{\mathbf{Z}}_s^{(n)} \in \Theta) - \mathbb{P}_{nf(s)}(\overline{\mathbf{Z}}_s^{(n)} \in \Theta)| \leq \exp(-\gamma'' n^{\mathbf{w}\mathbf{w}'}) + \exp(-n^{\mathbf{C}/2}). \quad (55)$$

We have

$$\sup_{x \in [0, a_n]} |\widehat{\mathbf{Z}}_s^{(n)}(x) - e^{i \arg(t-s)} \mathbf{Z}^{(n)}(x)| \leq d = O_{\mathcal{D}'[a]}(n^{\mathbf{w}\mathbf{w}' + 3\mathbf{w}/2 - \mathbf{B} - 1/2}). \quad (56)$$

Hence, using (50) and (51), if $\Delta^{\mathbf{DT}}[s, \gamma_n] \in \Theta_n^{(2)}$, then

$$\left\{ \left| \frac{\widehat{\mathbf{Z}}_{s_0}^{(n)}(a_n) - s_0}{a_n} - e^{i \arg(t-s)} F_{\mathbf{C}_{\text{bis}}^{\mathbf{CT}}, 0}(\widehat{\mathbf{Z}}_{s_0}^{(n)}(0)) \right| \leq y_n/a_n + d/a_n \right\},$$

and

$$\left\{ \sup_{x \in [0, a_n]} \left| (\widehat{\mathbf{Z}}_{s_0}^{(n)}(x) - \widehat{\mathbf{Z}}_{s_0}^{(n)}(0)) - (x-0)e^{i \arg(t-s)} F_{\mathbf{C}_{\text{bis}}^{\mathbf{CT}}, 0}(\widehat{\mathbf{Z}}_{s_0}^{(n)}(0)) \right| \leq y_n + d \right\}.$$

By (49), this occurs with a probability exponentially close 1 under $\mathbb{P}_{nf(s)}$, and then this is also true for $\overline{\mathbf{Z}}^{(n)}$ under $\mathbb{P}_{nf(s)}$ thanks to (55), and then for $\overline{\mathbf{Z}}^{(n)}$ under \mathbb{P}_{nf} thanks to (47).

We are now in situation to use Corollary 16 on the process $\overline{\mathbf{Z}}^{(n)}$ under \mathbb{P}_{nf} . The corresponding value of $\max\{a_n, c_n, c'_n\}$ is $M_n := \max\{n^{\mathbf{w}-1/2}, (y_n + O_{\mathcal{D}'[a]}(n^{\mathbf{w}\mathbf{w}' + 3\mathbf{w}/2 - \mathbf{B} - 1/2}))/a_n\} = \max\{n^{\mathbf{w}-1/2}, O_{\mathcal{D}'[a]}(n^{\mathbf{w}\mathbf{w}' - \mathbf{w}/2}), O_{\mathcal{D}'[a]}(n^{\mathbf{w}\mathbf{w}' + \mathbf{w}/2 - \mathbf{B}})\}$ and the probability b_n and d_n are smaller than e^{-n^c} for some $c > 0$, for n large enough. We want M_n to be as small as possible. For this we choose \mathbf{w}' close to 0, and since $\mathbf{B} < 1/4$, the maximum is obtained by taking $\mathbf{w} = \mathbf{B}^-$. In this case, $\max\{a_n, c_n, c'_n\} = n^{-\mathbf{B}/2^-}$. Now, the conclusion of the theorem holds if $-\mathbf{B}/2^- < -\alpha$, $\mathbf{B} - 1/2 < -\alpha$. Hence, α must be chosen in $(0, 1/8)$ for the existence of $\mathbf{B}, \varepsilon', \mathbf{w}$ satisfying all the requirements of the present proof and then at any time less or equal to $\text{Time}_{s,t}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}}$ as long as $\text{Pos}_{s,t}^{\mathbf{X}}(x\sqrt{n})$ is outside $B(t, n^{\mathbf{B}-1/2})$. By what is said above, at time $\text{Time}_{s,t}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}}$, the traveller is in $B(t, n^{-\alpha})$ with probability $1 - \exp(-n^c)$; therefore, by Proposition (10) after this time, the traveller will come closer to $\{t\}$. This implies the first assertion of the theorem.

We now pass to the proof of the second assertion. Since $n^{-\alpha} = n^{-1/8^-} \gg n^{\mathbf{B}-1/2} = n^{-1/4^-}$ we must control the time needed “to traverse $B(t, n^{-\alpha})$ ” (that is, to reach the target once in $B(t, n^{-\alpha})$). The argument below Equation (46) allows one to see that we only need to control the time needed to enter in $B(t, n^{\mathbf{B}-1/2})$ from time $\text{Time}_{s,t}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}}$ (when inside $B(t, n^{-\alpha})$) is smaller than $dn^{-\alpha+1/2}$. For this, we observe that Lemma 25 can still be used, as well as the sets $\Theta_n^{(2)}$ above (the approximation by the differential equation is still valid as long as the traveller is at distance at least $n^{\mathbf{B}-1/2}$ but won't be useful here). The sum of the length of the increments after c_n steps (once in $B(t, n^{-\alpha})$ with $c_n \geq \gamma_n$ is at least $dc_n n^{-1/2} \pm \frac{c_n}{\gamma_n} y_n$ for a constant d (that is $dc_n n^{-1/2}$ at the first order) with probability exponentially close to 1. Hence, thanks to Proposition 10 point (4) the number of steps needed to traverse a distance at most $n^{-\alpha}$ is at most $dn^{-\alpha+1/2}$ for a constant d . \square

4.1.2 Cross navigation

Theorem 29 *Let $\mathbf{X} \in \{\mathbf{CY}, \mathbf{CT}\}$ and $\theta \leq \pi/3$. Let $\mathcal{D}'[a] = \{(s, t) \in \mathcal{D}^2, \Gamma(s, t) \in \mathcal{D}[a]\}$. For any $\alpha \in (0, 1/8)$, there exists $c = c(\alpha)$,*

$$\mathbb{P}_{nf} \left(\sup_{x \in [0, \tau_{s,t}^{\mathbf{X}}]} \left| \text{Pos}_{s,t}^{\mathbf{X}}(x\sqrt{n}) - \text{Pos}_{s,t}^{\infty, \mathbf{C}_{\text{bis}}^{\mathbf{X}}, \mathbf{C}_{\text{bor}}^{\mathbf{X}}}(x) \right| \geq n^{-\alpha} \right) = O_{\mathcal{D}'[a]}(\exp(-n^c)),$$

where $\tau_{s,t}^{\mathbf{X}} = \text{Time}_{s,I(s,t)}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}} + \text{Time}_{I(s,t),t}^{\mathbf{C}_{\text{bor}}^{\mathbf{X}}}$. Moreover, there exists $d > 0$ such that for n large enough

$$\mathbb{P}_{nf} \left(|\text{Nb}_{s,t}^{\mathbf{X}}/\sqrt{n} - \tau_{s,t}^{\mathbf{X}}| \geq dn^{-\alpha} \right) = O_{\mathcal{D}'[a]}(\exp(-n^c)).$$

Again, for $\mathbf{X} \in \{\mathbf{CT}, \mathbf{CY}\}$, $\tau_{s,t}^{\mathbf{X}}$ is implicitly known since

$$\begin{aligned} \text{Time}_{s,I(s,t)}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}} &= \inf \left\{ x, \int_0^x \frac{e^{i \arg(I(s,t)-s)} \mathbf{C}_{\text{bis}}^{\mathbf{X}} du}{\sqrt{f(s + e^{i \arg(I(s,t)-s)u})}} = I(s,t) - s \right\} \\ \text{Time}_{I(s,t),t}^{\mathbf{C}_{\text{bor}}^{\mathbf{X}}} &= \inf \left\{ x, \int_0^x \frac{e^{i \arg(t-I(s,t))} \mathbf{C}_{\text{bor}}^{\mathbf{X}} du}{\sqrt{f(I(s,t) + e^{i \arg(t-I(s,t))u})}} = t - I(s,t) \right\}. \end{aligned}$$

When $f = c$ is constant, this gives

$$\tau_{s,t}^{\mathbf{X}} = \sqrt{c} \left(\frac{|I(s,t) - s|}{\mathbf{C}_{\text{bis}}^{\mathbf{X}}} + \frac{|t - I(s,t)|}{\mathbf{C}_{\text{bor}}^{\mathbf{X}}} \right).$$

Proof of Theorem 29. We here consider the case $\mathbf{X} = \mathbf{CT}$, the case $\mathbf{X} = \mathbf{CY}$ being similar.

Recall the contents of Lemma 27, and of the two Sections 3.9.1 and 3.9.2. The proof starts as for the proof of Theorem 28 using also Proposition 10, and again we consider only the case where $|s - t| > n^{\mathbf{B}-1/2}$ and the portion of the traveller trajectory which is outside the ball $B(t, n^{\mathbf{B}-1/2})$ as in the proof of Theorem 28. Consider, as defined in (45), the set

$$\Omega_{n,\varepsilon} = \left\{ \text{Maxball} \left[n^{\mathbf{B}-1/2} \right] \leq n^{(2+\varepsilon)\mathbf{B}}, \text{Navmax}[\theta] < n^{\mathbf{C}-1/2} \right\}$$

for $\mathbf{C} > 0, \mathbf{B} > 0, \varepsilon > 0$ that will be fixed later on.

