Ladder variables, Internal structure of Galton-Watson trees and Finite branching random walks

Jean-François Marckert,
Abdelkader Mokkadem

Université de Versailles
45 Avenue des Etats Unis
78035 Versailles Cedex
{marckert,mokkadem}@math.uvsq.fr

Abstract

In this paper, we consider Galton-Watson trees conditioned by the size. We show that the number of k-ancestors (ancestors that have k children) of a node u is (almost) proportional to its depth. The k,j-ancestors are also studied. The methods rely on the study of ladder variables on an associated random walk. We also give application to finite branching random walks.

1 Introduction and main results

We consider Galton-Watson branching process with offspring N, starting with 1 individual in generation 0. N is a non-negative, integer-valued random variable that satisfies:

$$\begin{cases} \mathbb{E}(N)=1, \quad 0<\mathrm{Var}(N)=\sigma^2<+\infty \\ \text{there exists a constant } \alpha>0 \text{ s.t. } \mathbb{E}\big(e^{\alpha N}\big)<+\infty. \end{cases} \tag{1}$$

Following Petrov [16] p. 56, there exist two positive constants g and T (that depend only on the distribution of N) such that:

$$\mathbb{E}\big(\exp(t(N-1))\big)\le \exp(gt^2/2) \quad \text{for} \quad |t|\le T.$$

The constant g will play an important role in the sequel of the paper.

We write ζ for the family tree of this branching process and Ω the probability space of all trees with the law induced by N.

In the present paper, we study some properties of Galton-Watson trees conditioned by their total progeny. We note Ω_n the space of size n trees endowed by the conditional law given $|\zeta|=n$. The law on Ω_n will be denoted by $\mathbb{P}_n$. It is now well known that each family of simply generated trees (in the sense of Meir & Moon [14]) can be viewed as critical Galton-Watson trees conditioned by the size (we refer to Kolchin [8]) and that considering only critical offspring distribution is not a restrictive assumption (when we condition by $|\zeta|=n$), see Kennedy [7].

We note $(p_i)_{i\ge 0}=(\mathbb{P}(N=i))_{i\ge 0}$ the offspring distribution. Let u be a node of ζ (we note $u\in\zeta$).

We consider the following random variables:

$a_k(u)$, the number of ancestors of u (u excluded) that have k children (we call these nodes k-ancestors of u).

$a_{k,j}(u)$, the number of ancestors of u (u excluded) that have k children, and from which u is a descendant of the jth one (thus j goes from 1 to k and the children are ordered from the leftmost one to the rightmost one). We call these ancestors, k,j-ancestors of u.

$h(u)$, the depth of the node u (that is $h(u)=d(u,root)$).

1.1 Internal structure of finite Galton-Watson trees

A first aim of this paper is to show that $a_k(u)$ is roughly speaking proportional to $h(u)$, and this "uniformally" in u:

Theorem 1 *Let β be a positive real number. For any $\gamma > \sqrt{(\beta+4)\sqrt{(\beta+7/2)g}/2}$,*

$$\mathbb{P}_n\Big(\sup_{u\in\zeta,\, k} \big|a_k(u) - kp_k h(u)\big| > \gamma n^{1/4} \ln n \Big) = o(n^{-\beta}).$$

Note that this result is not trivial since $h(u) \gg n^{1/4}\ln n$ for "almost" all the nodes. A partial explanation of this theorem follows: it is known that the number of nodes having k children in a tree of Ω_n is about $p_k n$. This implies, that about $kp_k n$ nodes have a father with k children. Taking one ancestor at random (that is a node having some children) will provide a node with k children with probability (asymptotically) kp_k.

This distribution $(kp_k)_k$ arises also in the size-biased trees (GW tree conditioned to survive forever). In this size-biased trees, when $m = 1$, the condition "survive forever" gives a tree with one infinite branch; on this branch, each node has k children with probability kp_k (we refer to Lyons & Peres [12]). Our Theorem 1 is quite similar since it "describes" the offspring distribution in large branches.

One can refine this theorem:

Theorem 2 *Let β be a positive real number. For any $\gamma > \sqrt{(\beta+5)\sqrt{(\beta+9/2)g}/2}$,*

$$\mathbb{P}_n\Big(\sup_{u\in\zeta,\, k,\, j} \big|a_{k,j}(u) - p_k h(u)\big| > \gamma n^{1/4} \ln n \Big) = o(n^{-\beta}).$$

These two theorems imply that $a_{k,j}$ behaves as a_k/k.

As an interesting application of Theorem 2, one can generalize a remark of Flajolet & Odlyzko [4] about the right-height and left-height in binary trees ($p_0 = p_2 = 1/2$). The right height R_n (resp. the left height L_n) is defined as the maximal number of right steps (resp. of left steps) in all the branches of ζ:

$$R_n = \max_{u\in\zeta} a_{2,1}(u) \quad \text{and} \quad L_n = \max_{u\in\zeta} a_{2,2}(u)$$

(the maxima are taken on all the nodes but they are reached on leaves). Flajolet & Odlyzko [4] note that in size n binary trees, the height H_n of the tree is "almost" the sum of R_n and L_n, more precisely,

$$\frac{H_n}{\sqrt{n}} - \frac{R_n}{\sqrt{n}} - \frac{L_n}{\sqrt{n}} \overset{proba}{\longrightarrow} 0 \quad \text{and} \quad \frac{L_n}{\sqrt{n}} - \frac{R_n}{\sqrt{n}} \overset{proba}{\longrightarrow} 0. \tag{2}$$

This remark says that the highest branch(es) in ζ contains as many right steps than left ones (up to $o(\sqrt{n})$); this says also that the highest branch(es) contains more left-steps (and right steps) than the other ones.

Theorem 2 is more precise and says that in each branch the number of right steps (or left steps) is "proportional" to the height of the branch (the ratio being the same one in all the branches); in particular the highest branch contains the maximum number of right steps (and left ones).

