

States spaces of the snake and of its tour – Convergence of the discrete snake

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Abstract

In this paper, we show that the states space of the Brownian snake and the states space of its tour are homeomorphic. We prove that the tour of the discrete snake (built on a geometrical Galton-Watson tree of size n) converges weakly to the tour of the Brownian snake. As a consequence, we obtain the weak convergence of the discrete snake to the Brownian snake. In a last part, we show the weak convergence of the “geometrical width” of the discrete snake to the one of the Brownian snake.

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Notations:

- We denote by $\check{X}(t_1, t_2)$ the minimum value of the process X on $[t_1, t_2]$
- We denote by $\llbracket a, b \rrbracket$ the set $\mathbb{N} \cap [a, b]$
- We denote by $\lfloor x \rfloor = \sup\{t \in \mathbb{N}, t \leq x\}$ and $\lceil x \rceil = \inf\{t \in \mathbb{N}, t \geq x\}$ the largest integer smaller than x (resp. the smallest integer larger than x). We set $\{x\} = x - \lfloor x \rfloor$.

1 Introduction and framework

1.1 Aim and contents

We consider the path-valued process called the Brownian snake, conditioned so that its life time process is a normalized Brownian excursion. We note \mathcal{S} its states space. We introduce the tour of the Brownian space and its states space T (the two spaces are endowed with the topology of the uniform distance).

Our first aim is to show that the two spaces \mathcal{S} and T are homeomorphic. In other words, any sentence in the space \mathcal{S} has its equivalent translation in the space T . In particular, any weak convergence in T is equivalent to a weak convergence in \mathcal{S} . As example, we provide a large deviation principle on the Brownian snake (in Section 2.2) by transferring the large deviation result of Serlet [26] on the tour of the Brownian snake.

This strong connexion between the snake and its tour is quite similar to the result of Aldous [2] (Theorem 20 p.271). He shows that the convergence of a family of *graph-theoretic trees* is equivalent to the one of the associated family tour.

Our second aim is to prove that the Brownian snake is a weak limit of a discrete analogue. More precisely, with the help of a finite branching random walk, we define what we call the discrete snake. We prove that the tour of the discrete snake converges to the tour of the Brownian snake in T (in a recent independent work, Chassaing & Schaeffer [8] obtain a quite similar result; they show in the Skorohod space the convergence of the tour of a branching random walk with geometrical offspring and displacement i.i.d., uniform on $\{-1, 0, 1\}$, to the tour of the Brownian snake).

Thanks to our homeomorphism theorem, we deduce the convergence of the discrete snake to the Brownian snake in \mathcal{S} (Section 3).

In Section 4, we retrieve the relation between ISE (of Aldous [3]) and the tour of the Brownian snake.

Finally, in Section 5, we show the weak convergence of the discrete snake width at any fixed level to the one of the Brownian snake.

As related results, one must cite the works of Durrett & al. [10] and Kesten [14] who study the same object: branching random walk whose family tree is conditioned by the size. They give limit results for the maximum displacement of the branching random walk (depending on the tail of the increment distribution). One can also mention the work of Kesten [13] who considers BRW whose family tree is conditioned on survival till time βn (for some fixed β).

1.2 Settings

1.2.1 Finite branching random walk

We consider geometrical Galton–Watson branching process, starting with 1 individual in generation 0 and with offspring distribution

$$(p_i)_{i \geq 0} = (p(1-p)^i)_{i \geq 0}.$$

In the sequel, N will denote a random variable $(p_i)_{i \geq 0}$ distributed. We write τ for the family tree of this branching process and Ω the probability space of all trees with the law induced by N . We note Ω_n the space plan trees with $n + 1$ nodes (we choose $n + 1$ to work on excursions of size $2n$ and avoid tedious normalizations). The law \mathbb{P}_n induced by the condition $|\tau| = n + 1$ on Ω is the uniform distribution on Ω_n (see Kolchin [15]). The probability space (Ω_n, \mathbb{P}_n) is the set of trees with $n + 1$ nodes. We superimpose a random walk on each path from the root down in the tree

of Ω . With each node u different from the root, we associate an \mathbb{R}^d -valued random variable $y(u)$, called the value of u . We assume that the random variables $y(u)$ are identically distributed. Let v_1^u, \dots, v_N^u be the (ordered) children of u . The joint distribution of

$$(N, (y(v_1^u), \dots, y(v_N^u)))$$

is quite arbitrary; the values attached to the children of different parents are independent but the ones attached to brothers may be dependent. Let u be a node and $h(u)$ its depth. Consider $(u_0 = \text{root}, u_1, \dots, u_{h(u)} = u)$ the path from the root to u ; we associate to u a trajectory of killed random walk $\Phi_u = (\Phi_u(j))_{j \in \llbracket 0, h(u) \rrbracket}$ defined by

$$\Phi_u(0) = 0, \quad \Phi_u(j) = \sum_{i=1}^j y(u_i), \text{ for } j \in \llbracket 1, h(u) \rrbracket.$$

We call branching random walk the union of the trajectories Φ_u :

$$B = \{\Phi_u, u \in \tau\}.$$

We condition the size of the underlying tree to be equal to $n + 1$ ($\tau \in \Omega_n$). The set of trajectories is then denoted by B_n and called *finite branching random walk*. The set of the points of the trajectories of B_n is:

$$B'_n = \{(\Phi_u(h(u)), h(u)), u \in \tau\}.$$

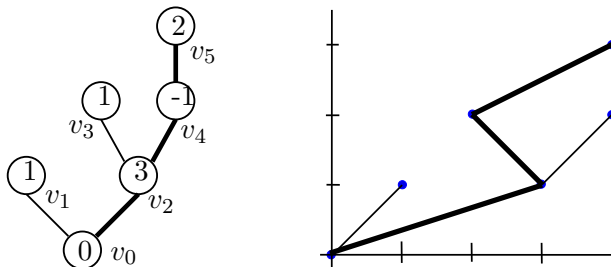


Figure 1. : A valued tree and its associated BRW

Depth first traversal of the BRW

We recall the depth first procedure. Let τ be an ordered tree with $n + 1$ nodes. We define a function (see Aldous [1] p.260):

$$\tilde{f} : \llbracket 0, 2n \rrbracket \longrightarrow \{\text{nodes of } \tau\},$$

which we regard as a walk around τ , as follows:

$$\tilde{f}(0) = \text{root}.$$

Given $\tilde{f}(i) = v$, choose, if possible, the most left child w of v which has not already been visited, and set $\tilde{f}(i + 1) = w$. If not possible, let $\tilde{f}(i + 1)$ be the parent of v .

The depth first walk (DFW): For $\tau \in \Omega_n$, we call the DFW of τ , the process V_n defined by:

$$V_n(i) = d(\text{root}, \tilde{f}(i)), \quad 0 \leq i \leq 2n.$$

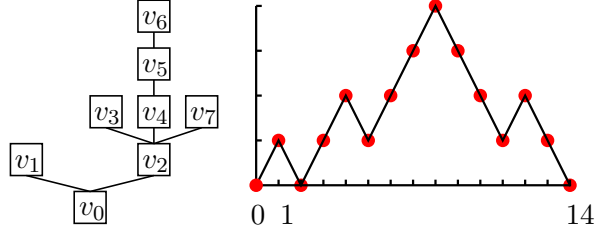


Figure 2. : A tree and its associated DFW

For i from 0 to n , set v_i the i th new node visited by the depth first procedure on $\tau \in \Omega_n$ ($v_0 = \text{root}$).

The depth branching random walk: B'_n can be rewritten:

$$B'_n = \{(R_n(k), V_n(k)), k \in \llbracket 0, 2n \rrbracket\},$$

where the process R_n is defined by

$$R_n(k) = \Phi_{\tilde{f}(k)}(h(\tilde{f}(k))).$$

We call R_n the depth branching random walk.

Remark 1: V_n is sometimes described as follows: $V_n(k)$ is the height at time k of a fly that is (clockwise) walking around the tree, 1 edge per time unit (the fly is on the root at time 0 and at time $2n$). Consider the successive values $y(u)$ of met nodes as abscissa displacements: $R_n(k)$ is the abscissa of the fly at time k .

1.2.2 The discrete snake

We interpolate piecewise V_n between integral points:

$$V_n(x) = V_n(\lfloor x \rfloor) + \{x\}(V_n(\lfloor x + 1 \rfloor) - V_n(\lfloor x \rfloor)), \text{ for } x \in [0, 2n]. \quad (1.1)$$

By analogy with the Brownian snake, we call discrete snake the process

$$(W_n(x, \cdot), V_n(y))_{(x,y) \in [0, 2n]^2}$$

where for each $x \in [0, 2n]$, $W_n(x, \cdot)$ is a stopped continuous process defined as follows:

– For $k \in \llbracket 0, 2n \rrbracket$ and $t \in [0, V_n(k)]$, $W_n(k, \cdot)$ is the process that interpolates piecewise the random walk $\Phi_{\tilde{f}(k)}$:

$$W_n(k, t) = \Phi_{\tilde{f}(k)}(\lfloor t \rfloor) + \{t\}(\Phi_{\tilde{f}(k)}(\lfloor t + 1 \rfloor) - \Phi_{\tilde{f}(k)}(\lfloor t \rfloor)).$$

– For $x \in [0, 2n]$ and $t \in [0, V_n(x)]$,

$$W_n(x, t) = \begin{cases} W_n(\lfloor x \rfloor, t) & \text{if } V_n(\lfloor x + 1 \rfloor) < V_n(\lfloor x \rfloor) \\ W_n(\lfloor x + 1 \rfloor, t) & \text{if } V_n(\lfloor x + 1 \rfloor) > V_n(\lfloor x \rfloor). \end{cases}$$

