Non-crossing Trees are Almost Conditioned Galton-Watson trees[1]

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ABSTRACT: A non-crossing tree (NC-tree) is a tree drawn on the plane having as vertices a set of points on the boundary of a circle, and whose edges are straight line segments that do not cross. In this paper, we show that NC-trees with size n are conditioned Galton–Watson trees. As corollaries, we give the limit law of depth-first traversal processes and the limit profile of NC-trees.

1 Preliminaries

A non-crossing tree (NC-tree) is a tree drawn on the plane having as vertices a set of points on the boundary of a circle, and whose edges are straight line segments that do not cross. The points are labeled, clockwise from 1 to n, so that two NC-trees are considered to be different even if they differ by a rotation, or a symmetry. Consider Ω_n the set of non-crossing trees of size n with node 1 as root. The cardinality D_n of Ω_n is well known (see e. g. [6, 11]) and following [6] it can be established under using the following combinatorial decomposition of a NC-tree:
A NC-tree consists of a root, which is attached to a (possibly empty) sequence of butterflies, where a butterfly is a (ordered) pair of NC-trees, that share a common root.

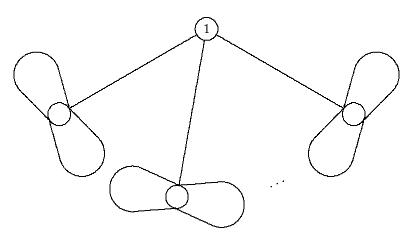

Figure 1 : The combinatorial decomposition of a non-crossing tree.

This combinatorial decomposition can be translated via symbolic methods immediately to an algebraic equation for the generating functions $D(z)$, resp. $B(z)$ of the numbers D_n and B_n of NC-trees, resp. butterflies of size n:

$$D(z) = \frac{z}{1 - B(z)}, \quad B(z) = \frac{D^2(z)}{z}.$$

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The coefficients D_n of the resulting algebraic equation for $D(z)$ can be extracted via Lagrange inversion formula and one obtains

$$D_n = \binom{3n-3}{n-1}/(2n-1).$$

Here we endowed the space Ω_n with the uniform law, each tree having then probability D_n^{-1}.

Using above combinatorial decomposition of NC-trees, in Panholzer [13] the distribution of the height $h(j)$ of the node j was studied and appears to be related to the Brownian excursion. In the present paper we give an explanation of this fact, proving that non-crossing trees behave almost like conditioned Galton–Watson trees. The methods are mainly probabilistic. In section 2, we study the shape of non-crossing trees and exhibit a Galton–Watson like tree, which has the same shape. In section 3, we show that the results known on conditioned GW-trees apply for NC-trees. For example we give the limit contour and the limit profile of NC-trees.

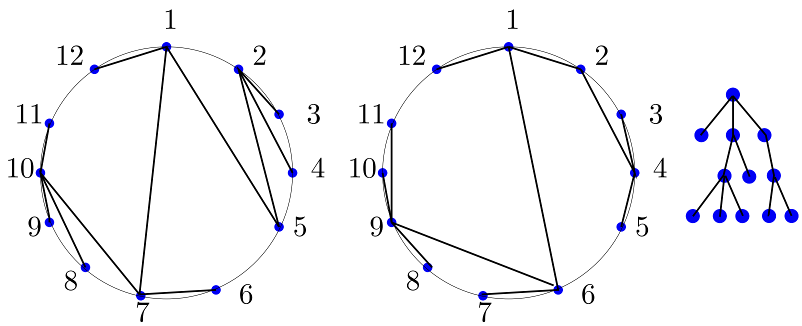

Figure 2 : Two size 12 non-crossing trees and their non-circle representation.

2 A Galton–Watson description of non-crossing trees

Consider a tree τ in Ω_n and consider its depth first traversal (see section 3.1). Note $lab(i)$ the label of the i^{th} node visited during the depth first traversal and N_i the number of children of node $lab(i)$ (we have then $lab(1) = 1$). We call the sequence $(N_i)_{i=1,\dots,n}$ the shape of the tree τ. The shape gives the number of neighbors of each vertex (the circle-representation will be here forgotten), and then describes entirely the tree τ. An important (obvious) remark is that several NC-trees may have the same shape. The number of NC-trees with a given shape is obtained in Lemma 2; this is an important point of the proof of our main Theorem 1.