In fact, Theorem 2 applies to any kind of simple trees and provides some extensions to (2). Set

$$A_k = \max_{u\in\zeta} a_k(u) \quad \text{and} \quad A_{k,j} = \max_{u\in\zeta} a_{k,j}(u).$$

A_k is the maximal number of nodes of type k contained in a branch, $A_{k,j}$ is the maximal number of nodes of type k, j contained in a branch.

Proposition 1 *Let β be a positive real number. Set $m = \sum_{k\geq 1} k\sqrt{p_k}$;*

for any $\gamma > m\sqrt{(\beta+5)\sqrt{(\beta+9/2)g}/2}$,

$$\mathbb{P}_n\Big(\big|H_n - \sum_{k,j} A_{k,j}\big| > \gamma n^{1/4}\ln n\Big) = o(n^{-\beta}).$$

In other words, the highest branch in the tree contains (up to some crumbs) the maximum number of nodes of each type (among all the branches). This proposition implies (2).
Since $\sum_{k,j} A_{k,j} \geq \sum_k A_k \geq H_n$, Proposition 1 also yields:

$$\mathbb{P}_n\Big(\big|H_n - \sum_{k} A_{k}\big| > \gamma n^{1/4}\ln n\Big) = o(n^{-\beta}). \tag{3}$$

Remark : Other results of this kind can be derived. For example, for any $\beta > 0$, if γ is large enough,

$$\mathbb{P}_n\Big(\big|p_k H_n - A_{k,j}\big| > \gamma n^{1/4}\ln n\Big) = o(n^{-\beta}). \tag{4}$$

The limit of H_n is known (see [1, 2, 4, 13]):

$$\lim_n \frac{H_n}{\sqrt{n}} \xrightarrow[n]{(law)} \frac{2}{\sigma}\max\{e(t), 0\leq t\leq 1\},$$

where $\big(e(t)\big)_{t\in[0,1]}$ is the Brownian excursion. One obtains as a corollary of (4):

$$\frac{A_{k,j}}{\sqrt{n}} \xrightarrow[n]{(law)} \frac{2p_k}{\sigma}\max\{e(t), 0\leq t\leq 1\}.$$

Theorem 1 says that the number of k-ancestors of u is roughly speaking $kp_k h(u)$. One can go further. Choose a node u. Note $u_0 = root, u_1, \ldots, u_{h(u)} = u$ the path from the root to u. A natural question is the following one: what is the depth of the jth k-ancestor of u in the sequence $u_0, \ldots, u_{h(u)}$? For $j \in [\![1, a_k(u)]\!]$, the depth of the jth k-ancestor of u is:

$$\beta_k^{(u)}(j) = \min\{i | a_k(u_{i+1}) = j\}.$$

The following proposition gives informations on the distribution of the depths of k-ancestors.

Proposition 2 *Let β be a positive real number. For any $\gamma > \sqrt{(\beta+4)\sqrt{(\beta+7/2)g}/2}$,*

$$\mathbb{P}_n\Big(\sup\Big\{\big|kp_k\beta_k^{(u)}(j) - j\big|, u\in\zeta, k\geq 1, 1\leq j\leq a_k(u)\Big\} > \gamma n^{1/4}\ln n\Big) = o(n^{-\beta}). \tag{5}$$

Proof : One has $a_k(u_{\beta_k^{(u)}(j)}) = j-1$ and $h(u_{\beta_k^{(u)}(j)}) = \beta_k^{(u)}(j)$. The left hand side of (5) is equal to:

$$\mathbb{P}_n\Big(\sup\Big\{\big|kp_k h\big(u_{\beta_k^{(u)}(j)}\big) - a_k\big(u_{\beta_k^{(u)}(j)}\big) - 1\big|, u\in\zeta, k\geq 1, 1\leq j\leq a_k(u)\Big\} > \gamma n^{1/4}\ln n\Big).$$

Then Proposition 2 is a consequence of Theorem 1. □

Remark

Our theorems can be stated as follows: For any $\nu > 0$, there exists a constant $C > 0$ such that if n is large enough

$$\mathbb{P}_n\Big(\sup_{u\in\zeta,\, k} \big|a_k(u) - kp_k h(u)\big| \geq n^{1/4+\nu}\Big) \leq \exp(-Cn^{\nu}).$$

In order to obtain the results given above, we use some connections between trees and discrete excursions. Such relations are well known and go back to Harris [5] and Kendall [6]. In particular, Kendall has shown a relationship between branching processes and the theory of queues. The discrete excursion that we consider is the depth first queue process (see section 2.1) that can be interpreted as the process of the remaining clients in a queue. It turns out that the a_k and $a_{k,j}$ can be viewed as functionals of the right minima of the depth first queue process.

A conditioning argument (section 2.3) allows to transfer the study of right-minima on this excursion to the study of right-minima on (non-conditioned) random walk. After a symmetry of the path, the right minima become ladder variables (in section 2.4). This leads us to prove new results on ladder variables that are among the key tools in our proofs of the previous theorems. Since these results seem to be unknown, we present them in the following subsection.

1.2 Ladder variables

Consider a random walk $(W^{\bullet}(i))_{i\geq 0}$ defined as

$$W^{\bullet}(0) = 0, \quad W^{\bullet}(k) = \sum_{j=1}^{k} w(j), \text{ for } k \geq 1,$$

where the random variables $(w(j))_{j\geq 1}$ are i.i.d., with mean 0. Moreover, we assume that the distribution of $w(1)$ is supported by $\{-1, 0, 1, \dots\}$ and is given by

$$\mathbb{P}(w(1) = i) = \tilde{p}_i \text{ for } i \geq -1.$$

Note $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ the weak ladder epochs for $W^{\bullet}$:

$$\tau_k = \inf\big\{ j > \tau_{k-1}, W^{\bullet}(j) \geq W^{\bullet}(\tau_{k-1})\big\}, \text{ for } k \geq 1. \tag{6}$$

We consider the r.v. $w(\tau_m)$, for $m \geq 1$. This is the increment of the mth ladder epoch; we call these variables the *ladder increments*. The distribution of $w(\tau_m)$ does not depend on m. Note also that $w(\tau_1)$ is not the standard weak ladder height $W^{\bullet}(\tau_1)$.