– For $x \in [0, 2n]$ and $t \in]V_n(x), +\infty[$, we set

$$W_n(x, t) = W_n(x, V_n(x)). \quad (1.2)$$

Note that

$$W_n(\lfloor x \rfloor, t) = W_n(\lfloor x + 1 \rfloor, t) \text{ for } 0 \leq t \leq V_n(\lfloor x \rfloor) \wedge V_n(\lfloor x + 1 \rfloor)$$

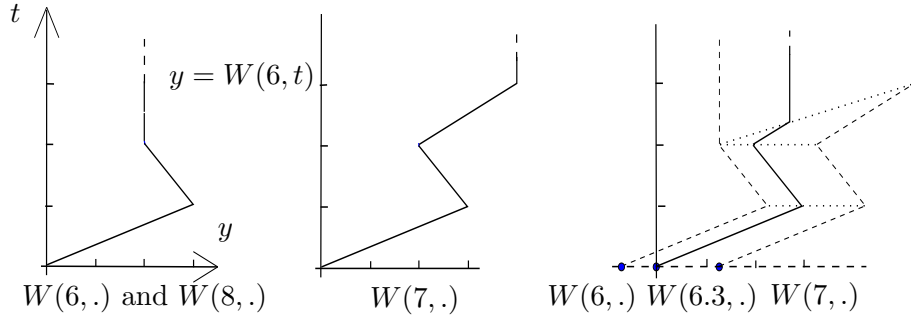


Figure 3. : 3 stopped paths associated to the BRW of Figure 1.

1.2.3 The tour of the discrete snake

We define the tour of the discrete snake as the process $(R_n(x), V_n(y))_{(x,y) \in [0,2n]^2}$ where V_n is given in (1.1) and

$$R_n(x) = R_n(\lfloor x \rfloor) + \{x\}(R_n(\lfloor x + 1 \rfloor) - R_n(\lfloor x \rfloor)), \text{ for } x \in [0, 2n[.$$

1.2.4 Normalizations

We set

$$\begin{aligned} v_n(s) &= \frac{V_n(2ns)}{\sqrt{n}} && \text{for } s \in [0, 1] \\ r_n(s) &= \frac{R_n(2ns)}{n^{1/4}} && \text{for } s \in [0, 1] \\ w_n(s, t) &= \frac{W_n(2ns, t\sqrt{n})}{n^{1/4}} && \text{for } (s, t) \in [0, 1] \times [0, +\infty[. \end{aligned}$$

We call the processes (w_n, v_n) and (r_n, v_n) the normalized discrete snake and the normalized tour of the discrete snake respectively.

Notice that w_n enjoys the property (1.2):

$$w_n(s, t) = w_n(s, v_n(s)) \text{ for } t \geq v_n(s).$$

1.2.5 States spaces of the snakes and of the tours

States space of the snakes

It is the states space of the Brownian snake. We note $C([0, +\infty[, \mathbb{R}^d)$ the space of all continuous functions from $[0, +\infty[$ to \mathbb{R}^d . We call stopped path in \mathbb{R}^d a pair, $(\nu, \zeta) \in C([0, +\infty[, \mathbb{R}^d) \times [0, +\infty[$ such that for every $t \geq \zeta, \nu(t) = \nu(\zeta)$. We denote by \mathcal{W} the space of all stopped paths in \mathbb{R}^d . We call ζ the lifetime of the stopped path (ν, ζ) and $\nu(\zeta)$ the “terminal point” of the path ν .

We note \mathcal{S} the subspace of $C([0, 1], C([0, +\infty[, \mathbb{R}^d)) \times C([0, 1], [0, +\infty[)$ of functions (ν, ζ) that satisfy the three following conditions:

$$(\nu(s, \cdot), \zeta(s)) \in \mathcal{W} \text{ for any } s \in [0, 1] \tag{1.3}$$

$$\zeta(0) = \zeta(1) = 0$$

$$\text{For any } 0 \leq s \leq s' \leq 1, \quad \nu(s, t) = \nu(s', t) \quad \text{for all } t \leq \min_{u \in [s, s']} \zeta(u). \quad (1.4)$$

(1.4) is the snake property.

We endow the space \mathcal{S} with the distance:

$$d_S((\nu_1, \zeta_1), (\nu_2, \zeta_2)) = \max \{ \|\nu_1 - \nu_2\|_\infty, \|\zeta_1 - \zeta_2\|_\infty \}$$

where

$$\|\nu_1 - \nu_2\|_\infty = \sup_{(s, t) \in [0, 1] \times [0, +\infty[} |\nu_1(s, t) - \nu_2(s, t)|.$$

The normalized snake $(w_n(s, \cdot), v_n(s'))_{(s, s') \in [0, 1]^2}$ is an element of \mathcal{S} .

States space of tours

We note T the subspace of $C([0, 1], \mathbb{R}^d) \times C([0, 1], [0, +\infty[)$ of functions (f, ζ) that satisfy:

$$\zeta(0) = \zeta(1) = 0$$

and

$$\text{for any } 0 \leq s \leq s' \leq 1, \quad f(s) = f(s') \quad \text{if} \quad \zeta(s) = \zeta(s') \quad \text{and} \quad \min_{s \leq u \leq s'} \zeta(u) = \zeta(s). \quad (1.5)$$

When $\min_{s \leq u \leq s'} \zeta(u) = \zeta(s) = \zeta(s')$, “we are” on the same node of the arborescent structure coded by ζ . (1.5) ensures that $f(s')$ is equal to $f(s)$ on this “node”.

The metric on T is given by

$$d_T((f_1, \zeta_1), (f_2, \zeta_2)) = \max \{ \|f_1 - f_2\|_\infty, \|\zeta_1 - \zeta_2\|_\infty \}.$$

By construction, the normalized tour of the snake (r_n, v_n) is an element of T .

T is the states space of the tour of the Brownian snake whose lifetime process is the normalized Brownian excursion (we will say briefly the Brownian snake).

For commodity, we recall the

Definition of the Brownian snake

We refer to [17, 18, 19, 26] for a detailed exposition on the Brownian snake (and the related processes called superprocesses).

Let $(e(s), s \in [0, 1])$ be the normalized Brownian excursion. The Brownian snake driven by $v = \sqrt{2}e$ is the process $(w(\cdot, \cdot), v(\cdot))$ in S whose law is defined by the following:

- For every s , $w(s, 0) = 0$
- The conditional law of $w(s, \cdot)$ knowing v is the law of an homogeneous Markov process whose transition kernel is described as follows: for $0 \leq s \leq s' \leq 1$,
 - ★ $w(s', t) = w(s, t)$ for all $t \leq \check{v}(s, s')$
 - ★ $(w(s', \check{v}(s, s') + t)_{t \in [0, v(s') - \check{v}(s, s')]} - w(s', \check{v}(s, s')))$ is independent of $w(s, \cdot)$ and distributed as a Brownian motion in \mathbb{R}^d starting from $w(s, \check{v}(s, s'))$ and stopped at time $v(s') - \check{v}(s, s')$.

Note The standard Brownian snake is defined with $v = e$ but, for convenience, we take $v = \sqrt{2}e$.

Tour of the Brownian snake

The tour of the Brownian snake is the process (r, v) in T defined by

$$r(s) = w(s, v(s)) \text{ for } s \in [0, 1],$$

where $v = \sqrt{2}e$. The conditional law of the process r with respect to v is Gaussian with mean 0 and covariance function:

$$\mathbb{E}(r(s) \ ^t r(t)) = \check{v}(s, t) I_d.$$

2 The homeomorphism

2.1 Statement and proof of the homeomorphism theorem

Let (ν, ζ) be an element of \mathcal{S} . Define the application \mathcal{H} by $\mathcal{H}(\nu, \zeta) = (f, \zeta)$ where

$$f(s) = \nu(s, \zeta(s)) \text{ for any } s \in [0, 1].$$

Let (f, ζ) be in T ; we define the application $\overline{\mathcal{H}}$ by $\overline{\mathcal{H}}(f, \zeta) = (\nu, \zeta)$ where

$$\nu(s, t) = f(\rho(s, t, \zeta)) \text{ for any } s \in [0, 1] \text{ and } t \geq 0$$

and

$$\rho(s, t, \zeta) = \begin{cases} \sup\{\alpha \leq s, \zeta(\alpha) = t\} & \text{if } 0 \leq t \leq \zeta(s), \\ s & \text{if } t \geq \zeta(s). \end{cases}$$

Theorem 2.1. (i) \mathcal{H} is an application from \mathcal{S} to T and $\overline{\mathcal{H}}$ is an application from T to \mathcal{S} .

(ii) \mathcal{H} is bijective and $\mathcal{H}^{-1} = \overline{\mathcal{H}}$.

(iii) \mathcal{H} is an homeomorphism from \mathcal{S} on T .

Proof : We begin with some notations and technical considerations:

• We set:

$$\rho^*(s, t, \zeta) = \begin{cases} \inf\{\alpha \geq s, \zeta(\alpha) = t\} & \text{if } 0 \leq t \leq \zeta(s), \\ s & \text{if } t \geq \zeta(s). \end{cases}$$

• If $0 \leq t \leq \zeta(s)$, we define the set $\hat{\rho}(s, t, \zeta)$ by $s' \in \hat{\rho}(s, t, \zeta)$ iff $\zeta(s') = t$ and the infimum of ζ between s' and s equals t ; if $t > \zeta(s)$, we set $\hat{\rho}(s, t, \zeta) = \{s\}$. We note $\rho^{\inf}(s, t, \zeta) = \inf \hat{\rho}(s, t, \zeta)$ and $\rho^{\sup}(s, t, \zeta) = \sup \hat{\rho}(s, t, \zeta)$ (notice that ρ , ρ^* , ρ^{\inf} and ρ^{\sup} are in $\hat{\rho}$).