Consider the two following offspring distributions $\mu = (\mu_k)_{k\geq 0}$ and $\lambda = (\lambda_k)_{k\geq 1}$ defined by:

$$\mu_k = \frac{4(k+1)}{3^{k+2}}, \text{ for } k = 0, 1, 2, \dots, \tag{1}$$

$$\lambda_k = 2 \times 3^{-k}, \text{ for } k = 1, 2, 3, \dots. \tag{2}$$

A Galton–Watson process $(Z_i)_{i\geq 0}$ (with respect to the distribution λ and μ) is defined as: $Z_0 = 1$, Z_1 is λ distributed, and, for any $k > 1$,

$$Z_k = \sum_{j=1}^{Z_{k-1}} Y_j^{(k-1)},$$

where the $Y_j^{(k)}$ are i.i.d. (and independent of Z_1), μ distributed.

In terms of the tree, Z_k represents the number of individuals at generation k (at level k in the tree). $Y_j^{(k-1)}$ is the number of children of the j-th individual of level $k-1$. Consider now t the family tree representation of this Galton–Watson process (that is simply the genealogical tree of the individuals that compose the GW process); we note Ω' the space of these GW trees and Ω'_n the space of trees of size n with the law induced by the conditioning $|t| = n$ on Ω'. As above, we consider the depth first traversal of a tree t from Ω'_n; we note N'_i the number of children of the i-th node visited during the depth first procedure. The sequence $(N'_i)_{i=1,\ldots,n}$ is then the shape of t. We have:

Theorem 1 *(The main result)*

$$(N'_1, \ldots, N'_n) \stackrel{(d)}{=} (N_1, \ldots, N_n).$$

This Theorem is the core of our work: it implies (together with the description of the offspring distribution law) all the new results on the properties of NC-trees given in the present paper.

Note: In [4], Deutsch and Noy have proved, that the mean number of nodes with out-degree d in NC-trees are asymptotically given by $\left(\frac{4(d+1)}{3^{d+2}}\right) n$. Using the law of large numbers, if non-crossing trees would be GW-trees, $\mu_d = \frac{4(d+1)}{3^{d+2}}$ would be the only choice for the offspring distribution. Unfortunately, NC-trees are not GW trees with offspring μ, and so, they are not GW trees; but, Theorem 1 says that NC-trees are almost GW trees: only the root distribution follows an other law than μ.

Note: The progeny distribution μ is the convolution of two geometrical laws with parameter 1/3. As a referee pointed out, this can be explained in the following way: From the combinatorial decomposition, as described in section 1, follows that NC-trees are sequences of butterflies, where butterflies have a counting generating function $B(z) = \frac{z}{(1-B(z))^2}$. There is then the general fact, that a combinatorial model of trees with counting generating function $T = z\varphi(T)$ when conditioned upon size n is equivalent to a Galton–Watson tree (also conditioned upon size n) with the probability generating function of a progeny being $\varphi(u) := \varphi(xu)/\varphi(x)$ with any x. This leads for butterflies to a progeny generating function $\phi(u) := \frac{(1-x)^2}{(1-ux)^2}$, and with the condition $\varphi'(1) = 1$ (or $x\varphi'(x) = \varphi(x)$) we obtain precisely the probability generating function of the convolution of two geometrical laws with parameter 1/3.