Lemma 1 *The distribution of $w(\tau_1)$ is given by*

$$q_k \stackrel{def}{=} \mathbb{P}(w(\tau_1) = k) = (k+1)\tilde{p}_k \text{ for } k \geq 0.$$

Note that if $\mathbb{E}\big(w^{m+1}(1)\big)$ exists, then $\mathbb{E}\big(w^m(\tau_1)\big) = \mathbb{E}\big(w^{m+1}(1)\big) + \mathbb{E}\big(w^m(1)\big)$ for $m > 0$; in particular $\mathbb{E}\big(w(\tau_1)\big) = \mathrm{Var}\big(w(1)\big)$.
The following lemma gives the conditional distribution of $W^{\bullet}(\tau_1)$ with respect to $w(\tau_1)$.

Lemma 2 *For $0 \leq r \leq k$, if $\tilde{p}_k > 0$,*

$$\mathbb{P}(W^{\bullet}(\tau_1) = r | w(\tau_1) = k) = \frac{1}{k+1}.$$

This is an interesting result for ruin problem or gambling problem. When we play at a fair game with the constant bet 1 but random win, the probability that our fortune becomes nonnegative in winning k is $k+1$ times the probability to win k. Lemma 2 specifies that at this time, our fortune is a random variable uniformally distributed on $[\![0,k]\!]$.

1.3 Application to finite branching random walks

We superimpose a random walk on each path from the root down in the tree of Ω. We associate with each node u a real random variable $y(u)$, called the value of u; the value of the root is 0. If u has $k(u)$ children, then the joint distribution of the values associated to these $k(u)$ (ordered) individuals $v_1^u, \ldots, v_{k(u)}^u$ is given by:

$$\Big(k(u), \big(y(v_1^u), \ldots, y(v_{k(u)}^u)\big)\Big) \stackrel{(law)}{=} \Big(N, \big(X_1^{(N)}, \cdots, X_N^{(N)}\big)\Big), \tag{7}$$

where the distribution of $\big(N, (X_1^{(N)}, \cdots, X_N^{(N)})\big)$ is quite general. Notice that the law of $X_i^{(N)}$ depends on i but also on N. As usual, we assume that the values attached to the children of different parents are independent.

Let u be a node and $(u_0 = root, u_1, \cdots, u_{h(u)} = u)$ be the path from the root to u. We associate to u a trajectory of killed random walk $W_u = (W_u(j))_{j \in [\![0,h(u)]\!]}$ defined by:

$$W_u(0) = 0, \quad W_u(j) = \sum_{i=1}^{j} y(u_i), \text{ for } 1 \le j \le h(u).$$

We call branching random walk the tree-shape process which is the union of the trajectories W_u:

$$B = \Big\{W_u, u \in \zeta\Big\}.$$

In this paper we consider the case where the underlying tree is from Ω_n. In this case the set of trajectories is denoted by B_n and called *finite branching random walk*. The set of the points of the trajectories of B_n is:

$$B_n' = \Big\{\big(h(u), W_u(h(u))\big), u \in \zeta\Big\}.$$

We note $\mathcal{B}_n$ the probability space of branching random walks on Ω_n endowed by the probability induced by the offspring-displacement distribution (7). The law on $\mathcal{B}_n$ will be denoted again by $\mathbb{P}_n$.

We suppose that N satisfies (1). Moreover, we assume that

$$\begin{cases} (i) & \text{There exist } a > 0, b > 0 \text{ such that, for any } (j,k) \in {\mathbb{N}^\star}^2, j \le k, \\ & \mathbb{E}\big(\exp(a|X_j^{(k)} - E(X_j^{(k)})|)\big) < b. \\ (ii) & \displaystyle\sum_k \sum_{j=1}^{k} \sqrt{p_k}\, |\mathbb{E}\big(X_j^{(k)}\big)| < +\infty. \end{cases} \tag{8}$$

Denote by Λ the "drift"

$$\Lambda = \sum_{k \ge 1} \sum_{j=1}^{k} p_k\, \mathbb{E}(X_j^{(k)}).$$

The following proposition reveals that $W_u(h(u))$ is concentrated around $\Lambda h(u)$.

Proposition 3 *Let β be a positive real number. Under condition (8), there exists a constant $\gamma_0 > 0$ such that for any $\gamma > \gamma_0$,*

$$\mathbb{P}_n\Big(\sup_{u\in\zeta} \big|W_u(h(u)) - \Lambda h(u)\big| > \gamma n^{1/4} \ln n \Big) = o(n^{-\beta}).$$

Remarks :
1) $(8, ii)$ is a weak condition: since by (1) N has exponential moments, the sequence $(p_k)_k$ goes to 0 at least geometrically fast. Hence, the sequence of expectations $\mathbb{E}(X_j^{(k)})$ can goes to $+\infty$.
2) Proposition 3 also holds when the r.v. $X_i^{(N)}$ take their values in $\mathbb{R}^d$ even if the coordinates are dependent.
3) The error term $n^{1/4} \ln n$ is natural. $W_u(h(u))$ is the sum of $h(u)$ r.v. and $h(u)$ is about $n^{1/2}$.