The following properties are straightforward:

Properties : 1) Fix $s \in [0, 1]$. The application

$$t \longmapsto \rho(s, t, \zeta) \text{ for } t \geq 0$$

is nondecreasing (increasing in $[0, \zeta(s)]$, constant in $[\zeta(s), +\infty[$).

2) f is constant on the set $\hat{\rho}(s, t, \zeta)$.

Proof of the first assertion of (i).

Since $\mathcal{H}(\nu, \zeta)$ is continuous we just have to prove (1.5). Since $(\nu, \zeta) \in \mathcal{S}$, if $\zeta(s) = \zeta(s') = \check{\zeta}(s, s')$, it follows from (1.4) that $\nu(s', \zeta(s')) = \nu(s, \zeta(s))$ and thus $f(s) = f(s')$.

Proof of the second assertion of (i).

We first have to prove that if $t \leq \min_{u \in [s, s']} \zeta(u)$ then $\nu(s, t) = \nu(s', t)$. By definition

$$\nu(s, t) = f(\rho(s, t, \zeta)) \text{ and } \nu(s', t) = f(\rho(s', t, \zeta))$$

– if $t < \min_{u \in [s, s']} \zeta(u)$ then $\rho(s, t, \zeta) = \rho(s', t, \zeta)$.

– if $t = \min_{u \in [s, s']} \zeta(u)$, then $\hat{\rho}(s, t, \zeta) = \hat{\rho}(s', t, \zeta)$. We conclude by property (2).

It remains to prove that $s \mapsto \nu(s, \cdot)$ is continuous at s_0 for any s_0 in $[0, 1]$. For this, we first prove that ν is continuous in (s_0, t_0) for any (s_0, t_0) in $[0, 1] \times [0, +\infty[$.

Notice that $(s, t) \mapsto \rho(s, t, \zeta)$ is not continuous. To overcome this difficulty, we prove that

$$\lim_{(s,t) \rightarrow (s_0,t_0)} \min \{ |a - b|, a \in \hat{\rho}(s, t, \zeta), b \in \hat{\rho}(s_0, t_0, \zeta) \} = 0; \quad (2.6)$$

the continuity of ν at (s_0, t_0) follows then from the uniform continuity of f and the property (2).

• 1) Case $t_0 > \zeta(s_0)$. For (s, t) sufficiently close to (s_0, t_0) we have $\hat{\rho}(s, t, \zeta) = s$ and (2.6) is obvious.

• 2) Case $t_0 < \zeta(s_0)$: In this case, $\rho(s_0, t_0, \zeta) < \rho^*(s_0, t_0, \zeta)$. We first prove (2.6) for $t \geq t_0$ and then for $t \leq t_0$.

a) $t \geq t_0$. For any s in the compact $I = [\rho(s_0, t_0, \zeta) + \varepsilon, \rho^*(s_0, t_0, \zeta) - \varepsilon]$, $\zeta(s) > t_0$. Set $t_1 = \inf_{s \in I} \zeta(s)$. We have $t_1 > t_0$. For any $(s, t) \in I \times [t_0, t_1]$, $\rho(s_0, t_0, \zeta) \leq \rho(s, t, \zeta) \leq \rho(s_0, t_0, \zeta) + \varepsilon$

b) $t \leq t_0$. $\rho^{\inf}(s_0, t_0, \zeta)$ is not the abscissa of a local minimum of ζ . Thus, for any $\varepsilon > 0$, there exists s_2 such that $\rho^{\inf}(s_0, t_0, \zeta) - \varepsilon < s_2 < \rho^{\inf}(s_0, t_0, \zeta)$ and $\zeta(s_2) = t_2 < t_0$. For any $(s, t) \in [\rho^{\inf}(s_0, t_0, \zeta), \rho^{\sup}(s_0, t_0, \zeta)] \times [t_2, t_0]$, $\rho^{\inf}(s_0, t_0, \zeta) - \varepsilon \leq \rho(s, t, \zeta) \leq \rho^{\inf}(s_0, t_0, \zeta)$.

• 3) Case $t_0 = \zeta(s_0)$. Consider $\varepsilon > 0$ and $I_\varepsilon = [s_0 - \varepsilon, s_0 + \varepsilon]$. Let (s^*, t^*) s.t. $t^* = \zeta(s^*) = \min_{I_\varepsilon} \zeta$.

a) If $t^* < \zeta(s_0)$ we have $s \leq \rho^*(s, t, \zeta) \leq s^*$ for any (s, t) such that $t \geq t^*$, $s < s^*$ and $s \in I_\varepsilon$; thus $\rho^*(s, t, \zeta) \in I_\varepsilon$. In the same way, if $s \geq s^*$ and $s \in I_\varepsilon$, we have $s^* \leq \rho(s, t, \zeta) \leq s$ and $\rho(s, t, \zeta) \in I_\varepsilon$.

b) If $t^* = \zeta(s_0)$, $\zeta(s_0)$ is a local minimum. For $t \geq t_0$ and $s \in I_\varepsilon$, $\rho(s, t, \zeta)$ or $\rho^*(s, t, \zeta)$ is in I_ε . The case $t \leq t_0$ is solved in 2)b).

Thus, (2.6) is proved and the function $(s, t) \mapsto \nu(s, t)$ is continuous.

Now, in $[0, 1] \times [0, \|\zeta\|_\infty]$, this function is uniformly continuous; by classical argument, it follows that $\lim_{s \rightarrow s_0} \sup_{t \in [0, \|\zeta\|_\infty]} |\nu(s, t) - \nu(s_0, t)| = 0$. On the other hand, $\lim_{s \rightarrow s_0} \sup_{t \in [\|\zeta\|_\infty, \infty[} |\nu(s, t) - \nu(s_0, t)| =$

$\lim_{s \rightarrow s_0} |\nu(s, \zeta(s)) - \nu(s_0, \zeta(s_0))| = 0$. Thus, $s \mapsto \nu(s, \cdot)$ is continuous.

Proof of (ii): – First we prove that if $(\nu, \zeta) \in \mathcal{S}$ then $\overline{\mathcal{H}}(\mathcal{H}(\nu, \zeta)) = (\tilde{\nu}, \zeta)$ where $\tilde{\nu} = \nu$.

$\mathcal{H}(\nu, \zeta) = (f, \zeta)$ where $f(s) = \nu(s, \zeta(s))$. By $\overline{\mathcal{H}}$,

$$\tilde{\nu}(s, t) = f(\rho(s, t, \zeta)) = \nu(\rho(s, t, \zeta), \zeta(\rho(s, t, \zeta))) = \nu(\rho(s, t, \zeta), t).$$

By definition of ρ , $t = \min\{\zeta(u), \rho(s, t, \zeta) \leq u \leq s\}$. By (1.4), one has:

$$\tilde{\nu}(s, t) = \nu(s, t).$$

– Let us prove that $\mathcal{H}(\overline{\mathcal{H}}(f, \zeta)) = (\tilde{f}, \zeta)$ where $\tilde{f} = f$.

Note $\overline{\mathcal{H}}(f, \zeta) = (\tilde{\nu}, \zeta)$ where

$$\tilde{\nu}(s, t) = f(\rho(s, t, \zeta)).$$

So $\tilde{f}(s) = \tilde{\nu}(s, \zeta(s)) = f(\rho(s, \zeta(s), \zeta)) = f(s)$

Proof of (iii): We first prove that \mathcal{H} is 1-Lipschitz. For $i \in \{1, 2\}$, let (f_i, ζ_i) be the image by \mathcal{H} of (ν_i, ζ_i) . One has

$$\begin{aligned} d_T((f_1, \zeta_1), (f_2, \zeta_2)) &= \max\left\{ \sup_{s \in [0, 1]} |\nu_1(s, \zeta_1(s)) - \nu_2(s, \zeta_2(s))|, \|\zeta_1 - \zeta_2\|_\infty \right\} \\ &\leq d_S((\nu_1, \zeta_1), (\nu_2, \zeta_2)). \end{aligned}$$

Now, we prove that \mathcal{H}^{-1} is continuous.

Let $(f_n, \zeta_n)_n$ be a sequence of elements of T that converges (uniformly) to (f, ζ) . Note (ν_n, ζ_n) their images by \mathcal{H}^{-1} (and (ν, ζ) for $\mathcal{H}^{-1}(f, \zeta)$). Let us show that $d_S((\nu, \zeta), (\nu_n, \zeta_n)) \rightarrow 0$. We just have to prove the uniform convergence of ν_n to ν .

Set

$$M = \max\{\sup_n \|\zeta_n\|_\infty, \|\zeta\|_\infty\}.$$

Since the sequence $(\zeta_n)_n$ converges uniformly to ζ on $[0, 1]$, M is finite.

A) Let us prove first that ν_n converges uniformly to ν on $K = [0, 1] \times [0, M]$. By contradiction, let us assume that there exist an $\varepsilon > 0$, a subsequence of ν_n (still called ν_n) and a sequence (s_n, t_n) of points of K such that

$$\forall n, \quad |\nu_n(s_n, t_n) - \nu(s_n, t_n)| > \varepsilon. \quad (2.7)$$

Since K is compact, there exists a subsequence noted (s_n, t_n) again that has a limit $(s', t') \in K$. Since ν is continuous one has:

$$\lim_n \nu(s_n, t_n) = \nu(s', t'). \quad (2.8)$$

It follows from (2.7) and (2.8) that $\nu_n(s_n, t_n)$ does not converge to $\nu(s', t')$ and thus $f_n(\rho(s_n, t_n, \zeta_n))$ does not converge to $f(\rho(s', t', \zeta))$. Since f_n goes to f (uniformly) this implies that the set of limit points of the sequence (ρ_n) (defined by $\rho_n \stackrel{def}{=} \rho(s_n, t_n, \zeta_n)$) is not included in the class $\hat{\rho}(s', t', \zeta)$.