When t is far from $\text{Cross}(t)$, that is if $d(t, \text{Cross}(s_0)) \geq n^{\mathbf{w}+\mathbf{C}-1/2}$, thanks to (41) the \mathbf{CT} coincides exactly with \mathbf{DT} with direction $\theta K_{s,t}$ for $n^{\mathbf{w}}$ stages, provided that $S \in \Omega_{n,\varepsilon}$ (since only $n^{\mathbf{w}}$ stages of size at most $\text{Navmax}[\theta]$ are concerned). Therefore, all the properties and bounds obtained for this case, particularly in the proof of Theorem 22 holds true here also. Hence, the traveller will stay close to $\mathbf{B}_{K_{s,t}}$ the bisecting line of $\text{Sect}[K_{s,t}, s]$, its fluctuation around this line being larger than $n^{-\alpha}$ for $\alpha \in (0, 1/8)$ with probability exponentially small. Therefore, the trajectory of the traveller from its starting point till a neighbourhood of $I_{s,t}$ will be the solution of $\text{ODE}(\mathbf{C}_{\text{bis}}^{\mathbf{CT}}, \arg(I_{s,t} - s), s_0)$.

Assume now that the traveller satisfies $d(t, \text{Cross}(s_0)) \leq n^{\mathbf{w}+\mathbf{C}-1/2}$ (this can occur at the beginning of its trip or after some sequences of $n^{\mathbf{w}}$ consecutive stages). Recall the considerations in Section 3.9.2 and also observe Figure 14. If for some j , $d(t, \text{Cross}(s_j)) \leq n^{\mathbf{w}+\mathbf{C}-1/2}$, then this will remain true till the traveller enters in the final ball $B(t, n^{\mathbf{B}-1/2})$. Let us describe more precisely, what happens when $d(t, \text{Cross}(s_j))$ is small. Denote by \mathbf{D} the parallel to the branch of $\text{Cross}(s_i)$ being close to t , passing via t . In order to control the position of the traveller, knowing that it is close to \mathbf{D} , an orthogonal projection on \mathbf{D} is used. The progression of this projection on \mathbf{D} measures the progress of the traveller toward t . We will not enter into the details, everything works with respect to the orthogonal projection on \mathbf{D} as in the case of directed navigation, since the projection ξ owns also some exponential moments (this is the main tool hidden in all concentration

results). Therefore, once close to Δ , the movement of the traveller will asymptotically be ruled by $\text{ODE}(\mathbf{C}_{\text{bor}}^{\mathbf{CT}}, \arg(t - I_{s,t}), s_0)$. Now, taking into account that $\mathbf{w} + \mathbf{C} - 1/2 \leq \mathbf{B} - 1/2$, then necessarily the traveller will enter in the ball $B(t, n^{\mathbf{B}-1/2})$ where it will make a negligible number of steps, with negligible fluctuations (see discussion below Equation (46)). \square

4.2 Analysis of the traveller costs

In Section 1.4, we introduced a notion of cost related to the traveller journey from s to t , associated with an elementary cost function H , and navigation \mathbf{X} :

$$\text{Cost}_H^{\mathbf{X}}(s, t) := \sum_{j=1}^{\text{Nb}(s,t)} H(\Delta_j^{\mathbf{X}}). \quad (57)$$

When H is the modulus function $H : x \mapsto |x|$, then $\text{Cost}_H^{\mathbf{X}}(s, t) = |\text{Path}^{\mathbf{X}}(s, t)|$, when $H : x \mapsto 1$ then $\text{Cost}_H^{\mathbf{X}}(s, t) = \text{Nb}(s, t)$, already discussed in Theorems 28 and 29. We will in the sequel be only interested in the case $H_o : x \mapsto |x|^o$ for some $o \geq 0$. Other functions could certainly be studied following the same steps, but the present case covers the applications we have (discussions around the cases $o \in [2, 4]$ appear in Section 2.5 and in [LWW01]).

Let us discuss a bit at the intuitive level. We have seen that under \mathbb{P}_{nf} , locally around position s_0 , a stage $\Delta^{\mathbf{DT}}$ is close in distribution to $\frac{\Delta_1^{\mathbf{DT}}}{\sqrt{nf(s_0)}}$ a rescaled stage under \mathbb{P}_1 . Hence, $H(\Delta^{\mathbf{DT}})$, for a regular function H , is close to $H(\Delta_1^{\mathbf{DT}}/\sqrt{nf(s_0)})$. Two main points have to be noticed so far: first, the contribution to the cost $H(\Delta^{\mathbf{DT}})$ depends on the position of the traveller, which implies that a differential equation will appear somewhere. The other point is that at the first order $H(\Delta^{\mathbf{DT}})$ depends asymptotically (on n) on the behaviour of H near 0. This yields two leading ideas in our analysis. The first one, is that it is important to consider the pair

$$((\text{Pos}^{\mathbf{X}}(s, t, i), \text{cost}^{\mathbf{X}}(s, t, i)), i = 1, \dots, k),$$

where $\text{cost}^{\mathbf{X}}(s, i) = \sum_{j=1}^i H(\Delta_j^{\mathbf{X}})$ since the cost can not be studied independently of the position. The other idea consists in considering only functions H that are regular near 0 (as the functions $H_o, o \geq 0$). This is needed to have some chances to get “regular” limit theorems for the cost.

Another remark concerns the case $H : x \mapsto 1$, in which case $\text{cost}^{\mathbf{X}}(s, j) = j$ amounts to counting the number of stages according to the number of stages! In this case, defining

$$\mathcal{C}^{(n)}(j/\sqrt{n}) := \text{cost}(s, j)/\sqrt{n} = j/\sqrt{n}, \quad \text{for } j \geq 0,$$

and by linear interpolation between the points $(j/\sqrt{n}, j \geq 0)$, one sees that independently of the navigation algorithm used, on any interval $[0, \lambda]$, $(\mathcal{C}^{(n)}, n \geq 0)$ converges uniformly to $c_{\text{sol}} := y \mapsto y$. Therefore, the pair $(\mathcal{C}^{(n)}, \text{Pos}_{s,t}^{\mathbf{X}}(\cdot, \sqrt{n}))$ formed by the rescaled cost and position function converges to $(c_{\text{sol}}, \text{Pos}_{s,t}^{\infty, c})$ under the same conditions under those $\text{Pos}_{s,t}^{\mathbf{X}}(\cdot, \sqrt{n})$ converges to $\text{Pos}_{s,t}^{\infty, c}$. Even if at the first glance this convergence seems to entail that of $\text{Nb}^{\mathbf{X}}(s, t)$ this is not immediately the case as observed in the proofs of Theorem 28 and 29, since the convergence of the pair $(\mathcal{C}^{(n)}, \text{Pos}_{s,t}^{\mathbf{X}}(\cdot, \sqrt{n}))$ to

$(c_{\text{sol}}, \text{Pos}_{s,t}^{\infty,c})$ was not sufficient to deduce the convergence of hitting times. Here the same phenomena arise for other cost functions.

Let us now come back to the case where H is a power function of type H_o , for some $o > 0$. By the scaling argument, we have for any $c > 0$, $H_o(\Delta_c^{\mathbf{X}}) \stackrel{(d)}{=} |\Delta_1^{\mathbf{X}}|^o / c^{o/2}$, and then, further, for i.i.d. random variables $(\Delta_c^{\mathbf{X}}(j), j \geq 1)$,

$$\sum_{j=1}^{a_n} H_o(\Delta_c^{\mathbf{X}}(j)) \stackrel{(d)}{=} \sum_{j=1}^{a_n} |\Delta_1^{\mathbf{X}}(j)|^o / c^{o/2}.$$

Since the r.v. $|\Delta_1^{\mathbf{X}}(j)|$ have exponential moments (see Section 3.3.2) so do $|\Delta_1^{\mathbf{X}}(j)|^o$ for any $o > 0$. Therefore, by the law of large numbers, for any $a_n \rightarrow +\infty$,

$$\sum_{j=1}^{a_n} |\Delta_1^{\mathbf{X}}(j)|^o / a_n \xrightarrow[n]{a.s.} \mathbb{E}(|\Delta_1^{\mathbf{X}}(j)|^o)$$

and Petrov's Lemma allowing to control the deviation around the mean can be used (see Section 3.3.2). Hence, at the first order, $\sum_{j=1}^{a_n} H_o(|\Delta_c^{\mathbf{X}}(j)|)$ is close to $\frac{a_n}{c^{o/2}} \mathbb{E}(|\Delta_1^{\mathbf{X}}(j)|^o)$, and this for any c , including the case where $c = c_n$ depends on n . Under \mathbb{P}_{nf} , if the traveller is at position s , then $|\Delta_{nf}^{\mathbf{X}}(1)|^o$ is close in distribution to $|\Delta_1^{\mathbf{X}}|^o / (nf(s))^{o/2}$, since $\text{Nb}(s, t)$ has order \sqrt{n} , set

$$\mathbf{C}^{(n), \mathbf{X}}(j/\sqrt{n}) := \text{cost}^{\mathbf{X}}(s, j) n^{o/2-1/2}.$$