2 Proofs

2.1 The depth first queue process and the height process

The depth first order: Let ζ be an ordered tree with n nodes. We define a function (see Aldous [1] p.260):

$$\tilde{f} : \{0, \cdots, 2n-2\} \longrightarrow \{\text{nodes of } \zeta\},$$

which we regard as a walk around ζ, as follows:

$$\tilde{f}(0) = \text{root}.$$

Given $\tilde{f}(i) = v$, choose, if possible, the leftmost child w of v which has not already been visited, and let $\tilde{f}(i+1) = w$. If it is not possible, let $\tilde{f}(i+1)$ be the parent of v. For i from 0 to $n-1$, let v_i be the ith new node visited by the depth first procedure on $\zeta \in \Omega_n$ ($v_0 = root$) and

$$\xi_i = \text{ the outdegree of } v_i = \text{the number of children of } v_i.$$

The depth first queue process (DFQP): For a tree $\zeta \in \Omega_n$, the DFQP, S_n is defined by $S_n(0) = 0$ and:

$$S_n(j) = \sum_{i=0}^{j-1} (\xi_i - 1) \text{ for any } 1 \le j \le n. \tag{9}$$

The height process: The height process of $\zeta \in \Omega_n$, h_n, is defined by

$$h_n(i) = h(v_i) \quad \text{for } 0 \le i \le n-1.$$

The DFQP as well as the height process characterizes the associated tree.

2.2 Study of the ancestors via the DFQP

Ancestors

It is proved in several papers [9, 10, 11, 13] that for any $l \in [\![0, n-1]\!]$,

$$\begin{aligned} h_n(l) &= \sum_{j=1}^{l} \mathbb{1}_{\big\{ \min_{0\le k\le l-j} \{S_n(j+k) - S_n(j-1)\} \ge 0 \big\}} \\ &= \#\big\{ j | 0 \le j \le l-1, \min_{0\le k\le l-j} \{S_n(j+k)\} = S_n(j) \big\}. \end{aligned} \tag{10}$$

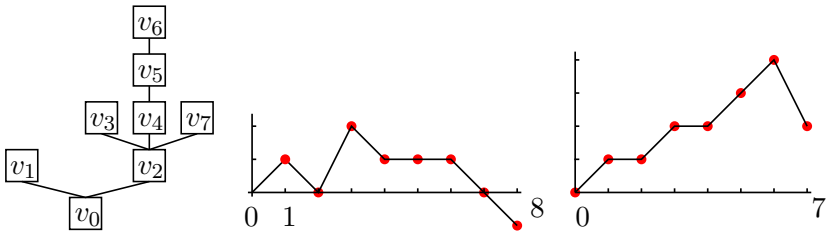

Figure 1 : A tree and its associated DFQP and height process.

We say that $S_n(j)$ is a right minimum of S_n on $[\![0, l-1]\!]$ if

$$\min_{0\le k\le l-j}\{S_n(j+k)\} = S_n(j).$$

(10) means that $h_n(l)$ is the number of "right" minima of $(S_n(i))_{i\in[\![0,l]\!]}$ on the interval $[\![0, l-1]\!]$. This connection between the height process and the DFQP is essential. We state and prove it in a slightly different way.

Lemma 3 *For any $l \in [\![0, n-1]\!]$ and $j \in [\![0, l-1]\!]$, $S_n(j)$ is a right minimum of S_n on $[\![0, l-1]\!]$ if and only if v_j is an ancestor of v_l.*

For commodity, when $S_n(j)$ is a right minimum of S_n on $[\![0, l-1]\!]$, we will say that v_j is a right minimum on $[\![0, l-1]\!]$.

Proof : Our proof is based on a queue interpretation. We consider a queue with one server where the rule service is: last in, first out; the server serves one client per unit time. Now, consider a tree $t \in \Omega_n$ with nodes $(v_0, \cdots, v_{n-1})$ (sorted according to their first visit time during the depth first procedure). The nodes are the clients. The root, v_0, arrives at time 0. The sons of v_i arrive at time $i+1$ from the rightmost one, to the leftmost one (the leftmost one arrives the last, so it will be served the first). At time i, the client who is at the head of the queue gets its service and disappears from the queue at time $i+1$.

The queue size in function of time is $\big(S_n(i)+1\big)_{i=0,\cdots,n}$, where S_n is the DFQP; actually, when the client i is served (-1), he is replaced by its sons $(+\xi_i)$, and the queue size at time 0 is 1.

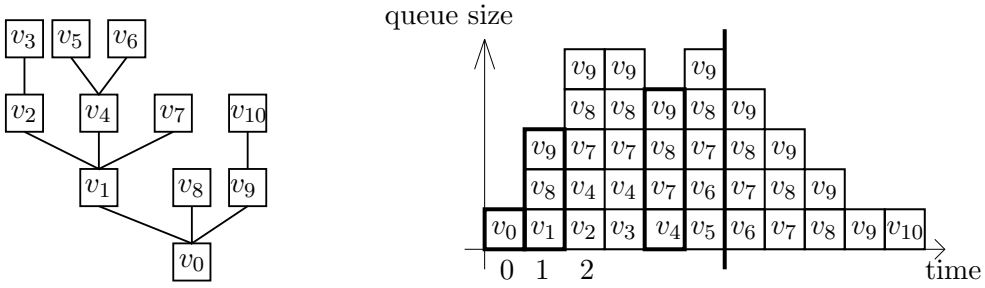

Figure 2 *: The LIFO queue associated to a tree*

As an illustration, consider on Figure 2 the node v_5. It it is served at time 5 (which is natural, by construction). In the tree, its depth is three since he has three ancestors v_4, v_1 and v_0. The times when its ancestors are served are the times of right minima before time v_5.