Since $[0, 1]$ is compact, there exists a subsequence of ρ_n that converges to say $\bar{\rho}$, and $\bar{\rho} \notin \hat{\rho}(s', t', \zeta)$. By the continuity of ζ and the uniform convergence of ζ_n , we have $\zeta(\bar{\rho}) = t'$. We now prove that this fact is not possible. It is clear that $\bar{\rho}$ can only satisfy either $\bar{\rho} < \rho^{\inf}(s', t', \zeta)$ or $\bar{\rho} > \rho^{\sup}(s', t', \zeta)$. By symmetry we just consider the first possibility. Necessarily, since $\bar{\rho} \notin \hat{\rho}(s', t', \zeta)$,

$$\inf_{\bar{\rho} \leq u \leq \rho^{\inf}} \zeta(u) < t'.$$

ζ has a minimum $t'' < t'$ in the set $[\bar{\rho}, \rho^{\inf}]$; note $s'' \in]\bar{\rho}, \rho^{\inf}[$ such that $\zeta(s'') = t''$.

For n sufficiently large:

- 1) (s_n, t_n) is sufficiently close to (s', t') so that $s_n > s''$, $t_n > (t' + t'')/2$.
- 2) $\zeta_n(s'') < (t' + t'')/2$

For these two reasons, for n sufficiently large, $\rho_n \geq s''$. It follows that ρ_n can not converge to $\bar{\rho}$ which is strictly smaller than s'' .

B) On $[0, 1] \times [M, +\infty[$, we have $\nu_n(s, t) = f_n(\zeta_n(s))$ and $\nu(s, t) = f(\zeta(s))$.

The uniform convergence of ν_n to ν on $[0, 1] \times [M, +\infty[$ is a consequence of the uniform convergence of f_n and ζ_n to f and ζ (respectively) and the uniform continuity of f on $[0, 1]$.

This ends the proof of (iii) and then also the one of Theorem 2.1. \square

2.2 An application to large deviations of Brownian snake

Consider the standard Brownian snake $(w(\cdot, \cdot), e(\cdot))$ with lifetime the normalized Brownian excursion. The homeomorphism theorem allows us to transfer the large deviation result of Serlet [26] for the process $((\varepsilon(w(s, e(s))), \varepsilon^{2/3}e(s)), s \in [0, 1])$ to the process $((\varepsilon(w(s, \varepsilon^{-2/3}t)), \varepsilon^{2/3}e(s)), s \in [0, 1], t \in [0, +\infty[)$.

For $(\nu, \zeta) \in \mathcal{S}$, set $f(s) = \nu(s, \zeta(s))$ (for $s \in [0, 1]$). We define the function I on \mathcal{S} by:

$$I(\nu, \zeta) = \frac{1}{2} \int_0^1 \dot{\zeta}_s^2 ds + \frac{1}{4} \int_0^1 \frac{|\dot{f}(s)|^2}{|\dot{\zeta}_s|} ds$$

when f and ζ are absolutely continuous and $I(\nu, \zeta) = +\infty$ otherwise (\dot{f} and $\dot{\zeta}$ are the almost everywhere derivatives of f and ζ). The following theorem is an immediate consequence of Serlet Theorem 1 (in [26]); the function I is equal to $J \circ \mathcal{H}$ where J is the good rate function on T given by Serlet.

Theorem 2.2. *The processes $((\varepsilon(w(s, \varepsilon^{-2/3}t)), \varepsilon^{2/3}e(s)), s \in [0, 1], t \in [0, +\infty[)$ satisfy a large deviation principle as $\varepsilon \downarrow 0$ with speed $\varepsilon^{-4/3}$ and good rate function I :*

- for every open subset V of S ,

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{4/3} \log \mathbb{P}((\varepsilon(w(s, \varepsilon^{-2/3}t)), \varepsilon^{2/3}e(s))_{(s,t) \in [0,1] \times [0,+\infty[} \in V) \geq -\inf_V I.$$

- for every closed subset L of S

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{4/3} \log \mathbb{P}((\varepsilon(w(s, \varepsilon^{-2/3}t)), \varepsilon^{2/3}e(s))_{(s,t) \in [0,1] \times [0,+\infty[} \in L) \leq -\inf_L I.$$

3 Weak convergence of the discrete snake

3.1 Assumptions

We assume in the sequel of the paper that

- (i) $\mathbb{E}(y(u)) = 0$ where 0 is the null vector in \mathbb{R}^d .
- (ii) $\text{Var}(y(u)) = I_d$ where I_d is the $d \times d$ identity matrix.
- (iii) There exists $p > 6$ such that $\mathbb{E}|y(u)|^p < +\infty$.

We recall that the values $y(u)$ are identically distributed random variables. The values attached to the children of different parents are independent but the ones attached to brothers may be dependent.

Remark (ii) is not a restrictive assumption; if $\text{Var}(y(u)) = \Sigma$ where Σ is a positive matrix, then $\text{Var}(\Sigma^{-1/2}y(u)) = I_d$. So all our results remain valid up to scaling.

3.2 Results

Convergence of the normalized tour

Theorem 3.3. *The following weak convergence holds in T*

$$(r_n, v_n) \xrightarrow[n]{\text{weakly}} (r, v),$$

where (r, v) is the tour of the Brownian snake.

Proof: The tightness of the sequence $(v_n)_n$ follows the fact that v_n converges weakly to v (see Aldous [1], Marckert & Mokkadem [21]). The tightness of r_n is given by Theorem 3.5 (in subsection 3.3). Hence each of the marginals is tight and so, the couple (r_n, v_n) is tight. The convergence of the finite dimensional laws of (r_n, v_n) is given in Lemma 3.9 (in subsection 3.4). Hence, Theorem 3.3 is a consequence of Theorem 3.5 and Lemma 3.9. \square

Commentary on the assumption (i). If $\mathbb{E}(y(u)) \neq 0$, a simple consequence of Theorem 3.3 is that $(\frac{R_n}{n^{1/2}}, \frac{V_n}{n^{1/2}})$ converges to $(v\mathbb{E}(y(u)), v)$. This is the tour of a degenerate snake (conditionally to v , it is deterministic).

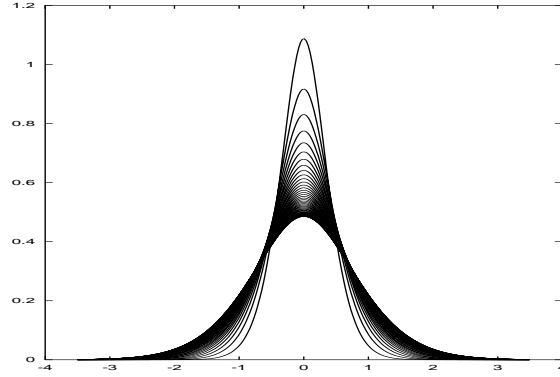


Figure 4. : Densities of $r(t)$ for $t \in [0, 1]$ when $d = 1$.

Convergence of the discrete snake

Thanks to Theorem 3.3 and to the homeomorphism Theorem 2.1, one has

Corollary 3.4. *The following weak convergence holds in S*

$$(w_n, v_n) \xrightarrow[n]{\text{weakly}} (w, v),$$

where (w, v) is the Brownian snake.

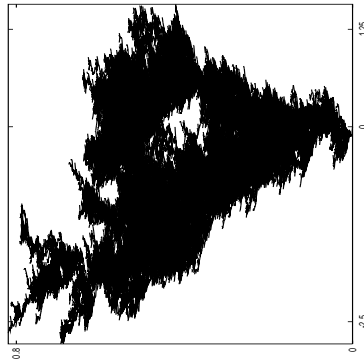


Figure 5. : Size 113905 BRW with normal $\mathcal{N}(0, 1)$ i.i.d. displacement.

3.3 Tightness of r_n (the proof)

To prove the tightness of the sequence (r_n) in $(C([0, 1], \mathbb{R}^d), \|\cdot\|_\infty)$ we use the moment criterion (see Billingsley [5] p.95). Since $p > 6$, it is sufficient to prove:

Theorem 3.5. *There exists a constant $\gamma(p)$ such that, for any $0 \leq s \leq t \leq 1$ and any $n \geq 1$,*

$$\mathbb{E}|r_n(s) - r_n(t)|^p \leq \gamma(p)|t - s|^{p/4-1/2}. \quad (3.9)$$

It is classical that if (3.9) holds for any $n \geq 1$ and any $\frac{1}{2n} \leq s \leq t \leq 1$ such that $2ns$ and $2nt$ are integers then it holds for any (s, t) such that $0 \leq s \leq t \leq 1$. So, we have only to prove Theorem 3.5 for such (s, t, n) . In this case, $R_n(2nt) - R_n(2ns)$ is a sum involving $V_n(2nt) +$

$V_n(2ns) - 2\check{V}_n(2ns, 2nt)$ copies of $y(u)$. Let us examine the dependence between these values. If $(V_n(2nt) \wedge V_n(2ns)) = \check{V}_n(2ns, 2nt)$, the values involved are independent since they concern values attached to only one branch. In the other case, among the $V_n(2nt) + V_n(2ns) - 2\check{V}_n(2ns, 2nt)$ involved variables, only two of them depend on each other (but are independent from the other ones): as a matter of fact, the involved values are the non common values attached to the branches $(root - \check{f}(2nt))$ and $(root - \check{f}(2ns))$. Only two of them concern “brothers”.