Proof of Theorem 1

For the remaining of the paper, we note $(x_1, \ldots, x_n)$ a possible shape (obtained during the depth first traversal) of a tree τ with n nodes. The sequence $(x_1, \ldots, x_n)$ must satisfy the set of conditions:

$$(A) = \left\{ \begin{array}{lcl} x_1 - 1 & \geq & 0 \\ x_1 + x_2 - 2 & \geq & 0 \\ \quad\vdots & \vdots & \vdots \\ x_1 + \cdots + x_{n-1} - (n-1) & \geq & 0 \\ x_1 + \cdots + x_n - n & = & -1. \end{array} \right. \tag{3}$$

Lemma 2 *On Ω_n the number of non-crossing trees that satisfies $(N_1, \ldots, N_n) = (x_1, \ldots, x_n)$ is*

$$\prod_{i \geq 2} (x_i + 1).$$

Proof : This result is a simple consequence of the work of Panholzer and Prodinger [14]. We copy the main arguments there:
Consider τ a non crossing tree and v a node from τ. The sons of v that have smaller label than v are called *left sons*, and the other ones, *right sons*. One can describe a non-crossing tree as a planted plane tree (ordered tree), where each node v_i apart from the root gets a mark from $\{0, \dots, |v_i|\}$ and the root gets mark 0. The mark of a node counts the number of left sons of this node. So two different NC-trees will be marked differently; conversely, if two marked trees differ by one (or several) mark, they represent different non-crossing trees.
Now, consider a tree with shape $(x_1, \dots, x_n)$. Each of the $\prod_{i\geq 2}(x_i + 1)$ ways to mark a planted tree with shape $(x_1, \dots, x_n)$ gives an existing non crossing tree. $\square$
Hence, if $(x_1, \dots, x_n)$ is a sequence of positive integers satisfying condition (A), we have:

$$\mathbb{P}_{\Omega_n}((N_1, \dots, N_n) = (x_1, \dots, x_n)) = D_n^{-1} \prod_{i\geq 2}(x_i + 1). \tag{4}$$

Proposition 3 (**Probability of a given shape on** Ω'_n) *Let $(x_1, \dots, x_n)$ be a sequence of positive integers satisfying condition (A); there exists a constant C_n (which does not depend on the x_i's), such that:*

$$\mathbb{P}_{\Omega'_n}((N'_1, \dots, N'_n) = (x_1, \dots, x_n)) = C_n^{-1} \prod_{i\geq 2}(x_i + 1).$$

Proof of Theorem 1: Assume that Proposition 3 is proved. Proposition 3 and formula (4) say that, for any fixed n, $\mathbb{P}_{\Omega'_n}((N'_1, \dots, N'_n) = (x_1, \dots, x_n))$ and $\mathbb{P}_{\Omega_n}((N_1, \dots, N_n) = (x_1, \dots, x_n))$ are proportional; since $\mathbb{P}_{\Omega_n}$ and $\mathbb{P}_{\Omega'_n}$ are both probabilities on the set of all rooted plane trees, these two probabilities are equal. Hence, Proposition 3 and formula (4) imply Theorem 1. $\square$

Proof of Proposition 3: First, we compute the probability for a tree in Ω' to have shape $(x_1, \dots, x_n)$.

$$\mathbb{P}_{\Omega'}((N'_1, \dots, N'_n) = (x_1, \dots, x_n)) \quad = \quad \mathbb{P}_{\Omega'}(Z_1 = x_1, Y_2 = x_2, \dots, Y_n = x_n)$$

where the random variables Y_k are i.i.d., μ distributed and independent from Z_1. Using, formula (1,2,3), we obtain:

$$\mathbb{P}_{\Omega'}((N'_1, \dots, N'_n) = (x_1, \dots, x_n)) = 2(\,4\,/\,3^3\,)^{n-1} \prod_{i\geq 2}(x_i + 1). \tag{5}$$

Note that the law λ has been chosen to provide the simplification of x_1 in the right hand term. Now, since $(x_1, \dots, x_n)$ is the shape of a tree from Ω_n (and so from Ω'_n, since each size n tree is in Ω'_n), we have:

$$\begin{aligned}
\mathbb{P}_{\Omega'_n}((N'_1, \dots, N'_n) = (x_1, \dots, x_n)) \quad &= \quad \frac{\mathbb{P}_{\Omega'}((N'_1, \dots, N'_n) = (x_1, \dots, x_n))}{\mathbb{P}_{\Omega'}(\Omega'_n)} \\
&= \quad C_n^{-1} \prod_{i\geq 2}(x_i + 1),
\end{aligned}$$

where $C_n^{-1} = \dfrac{2(\,4\,/\,3^3\,)^{n-1}}{\mathbb{P}_{\Omega'}(\Omega'_n)}$. $\square$

Note that these results entail that $\mathbb{P}_{\Omega'}(\Omega'_n) = 2(\,4\,/\,3^3\,)^{n-1} D_n$ which is equivalent to $\sqrt{\frac{3}{4\pi}}\, n^{-3/2}$ when n goes to $+\infty$.