Let us come back to the proof. First, note that by the queue procedure, the parents are served before their children. Consider v_k. By construction, v_k is served at time k. Note $t(v_k)$ the subtree of

t with root v_k. After v_k, the next $|t(v_k)|-1$ clients that are served are exactly the nodes of $t(v_k)$; but these nodes were not in the queue at time k. So, $S_n(k) \leq S_n(k+i)$, for all $i \in \{1, \cdots, |t(v_k)|-1\}$. Moreover, $S_n(k) > S_n(k+|t(v_k)|)$. As a matter-of-fact, at time $k+|t(v_k)|$, all the descendants of v_k have left the queue. The queue contains the same elements than at time k except v_k that has disappeared. It follows that $S_n(k)$ is a right minimum in the set $[\![1, k+|t(v_k)|-1]\!]$ but not in $[\![1, k+|t(v_k)|]\!]$.
Consider now v_j a node of t. All the ancestors of v_j are served before v_j and are right minima in $[\![0, l-1]\!]$. Moreover, a node v_i that is served before time l and that is not an ancestor of v_l is not a right minima in $[\![0, l-1]\!]$ since $S_n(i+|t(v_i)|) < S_n(i)$ and $i+|t(v_i)| < l$ (because in this case v_l is not in $t(v_i)$ and all the nodes of $t(v_i)$ are served before v_l; thus $l > i+|t(v_i)|$). Hence, the right minima of S_n in $[\![0, l-1]\!]$ are exactly the ancestors of v_l. □

Type of ancestors

Lemma 3 allows to give a good description of who are the ancestors of v_l. Note $A(l)$ the set of indices j for which $S_n(j)$ is a right minimum for S_n in the interval $[\![0, l-1]\!]$.

$$A(l) = \big\{ j | 0 \leq j \leq l-1, \min_{0 \leq k \leq l-j} \{S_n(j+k)\} = S_n(j) \big\}.$$

For the largest j in $A(l)$, set $j^\star = l$; for the other j in $A(l)$, set $j^\star = \inf\{k > j | k \in A(l)\}$.

Lemma 4 *For any* $0 \leq l \leq n-1$,

$$\begin{aligned} a_k(v_l) &= \#\{j \in A(l) \text{ s.t. } S_n(j+1) - S_n(j) = k-1\} \\ a_{k,i}(v_l) &= \#\{j \in A(l) \text{ s.t. } S_n(j+1) - S_n(j) = k-1, S_n(j^\star) - S_n(j) = k-i\}. \end{aligned} \tag{11}$$

Proof : The first equality follows from the fact that: the assertion $\big\{S_n(j)$ is a right minimum and $S_n(j+1) - S_n(j) = k-1\big\}$ is equivalent to the assertion $\big\{v_j$ is an ancestor of v_l and v_j has k children$\big\}$. Proof of the second equality: Let v_j be a k,i-ancestor of v_l. The next right minimum ($v_{j^\star}$) after v_j is the ith son of v_j. In the queue, when v_j is served, the queue size increases by $k-1$, $S_n(j+1) = S_n(j) + k - 1$. Then the sons of v_j (and their progenies) are served with respect to their order. When the first son of v_j is served, the queue size is $S_n(j+1) - 1$. When the second son of v_j is served, the queue size is $S_n(j+1) - 2$... When the $(i-1)$-th son of v_j is served, the queue size is $S_n(j^\star) = S_n(j+1) - (i-1) = S_n(j) + k - i$. Conversely, by the same arguments, the relation $S_n(j^\star) = S_n(j) + k - i$ implies that $v_{j^\star}$ is the ith son of v_j. □

2.3 Conditioning arguments

Consider a random walk $(Z_i)_{i \geq 0}$ with increments $(z_i)_{i \geq 0}$, i.i.d. such that $Z_0 = 0$, $\mathbb{E}(z_1) = 0$ and $0 < \mathrm{Var}(z_1) < +\infty$. Assume moreover that z_i takes its values on the set $\{-1, 0, 1, 2, \cdots\}$.
We denote by $\mathcal{W}_Z(n)$ the set of n length random walks $(Z_i)_{0 \leq i \leq n}$ with the law induced by the law of z_i and $\mathcal{E}_Z(n)$ the set of corresponding "excursions":

$$\{\omega \in \mathcal{E}_Z(n)\} \Leftrightarrow \{\omega \in \mathcal{W}_Z(n), Z_1(\omega) \geq 0, \cdots, Z_{n-2}(\omega) \geq 0, Z_{n-1}(\omega) = 0, Z_n(\omega) = -1\}.$$

We have (Otter's formula [15]):

$$\mathbb{P}(\mathcal{E}_Z(n)) = \frac{1}{n} \mathbb{P}(Z_n = -1).$$

Since z_i may have a lattice distribution, $\mathbb{P}(Z_n = -1)$ may be zero for some n. Assume that z_i takes its values on the lattice $-1 + kh$, where h is the maximal span. Note $(a(n))_n$ a sequence of indices

such that $a(n) \underset{n}{\rightarrow} +\infty$ and $\mathbb{P}(Z_{a(n)} = -1) \neq 0$. The local central limit theorem (see Port [17] p.706) says

$$\frac{\sqrt{a(n)}}{h}\mathbb{P}(Z_{a(n)} = -1) - \frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{1}{2\mathrm{Var}(z_1)a(n)}\Big) \longrightarrow 0, \text{ as } n \rightarrow +\infty,$$

and then

$$\mathbb{P}\big(\mathcal{E}_Z(a(n))\big) \sim \frac{h}{\sqrt{2\pi}\sigma a(n)^{3/2}}.$$

When ξ is lattice distributed, the total progeny in the GW process with offspring ξ is also lattice distributed and then the length of the height process and the length of the DFQP are also lattice distributed. The appearance of the set of indices $a(n)$ is necessary. Each of our theorems should be stated using $\mathbb{P}_{a(n)}$ instead of $\mathbb{P}_n$. For seek of simplicity we have removed the subsequence indices.

From random walk to excursion (the conditioning argument)

Let A_n be a subset of $\mathcal{W}_Z(n)$ and n such that $\mathbb{P}(Z_n = -1) \neq 0$. One has

$$\mathbb{P}(A_n|\mathcal{E}_Z(n)) \leq \frac{\mathbb{P}(A_n)}{\mathbb{P}(\mathcal{E}_Z(n))}.$$

Thus,

$$\mathbb{P}(A_n|\mathcal{E}_Z(n)) = O(n^{-3/2})\mathbb{P}(A_n). \tag{12}$$

Hence, if there exists a moderate deviations principle (or a large deviations principle) for a functional of a centered random walk, there exists an upper bound for the analogous principle on the associated excursion. The interest is that almost all r.v. are much more difficult to handle on $\mathcal{E}_Z(n)$ than on $\mathcal{W}_Z(n)$.