The following lemma is in Petrov [24], th. 2.10 p.62:

Lemma 3.6. *If X_n is a random walk with i.i.d. increments that have moment of order $q \geq 2$, there exists a constant $C(q)$ (that depends only on the law of the increments) such that for any $n \geq 0$*

$$\mathbb{E}|X_n|^q \leq C(q)n^{q/2}.$$

Applying usual argument, one can extend this property to a random walk owning at most two dependent variables (independent from the other ones). So one has:

$$\mathbb{E}|r_n(s) - r_n(t)|^p \leq C(p)\mathbb{E}|v_n(s) + v_n(t) - 2\check{v}_n(s, t)|^{p/2}.$$

Note

$$E_n(t, s, p) \stackrel{def}{=} \mathbb{E}|v_n(s) + v_n(t) - 2\check{v}_n(s, t)|^{p/2}.$$

We will prove the following Proposition that implies Theorem 3.5:

Lemma 3.7. *There exists a constant $C > 0$ such that for any $n \geq 0$ and any (s, t) , $\frac{1}{2n} \leq s \leq t \leq 1$, such that $2ns$ and $2nt$ are integers,*

$$E_n(t, s, p) \leq C(t - s)^{p/4 - 1/2}.$$

Since the progeny is geometrically distributed, V_n is a Bernoulli excursion with length $2n$ (see e.g [1, 12]). We recall what is a Bernoulli excursion. Consider RW_n the set of $2n$ sized simple random walks:

$$W \in RW_n \Leftrightarrow (W_0 = 0, W_k = W_{k-1} \pm 1 \text{ for } k \in \llbracket 1, 2n \rrbracket).$$

The set RW_n contains 2^{2n} trajectories and is endowed with the uniform distribution. We define Ex_n the set of $2n$ sized Bernoulli excursions by

$$W \in Ex_n \Leftrightarrow (W \in RW_n, W_{2n} = 0, \forall k \in \llbracket 0, 2n \rrbracket, W_k \geq 0).$$

We endow also Ex_n with the uniform distribution. It follows that the probability of an event A on Ex_n is equal to $\mathbb{P}(A|Ex_n)$ on RW_n (for simplicity \mathbb{P} will always denote the law on random walks, the law on excursions will appear as a conditional distribution). Thus,

$$E_n(t, s, p) = \sum_{(x, y, z) \in \mathbb{N}^3} \frac{\mathbb{P}(W_{2ns} = x, W_{2nt} = y, \check{W}(2ns, 2nt) = z, W \in Ex_n)}{\mathbb{P}(W \in Ex_n)} \left(\frac{x + y - 2z}{\sqrt{n}} \right)^{p/2}.$$

The cardinality of Ex_n is $\binom{2n}{n}/(n+1)$; so, there exists $c_1 > 0$ such that $\mathbb{P}(Ex_n) \geq c_1 n^{-3/2}$.

Let W be an element of RW_n that satisfies $(W_{2ns} = x, W_{2nt} = y, \check{W}(2ns, 2nt) = z, W \in Ex_n)$; we decompose it in three parts.

1) On $[0, 2ns]$ it is a nonnegative random walk that satisfies $W_{2ns} = x$; the probability of this event equals the one of $\{\tau_{-x-1} = 2ns + 1\}$ divided by $1/2$ (where $\tau_k = \inf\{j|W_j = k\}$).

- 2) On $[2nt, 2n]$, from the right to the left, this walk is a nonnegative walk that satisfies $W_{2n(1-t)} = y$; the probability of this event equals the one of $\{\tau_{-y-1} = 2n(1-t) + 1\}$ divided by $1/2$.
- 3) On $[2ns, 2nt]$ it is a discrete bridge with minimum equal to z (which takes values x and y respectively in 0 and $2n(t-s)$). The probability of this event is

$$P_{2n(t-s)}(x+y-2z) \frac{2(x+y-2z+1)}{2n(t-s)+x+y-2z+2} \quad (3.10)$$

where $P_k(l) = \mathbb{P}(W_k = l)$.

As a matter of fact, according to the reflexion principle,

$$\mathbb{P}_x(W_{2n(t-s)} = y, \check{W}(0, 2n(t-s)) = z) = P_{2n(t-s)}(2z-x-y) - P_{2n(t-s)}(2z-2-x-y)$$

where \mathbb{P}_x is the law of a random walk starting at x ($W_0 = x$). Using $P_k(l) = 2^{-k} \binom{k}{(k+l)/2}$, we obtain (3.10).

Using Otter formula [23]

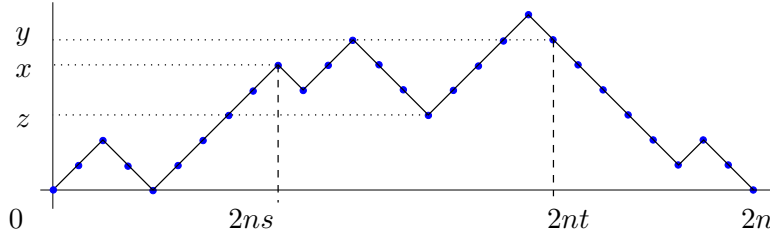


Figure 6. : The decomposition into three parts.

$$\mathbb{P}(\tau_k = n) = \frac{k}{n} P_n(k),$$

we obtain:

$$\begin{aligned} E_n(t, s, p) &= \sum_{(x,y,z) \in A} 8 \frac{x+1}{2ns+1} P_{2ns+1}(x+1) \frac{y+1}{2n(1-t)+1} P_{2n(1-t)+1}(y+1) \\ &\quad \times P_{2n(t-s)}(x+y-2z) \frac{(x+y-2z+1)}{2n(t-s)+x+y-2z+2} \frac{(x+y-2z)^{p/2}}{n^{p/4}} \frac{1}{\mathbb{P}(Ex_n)} \end{aligned}$$

where A is the set of (x, y, z) allowed: $0 \leq x \leq 2ns \wedge 2n(1-s), 0 \leq y \leq 2nt \wedge 2n(1-t), 0 \leq z \leq x \wedge y, x+y-2z \leq 2n(t-s)$ and parities assumptions.

Technical remarks :

(\star) we will just prove Lemma 3.7 for $|t-s| \leq 1/4$ which is sufficient.

($\star\star$) The processes $V_n = (V_n(k))_{k \in [0, 2n]}$ and $(V_n(2n-k))_{k \in [0, 2n]}$ are equal in distribution.

Using (\star), we can suppose that $0 \leq s \leq t \leq 3/4$. And then $(1-t)^{-1}$ is bounded by 4 on this interval.

• The following property is essential (Petrov [24], th. 2.22 p.76.):

Property : There exists a constant C such that for any $\lambda \geq 0$ and any $n \geq 0$,

$$\sup_y \mathbb{P}(y \leq W_n \leq y + \lambda) \leq C \frac{\lambda + 1}{\sqrt{n}}. \quad (3.11)$$

• We write $f_n(s, t, p) \ll g_n(s, t, p)$ if there exists a constant c , that may depend on p , such that $f_n(s, t, p) \leq cg_n(s, t, p)$ for any (s, t, n) , $0 \leq s \leq t \leq 3/4$, $n \geq 1$. As examples: $x + 1 \ll x$ on \mathbb{N}^* but not on \mathbb{N} ; $x(x + 1) \ll x^2$ holds on \mathbb{N} .

$E_n(t, s, p)$ satisfies:

$$E_n(t, s, p) \ll \sum_{(x, y, z) \in A} \frac{(x + 1)(y + 1)P_{2ns+1}(x + 1)P_{2n(1-t)+1}(y + 1)P_{2n(t-s)}(x + y - 2z)}{n^{3/2}s(t - s)} \frac{(x + y - 2z)^{\frac{p+2}{2}}}{n^{p/4}}$$

The sum on z is:

$$\sum_{z=0}^{x \wedge y} P_{2n(t-s)}(x + y - 2z) \frac{(x + y - 2z)^{p/2+1}}{n^{p/4}} = \sum_{w=|y-x|}^{x+y} P_{2n(t-s)}(w) \frac{w^{\frac{p+2}{2}}}{n^{p/4}}.$$

Now, we write

$$E_n(t, s, p) \leq A_n^{(0)}(s, t) + B_n(s, t) \quad (3.12)$$

where $A_n^{(0)}(s, t)$ is the contribution of $x = 0$ in the triple sum, so that

$$\begin{aligned} A_n^{(0)}(s, t) &\ll \sum_{y=0}^{2nt} \frac{P_{2ns+1}(1)P_{2n(1-t)+1}(y + 1)P_{2n(t-s)}(y)y^{p/2+2}}{s(t - s)n^{3/2}n^{p/4}} \\ B_n(s, t) &= \sum_{x=1}^{2ns} \frac{(x + 1)P_{2ns+1}(x + 1)}{ns(t - s)} \left(A_n^{(1)}(s, t) + A_n^{(2)}(s, t) \right) \end{aligned} \quad (3.13)$$

with

$$\begin{aligned} A_n^{(1)}(s, t) &= \sum_{k=3}^{\lceil 2nt/x \rceil} \sum_{y=(k-1)x}^{kx} (y + 1)P_{2n(1-t)+1}(y + 1) \sum_{w=|y-x|}^{x+y} P_{2n(t-s)}(w) \frac{w^{p/2+1}}{n^{p/4+1/2}} \\ A_n^{(2)}(s, t) &= \sum_{y=0}^{2x} (y + 1)P_{2n(1-t)+1}(y + 1) \sum_{w=|y-x|}^{x+y} P_{2n(t-s)}(w) \frac{w^{p/2+1}}{n^{p/4+1/2}} \end{aligned}$$

To end the proof of Lemma 3.7, we show

Lemma 3.8.

$$A_n^{(0)}(s, t) \ll (t - s)^{p/4} \quad (3.14)$$

$$A_n^{(1)}(s, t) \ll x(t - s)^{p/4+1} \quad (3.15)$$

$$A_n^{(2)}(s, t) \ll \frac{x^2}{\sqrt{n}}(t - s)^{p/4+1/2} \quad (3.16)$$

Proof of (3.14): Using (3.11) to bound $P_{2ns+1}(1)P_{2n(1-t)+1}(y+1)$, and using $\sum_{y=0}^{2nt} P_{2n(t-s)}(y)y^{p/2+2} \leq$

$\mathbb{E}|W_{2n(t-s)}|^{p/2+2}$, we obtain

$$A_n^{(0)}(s, t) \ll (t - s)^{p/4} (sn)^{-3/2}$$

since $s \geq 1/2n$, one has the announced result.