In order to extend Theorem 22 to the pair (Position, Cost), we need to introduce a system of ODE. We already saw that at least up to some decompositions by parts of the trajectories, the limiting position of the traveller was the solution of $\text{ODE}(\lambda, \nu, s)$ for some parameters (λ, ν) depending on the details of the studied navigation. For any $s_0 \in \mathcal{D}$, $c_0 \in \mathbb{R}$, $q \in \mathbb{R}$ consider the following system

$$\text{ODE}^{(2)}(\lambda, \nu, q, s_0, c_0) = \begin{cases} \rho(0) & = s_0 \\ \frac{\partial \rho(x)}{\partial x} & = F_{\lambda, \nu}(\rho(x)) = \frac{\lambda e^{i\nu}}{f(\rho(x))^{1/2}}, \\ \mathcal{C}(0) & = c_0 \\ \frac{\partial \mathcal{C}(x)}{\partial x} & = \frac{q}{f(\rho(x))^{o/2}}. \end{cases} \quad (58)$$

The existence and uniqueness of a solution (ρ, \mathcal{C}) to this system is guaranteed by the Cauchy-Lipschitz theorem. The function ρ is $\text{SOL}_{s_0}^{\lambda, \nu}$ since the conditions on ρ coincide with $\text{ODE}(\lambda, \nu, s_0)$. The function \mathcal{C} has clearly a simple integral representation using ρ and f :

$$\mathcal{C}(y) = c_0 + q \int_0^y \frac{dx}{f(\rho(x))^{o/2}}.$$

Since

$$\rho(y) = s_0 + \int_0^y F_{\lambda, \nu}(\rho(x)) dx = s_0 + \lambda e^{i\nu} \int_0^y \frac{dx}{f(\rho(x))^{1/2}}$$

in the case $o = 1$ this immediately leads to

$$\mathcal{C}(y) = c_0 + q\lambda^{-1}(\rho(y) - s_0)e^{-iy}. \quad (59)$$

When $o \neq 1$, ρ and \mathcal{C} are related by a linear formula only when f is constant since in this case ρ and \mathcal{C} are linear functions.

Limits in the directed case We explain in this case only the appearance of the limiting differential equation.

Lemma 30 *Let $\mathbf{X} = \mathbf{DT}$ and $\theta \leq \pi$, or $\mathbf{X} = \mathbf{DY}$ and $\theta \leq \pi/2$. The pair of processes $(\text{Pos}_{s,t}^{\mathbf{X}}(x\sqrt{n}), \mathbf{C}^{(n),\mathbf{X}}(x))_{x \in [0,\lambda]}$ satisfies the assumption of Corollary 16, for $b_n = n^{\mathbf{w}\mathbf{w}' - \mathbf{w}/2} / \sqrt{m_f}$, $c_n = \exp(-n^{\min(\mathbf{w}\mathbf{w}'/2, c)})$, $c'_n = c_n$, $d_n = b_n$. Therefore it converges to $(\text{Pos}_{s,t}^{\infty, \mathbf{C}_{\text{bis}}^{\mathbf{X}}}(x), \text{cost}^{\infty, \mathbf{X}}(x))_{x \in [0,\lambda]}$ solution ODE⁽²⁾ $(\mathbf{C}_{\text{bis}}^{\mathbf{X}}, 0, \mathbf{C}_{\text{bis}}^{\mathbf{X}, o}, s_0, 0)$ for $\lambda \in [0, \lambda(F_{\mathbf{C}_{\text{bis}}^{\mathbf{X}, o}}, s))$, where $\mathbf{C}_{\text{bis}}^{\mathbf{X}, o} = \mathbb{E}(|\Delta_1^{\mathbf{X}}|^o)$.*

Proof. The proof uses the idea of that of Theorem 22 (we will use below $\Theta_n^{(1)}$ and $\mathbf{Z}^{(n)}$, defined in its proof). Consider again the set $\Theta_n^{(1)}$ as defined in (39), and introduce the following Borelian subset of $\mathbb{C}^{\gamma n}$:

$$\Theta_n^{(4)} := \left\{ (\lambda_1, \dots, \lambda_{\gamma n}) \in \mathbb{C}^{\gamma n}, \sup_{l \in \llbracket 1, \gamma n \rrbracket} \left| \sum_{j=1}^l \left(n^{o/2-1/2} |\lambda_j|^o - \frac{\mathbf{C}_{\text{bis}}^{\mathbf{T}, o}}{\sqrt{n}(f(s_0))^{o/2}} \right) \right| \leq y_n \right\} \quad (60)$$

for (y_n) a sequence. Using $\mathbf{C}^{(n)}(a_n) = c_0 + n^{o/2-1/2} \sum_{l=1}^{\gamma n} |\Delta^{\mathbf{DT}}(s_0, l)|^o$ we get

$$\{\Delta^{\mathbf{DT}}[s_0, \gamma n] \in \Theta_n^{(4)}\} = \left\{ \sup_{l \in \llbracket 1, \gamma n \rrbracket} \left| \mathbf{C}_{s_0}^{(n)}(l/\sqrt{n}) - c_0 - \frac{l}{\sqrt{n}} \frac{\mathbf{C}_{\text{bis}}^{\mathbf{T}, o}}{f(s_0)^{o/2}} \right| \leq y_n \right\},$$

and everything works as in the proof of Theorem 22, in particular, using also the change of scale (22), $\mathbb{P}_{n, f(s_0)}(\Delta^{\mathbf{DT}}[s_0, \gamma n] \in \Theta_n^{(4)})$

$$\begin{aligned} &= \mathbb{P}_{n, f(s_0)} \left(\sup_{l \in \llbracket 1, \gamma n \rrbracket} \left| \sum_{j=1}^l \left(n^{o/2-1/2} |\Delta^{\mathbf{DT}}(s_0, j)|^o - \frac{\mathbf{C}_{\text{bis}}^{\mathbf{T}, o}}{\sqrt{n}(f(s_0))^{o/2}} \right) \right| \geq y_n \right) \\ &\leq \mathbb{P}_1 \left(\sup_{l \in \llbracket 1, \gamma n \rrbracket} \left| \sum_{j=1}^l \frac{|\Delta^{\mathbf{DT}}(s_0, j)|^o - \mathbf{C}_{\text{bis}}^{\mathbf{T}, o}}{\sqrt{\gamma n}} \right| \geq y_n f(s_0)^{o/2} \frac{\sqrt{n}}{\sqrt{\gamma n}} \right). \end{aligned}$$

Take $y_n = x_{\gamma n} \frac{\sqrt{n}}{\sqrt{\gamma n} m_f^{o/2}}$, for $x_n \sim n^{\mathbf{w}'}$ and then, the proof ends as the one of Theorem 22. Notice that the deviations of $\mathbf{C}^{(n)}$ are of the same order as that of $\mathbf{Z}^{(n)}$ in the proof of Theorem 22. Now, using that $\Delta^{\mathbf{DT}} \in \Theta_n^{(1)} \cap \Theta_n^{(4)}$ with a probability exponentially close to 1 and therefore the conclusion of the theorem holds true for the pair $(\mathbf{Z}^{(n)}, \mathbf{C}^{(n)})$. \square

Limits in the straight case Here $\mathbf{X} \in \{\mathbf{ST}, \mathbf{SY}\}$. Consider $(\text{SOL}_s^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}, \arg t-s}, C^{\mathbf{X}, o})$ the solution to the equation $\text{ODE}^{(2)}(\mathbf{C}_{\text{bis}}^{\mathbf{X}}, \arg(t-s), \mathbf{C}_{\text{bis}}^{\mathbf{X}, o}, s, 0)$, and consider again $\text{Time}_{s,t}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}}$ the hitting time of t by $\text{Pos}_{s,t}^{\infty, \mathbf{C}_{\text{bis}}^{\mathbf{X}}}$. The limiting cost will be $\mathbf{C}_{s,t}^{\mathbf{X}, o} := C^{\mathbf{X}, o}(\text{Time}_{s,t}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}})$. Notice that if $o = 1$, then $C^{\mathbf{X}, 1}(x) = \mathbb{E}(|\Delta_1^{\mathbf{X}}|) |\text{Pos}_{s,t}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}} - s|$.

Theorem 31 *Let $\mathbf{X} = \mathbf{SY}$ and $\theta \leq 2\pi/3$ or $\mathbf{X} = \mathbf{ST}$ and $\theta \leq \pi/2$. For any $\alpha \in (0, 1/8)$, any $\beta > 0$, for any $\lambda > 0$, for n large enough*

$$\mathbb{P}_{nf} \left(\sup_{(s,t) \in \mathcal{D}, [s,t] \subset \mathcal{D}[a]} \left| \frac{\text{Cost}_{H_o}^{\mathbf{X}}(s,t)}{n^{1/2-o/2}} - \mathbf{C}_{s,t}^{\mathbf{X}, o} \right| \geq n^{-\alpha} \right) \leq n^{-\beta}$$

The proof follows the step of that of Theorem 28 and in particular: first, a proof using s and t fixed is obtained, then, the proof is extended to a sub-grid of $\mathcal{D}[a]^2$ and then to all pairs thanks to the arguments of the Section 4.3.