2.4 Right minima and ladder variables on an associated random walk

We first work on a random walk $(W(i))_{i\geq 0}$ (with $W(0) = 0$) that has i.i.d. increments with distribution $(\tilde{p}_j)_{j\geq -1} = (p_{j+1})_{j\geq -1}$. We note $RM(l)$ the number of right minima of $(W(i))_i$ on $[\![0, l-1]\!]$:

$$RM(l) = \#\big\{j|0 \leq j \leq l-1, \min_{0\leq k\leq l-j}\{W(j+k)\} = W(j)\big\}. \tag{13}$$

We define the analogous variables of $A(l)$, a_k, $a_{k,j}$.

$$B(l) = \big\{j|0 \leq j \leq l-1, \min_{0\leq k\leq l-j}\{W(j+k)\} = W(j)\big\}.$$

For the largest j in $B(l)$, set $j' = l$; for the other j in $B(l)$, set $j' = \inf\{k > j | k \in B(l)\}$.

$$\begin{aligned} b_k(l) &= \#\big\{j \in B(l) \text{ s.t. } W(j+1) - W(j) = k-1\big\}, \\ b_{k,i}(l) &= \#\big\{j \in B(l) \text{ s.t. } W(j+1) - W(j) = k-1, W(j') - W(j) = k-i\big\}. \end{aligned} \tag{14}$$

With the random walk $(W(j))_{j\in[\![0,l]\!]}$, we associate the random walk $(W^\bullet(j))_{j\in[\![0,l]\!]}$ defined by:

$$W^\bullet(j) = W(l) - W(l-j) \text{ for any } j \in [\![0, l]\!]. \tag{15}$$

The graph of the trajectory $W^\bullet$ is the symmetric of the graph of W according to the center $(l/2, W(l)/2)$. The increments of $(W^\bullet(i))_{i\geq 0}$ are also $(\tilde{p}_j)_{j\geq -1}$ distributed.

Set $R^\bullet(l)$ the number of records (in the large sense) of the sample path $(W^\bullet(j))_{j\in[\![0,l]\!]}$:

$$R^\bullet(l) = \#\big\{j, 1 \leq j \leq l, \max_{0\leq k\leq j}\{W^\bullet(k)\} = W^\bullet(j)\big\}.$$

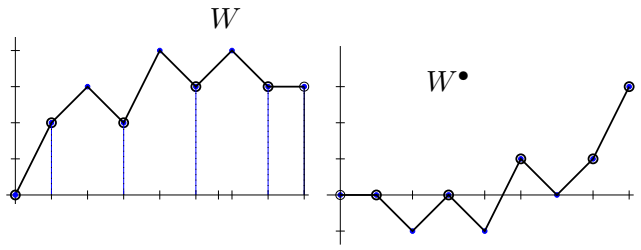

Figure 3 *: Effect of the symmetry on W.*

By (15), we obtain:

$$\begin{aligned} R^\bullet(l) &= RM(l), && (16)\\ \max_{0\le i\le l} W^\bullet(i) &= W(l) - \min_{0\le i\le l} W(i). && (17)\\ \big|\max_{0\le j\le l}\{W^\bullet(j)\} - R^\bullet(l)\frac{\sigma^2}{2}\big| &= \big|W(l) - \min_{0\le i\le l}\{W(i)\} - RM(l)\frac{\sigma^2}{2}\big|. && (18)\end{aligned}$$

The random variables b_k and $b_{k,j}$ can be also translated on $W^\bullet$. Note $0=\tau_0<\tau_1<\cdots<\tau_{R^\bullet(l)}<\cdots$ the weak ladder epochs for $W^\bullet$ as in (6), and set

$$\begin{aligned} c_k(l) &= \#\big\{\tau_j, 0<\tau_j\le l, w(\tau_j)=k-1\big\}, && (19)\\ c_{k,i}(l) &= \#\big\{\tau_j, 0<\tau_j\le l, w(\tau_j)=k-1, W^\bullet(\tau_j)-W^\bullet(\tau_{j-1})=k-i\big\}.\end{aligned}$$

One has:

$$c_k(l)=b_k(l), \qquad c_{k,j}(l)=b_{k,j}(l). \tag{20}$$

Summary table

In this table, we summarize the correspondences between the studied r.v..

S_n	W	$W^\bullet$
$h_n(l)$	$RM(l)$	$R^\bullet(l)$
$S_n(l)$	$W(l)-\min_{j\le l} W(j)$	$\max_{j\le l} W^\bullet(j)$
a_k	b_k	c_k
$a_{k,j}$	$b_{k,j}$	$c_{k,j}$

Even if the r.v. in the second and third columns are very similar functionals of the path, the ones in the third column are easier to handle. As a matter of fact, since ladder epochs are stopping times, they can be studied using Markov properties of the path. This is the reason why the path $W^\bullet$ is introduced.

2.5 Ladder variables distributions: proof of Lemmas 1 and 2

We note $\lambda_r=\mathbb{P}(W^\bullet(\tau_1)=r)$; for any $r\ge 0$,

$$\lambda_r=\tilde{p}_r+\frac{\tilde{p}_{-1}}{1-\lambda_0}\lambda_{r+1}. \tag{21}$$

A straightforward computation gives $\lambda_0 = 1 - \tilde{p}_{-1}$ and

$$\lambda_r = \tilde{p}_r + \lambda_{r+1} = \mathbb{P}(W^\bullet(1) \geq r) \text{ for } r \geq 1, \tag{22}$$

see Feller [3], chap. XII and p. 425-426.