3 Corollaries

Let us define classical processes associated to the trees.

3.1 Processes associated to non-crossing trees

The processes that we consider are defined on integer values; in order to prove convergence of these processes, we will consider that they are interpolated between these values. The weak convergence considered is the weak convergence in $C([0,1], \|.\|_\infty)$ for the depth processes and the weak convergence in $C([0,+\infty[, \|.\|_\infty)$ for the profile.

The depth first search:

Let τ be an ordered tree with n nodes. We define a function (see Aldous [1]):

$$\tilde{f} : \{0, \ldots, 2n-2\} \longrightarrow \{\text{nodes of } \tau\},$$

which we regard as a walk around τ, as follows:

$$\tilde{f}(0) = \text{root}.$$

Given $\tilde{f}(i) = v$, choose, if possible, the most left child w of v which has not already been visited, and let $\tilde{f}(i+1) = w$. If not possible, let $\tilde{f}(i+1)$ be the parent of v.

The contour: For a tree $\tau \in \Omega_n$, we call the contour of τ, the process V_n defined by:

$$V_n(i) = d(root, \tilde{f}(i)), \quad 0 \leq i \leq 2n-2,$$

where d is the usual distance between two nodes in a tree, that counts the number of edges on the direct path between these two nodes (thus $V_n(i)$ is also the height of the node $\tilde{f}(i)$). For i from 0 to $n-1$, set v_i as the i-th new node visited by the depth first procedure on $\tau \in \Omega_n$ ($v_0 = root$) and

$$\xi_i := \text{ the outdegree of } v_i = \text{the number of children of } v_i.$$

The depth first queue process (DFQP): For a tree $\tau \in \Omega_n$, the DFQP, S_n is defined by $S_n(0) = 0$ and:

$$S_n(j) = \sum_{i=0}^{j-1} (\xi_i - 1) \text{ for any } 1 \leq j \leq n. \tag{6}$$

The profile: For a tree $\tau \in \Omega_n$, the profile $(L_k)_{k \geq 0}$ is defined by:

$$L_k = \#\{i \,|\, d(v_i, root) = k\}.$$

The profile is then the sequence of the numbers of individuals on each level of the tree.

These three processes give a better understanding of the behavior of the shape of the tree. The contour is geometrically very close to the usual representation of trees. For example, the maximum of the contour is also the height of the tree, the path length (as defined below) can be computed via the area under the contour. The interest of the DFQP is that, in the case of GW trees, it is a simple random walk conditioned to be an excursion. Usually, its study is then simpler that the one of the contour. Using [12], one can express the contour via the DFQP, and the properties of these two processes are very close. The profile of the tree contains the information about the repartition of the individuals on the levels of the tree. Hence, the width of the tree (that is the maximum number of nodes on a level of the tree) is simply the maximum of the profile. The first two processes realize a one to one correspondence with the associated tree.

3.2 Degree of the root

The process $(S_n(i))_i$ allows to compute the real law of the root degree (which is not λ since the tree is size conditioned). Note T_k the hitting time of $-k$ by a random walk with i.i.d. increment, distributed as X-1 where X is μ distributed.

$$\mathbb{P}_{\Omega'_n}(Z_1 = k) = \lambda_k \frac{\mathbb{P}(T_{-k} = n-1)}{\sum_{j\geq 1} \lambda_j \mathbb{P}(T_{-j} = n-1)} = \lambda_k \frac{\frac{k}{n-1}\mathbb{P}(S_{n-1} = k)}{\sum_{j\geq 1} \lambda_j \frac{j}{n-1} \mathbb{P}(S_{n-1} = j)} \tag{7}$$