Proof of (3.15): In the involved sum, $x \geq 1$. So $x + 1 \ll x$.

$$A_n^{(1)}(s, t) \leq \sum_{k=3}^{\lfloor 2nt/x \rfloor} \sum_{y=(k-1)x}^{kx} (kx+1)P_{2n(1-t)+1}(y+1) \sum_{w=(k-2)x}^{(k+1)x} P_{2n(t-s)}(w) \frac{w^{p/2+1}}{n^{p/4+1/2}}$$

we use then that $kx+1 \ll kx$ and that, in the sum on w , $k \leq w/x+2 \ll w/x$; it yields:

$$A_n^{(1)}(s, t) \ll \sum_{k=3}^{\lfloor 2nt/x \rfloor} \sum_{y=(k-1)x}^{kx} xP_{2n(1-t)+1}(y+1) \sum_{w=(k-2)x}^{(k+1)x} P_{2n(t-s)}(w) \frac{w^{p/2+1}w/x}{n^{p/4+1/2}}$$

According to (3.11)

$$\sum_{y=(k-1)x}^{kx} P_{2n(1-t)+1}(y+1) = P_{2n(1-t)}(\llbracket (k-1)x+1, kx+1 \rrbracket) \ll x/\sqrt{n}$$

$$A_n^{(1)}(s, t) \ll x \sum_{w \geq 0} P_{2n(t-s)}(w) \frac{w^{p/2+2}}{n^{p/4+1}} \ll x(t-s)^{p/4+1} C(p/2+2),$$

where $C(p/2+2)$ is the constant arising in Lemma 3.6.

Proof of (3.16): Using the same kind of inequalities, we have:

$$A_n^{(2)}(s, t) \ll (x+1)P_{2n(1-t)+1}(\llbracket 1, 2x+1 \rrbracket)(t-s)^{p/2+1/2}.$$

This ends the proof of Lemma 3.8. \square .

Plugging (3.15) and (3.16) in (3.13) and using Lemma 3.6, we obtain:

$$B_n(s, t) \ll (t-s)^{p/4-1/2} \tag{3.17}$$

Using (3.14), (3.17) and (3.12), one has:

$$E_n(s, t, p) \ll (t-s)^{p/4-1/2}$$

that ends the proof of Lemma 3.7 and also the one of Theorem 3.9. \square

3.4 Proof of the convergence of the finite dimensional laws

Note

$${}^t\Theta = ({}^t\theta_1, \dots, {}^t\theta_k) \text{ and } {}^t\Theta' = (\theta'_1, \dots, \theta'_{k'}),$$

where $\theta_i \in \mathbb{R}^d$ and $\theta'_i \in \mathbb{R}$. We consider two sequences $0 \leq t_1 \leq \dots \leq t_k \leq 1$ and $0 \leq t'_1 \leq \dots \leq t'_{k'} \leq 1$, and

$$\Psi_n({}^t\Theta, {}^t\Theta') = \mathbb{E} \left(\exp i \left(\sum_{l=1}^k {}^t\theta_l r_n(t_l) + \sum_{j=1}^{k'} \theta'_j v_n(t'_j) \right) \right)$$

and

$$\Psi({}^t\Theta, {}^t\Theta') = \mathbb{E} \left(\exp i \left(\sum_{l=1}^k {}^t\theta_l r(t_l) + \sum_{j=1}^{k'} \theta'_j v(t'_j) \right) \right)$$

the characteristic functions of the finite dimensional laws of the couples of processes (r_n, v_n) and (r, v) . Theorem 3.3 is a consequence of

Lemma 3.9. For any sequences $0 \leq t_1 \leq \dots \leq t_k \leq 1$ and $0 \leq t'_1 \leq \dots \leq t'_{k'} \leq 1$, for all $\theta_1, \dots, \theta_k$ in \mathbb{R}^d and all $\theta'_1, \dots, \theta'_{k'}$ in \mathbb{R} ,

$$\Psi_n({}^t\Theta, {}^t\Theta') \xrightarrow[n]{} \Psi({}^t\Theta, {}^t\Theta').$$

Proof: Since

$$\mathbb{E}\left(\frac{|R_n(2ns) - R_n(\lfloor 2ns \rfloor)|}{n^{1/4}}\right) \leq \mathbb{E}|y(u)|n^{-1/4}$$

and

$$\mathbb{E}\left(\frac{|V_n(2ns) - V_n(\lfloor 2ns \rfloor)|}{n^{1/2}}\right) \leq n^{-1/2}$$

then one can assume that $2nt_j$ and $2nt'_j$ are integers (for all considered j). The conditional covariance matrix of ${}^t(r(t_1), \dots, r(t_k))$ is

$$\Gamma = (\tilde{v}(t_i, t_j))_{1 \leq i \leq k, 1 \leq j \leq k} \otimes I_d.$$

$A \otimes B$ is the tensor product (or the Kronecker product) of the matrices A and B .

One has

$$\Psi({}^t\Theta, {}^t\Theta') = \mathbb{E}(\exp(-{}^t\Theta\Gamma\Theta/2) \exp(i\theta'_1 v(t'_1) + \dots + i\theta'_{k'} v(t'_{k'}))).$$

Now, we write

$$\Psi_n({}^t\Theta, {}^t\Theta') = \mathbb{E}\left(\mathbb{E}\left(\exp i\left(\sum_{l=1}^k {}^t\theta_l r_n(t_l)\right) \middle| v_n\right) \exp i\left(\sum_{j=1}^{k'} \theta'_j v_n(t'_j)\right)\right) \quad (3.18)$$

In a first step, we study:

$$\mathbb{E}\left(\exp i\left(\sum_{l=1}^k {}^t\theta_l r_n(t_l)\right) \middle| v_n\right).$$

Let us investigate the dependence between the $r_n(t_l)$. $R_n(2nt_i)$ (resp. $R_n(2nt_j)$) involves $V_n(2nt_i)$ (resp. $V_n(2nt_j)$) copies of $y(u)$. $V_n(2nt_i) \wedge V_n(2nt_j)$ are common (the ones which are attached to the nodes at depth smaller than $V_n(2nt_i) \wedge V_n(2nt_j)$). The values attached to the depth $(V_n(2nt_i) \wedge V_n(2nt_j)) + 1$ are different but may be dependent (since they are attached to brothers). The other involved random variables are independent. If we consider now the k -uple $(R_n(2nt_i))_{i \in [1, k]}$ and the copies of $y(u)$ used to “compute” them, at most $2(k-1)$ copies of $y(u)$ concern brothers. We note $B_k(i)$ the set of nodes belonging to the branch ($root - \tilde{f}(2nt_i)$) that has at least one brother in another branch $(root - \tilde{f}(2nt_j))_{j \neq i}$.

In order to suppress the problem of dependence between the variables $(R_n(2nt_i))_{i \in [1, k]}$ (it is intuitively obvious that these $2(k-1)$ random variables do not matter when n goes to $+\infty$), we define a sequence $(\tilde{R}_n(2nt_i))_{i \in [1, k]}$ as $\tilde{R}_n(2nt_i) = R_n(2nt_i) - Y_k(i)$ where $Y_k(i) = \sum_{v \in B_k(i)} y(v)$.

All the values used to compute the random variables $\tilde{R}_n(2nt_i)_i$ are independent. The number of common values used to compute $\tilde{R}_n(2nt_i)$ and $\tilde{R}_n(2nt_j)$ is $V_n(2nt_i) \wedge V_n(2nt_j) - b(i, j)$ where

$$b(i, j) = \#B_k(i) \cap B_k(j).$$

Hence,

$$\mathbb{E}|r_n(t_j) - \tilde{r}_n(t_j)| \leq \frac{2(k-1)\mathbb{E}|y(u)|}{n^{1/4}}$$

for $j = 1, \dots, k$. So, we can replace r_n by \tilde{r}_n .