We decompose q_k as follows:

$$q_k = \sum_{i=-k}^{0} \mathbb{P}\left(W^\bullet(\tau_1 - 1) = i \cap w(\tau_1) = k\right).$$

$W^\bullet(\tau_1 - 1) = 0$ implies that $\tau_1 = 1$; so

$$\mathbb{P}\left(W^\bullet(\tau_1 - 1) = 0 \cap w(\tau_1) = k\right) = \tilde{p}_k.$$

For i in $[\![1, k]\!]$ consider the event $E_{i,k} = \{W^\bullet(\tau_1 - 1) = -i \cap w(\tau_1) = k\}$.

Let $W^\bullet$ be a trajectory in $E_{i,k}$. During the times $[\![1, \tau_1 - 1]\!]$, $W^\bullet$ has visited the sites $-1, \cdots, -i$ (and maybe other negative sites).
We note $r_1^{(1)} < \cdots < r_1^{(\kappa_1)}$ the visit times of -1,
$r_2^{(1)} < \cdots < r_2^{(\kappa_2)}$ the visit times of -2 after the last visit time of -1 (we have $r_1^{(\kappa_1)} < r_2^{(1)}$) and so on.
$r_i^{(1)} < \cdots < r_i^{(\kappa_i)}$ the visit times of $-i$ after the last visit time of $-i+1$ (we have $r_{i-1}^{(\kappa_{i-1})} < r_i^{(1)}$).
Note that for all $l \geq 1$, $\kappa_l \geq 1$.
Between the time 0 and $\tau_1 - 1$, the trajectory $W^\bullet$ can be decomposed into i pieces $V_1, \ldots, V_i$ in the following way:

$$\begin{aligned} V_1(m) &= W^\bullet(m) \text{ for } m \in [\![0, r_1^{(\kappa_1)}]\!] \\ V_j(m) &= W^\bullet(m) \text{ for } m \in [\![r_{j-1}^{(\kappa_{j-1})}, r_j^{(\kappa_j)}]\!]. \end{aligned}$$

For $j \in [\![1, i]\!]$, define V_j' as:

$$\begin{aligned} V_1' &= V_1 \\ V_j'(m) &= V_j\big(m + r_{j-1}^{(\kappa_{j-1})}\big) + j - 1 \text{ for } m \in [\![0, r_j^{(\kappa_j)} - r_{j-1}^{(\kappa_{j-1})}]\!] \text{ and } j \in [\![2, i]\!]. \end{aligned}$$

Using the strong Markov property of the path, the V_j's are independent and have the same distribution. The probability of $W^\bullet$ is the product of the probabilities of the V_j's multiplied by $\tilde{p}_k$ (the probability of the last jump).

The probability of a trajectory V_j' with $\kappa_j = k_j$ is the probability to realize a random walk which starts by a step -1, followed by k_j passages in -1. Thus the restricted trajectory $V_j'(m) + 1\big|_{m \geq 1}$ is identical to a trajectory of a random walk starting from 0 and having its $k_j - 1$ first weak ladder heights equal to 0. Hence,

$$\mathbb{P}(V_j', \kappa_j = k_j) = \tilde{p}_{-1}\, \lambda_0^{k_j - 1}.$$

So

$$\mathbb{P}(E_{i,k}) = \tilde{p}_k \sum_{(k_1, \cdots, k_i) \in \mathbb{N}^{\star i}} \tilde{p}_{-1}^{\,i} \prod_{j=1}^{i} \lambda_0^{k_j - 1} = \tilde{p}_k.$$

This implies Lemma 1 and also Lemma 2. $\square$

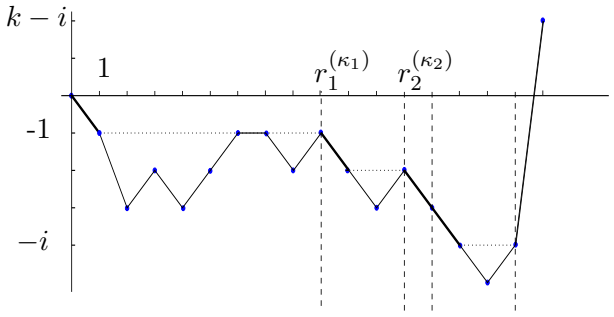

Figure 4 *Decomposition of the paths*

2.6 Proof of Theorems 1 and 2

We recall two lemmas about random walk properties that are straightforward corollaries of Theorems 2.2, 2.4, and 2.6 from Petrov [16].

Lemma 5 *Let $(Z_i)_{i\geq 0}$ be a random walk with increment z_i, i.i.d., centered, with $\mathbb{E}(z_i^2) < +\infty$. Then, for every x:*

$$\mathbb{P}\big(\max_{0\leq k\leq n}\{Z_k\} \geq x\big) \leq 2\,\mathbb{P}\big(Z_n \geq x - \sqrt{2(n-1)\mathbb{E}(z_i^2)}\big).$$

Lemma 6 *Moreover, if there exist two positive constants g_z and T_z such that*

$$\mathbb{E}(\exp(tz_1)) \leq \exp\big(\frac{g_z t^2}{2}\big) \quad \textit{for} \quad |t| \leq T_z,$$

then

$$\begin{aligned}\mathbb{P}(Z_n \geq x) &\leq \exp(-\tfrac{x^2}{2ng_z}) \quad \textit{if } 0 \leq x \leq ng_zT_z\\ \mathbb{P}(Z_n \geq x) &\leq \exp(-\tfrac{T_z x}{2}) \quad \textit{if } x \geq ng_zT_z.\end{aligned}$$

To prove Theorems 1 and 2, we begin with two technical lemmas about "deviations" of the number of ladder epochs.