This formula comes from the decomposition of the DFQP in two parts before and after the first step due to the root. Using the central local limit theorem (see [15], page 706), we find the asymptotic law for the root degree when n goes to $+\infty$:

$$\mathbb{P}_{\Omega'_n}(Z_1 = k) \underset{n}{\longrightarrow} \frac{2}{3} k\lambda_k = \mu_{k-1} \text{ for } k \geq 1, \text{uniformly in } k.$$

This translates the fact that the degree of the root (that is, its outdegree) has the same law as the one of the other nodes (indegree + outdegree).

3.3 Differences with usual conditioned Galton–Watson trees

It is well known that the contour or other discrete excursions associated to Galton–Watson trees of size n converge, suitably normalized, to the Brownian excursion. But the trees in Ω'_n are not exactly Galton–Watson trees, because the law of Z_1 is different of the one of the other nodes. At the first glance, one may think, that limit theorems valid in the case of pure Galton–Watson trees apply for non-crossing trees, since the law of Z_1 does not seem to be important; but one can easily find lattice cases, where changing the law of the root degree (under the condition size=n) changes the fact whether the trees exists or not.

Let us give in two points a formal argument that NC-trees behave like GW trees:

• **In Ω'_n, the largest subtree has size $n - o(n)$:**
we note with Ω''_n the set of Galton–Watson trees with offspring distribution μ (now valid also for the root), conditioned to have size n. We note with a tilde, the random variables on Ω''_n (which are yet defined on Ω'_n). Note L the size of the largest subtree of the root in Ω'_n ($\tilde{L}$ for the same object in Ω''_n). The limit law of $\tilde{Z}_1$ is given by

$$\lim \mathbb{P}_{\Omega''_n}(\tilde{Z}_1 = k) \overset{def}{=} \gamma_k = k\mu_k, \text{ for } k \geq 1 \tag{8}$$

(this is obtained using the same computation as in (7)). Now, the contour of trees from Ω''_n converges to the Brownian excursion (see Aldous [1] or Marckert and Mokkadem [12]). It follows that:

$$\tilde{L}/n \xrightarrow{proba} 1;$$

one refers also to Gourdon [8] for combinatorial arguments (he proved that the largest subtree of the root has size $n - o(n)$). Let us prove that this is also true under the condition $\tilde{Z}_1 = k$ (for any value of $k \geq 1$):

Suppose there exists a $k \geq 1$ and an $\varepsilon > 0$ such that

$$\liminf \mathbb{P}_{\Omega''_n}\big(\tilde{L}/n \geq 1 - \varepsilon | \tilde{Z}_1 = k\big) < 1.$$

Using (8) this implies, that

$$\liminf \mathbb{P}_{\Omega''_n}\big((\tilde{L}/n \geq 1-\varepsilon)\cap(\tilde{Z}_1 = k)\big) < \gamma_k \tag{9}$$

Since $\liminf_n \mathbb{P}_{\Omega''_n}(\tilde{Z}_1 = k) = \limsup_n \mathbb{P}_{\Omega''_n}(\tilde{Z}_1 = k) = \gamma_k$, relation (9) implies that

$$\liminf_n \mathbb{P}_{\Omega''_n}\big(\tilde{L}/n \geq 1-\varepsilon\big) < 1$$

which is false. Hence, it is established that, for any $k \geq 1$ and any $\varepsilon > 0$,

$$\liminf_n \mathbb{P}_{\Omega''_n}\big(\tilde{L}/n \geq 1-\varepsilon | \tilde{Z}_1 = k\big) = 1.$$