To get the proof for (s, t) fixed, the proof is similar to that of Theorem 28: the limiting contribution to the cost of the stages of the traveller outside the final ball $B(t, n^{-\alpha})$ is provided by the solution of a differential equation. Then, when the traveller enter in the final balls $B(t, n^{-\alpha})$ and then in $B(t, n^{\mathbf{B}-1/2})$, we use again that these final contributions are negligible and affect the total cost up to a negligible amount (smaller than $n^{-\alpha}$ with a huge probability).

Limits in the cross case Here $\mathbf{X} \in \{\mathbf{CT}, \mathbf{CY}\}$. This time again, one must use the decomposition of the limiting path at the point $I(s, t)$. From s to $I(s, t)$, denote by $(\text{SOL}_s^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}, \arg I(s,t)-s}, C_1^{\mathbf{X}, o})$ the solution of $\text{ODE}^{(2)}(\mathbf{C}_{\text{bis}}^{\mathbf{X}}, \arg(I(s, t) - s), \mathbf{C}_{\text{bis}}^{\mathbf{X}, o}, s_0, 0)$, on the time interval $[0, \text{Time}_{s, I(s,t)}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}}]$; between $I(s, t)$ and t , let $(\text{SOL}_{I(s,t)}^{\mathbf{C}_{\text{bor}}^{\mathbf{X}}, \arg t - I(s,t)}, C_2^{\mathbf{X}, o})$ be the solution of $\text{ODE}^{(2)}(\mathbf{C}_{\text{bor}}^{\mathbf{X}}, \arg(t - I(s, t)), \mathbf{C}_{\text{bis}}^{\mathbf{X}, o}, I(s, t), 0)$ on the time interval $[0, \text{Time}_{I(s,t), y}^{\mathbf{C}_{\text{bor}}^{\mathbf{X}}}]$. The limiting cost will be in this case

$$\mathbf{C}_{s,t}^{\mathbf{X}, o} = C_1^{\mathbf{X}, o}(\text{Time}_{s, I(s,t)}^{\mathbf{C}_{\text{bis}}^{\mathbf{X}}}) + C_2^{\mathbf{X}, o}(\text{Time}_{I(s,t), t}^{\mathbf{C}_{\text{bor}}^{\mathbf{X}}})$$

Theorem 32 *For any $\theta \leq \pi/3$, for any $\mathbf{X} \in \{\mathbf{CT}, \mathbf{CY}\}$, any $\alpha \in (0, 1/8)$, any $\beta > 0$, for any $\lambda > 0$, for n large enough*

$$\mathbb{P}_{nf} \left(\sup_{(s,t) \in \mathcal{D}, [s,t] \subset \mathcal{D}[a]} \left| \frac{\text{Cost}_{H_o}^{\mathbf{X}}(s,t)}{n^{1/2-o/2}} - \mathbf{C}_{s,t}^{\mathbf{X}, o} \right| \geq n^{-\alpha} \right) \leq n^{-\beta}$$

4.3 Globalisation of the bounds

We have obtained some bounds for the position, and for some cost functions associated with a traveller going from s to t . We here desire to prove some uniform bounds, for all (s, t) taking altogether (in the main theorems a supremum on (s, t) lies inside the considered probabilities). The number of possible pairs (s, t) being infinite, the union bound is not sufficient here. Since we have some exponential bounds in most of our theorems concerning one trajectory, these bounds resist to

the union bound of a polynomial number of terms. We adopt a two points strategy to fill this gap. First, we take as possible starting points and targets, the points of a sub-grid of $\mathcal{D}[a]$

$$\text{Grid}_n(c_0, a) := n^{-c_0} \mathbb{Z}^2 \cap \mathcal{D}[a].$$

The cardinality of $\text{Grid}_n(c_0, a)$ being $O(n^{2c_0})$, by the union bound, any theorem of the form:

$$\text{for all } s, t \in \mathcal{D}[a], \quad \mathbb{P}_{nf} (h(\text{Path}_{s,t}^{\mathbf{X}}) \in A) = O(u_n)$$

where u_n does not depend on s, t , and where h is any function of the paths has the following corollary

$$\mathbb{P}_{nf} \left(\sup_{(s,t) \in \text{Grid}_n(c_0, a)} h(\text{Path}_{s,t}^{\mathbf{X}}) \in A \right) = O(n^{2c_0} u_n).$$

In other words, if the probability concerning one path is exponentially small, then it is still exponentially small when considering altogether all paths starting and ending in $\text{Grid}_n(c_0, a)$.

The second point of our strategy is the following. Take any s and t in $\mathcal{D}[a]$ say in the complementary of $\text{Grid}_n(c_0, a)$. Can we compare the path of the traveller from s to t , to the path of a traveller starting and ending to some points s_g and t_g close to s and t and being in $\text{Grid}_n(c_0, a)$? The answer is a bit complicated. We will show that with a probability close to 1, any trajectory from any points s to t can be split in a finite number of pieces, and with a huge probability bounded by 12, such that on each piece, the path of the traveller coincides with a portion of the path of a traveller starting and ending on the grid. This will be sufficient to conclude, since the theorems we have control the behaviour of the path of a traveller all along, and then, on the aforementioned portions without any additional effort.

In order to do so, we introduce $\text{Squ}_n^*(c_0)$ the set of squares of the plane with side length n^{-c_0} , having their vertices in $n^{-c_0} \mathbb{Z}^2$ and at least one of them in $\text{Grid}_n(c_0, a)$.

Additionally, consider four tilings of the plane with squares with length

$$a_n = n^{\mathbf{B}-1/2},$$

the three last ones being obtained from the first one by the translation of $a_n/2$, $ia_n/2$ and $a_n(1+i)/2$, respectively. By $\text{Squ}_a^1(\mathbf{B}), \text{Squ}_a^2(\mathbf{B}), \text{Squ}_a^3(\mathbf{B}), \text{Squ}_a^4(\mathbf{B})$ we denote the subsets of the squares of each of these tilings having a distance to $\mathcal{D}[a]$ smaller than a_n and by $\text{Squ}_a(\mathbf{B})$ their union. If n is large enough, the union of the squares of $\text{Squ}_a(\mathbf{B})$ contains $\mathcal{D}[a]$ and are included in the interior of \mathcal{D} (for example in $\mathcal{D}[a/2]$), and observe also that any disk $B(x, a_n/4)$ with $x \in \mathcal{D}[a]$ is totally included in a square of $\text{Squ}_a(\mathbf{B})$.

With any $(s, t) \in \mathcal{D}[a]$ we will associate two points (s_g, t_g) belonging to $\text{Grid}_n(c_0, a)$ as follows. First t_g is the upper left corner of the square of $\text{Squ}_n^*(c_0)$ containing t (therefore t_g depends on n and c_0); if t is in $\text{Squ}^*(c_0)$ then take $t_g = t$. The point s_g is given by the following lemma.

Lemma 33 *Let $s \in \mathcal{D}$, and consider one of the navigation processes presented in this paper using as set of stopping places S . For any $\rho < r_{\min}(S) := \min\{|x - y|, x, y \in S, x \neq y\}$ and for any pair (s', t) of $S \times \mathcal{D}$ there exists a point s_g in $\rho \mathbb{Z}^2 \cap \mathcal{D}$ such that $\mathbf{X}(s_g, t) = \mathbf{X}(s, t) = s'$, in other words the path from s and from s_g to t coincides from the second stage.*

Proof. First observe that around s' there is a disk of radius $r_{\min}(S)$ containing no other vertex of S . For a straight navigation process let us define $R = \overrightarrow{\text{Cam}}(s', s'-t)(r_{\min}(S)/2)$ and for a cross navigation process, let us define $R = \text{Cam}[s, -k](r_{\min}(S)/2)$ where k is such that $t \in \text{Sect}[s', k]$. By construction

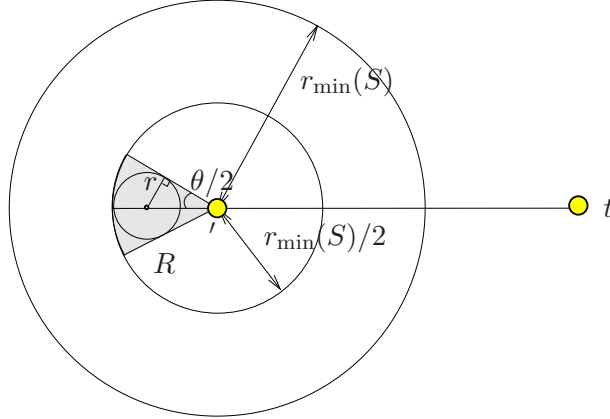


Figure 15: Illustration of the proof of Lemma 33

any point s of R is such that $\mathbf{X}(s, t) = s'$. R contains a disk of radius $r = \frac{r_{\min}(S)}{2} \frac{\sin(\theta/2)}{1+\sin(\theta/2)} > \frac{\theta}{4\pi}$ (see Figure 15). This disk intersects at least one point of $\rho\mathbb{Z}^2 \cap \mathcal{D}$. \square

Lemma 34 For any $c > 0$,

$$\mathbb{P}_{nf}(r_{\min} \leq n^{-c}) = O(n^{-2c+2}).$$

Proof. With a probability exponentially close to 1, $N = \#\mathbf{S}_{nf}$ is smaller than $c_1 n$ for some $c_1 > 0$ (see the proof of Lemma 18). Conditionally on $N = m$, by the union bound

$$\mathbb{P}_{nf}(r_{\min} \leq \lambda \mid N = m) \leq m^2 \mathbb{P}_{nf}(|V_1 - V_2| \leq \lambda)$$

for V_1, V_2 independent with density f on \mathcal{D} . Conditioning on V_1 , it is easily seen that this probability is smaller than $c_2 \lambda^2$ (for some c_2 independent on λ). Finally, we get $\mathbb{P}_{nf}(r_{\min} \leq \lambda) \leq c_1^2 n^2 c_2 \lambda^2 + \mathbb{P}(|N| \geq c_1 n)$, which leads easily to the stated result. \square

A consequence of the two previous lemmas is that if c_0 is large enough, with a probability $O(n^{-2c_0+2})$ any path from s to t coincides with a path from a point s_g to t for a point s_g in $\text{Grid}_n(c_0, a)$ up to the starting position (and then, up to the first step, smaller than Navmax, which will then be uniformly negligible at the scale we are working in). Hence, to approximate a path from s to t with a path from s_g to t_g , most of the difficulties come from the target.