Now, we study

$$\mathbb{E}\left(\exp i\left(\sum_{l=1}^k {}^t\theta_l \tilde{r}_n(t_l)\right) | v_n\right).$$

Knowing v_n , $({}^t\tilde{r}_n(t_1), \dots, {}^t\tilde{r}_n(t_k))$ is a random vector with covariance $kd \times kd$ matrix:

$$\tilde{\Gamma}_k(v_n) = \left((\check{v}_n(t_i, t_j))_{(i,j) \in \llbracket 1, k \rrbracket^2} - n^{-1/4} (b(i, j))_{(i,j) \in \llbracket 1, k \rrbracket^2} \right) \otimes I_d$$

Knowing v_n , one has

$$\text{Var}\left(\sum_{l=1}^k {}^t\theta_l \tilde{r}_n(t_l)\right) = {}^t\Theta \tilde{\Gamma}_k(v_n) \Theta$$

We note $m(V_n)$ the number of nodes in the set of “branches”

$$Br = \bigcup_{i \in \llbracket 1, k \rrbracket} \{(root, \tilde{f}(2nt_i)) \setminus B_k(i)\}.$$

One has:

$$m(V_n) \leq \sum_{i=1}^k V_n(2nt_i) \leq k \max V_n.$$

Hence ${}^t\theta_1 \tilde{r}_n(t_1) + \dots + {}^t\theta_k \tilde{r}_n(t_k)$ can be expressed in function of the $m(V_n)$ i.i.d. $(y(u_i))$, where $u_i \in Br$.

$${}^t\theta_1 \tilde{r}_n(t_1) + \dots + {}^t\theta_k \tilde{r}_n(t_k) = n^{-1/4} \sum_{i=1}^{m(V_n)} {}^t\beta_j y_j$$

where the $(y_j)_j$ are independent copies of $y(u)$. Each β_j is a sum of a subset of $\{\theta_1, \dots, \theta_k\}$. We set

$$Z_j = {}^t\beta_j y_j$$

and

$$\sigma_j^2 = \text{Var}(Z_j) = {}^t\beta_j \beta_j = |\beta_j|^2.$$

These variances are bounded by

$$a = \left(\sum_{i=1}^k |\theta_i| \right)^2.$$

Moreover, one has:

$$\mathbb{E}(|Z_j|^{2+\delta}) \leq \mathbb{E}(|y_1|^{2+\delta}) \left(\sum_{j=1}^k |\theta_j| \right)^{2+\delta} \stackrel{def}{=} b.$$

We have

$$\begin{aligned} \left| \mathbb{E}\left(\exp\left(i\frac{Z_j}{n^{1/4}}\right)\right) - \exp\left(-\frac{\sigma_j^2}{2n^{1/2}}\right) \right| &= \left| 1 + i\mathbb{E}\left(\frac{Z_j}{n^{1/4}}\right) - \frac{\sigma_j^2}{2n^{1/2}} + \varepsilon(n, j) - \exp\left(-\frac{\sigma_j^2}{2n^{1/2}}\right) \right| \\ &\leq \left| \exp\left(-\frac{\sigma_j^2}{2n^{1/2}}\right) - 1 + \frac{\sigma_j^2}{2n^{1/2}} \right| + |\varepsilon(n, j)|. \end{aligned}$$

The first term in the right hand side is bounded as follows

$$\left| \exp\left(-\frac{\sigma_j^2}{2n^{1/2}}\right) - 1 + \frac{\sigma_j^2}{2n^{1/2}} \right| \leq \frac{\sigma_j^4}{8n} \leq \frac{a^2}{8n}.$$

For the second one, since for all real x , $|\exp(ix) - 1 - ix + \frac{x^2}{2}| \leq \min\{\frac{|x|^3}{6}, x^2\}$, (see [4] p.353), we have

$$|\varepsilon(n, j)| \leq \mathbb{E}\left(\min\left\{\frac{Z_j^2}{n^{1/2}}, \frac{|Z_j|^3}{6n^{3/4}}\right\}\right).$$

Since for any real valued two functions F et G , and any subset A of \mathbb{R} , one has

$$\min\{F(x), G(x)\} \leq F(x) \mathbb{1}_A(x) + G(x) \mathbb{1}_{\mathbb{C}A}(x) \quad \text{for any } x,$$

we have:

$$|\varepsilon(n, j)| \leq \mathbb{E}\left(\frac{Z_j^2}{n^{1/2}} \mathbb{1}_{|Z_j| \geq n^{1/8}}\right) + \mathbb{E}\left(\frac{|Z_j|^3}{6n^{3/4}} \mathbb{1}_{|Z_j| \leq n^{1/8}}\right).$$

Since Z has a finite moment of order $2 + \gamma$,

$$\mathbb{E}\left(\frac{Z_j^2}{n^{1/2}} \mathbb{1}_{|Z_j| \geq n^{1/8}}\right) \leq \frac{\mathbb{E}|Z_j|^{2+\gamma}}{n^{1/2+\gamma/8}} \leq \frac{b}{n^{1/2+\gamma/8}}$$

and

$$\mathbb{E}\left(\frac{|Z_j|^3}{n^{3/4}} \mathbb{1}_{|Z_j| \leq n^{1/8}}\right) \leq \frac{a}{n^{5/8}}$$

According to Lemma 1 of Billingsley p.367 [4] and noting that

$$\sum_{j=1}^{m(V_n)} \frac{\sigma_j^2}{2n^{1/2}} = {}^t\Theta \tilde{\Gamma}_k(v_n) \Theta / 2,$$

$$\begin{aligned} \left| \mathbb{E}\left(\exp\left(i \sum_{j=1}^{m(V_n)} \frac{Z_j}{n^{1/4}}\right) | v_n\right) - \exp\left(-{}^t\Theta \tilde{\Gamma}_k(v_n) \Theta / 2\right) \right| &\leq \sum_{j=1}^{m(V_n)} \left| \mathbb{E}\left(\exp\left(i \frac{Z_j}{n^{1/4}}\right)\right) - \exp\left(-\frac{\sigma_j^2}{2n^{1/2}}\right) \right| \\ &\leq k(\max V_n) \left(\frac{a}{n^{5/8}} + \frac{b}{n^{1/2+\gamma/8}} + \frac{a^2}{8n} \right). \end{aligned} \quad (3.19)$$

Using (3.18) and (3.19) one has

$$\begin{aligned} &\left| \Psi_n({}^t\Theta, {}^t\Theta') - \mathbb{E}\left(\exp\left(-{}^t\Theta \Gamma_k(v_n) \Theta / 2 + i \sum_{j=1}^{k'} \theta'_j v_n(t'_j)\right)\right) \right| \\ &\leq \left| \mathbb{E}(k(\max v_n) \left(\frac{a}{n^{1/8}} + \frac{b}{n^{\gamma/8}} + \frac{a^2}{8n^{1/2}} \right) \exp i \left(\sum_{j=1}^{k'} \theta'_j v_n(t'_j) \right)) \right| \end{aligned}$$

The modulus of the exponential factor is bounded by 1. So, the right hand side goes to 0, since $\mathbb{E}(\max v_n)$ converges to $\mathbb{E}(\max \sqrt{2}e)$ (see Flajolet and Odlyzko [12], where it is proved that the moments of the height of plan trees converge to the ones of $\max e$).

Now, when n goes to $+\infty$, $\mathbb{E}\left(\exp\left(-{}^t\Theta \Gamma_k(v_n) \Theta / 2 + i \sum_{j=1}^{k'} \theta'_j v_n(t'_j)\right)\right)$ goes to $\Psi({}^t\Theta, {}^t\Theta')$ since it is the expectation of a bounded continuous function of v_n . \square

4 Relation with ISE

ISE, acronym for *integrated superBrownian excursion*, has been introduced by Aldous [3] as a natural limit distribution of mass. ISE appears to be the limit measure of numerous arborescent phenomenon (for applications and further references, see [7, 9, 27]).

Aldous considers branching random walk on a Galton-Watson underlying tree (with mean 1 and variance $\sigma^2 > 0$, conditioned to have size n). The displacements are independent, nodes by nodes, and are uniformly distributed on the lattice \mathbb{Z}^d (or on the sphere $S(0, 1)$ in \mathbb{R}^d). Now, he attributes to each node a weight $1/n$. The branching random walk induces a mass distribution on \mathbb{R}^d (a node is located by its coordinates in \mathbb{R}^d , the coordinate v_n is not considered). After normalization this distribution of mass converges to a limit mass on \mathbb{R}^d (which is μ_{ISE}).

In our settings, this means that the following sequence of measures on \mathbb{R}^d

$$\mu_n = \frac{1}{n+1} \sum_{k=0}^n \delta_{a_k(n)} \quad (4.20)$$

where

$$a_k(n) = \frac{\sqrt{2}\Phi_{v_k}(h(v_k))}{n^{1/4}},$$

converges weakly to the random measure μ_{ISE} (on \mathbb{R}^d). Hence, by definition

$$\mu_{ISE} = \lim_{weak} \mu_n. \quad (4.21)$$

Below, we give a simple proof of a known description of ISE (Le Gall [17]) using the first marginal r of the Brownian snake:

Proposition 4.10. *For any bounded uniform continuous function f from \mathbb{R}^d to \mathbb{R} , one has*

$$\langle \mu_{ISE}, f \rangle = \int_0^1 f(\sqrt{2}r(t))dt.$$

Proof : Consider r_n , the first marginal of the normalized tour and the discrete associated measure:

$$\lambda_n = \frac{1}{2(n+1)} \sum_{j=1}^{2n} \delta_{\sqrt{2}r_n(j/2n)}.$$

λ_n allows us to express μ_{ISE} in term of the tour process since μ_n and λ_n have the same behavior:

$$\langle \mu_n, f \rangle - \langle \lambda_n, f \rangle \xrightarrow{n} 0. \quad (4.22)$$

Proof of (4.22): For v_k a node of τ different from the root, we note v_k^ϕ the father of v_k and

$$a_k^\phi(n) = \frac{\sqrt{2}\Phi_{v_k^\phi}(h(v_k^\phi))}{n^{1/4}}.$$

Note

$$\lambda'_n = \frac{1}{n+1} \sum_{k=1}^n \frac{\delta_{a_k(n)} + \delta_{a_k^\phi(n)}}{2}.$$

The following equality holds

$$\lambda'_n = \lambda_n. \quad (4.23)$$

As a matter of fact, let us express the number of times that δ_v appears for a node v :

– in the measure λ_n : it appears $\text{outdegree}(v) + 1$ times except if v is the root; in this case, it appears $\text{degree}(\text{root})$.