Now, let us prove that

$$L/n \xrightarrow{proba} 1.$$

Let ε be a positive real number. Since, Ω''_n and Ω'_n under the condition $degree(root) = k$ have the same law, we have:

$$\begin{aligned}
\mathbb{P}_{\Omega'_n}\Big(L/n > 1-\varepsilon\Big) &= \sum_{k\geq 1}\mathbb{P}_{\Omega'_n}\Big(L/n > 1-\varepsilon|Z_1 = k\Big)\mathbb{P}_{\Omega'_n}(Z_1 = k)\\
&= \sum_{k\geq 1}\mathbb{P}_{\Omega''_n}\Big(\tilde{L}/n > 1-\varepsilon|\tilde{Z}_1 = k\Big)\mathbb{P}_{\Omega'_n}(Z_1 = k).
\end{aligned}$$

The k-th term of the last sum converges to μ_{k-1}. Using (7) and the fact that the approximation given by the local limit theorem is uniform in k, we have for n large enough, that

$$\mathbb{P}_{\Omega''_n}\big(\tilde{L}/n > 1-\varepsilon|\tilde{Z}_1 = k\big)\mathbb{P}_{\Omega'_n}(Z_1 = k) \leq 2k\lambda_k.$$

Hence, the terms of the sum are dominated independently of n; using Lebesgue theorem with the same name, it follows that

$$\lim_n \mathbb{P}_{\Omega'_n}\big(L/n > 1-\varepsilon\big) = 1.$$

• **The largest subtree is a simple Galton–Watson tree:**
Consider τ_1 the largest subtree of τ in Ω'_n. τ_1 is a simple Galton–Watson tree. Its depth first processes converge to the Brownian excursion (with the usual normalization, since its size is $n - o(n)$). Since the other subtrees have cumulative size $o(n)$, their contributions to the limit are null. It follows that the limit depth processes of τ and the one of τ_1 are equal. Using the same argument, the profile of the trees τ and τ_1 have the same limit.

3.4 Limit of the processes

The propositions given in this section are consequences of Section 3.3.

Under μ, the mean is 1 and the variance is $\sigma^2 = 3/2$. The following results concerning conditioned Galton–Watson trees apply for NC-trees:

Proposition 4 *(***Contour of non-crossing trees***) Set $(V_n(k))_{k=0,\dots,2n-2}$ the contour of the tree of Ω_n. We have*

$$\Big(\frac{V_n(2nt)}{\sqrt{n}}\Big)_{t\in[0,1]} \xrightarrow[n]{weakly} \Big(2\sqrt{\frac{2}{3}}\,e(t)\Big)_{t\in[0,1]}$$

where $\big(e(t)\big)_{t\in[0,1]}$ is a standard normalized Brownian excursion.

Note H_n (resp. PL_n) the height of a tree τ in Ω_n (resp. the path length, that is the sum of the level of each node in τ): Proposition 4 implies that

Corollary 5

$$\frac{H_n}{\sqrt{n}} \xrightarrow[n]{law} \sqrt{\frac{8}{3}} \max\{e(t), 0 \leq t \leq 1\}$$

and

$$\frac{PL_n}{n^{3/2}} \xrightarrow[n]{law} \sqrt{\frac{8}{3}} \int_0^1 e(t)\, dt.$$

The law of these functionals of the Brownian excursion are known (see Kennedy [9] and Louchard [10]). The distribution of $\max\{e(t), 0 \leq t \leq 1\}$ is the classical theta distribution arising as a limit of a lot of functional of random walks and trees (see [3] for a list of examples) and $\int_0^1 e(t)\, dt$ is the Brownian excursion area (Airy distributed). The result of Panholzer [13] about the limit height of a given node, in NC-trees appears also to be a corollary of the present Proposition.
The limit mean of H_n and PL_n have been computed by Deutsch and Noy [4]; they find:

$$\lim_n \frac{\mathbb{E}(H_n)}{\sqrt{n}} = \frac{2}{3}\sqrt{3\pi}, \qquad \lim_n \frac{\mathbb{E}(PL_n)}{n^{3/2}} = \sqrt{\pi/3}.$$

This is coherent with our results (even if our results do not imply them, since law convergence does not imply moments convergence).