Given a source s , a *section* of the $\text{Path}(s, t) = (s_0, \dots, s_{\text{Nb}(s,t)})$ is a sub-path $(s_{j_1}, \dots, s_{j_2})$ for which $\mathbf{X}(s, t, j) = \mathbf{X}(s_{j-1}, t_g)$ for each $1 \leq j_1 < j < j_2$, that is a part of $\text{Path}(s, t)$ matching a path toward t_g . Given a source s and a target t , a $\text{Squ}_a(\mathbf{B})$ -black box is a sub-path $(s_{j_1}, \dots, s_{j_2})$ of $\text{Path}(s, t)$ such that s_{j_1} and s_{j_2} are both in the same square of $\text{Squ}_a(\mathbf{B})$. For $k \in \mathbb{N}$, we will say that a path admits a (k, \mathbf{B}, c_0) decomposition if it can be decomposed into at most k sections and k $\text{Squ}_a(\mathbf{B})$ black boxes.

Proposition 35 *Let $\mathbf{B} > 0$ and $\rho > 0$ be fixed. For any $\mathbf{X} \in \{\mathbf{CY}, \mathbf{CT}\}$ and $\theta \leq \pi/3$, or $\mathbf{X} = \mathbf{SY}$ and $\theta \leq \pi/2$, or $\mathbf{X} = \mathbf{ST}$ and $\theta \leq 2\pi/3$, there exists $c_0 > 0$ such that,*

$$\mathbb{P}_{n,f} (\forall s, t \in \mathcal{D} \text{ such that } Z(s, t) \in \mathcal{D}[a], \text{Path}^{\mathbf{X}}(s, t) \text{ admits a } (13, \mathbf{B}, c_0) \text{ decomposition}) \geq 1 - o(n^{-\rho}),$$

where $Z(s, t) = \Gamma(s, t)$ for $\mathbf{X} \in \{\mathbf{CY}, \mathbf{CT}\}$ and $Z(s, t) = [s, t]$ for $\mathbf{X} \in \{\mathbf{ST}, \mathbf{SY}\}$ is the limiting trajectory for the respective navigations.

Before proving this proposition, let us examine how it entails, together with the already proved results concerning one trajectory, the theorems of this paper.

4.4 Consequences

Thanks to the above discussion, the results concerning one trajectory as Theorems 28 and 29 can be extended as follows: a supremum on (s, t) in $\text{Grid}(c_0, a)$ can be added inside the probabilities present in these theorems. It remains to see what is the price to take the supremum on all (s, t) . We saw above that with a polynomially close to 1 probability, for any (s, t) , $\text{Path}(s, t)$ can be decomposed in at most 13 sections, and 13 black boxes. On each section, $\text{Path}(s, t)$ coincides with a section of $\text{Path}(s, t_g)$. Therefore the fluctuations of the position functions (or the cost functions) on each of these sections are smaller than $n^{-\alpha}$ with a probability exponentially close to 1, since this is the case for the trajectories between points of $\text{Grid}_n(c_0, a)$. To see this, assume for example that

$$\sup_{x \in [0, \lambda]} \left| \text{Pos}_{s,t}^{\mathbf{X}}(x\sqrt{n}) - \text{Pos}_{s,t}^{\infty, \mathbf{C}_{\text{bis}}^{\mathbf{X}}}(x) \right| \leq \varepsilon.$$

Therefore, for any increasing sequence x_1, \dots, x_{2k} (where $x_{2i-1}\sqrt{n}$ and $x_{2i}\sqrt{n}$ have to be understood as the beginning and ending time of the sections), one has

$$\sup_{0 \leq x_1 \leq \dots \leq x_{2k} \leq \lambda} \left| \sum_{i=1}^k \left| \text{Pos}_{s,t}^{\mathbf{X}}(x_{2i}\sqrt{n}) - \text{Pos}_{s,t}^{\mathbf{X}}(x_{2i-1}\sqrt{n}) \right| - \sum_{i=1}^k \left| \text{Pos}_{s,t}^{\infty, \mathbf{C}_{\text{bis}}^{\mathbf{X}}}(x_{2i}) - \text{Pos}_{s,t}^{\infty, \mathbf{C}_{\text{bis}}^{\mathbf{X}}}(x_{2i-1}) \right| \right| \leq k\varepsilon;$$

Hence, a global control of the path ensures a good control of the sections, provided that the number of sections is small. This is the case here, since we have 13 sections at most with a large probability. Now, the contribution of the black boxes (the stages between times $x_{2i}\sqrt{n}$ and $x_{2i+1}\sqrt{n}$) have to be controlled. For this, notice first that the space fluctuations for any black box $(s_{j_1}, \dots, s_{j_2})$ is small: $\max\{|s_j - s_{j_1}|, j \in \llbracket j_1, j_2 \rrbracket\} \leq ca_n$ for a constant c depending only on θ ; indeed, since s_{j_1} and s_{j_2} are in the same square \square of $\text{Squ}_a(\mathbf{B})$, and since at each stage the traveller gets closer to the target, and since the angle between its local direction and the direction to the target is smaller than θ , it must stay in a domain with area at most $O(a_n^2)$ (which is included in the ball $B(\circ, ca_n)$ where \circ is the centre of \square). Moreover, let $l(\mathbf{X})$ be the maximum length (number of stages) of all black boxes for the algorithm $\mathbf{X} \in \{\mathbf{CT}, \mathbf{CY}, \mathbf{ST}, \mathbf{SY}\}$:

$$l(\mathbf{X}) := \max\{j_2 - j_1, (s_{j_1}, \dots, s_{j_2}) \text{ black-box}\}$$

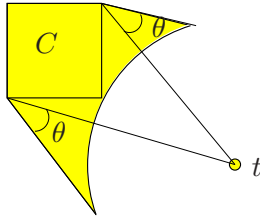


Figure 16: Area around \square containing the range of the black-box $(s_{j_1}, \dots, s_{j_2})$

is then bounded by $\text{Maxball}(ca_n, a)$, and then

$$\mathbb{P}(l(\mathbf{X}) \geq a_n^2 n^{1+\varepsilon}) = \mathbb{P}(l(\mathbf{X}) \geq n^{2\mathbf{B}+\varepsilon}) \leq e^{-n^d}$$

for some $d > 0$, if n is large enough. Since $2\mathbf{B} + \varepsilon$ can be chosen smaller than $1/2$ in all proofs, at most k black box, of negligible size $n^{2\mathbf{B}+\varepsilon}$ are concerned. The contribution of these black boxes to the time normalisation of the processes is negligible, as well as the space normalisation: at most $a_n^2 n^{1+\varepsilon}$ contributions of size $n^{\mathbf{C}}$ with size smaller than Navmax or $H(\text{Navmax}/\sqrt{n})$ are (uniformly for all (s, t)) negligible.

4.5 Proof of Proposition 35

In order to bound the number of black boxes on a trajectory, we need to understand when a navigation decision differs when navigating to t or to t_g . We will see that this is related to some geometrical features of some straight lines, called below “rays”, issued from the points of S .

Let us introduce the notion of *navigation-star* and *standard constellation*. A *star* is a collection of half-lines (we also use the term *rays*) issued from the same point called the *centre*. A *navigation-star* of $s \in S$ is a cyclic ordered list (r_1, r_2, \dots, r_k) of half-lines starting at s such that for any two points t, t' of \mathcal{D} between two consecutive half-lines then if $\mathbf{X}(s, t) \neq t$ and $\mathbf{X}(s, t') \neq t'$ then $\mathbf{X}(s, t) = \mathbf{X}(s, t')$; in other words, if t and t' are far enough from s (at distance at least 2Navmax suffices) then the first stop starting from s is the same whether if the target point is t or t' (see Figure 17).