– in the measure λ'_n : it appears as $\delta_{a_k(n)}$ for a unique k and appears as $\delta_{a_k^\phi(n)}$ as many times as v is father (except for the root that appears only as many times he is father, that is $\text{degree}(\text{root})$ times). Thus, (4.23) is proved.

Then

$$\left| \langle \mu_n - \lambda_n, f \rangle \right| = \frac{1}{n+1} \left| f(a_0(n)) + \sum_{k=1}^n \frac{f(a_k(n)) - f(a_k^\phi(n))}{2} \right|.$$

Thanks to the facts that f is uniformly continuous and

$$\sup_k \frac{a_k(n) - a_k^\phi(n)}{n^{1/4}} \xrightarrow[n]{\text{proba}} 0,$$

we have

$$\left| \langle \mu_n - \lambda'_n, f \rangle \right| \xrightarrow[n]{\text{proba}} 0. \quad (4.24)$$

By (4.23), this ends the proofs of (4.22).

On the other hand,

$$\begin{aligned} \langle \lambda_n, f \rangle - \int_0^1 f(\sqrt{2}r_n(t))dt &= \frac{1}{2n} \sum_{k=1}^{2n} f(\sqrt{2}r_n(k/2n)) - \int_0^1 f(\sqrt{2}r_n(t))dt \\ &= \sum_{k=1}^{2n} \int_{(k-1)/2n}^{k/2n} f(\sqrt{2}r_n(k/2n)) - f(\sqrt{2}r_n(t))dt \end{aligned}$$

since f is uniformly continuous and since $\sup_k \sup_{t \in [(k-1)/2n, k/2n]} |r_n(k/2n) - r_n(t)|$ goes to 0 in probability we have

$$\langle \lambda_n, f \rangle - \int_0^1 f(\sqrt{2}r_n(t))dt \xrightarrow[n]{\text{proba}} 0. \quad (4.25)$$

Now, since r_n converges weakly to r , one has by a continuity argument:

$$\int_0^1 f(\sqrt{2}r_n(t))dt \xrightarrow[n]{(d)} \int_0^1 f(\sqrt{2}r(t))dt. \quad (4.26)$$

Proposition 4.10 follows from (4.21), ... , (4.26). \square

5 Geometrical width

For each $\lambda > 0$, we define the *geometrical width* at height λ of the normalized discrete snake (or of the branching random walk) by:

$$l_n(\lambda) = \sup_{x, x' \in v_n^{-1}(\lambda)} |r_n(x) - r_n(x')|;$$

in the unidimensional case,

$$l_n(\lambda) = \max_{x \in v_n^{-1}(\lambda)} r_n(x) - \min_{x \in v_n^{-1}(\lambda)} r_n(x).$$

Let

$$l(\lambda) = \sup_{x, x' \in v^{-1}(\lambda)} |r(x) - r(x')|.$$

The main result of this part is

Proposition 5.11. *For any $\lambda > 0$,*

$$l_n(\lambda) \xrightarrow[n]{(d)} l(\lambda).$$

The limit process $(l(\lambda))_{\lambda > 0}$ is of course of interest. One can see that the finite dimensional laws $(l_n(\lambda_1), \dots, l_n(\lambda_k))$ converge to the ones of $(l(\lambda_1), \dots, l(\lambda_k))$ (applying argument used to prove Proposition 5.11). However, the tightness of l_n seems hard to be proved in any suitable space.

The geometrical width has been well studied for non conditioned branching random walk in the case $\mathbb{E}(N) > 1$. In this supercritical case, the size of the underlying GW tree is infinite with positive probability. The geometrical progression of the number of individuals generation after generation yields to a somewhere degenerate behavior of the branching random walk. The right

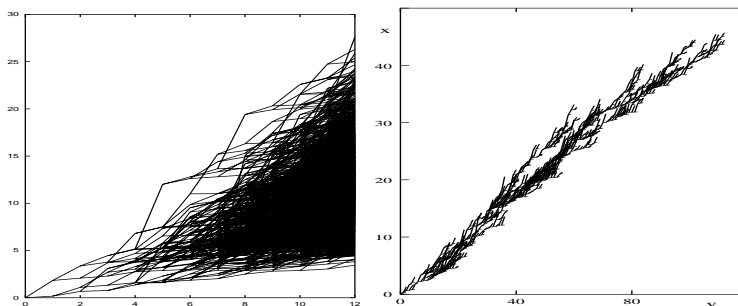


Figure 7. *A supercritical BRW and a finite BRW with drift*

most position RM_n (as well as left most position) of the particles in generation n satisfies the following property:

$$\frac{RM_n}{n} \xrightarrow[n]{a.s.} \gamma$$

where γ is a constant that depends on the law of the point process (see McDiarmid [20]). The case of finite branching random walk is drastically different.

In this paper, we assume N geometrically distributed and the displacement $y(u)$ identically distributed. One can guess (hope?) that these assumptions are not so restrictive that it can appear at first glance: one knows that the progeny law has no effect on the shape of GW trees conditioned by the size (only “microscopic effects”). One may think that what is true here is also true for BRW associated with GW trees with offspring distribution that owns exponential moments. In a recent paper [22], we give moderate deviations on the shape of BRW on general settings (N is not assumed to be geometrically distributed and the displacement law may depend on N).

In order to prove Proposition 5.11, we first prove

Lemma 5.12. Consider (f_n, g_n) a sequence of elements of $C([0, 1], \mathbb{R}^d) \times C([0, 1], \mathbb{R})$ that converges uniformly to (f, g) such that λ is not an extreme value for g . Then

$$\sup_{x, x' \in g_n^{-1}(\lambda)} |f_n(x) - f_n(x')| \xrightarrow{n} \sup_{x, x' \in g^{-1}(\lambda)} |f(x) - f(x')|.$$

Proof: Let us set $A_n = g_n^{-1}(\lambda)$ and $A = g^{-1}(\lambda)$. First we show that

$$\sup_{x \in A_n} d(x, A) \longrightarrow 0 \quad (5.27)$$

$$\sup_{x \in A} d(x, A_n) \longrightarrow 0. \quad (5.28)$$

Suppose that $\sup_{x \in A_n} d(x, A)$ does not go to 0. There exists an $\eta > 0$ and a subsequence A_n such that

$$\sup_{x \in A_n} d(x, A) > \eta$$

One may choose an element x_n in A_n such that $d(x_n, A) > \eta$. Since $[0, 1]$ is compact, there exists a subsequence of x_n that converges to a point x^* . This point x^* is in A (since $g_n(x_n) = \lambda$ and g_n goes uniformly to g); but, $d(x^*, A) \geq \eta$. So (5.27) is proved.

Now, let us prove (5.28). If x is in A , $g(x) = \lambda$. The fact that λ is not a local maximum or minimum ensures that g crosses the line $y = \lambda$ in any neighborhood of x . Now, by the uniform convergence of g_n to g , one sees that g_n has to cross the line $y = \lambda$ in a neighborhood of x when n goes to ∞ . Now, fix an $\varepsilon > 0$ and consider the family of intervals $[x - \varepsilon, x + \varepsilon]$ for $x \in A$. Since A is compact, it is covered by a finite number of such intervals. For n large enough, A_n intersects each one of these intervals and thus, $d(x', A_n) < 2\varepsilon$ for any x' in A . This ends the proof of (5.28).

End of the proof of the lemma: The uniform convergence yields:

$$\left| \sup_{x, x' \in A_n} |f_n(x) - f_n(x')| - \sup_{x, x' \in A} |f(x) - f(x')| \right| \longrightarrow 0.$$

Now fix $\varepsilon > 0$. By the uniform continuity of f , it comes from (5.27) that

$$\sup_{x, x' \in A_n} |f(x) - f(x')| \leq \sup_{x, x' \in A} |f(x) - f(x')| + \varepsilon$$

for n large enough. In the same way, it comes from (5.28) that

$$\sup_{x, x' \in A} |f(x) - f(x')| \leq \sup_{x, x' \in A_n} |f(x) - f(x')| + \varepsilon.$$

Thus,

$$\sup_{x, x' \in A_n} |f(x) - f(x')| \longrightarrow \sup_{x, x' \in A} |f(x) - f(x')|. \quad \square$$

Proof of Proposition 5.11: Note L the application defined on T by:

$$\begin{aligned} L : \quad T &\longrightarrow \mathbb{R} \\ (f, \zeta) &\longrightarrow \sup_{x, x' \in \zeta^{-1}(\lambda)} |f(x) - f(x')| \end{aligned}$$

L is continuous at every point (f, ζ) where λ is not a local extreme value for ζ (in virtue of lemma 5.12). The set of discontinuities of L in T is included in the set of couples (f, ζ) such that ζ has a local maximum λ . But, the probability that λ is a local maximum for the Brownian excursion is 0. The convergence $l_n(\lambda) \xrightarrow[n]{(d)} l(\lambda)$ follows from the convergence of the tour. \square

Note: The fact that v (or e) has λ for local maximum with probability 0 can be proved by showing first that the maximum of e on each interval $[a, b]$ (a, b rational) has a density (that can be computed, which is an exercise rather long). Similar arguments hold for local minimum.

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