Proposition 6 (The limit DFQP) *The DFQP converges to the Brownian excursion:*

$$\Big(\frac{S_n(nt)}{\sqrt{n}}\Big)_{t\in[0,1]} \xrightarrow[n]{weakly} \Big(\sqrt{\frac{3}{2}}\, e(t)\Big)_{t\in[0,1]}.$$

Proposition 7 (The limit profile of NC-trees) *The limit profile converges to the local time of Brownian excursion:*

$$\Big(\frac{Z_{\sqrt{n}x}}{\sqrt{n}}\Big)_{x\geq 0} \xrightarrow[n]{weakly} \Big(\sqrt{\frac{3}{8}}\, l(\sqrt{\frac{3}{8}}\, x)\Big)_{x\geq 0},$$

where $\big(l(x)\big)_{x\geq 0}$ is the local time (at time 1 and level x) of a normalized Brownian excursion (see e. g. [7] for such results for Galton–Watson trees).

As a simple corollary of Proposition 7, we have for the width of a tree, being the maximum number of nodes on a level of the tree:

Corollary 8 (The limit width)

$$\frac{\max_{k\geq 0} Z_k}{\sqrt{n}} \xrightarrow[n]{(law)} \sqrt{\frac{3}{8}} \max\{e(t), 0 \leq t \leq 1\},$$

since the local time of the Brownian excursion is a normalized excursion changed of time.
The following proposition contains as a by-product Propositions 4 and 7.

Proposition 9 (Joint convergence)

$$\Big(\frac{V_n(2n.)}{\sqrt{n}}, \frac{Z_{\sqrt{n}.}}{\sqrt{n}}\Big) \xrightarrow[n]{weakly} \Big(2\sqrt{\frac{2}{3}}\, e(.), \sqrt{\frac{3}{8}}\, l(\sqrt{\frac{3}{8}}\, .)\Big)$$

Proof : See Aldous [2], Th.3 p.86.□

Note: In fact, one can also prove that the five processes given in Marckert and Mokkadem [12] (the contour, the DFQP, the height process, the process of the height of nodes with a given out-degree, the process of the height of nodes being the root of a subtree of given type) converge all to the same Brownian excursion.

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References

[1] D. Aldous, (1993). *The continuum random tree. III.*, Ann. Probab. 21, No.1, 248-289.

[2] D. Aldous, (1998). *Brownian excursion conditioned on its local time.*, Electron. Commun. Probab. 3, 79-90.

[3] P. Chassaing, J.F. Marckert, (2001). *Parking functions, empirical processes, and the width of rooted labeled trees*, Elec. Jour. of Comb. 8, R14.

[4] E. Deutsch, M. Noy, *Statistics on Non-crossing Trees*, to appear in Discrete Math.

[5] M. Drmota, B. Gittenberger, (1997). *On the profile of random trees.*, Random Structures Algorithms 10, no. 4, 421-451.

[6] P. Flajolet, M. Noy, (1999). *Analytic combinatorics of non-crossing configurations.* Discrete Math. 204, No.1-3, 203-229.

[7] B. Gittenberger, G, Louchard, (1999). *The Brownian excursion multi-dimensional local time density*, J. Appl. Probab. 36, 350-373.

[8] X. Gourdon, (1998). *Largest component in random combinatorial structures.* Discrete Math. 180, No.1-3, 185-209.

[9] D.P. Kennedy, (1976). *The distribution of the maximum Brownian excursion*, J. Appl. Probab. 13, 371-376.

[10] G. Louchard, (1985). *The Brownian excursion area: a numerical analysis*, Comp. & Math. App. 10, No.6, 413-417.

[11] M. Noy, (1998). *Enumeration of noncrossing trees on a circle.* Discrete Math. 180, No.1-3, 301-313.

[12] J.F. Marckert, A. Mokkadem, (2001). *The depth first processes of Galton–Watson trees converge to the same Brownian excursion*, preprint 56 , LAMA, Université Versailles.

[13] A. Panholzer, *The height distribution of nodes in non-crossing trees*, to appear in Ars Comb.

[14] A. Panholzer, H. Prodinger, *Bijections for Ternary Trees and Non-Crossing Trees*, to appear in Discrete Math.

[15] S. C. Port, *Theoretical Probability for Applications*, Wiley, Chichester, 1994.