In order to build navigation stars we now define different types of rays; they will be used to control the decompositions of straight or cross navigation paths in the sequel:

- Given a point s of S and $k \in \llbracket 1, p\theta \rrbracket$, the k th *type-1 ray* around s is simply $r_1(s, k) = \text{HL}_k(s)$, as defined in (2).
- Let s be a point of S . We construct two rays associated with each points s' of S . Denote by $r_2^+(s, s')$ (resp. $r_2^-(s, s')$) the half-line obtained by a rotation of $\theta/2$ (resp. $-\theta/2$) with centre s of the half-line $[s, s']$. These rays are called *type-2 rays*.
- Let s be a point of S . For any pair (s', s'') of S^2 , we denote by $r_3(s, s')$ the half-line $[s, t)$ such that $[s, t)$ is orthogonal to (s', s'') . This rays are called *type-3 rays*.

Let us now examine some navigation stars for the different algorithms of interest (it is convenient to notice that if NS is a navigation star around s , then any set of rays containing NS is also a navigation star).

- **Cross navigations.** For cross navigations, the set of type-1 rays centred at s , in other word, $\text{Cross}(s)$, is a navigation star of s .
- **θ Straight Yao navigation.** Starting from s for a target t far enough, $\text{SY}(s, t)$ depends only on $\arg(t - s)$. When t is turning around s , $\text{SY}(s, t)$ changes when the nearest point of s in $\overrightarrow{\text{Sect}}(s, t) \cap S$ is changing. This potentially happens when the angle between (s, t) and (s, s') is $\pm\theta/2$ for a point s' in S close enough to s , in any case for s' such that $|s - s'| \leq \text{Navmax}$. Hence the set of type-2 rays centred at s is a navigation star of s for θ Straight Yao navigation.
- **θ Straight T navigation.** Additionally to the rays presented in the previous point, another case has to be treated. Notice that $\text{ST}(s, t)$ is the point of $\overrightarrow{\text{Sect}}(s, t) \cap S$ that has the closest to s orthogonal projection on the bisecting line of $\overrightarrow{\text{Sect}}(s, t) \cap S$. When t is turning around s , $\text{ST}(s, t)$ is potentially changing when the order of the orthogonal projections of the elements of $\overrightarrow{\text{Sect}}(s, t) \cap S$ on the bisecting line of $\overrightarrow{\text{Sect}}(s, t)$ is changing. This may happens when the line that passes via two elements s' and s'' of S , not too far from s (such that $|s - s'| \leq \text{Navmax}$ and $|s - s''| \leq \text{Navmax}$), is orthogonal to the line (s, t) . Hence the set of type-2 rays and type-3 rays centred at s is a navigation star of s for θ Straight T navigation.

A navigation star containing only the rays defined in the previous paragraph is called *standard*. A *constellation* (resp. *standard constellation*) of S is the union of the navigation stars (resp. standard navigation stars) of every $s \in S$.

Given $C_t \in \text{Squ}_n^*(c_0)$ and $C \in \text{Squ}_a(\mathbf{B})$, denote by $\text{Numb}_i(C, C_t)$ the number of type- i rays that intersects C and that are defined with points s, s' and s'' in $S \cap C$ (depending on the case, s' and s'' can be non necessary). Define also,

$$\text{Numb}^{\text{CT}}(C, C_t) = \text{Numb}^{\text{CY}}(C, C_t) = \text{Numb}_1(C, C_t), \quad (61)$$

$$\text{Numb}^{\text{SY}}(C, C_t) = \text{Numb}_2(C, C_t), \quad (62)$$

$$\text{Numb}^{\text{ST}}(C, C_t) = \text{Numb}_2(C, C_t) + \text{Numb}_3(C, C_t) \quad (63)$$

as one can guess in view of the above discussion. We also set $\text{Numb}(C, C_t) = \sum_{i=1}^3 \text{Numb}_i(C, C_t)$ a bound on all the $\text{Numb}^{\mathbf{X}}(C, C_t)$ for $\mathbf{X} \in \{\text{ST}, \text{SY}, \text{CT}, \text{CY}\}$.

Lemma 36 *Let $\mathbf{B} > 0$ and $c_0 > 0$ be fixed. For any algorithm $\mathbf{X} \in \{\text{ST}, \text{SY}, \text{CT}, \text{CY}\}$,*

$$\sup_{\substack{(C, C_t) \in \text{Squ}_a(\mathbf{B}) \times \text{Squ}_n^*(c_0) \\ d(C, C_t) \geq a_n}} \mathbb{P}_{nf}(\text{Numb}^{\mathbf{X}}(C, C_t) \geq 1) = O(n^{4\mathbf{B}+2-c_0}).$$

Proof. All along the proof, $C_t \in \text{Squ}_n^*(c_0)$ and $C \in \text{Squ}_a(\mathbf{B})$ are such that $d(C_t, C) \geq a_n$. We first control $\text{Numb}_1(C, C_t)$. Let $St(C_t)$ the set of points $s \in \mathcal{D}[a]$ such that $\text{Cross}(s)$ crosses C_t . This set forms also a cross, as represented on Figure 18.

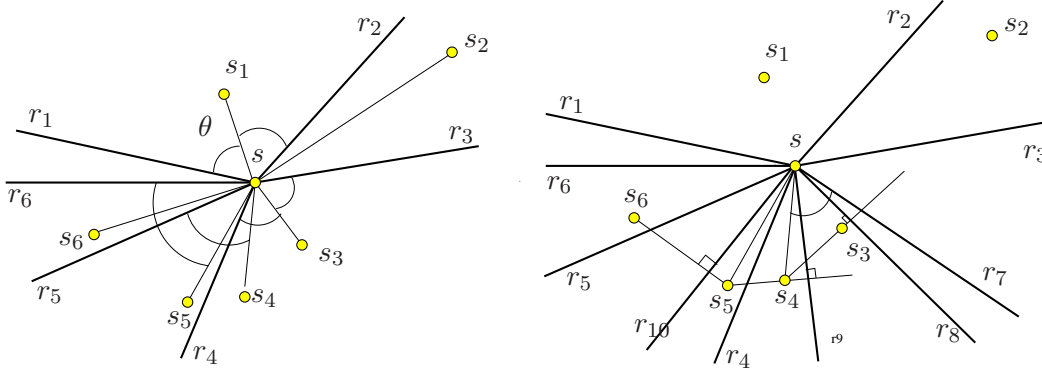


Figure 17: On the left, an example of a navigation-star in θ Straight Yao navigation. For a vertex t for enough between r_i and r_{i+1} , $\mathbf{SY}(s, t) = s_i$ (assuming that $r_7 = r_1$). On the right, an example of a navigation-star in θ Straight T navigation. In this case the situation is a bit more complex. For instance, $\mathbf{ST}(s, t) = s_3$ if t is between r_3 and r_7 or between r_8 and r_4 . When t is between r_7 and r_8 (and far enough) then $\mathbf{ST}(s, t) = s_4$.

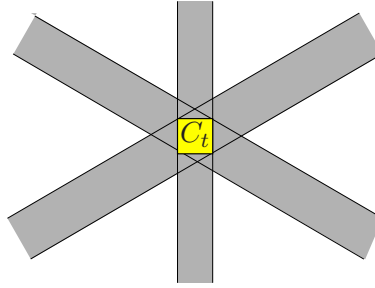


Figure 18: $St(C_t)$ for $p_\theta = 6$.

Clearly for any $i \in \{1, 2, 3\}$

$$\mathbb{P}_{nf}(\text{Numb}_i(C, C_t) \geq 1) \leq \mathbb{P}_{nM_f}(\text{Numb}_i(C, C_t) \geq 1).$$

We then obtain the bound under \mathbb{P}_{nM_f} instead than \mathbb{P}_{nf} . Now,

$$\mathbb{P}_{nM_f}(\text{Numb}_1(C, C_t) \geq 1) \leq \mathbb{P}_{nM_f}(\#(\mathbf{S} \cap C \cap St(C_t)) \geq 1) = O(n|C \cap St(C_t)|)$$

where $|A|$ denote the area of A (since, for an integer random variable X , $\mathbb{P}(X \geq 1) \leq \mathbb{E}(X)$). Observing that $St(C_t)$ is composed by p_θ stripes having width $O(n^{-c_0})$ and that C has diameter $\sqrt{2}n^{\mathbf{B}-1/2}$, we have $|C \cap St(C_t)| = O(n^{\mathbf{B}-1/2-c_0})$, and then

$$\mathbb{P}_{nf}(\text{Numb}_1(C, C_t) \geq 1) = O(n^{1/2+\mathbf{B}-c_0}). \quad (64)$$

Let us pass to the control of Numb_2 . For p the centre of C_t , we have $\mathbb{P}_{nf}(\text{Numb}_2(C, C_t) \geq 1)$

$$\leq \mathbb{P}_{nM_f} \left(\exists X_1, X_2 \in \mathbf{S} \cap C, \left| \arg(X_2 - X_1) - \arg(p - X_1) \right| \leq \frac{\theta}{2} \right) \leq \arctan \left(\frac{n^{-c_0}}{\sqrt{2}|p - X_1|} \right).$$

Since $|X_1 - p| > n^{\mathbf{B}-1/2}$, $\arctan\left(\frac{n^{-c_0}}{\sqrt{2}|p-X_1|}\right) \leq \arctan(n^{-c_0-\mathbf{B}+1/2}/\sqrt{2})$. Moreover using (31), $\mathbb{P}_{nM_f}(|\mathbf{S} \cap C| > n^{2\mathbf{B}+d}) \leq \exp(-n^{2\mathbf{B}+d})$ for any $d > 0$, provided that n is large enough. Recall now, that under \mathbb{P}_{nM_f} , knowing that $\#(\mathbf{S} \cap C) = k$, the set $\mathbf{S} \cap C$ is distributed as a sample of k i.i.d. uniform random variables in C . Therefore, for any function g and measurable set A ,

$$\begin{aligned} & \leq \mathbb{P}_{nM_f}(\#(\mathbf{S} \cap C) > n^{2\mathbf{B}+d}) \\ & \quad + \sum_{k \leq n^{2\mathbf{B}+d}} \mathbb{P}_{nM_f}(\exists X_1, X_2 \in \mathbf{S} \cap C, g(X_1, X_2) \in A | \#\mathbf{S} = k) \mathbb{P}_{nM_f}(\#\mathbf{S} = k) \\ & \leq \mathbb{P}_{nM_f}(\#(\mathbf{S} \cap C) > n^{2\mathbf{B}+d}) + (n^{2\mathbf{B}+d})^2 \mathbb{P}(g(U_1, U_2) \in A) \end{aligned}$$

for two i.i.d. uniform random variables U_1 and U_2 in C .

Therefore, $\mathbb{P}_{nM_f}(\text{Numb}_2(C, C_t) \geq 1)$ is bounded by:

$$O((n^{2\mathbf{B}+d})^2) \mathbb{P}\left(\left|\arg(U_2 - U_1) - \arg(p - U_1)\right| - \frac{\theta}{2} \leq \arctan\left(\frac{n^{-c_0}}{\sqrt{2}|p - U_1|}\right)\right) + \exp(-n^{2\mathbf{B}+d}).$$

A simple picture shows that U_1 given, U_2 must lie in a triangle included in C , with diameter smaller than $\sqrt{2}a_n$ and with an angle smaller than $O\left(\frac{n^{-c_0-\mathbf{B}+1/2}}{\sqrt{2}}\right)$. Since the density of U_2 is $1/a_n^2$, for any U_1 given in C , the probability is $O(n^{-c_0-\mathbf{B}+1/2}a_n/a_n^2) = O(n^{-c_0-2\mathbf{B}+1})$ (with a constant in the O sign uniform for U_1 in C). Then

$$\begin{aligned} \mathbb{P}_{nM_f}(\text{Numb}_2(C, C_t) \geq 1) & = O(n^{4\mathbf{B}+2d})O(n^{-c_0-2\mathbf{B}+1}) + \exp(-n^{2\mathbf{B}+d}) \\ & = O(n^{2\mathbf{B}+2d-c_0+1}). \end{aligned}$$

Taking d small enough:

$$\mathbb{P}_{nM_f}(\text{Numb}_2(C, C_t) \geq 1) = O(n^{2\mathbf{B}-c_0+2}).$$

Let us pass to the control of Numb_3 . The quantity $\mathbb{P}_{nM_f}(\text{Numb}_3(C, C_t) \geq 1)$ is bounded by

$$\mathbb{P}_{nM_f}\left(\exists X_1, X_2, X_3 \in \mathbf{S} \cap C, \left|\arg(X_3 - X_2) - \arg(p - X_1) - \frac{\pi}{2}\right| \leq \arctan\left(\frac{n^{-c_0-\mathbf{B}+1/2}}{\sqrt{2}}\right)\right)$$

Again, this is

$$O((n^{2\mathbf{B}+d})^3) \mathbb{P}\left(\left|\arg(U_3 - U_2) - \arg(p - U_1) - \frac{\pi}{2}\right| \leq \arctan\left(n^{-c_0-\mathbf{B}+1/2}\right)\right)$$

for some U_1, U_2 and U_3 i.i.d. uniform in C . For (U_1, U_2) given, U_3 must be in a subset of C with Lebesgue measure $O(n^{-c_0-\mathbf{B}+1/2}a_n)$. Taking into account the density of U_3 , we get

$$\mathbb{P}_{nM_f}(\text{Numb}_3(C, C_t) \geq 1) = O(n^{4\mathbf{B}-c_0+2})$$

for d chosen small enough.

We conclude this proof thanks to (61), (62), (63), and the union bound. \square

For any $C_t \in \text{Squ}_{c_0}^*$ let

$$\text{BB}(C_t) = \{C \in \text{Squ}_a(\mathbf{B}), \text{Numb}(C, C_t) \geq 1\}$$

be the number of squares in $\text{Squ}_a(\mathbf{B})$ from which is issued at least one ray crossing C_t . The notation BB is chosen to make apparent that the elements of $\text{BB}(C_t)$ will be considered as black boxes related to C_t later on.

Lemma 37 *Let $\mathbf{B} > 0$ be fixed and $\rho > 0$ be fixed. There exists a $c_0 > 0$ such that,*

$$\mathbb{P}_{nf}(\exists C_t \in \text{Squ}_{c_0}^*, \#\text{BB}(C_t) \geq 12) = o(n^{-\rho}).$$

In other words, for any navigation algorithm, the number of black boxes is smaller than 12 with a huge probability.

Proof. Notice that under \mathbb{P}_{nM_f} , since $\mathbf{S} \cap C_1$ and $\mathbf{S} \cap C_2$ are independent variables for $C_1 \cap C_2 = \emptyset$, for any fixed $j \in \{1, 2, 3, 4\}$, any C_t and i fixed, the family of variables $(\text{Numb}_i(C, C_t), C \in \text{Squ}_a(\mathbf{B})^j)$ are i.i.d.. Hence, for each $j \in \{1, 2, 3, 4\}$ the variables $(\mathbf{1}_{\sum_{i=1}^3 \text{Numb}_i(C, C_t) \geq 1}, C \in \text{Squ}_a(\mathbf{B})^j)$ are i.i.d. Bernoulli distributed. Let

$$\text{BB}_j(C_t) = \{C \in \text{Squ}_a(\mathbf{B})^j, \text{Numb}(C, C_t) \geq 1\}.$$

We have, by the union bound,

$$\begin{aligned} \mathbb{P}(\exists C_t \in \text{Squ}_{c_0}^*, \#\text{BB}_j(C_t) \geq 3) &\leq \sum_{C_t \in \text{Squ}_{c_0}^*} \binom{\text{Squ}_a(\mathbf{B})^j}{3} \mathbb{P}_{nM_f}(\text{Numb}(C, C_t) \geq 1)^3 \\ &= O(n^{3-6\mathbf{B}+2c_0}) \sup_{C \in \text{Squ}_a(\mathbf{B}), C_t \in \text{Squ}_{c_0}^*} \mathbb{P}_{nM_f}(\text{Numb}(C, C_t) \geq 1)^3. \end{aligned}$$

From Lemma 36 this last term is $O(n^{3(4\mathbf{B}+2-c_0)})$. Then, this is $o(n^{-\rho})$ for c_0 chosen large enough. Now, to conclude, write

$$\mathbb{P}_{nf}(\exists C_t \in \text{Squ}_{c_0}^*, \#\text{BB}(C_t) \geq 12) \leq \sum_{j=1}^4 \mathbb{P}_{nM_f}(\exists C_t \in \text{Squ}_{c_0}^*, \#\text{BB}_j(C_t) \geq 3). \quad \square$$

Proof of Proposition 35. First observe that rays are defined by at most 3 points belonging to a ball of radius at most Navmax . From Lemma 17 we know that $\text{Navmax} \leq n^{-1/2+\mathbf{B}}/4$ with high probability. Hence the probability that 12 rays cross a square C_t is bounded by

$$\mathbb{P}_{nM_f}(\exists C_t \in \text{Squ}_{c_0}^*, \#\text{BB}(C_t) \geq 12) + \mathbb{P}_{nf}(\text{Navmax} \geq n^{-1/2+\mathbf{B}}/4),$$

which is $O(n^{-\rho})$ as wished, for $\mathbf{B} > 0$ and c_0 large enough. This means that for any target t , for any point s outside these 12 squares plus the square of $\text{Squ}_a(\mathbf{B})$ that contains $B(t, n^{-1/2+\mathbf{B}})$: $\mathbf{X}(s, t) = \mathbf{X}(s, t_g)$. For each $\text{Path}(s, t)$ and each C of these (at most) 13 squares we consider the sub-path P_C (of $\text{Path}(s, t)$) between the first stage that is inside C and the last stage that is inside C . Each of these portions of the trajectory forms a black box.

For every point s' of the path $\text{Path}(s, t)$ outside each of these 13 black boxes, none of the rays of the navigation star of s' crosses the square C_t , hence $\mathbf{X}(s', t) = \mathbf{X}(s', t_g)$. Hence path is partitioned into at most 13 black boxes and at most 13 sections. \square

5 Appendix

Proof of Lemma 15. By hypothesis, for a certain triangular array $(c'_n(j), j = 1, \dots, \lfloor \lambda/a_n \rfloor)$ satisfying $|c'_n(j)| \leq c_n$ for any $j \leq \lfloor \lambda/a_n \rfloor$, the following representation of y_n holds:

$$y_n((j+1)a_n) = y_n(ja_n) + a_n G(y_n(ja_n)) + a_n c'_n(j).$$

Hence y_n appears to be a perturbed version of the explicit Euler scheme, used to approximate a solution of $\text{Eq}(G, z)$, which is defined by $z_n(0) = z$,

$$z_n((j+1)a_n) = z_n(ja_n) + a_n G(z_n(ja_n)), \quad j \leq \lfloor \lambda/a_n \rfloor.$$

We review here the standard argument leading to the conclusion, namely the convergence of (z_n) to y_{sol} with a bound on the speed of convergence, and derive the same result for (y_n) . First, by the Cauchy-Lipschitz lemma, $\text{Eq}(G, z)$ has a unique solution, denoted y_{sol} on $[0, \lambda]$ (for $\lambda < \lambda(G, z)$ the maximum domain on which one can extend this solution). Now define for any $x \in [0, \lambda]$

$$R_{a_n}(x) = \frac{y_{\text{sol}}(x+a_n) - y_{\text{sol}}(x)}{a_n} - G(y_{\text{sol}}(x)). \quad (65)$$

Assume that G is α -Lipschitz and bounded by β (it is bounded since it is the maximum of a Lipschitz function on the bounded domain D); then y_{sol} is clearly β -Lipschitz. We have

$$\sup_{x \in [0, \lambda - a_n]} |R_{a_n}(x)| = \sup_{x \in [0, \lambda - a_n]} \left| \int_x^{x+a_n} \frac{G(y_{\text{sol}}(u)) du}{a_n} - G(y_{\text{sol}}(x)) \right| \quad (66)$$

$$= \sup_{x \in [0, \lambda - a_n]} \left| \int_x^{x+a_n} \frac{G(y_{\text{sol}}(u)) - G(y_{\text{sol}}(x)) du}{a_n} \right| \quad (67)$$

$$\leq \sup_{x \in [0, \lambda - a_n]} \alpha \int_x^{x+a_n} \frac{|y_{\text{sol}}(u) - y_{\text{sol}}(x)| du}{a_n} \quad (68)$$

$$\leq \frac{\alpha\beta}{2} a_n. \quad (69)$$

Let ζ_n be a perturbation of the explicit Euler scheme:

$$\zeta_n((j+1)a_n) = \zeta_n(ja_n) + a_n G(\zeta_n(ja_n)) + \varepsilon_n(j), \quad j \leq \lfloor \lambda/a_n \rfloor.$$

We have

$$\zeta_n((j+1)a_n) - z_n((j+1)a_n) = \zeta_n(ja_n) - z_n(ja_n) + a_n (G(\zeta_n(ja_n)) - G(z_n(ja_n))) + \varepsilon_n(j)$$

and thus

$$|\zeta_n((j+1)a_n) - z_n((j+1)a_n)| \leq (1 + \alpha a_n) |\zeta_n(ja_n) - z_n(ja_n)| + |\varepsilon_n(j)|$$

and therefore, for any j

$$|\zeta_n(ja_n) - z_n(ja_n)| \leq (1 + \alpha a_n)^j |\zeta_n(0) - z_n(0)| + \sum_{m=0}^{j-1} (1 + \alpha a_n)^{j-1-m} |\varepsilon_n(m)|$$

which leads (using $(1 + \alpha a_n)^k \leq e^{ka_n}$ for any $k \geq 0$) that for j such that $j \leq \lfloor \lambda/a_n \rfloor + 1$,

$$\begin{aligned} |\zeta_n(ja_n) - z_n(ja_n)| &\leq e^{ja_n} \left(|\zeta_n(0) - z_n(0)| + \sum_{m=0}^{j-1} |\varepsilon_n(m)| \right) \\ &\leq e^{a_n(1+\lfloor \lambda/a_n \rfloor)} \left(|\zeta_n(0) - z_n(0)| + \sum_{m=0}^{\lfloor \lambda/a_n \rfloor} |\varepsilon_n(m)| \right). \end{aligned}$$

Now to end, observe that

$$y_{\text{sol}}((j+1)a_n) = y_{\text{sol}}(ja_n) + a_n G(y_{\text{sol}}(ja_n)) + a_n R_{a_n}(ja_n)$$

This is a perturbed version of the Euler explicit scheme, and then by (66), for n large enough,

$$\sup_{j, j \leq \lfloor \lambda/a_n \rfloor + 1} |y_{\text{sol}}(ja_n) - z_n(ja_n)| \leq e^{a_n(1+\lfloor \lambda/a_n \rfloor)} \sum_{m=0}^{\lfloor \lambda/a_n \rfloor} a_n^2 \alpha \beta / 2 \leq \lambda e^\lambda \alpha \beta a_n, \quad (70)$$

since $e^{a_n} < 2$ for n large enough.

The same work, using the comparison between z_n and y_n leads, for n large enough, to

$$\sup_{j, \lfloor \lambda/a_n \rfloor + 1} |y_n(ja_n) - z_n(ja_n)| \leq e^{a_n(1+\lfloor \lambda/a_n \rfloor)} \sum_{m=0}^{\lfloor \lambda/a_n \rfloor} a_n c'_n(j) \leq 2e^\lambda \lambda \max_{j \leq \lambda/a_n} |c'_n(j)| \leq 2\lambda e^\lambda c_n, \quad (71)$$

Finally, summing (70) and (71), we get

$$\sup_{x \in [0, \lambda]} |y_{\text{sol}}(x) - y_n(x)| \leq C_\lambda \max\{a_n, c_n\},$$

for $C_\lambda = \lambda e^\lambda (2 + \alpha \beta) + 3\beta$ (this is obtained at first for the points $x \in \{ja_n, j \leq \lfloor \lambda/a_n \rfloor + 1\}$, then for all x in the interval using that on an interval of size a_n the fluctuation of y_{sol} are bounded by $a_n \beta$, and those of y_n are bounded by $\beta a_n + c_n$. Then, it is easily seen that the map $\lambda \mapsto C_\lambda$ has the properties stated in the lemma. \square

Proof of Corollary 16. Let (\mathbf{Z}_n) be a sequence of processes satisfying the hypothesis. Define, for each n , a process \mathbf{Y}_n coinciding with \mathbf{Z}_n at the points $(ja_n, j = 0, \dots, \lfloor \lambda/a_n \rfloor + 1)$, and which is linear between them. Introduce the event

$$\Omega_n = \bigcap_{j=0}^{\lfloor \lambda/a_n \rfloor} \left\{ \left| \frac{\mathbf{Y}_n((j+1)a_n) - \mathbf{Y}_n(ja_n)}{a_n} - G(\mathbf{Y}_n(ja_n)) \right| \leq c_n \right\} \quad (72)$$

according to Lemma 15, $\sup_{x \in [0, \lambda]} |\mathbf{Y}_n(x) - y_{\text{sol}}(x)| \leq D_\lambda \max\{a_n, c_n\}$ on Ω_n , for a function $\lambda \mapsto D_\lambda$ bounded on compact sets; moreover by the union bound,

$$\mathbb{P}(\Omega_n) \geq 1 - (\lfloor \lambda/a_n \rfloor + 1) a_n d_n. \quad (73)$$

We then also get immediately $\mathbb{P} \left(\sup_{x \in [0, \lambda]} |\mathbf{Y}_n(x) - y_{\text{sol}}(x)| \leq \varepsilon \right) \xrightarrow[n]{n} 1$, for any $\varepsilon > 0$, that is the convergence in probability for the uniform norm of \mathbf{Y}_n to y_{sol} . Define now

$$\Omega'_n := \bigcap_{j=0}^{\lfloor \lambda/a_n \rfloor} \left\{ \sup_{x \in [ja_n, (j+1)a_n]} |\mathbf{Z}_n(x) - \mathbf{Z}_n(ja_n) - (x - ja_n)G(\mathbf{Z}_n(ja_n))| \leq c'_n \right\}.$$

Again, by the union bound $\mathbb{P}(\Omega'_n) \geq 1 - (\lfloor \lambda/a_n \rfloor + 1)a_n b_n$. Assume now that we are on $\Omega_n \cap \Omega'_n$. Since \mathbf{Y}_n is linear between in $[ja_n, (j+1)a_n]$, and since \mathbf{Z}_n and \mathbf{Y}_n coincide at the points $(ja_n, j \geq 0)$, then \mathbf{Z}_n satisfies also (72). Hence, for any $t \in [ja_n, (j+1)a_n]$,

$$\mathbf{Y}_n(x) = \mathbf{Z}_n(ja_n) + (x - ja_n)G(\mathbf{Z}_n(a_n)).$$

And thus, on Ω'_n , we have $|\mathbf{Z}_n(x) - \mathbf{Y}_n(x)| \leq c'_n$, for any $x \in [0, \lambda]$ and then, $|\mathbf{Z}_n(x) - y_{\text{sol}}(x)| \leq c'_n + D_\lambda \max\{a_n, c_n\}$ on $\Omega_n \cap \Omega'_n$. Therefore

$$\begin{aligned} \mathbb{P}\left(\sup_{x \in [0, \lambda]} |\mathbf{Z}_n(x) - y_{\text{sol}}(x)| \leq c'_n + D_\lambda \max\{a_n, c_n\}\right) &\geq 1 - (\lfloor \lambda/a_n \rfloor + 1)a_n(b_n + d_n) \\ &\geq 1 - (\lambda + a_n)(b_n + d_n). \quad \square \end{aligned}$$

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