Lemma 7 *Let β be a positive real number. For any $\gamma > \sqrt{2g\beta}$,*

$$\sup_{0\leq l\leq n} \mathbb{P}(R^\bullet(l) \geq \gamma n^{1/2}\ln n) = o(n^{-\beta}).$$

Proof: For any $0 \leq l \leq n$,

$$\mathbb{P}(R^\bullet(l) \geq \gamma n^{1/2}\ln n) \leq \mathbb{P}(R^\bullet(n) \geq \gamma n^{1/2}\ln n).$$

In order to simplify the notations, throughout this proof the indices $\gamma n^{1/2}\ln n$ have to be read $\lfloor \gamma n^{1/2}\ln n\rfloor$.

$$\begin{aligned}\mathbb{P}(R^\bullet(n) \geq \gamma n^{1/2}\ln n)) &\leq \mathbb{P}(\tau_{\gamma n^{1/2}\ln n} \leq n)\\ &\leq \mathbb{P}\big(\max_{0\leq j\leq \tau_{\gamma n^{1/2}\ln n}} W^\bullet(j) \leq \max_{0\leq j\leq n} W^\bullet(j)\big)\\ &\leq \mathbb{P}(W^\bullet(\tau_{\gamma n^{1/2}\ln n}) \leq \gamma n^{1/2}\sqrt{\ln n}) + \mathbb{P}(\max_{0\leq j\leq n} W^\bullet(j) > \gamma n^{1/2}\sqrt{\ln n}). \qquad (23)\end{aligned}$$

By Lemmas 5 and 6 and for n large enough, we have

$$\begin{aligned}\mathbb{P}(\max_{0\le j\le n} W^\bullet(j) > \gamma n^{1/2}\sqrt{\ln n}) &\le 2\mathbb{P}(W^\bullet(n) \ge \gamma n^{1/2}\sqrt{\ln n} - \sqrt{2\sigma^2(n-1)}) \\ &\le 2\exp\Big(-\frac{(\gamma n^{1/2}\sqrt{\ln n} - \sqrt{2\sigma^2(n-1)})^2}{2ng}\Big).\end{aligned}$$

The condition $\gamma > \sqrt{2g\beta}$ implies that the second term in the right hand side of (23) is $o(n^{-\beta})$.

Now, for the first term in the right hand side of (23), we use the fact that $W^\bullet(\tau_{\gamma n^{1/2}\ln n})$ is the sum of the $\lfloor \gamma n^{1/2}\ln n\rfloor$ first ladder heights. These ladder heights are i.i.d. random variables, distributed as $W^\bullet(\tau(1))$; thanks to Lemmas 1 and 2, one has:

$$\mathbb{P}(W^\bullet(\tau(1)) = k) = \mathbb{P}(w(1) \ge k) \text{ for } k \ge 0.$$

Thus, $W^\bullet(\tau(1))$ has exponential moments and its mean is $\sigma^2/2$. We write

$$\mathbb{P}\big(W^\bullet(\tau_{\gamma n^{1/2}\ln n}) \le \gamma n^{\frac{1}{2}}\sqrt{\ln n}\big) \le \mathbb{P}\left(W^\bullet(\tau_{\gamma n^{\frac{1}{2}}\ln n}) - \frac{\sigma^2}{2}\lfloor \gamma n^{\frac{1}{2}}\ln n\rfloor \le \gamma n^{\frac{1}{2}}\sqrt{\ln n} - \frac{\sigma^2}{2}\lfloor \gamma n^{\frac{1}{2}}\ln n\rfloor\right).$$

Using Lemma 6, one sees that this term is exponentially small. □

Lemma 8 *Let β be a positive real number. For any $\gamma > \sqrt{(\beta+1/2)\sqrt{\beta g}/2}$,*

$$\begin{aligned}\sup_{k>0}\sup_{0\le l\le n}\mathbb{P}\big(|c_k(l) - q_{k-1}R^\bullet(l)| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n\big) &= o(n^{-\beta}) \\ \sup_{k>0}\sup_{1\le j\le k}\sup_{0\le l\le n}\mathbb{P}\big(|c_{k,j}(l) - \tilde{p}_{k-1}R^\bullet(l)| > \gamma\sqrt{\tilde{p}_{k-1}}n^{1/4}\ln n\big) &= o(n^{-\beta}).\end{aligned}$$

Proof: For $r > 0$, set

$$\begin{aligned}d_k(r) &= \#\big\{\tau_j, 0 < j \le r, w(\tau_j) = k-1\big\}, \\ d_{k,i}(r) &= \#\big\{j, 0 < j \le r, w(\tau_j) = k-1, W^\bullet(\tau_j) - W^\bullet(\tau_{j-1}) = k-i\big\}.\end{aligned}$$

Hence, $c_k(l)$ deals with the ladder variables before time l and $d_k(r)$ with the r first ladder variables. By the strong Markov property and Lemma 1 and 2,

$$\begin{aligned}\mathbb{P}(w(\tau_j) = k-1) &= q_{k-1} = k\tilde{p}_{k-1}, \\ \mathbb{P}(w(\tau_j) = k-1, W^\bullet(\tau_j) - W^\bullet(\tau_{j-1}) = k-i) &= \tilde{p}_{k-1},\end{aligned}$$

$d_k(r)$ is binomial $B(r, q_{k-1})$ distributed and $d_{k,i}(r)$ binomial $B(r, \tilde{p}_{k-1})$ distributed.
We go on the proof of the first assertion of Lemma 8, the proof of the second one being similar. We write

$$\begin{aligned}\mathbb{P}(|c_k(l) - q_{k-1}R^\bullet(l)| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n) &\le \mathbb{P}(|c_k(l) - q_{k-1}R^\bullet(l)| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n, R^\bullet(l) \le \gamma_2 n^{1/2}\ln n) \\ &\quad + \mathbb{P}(R^\bullet(l) > \gamma_2 n^{1/2}\ln n).\end{aligned}$$

Choosing a number $\gamma_2 > \sqrt{2\beta g}$, the second term in the right hand side is $o(n^{-\beta})$ by Lemma 7. For the first one, we have

$$\mathbb{P}(|c_k(l) - q_{k-1}R^\bullet(l)| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n, R^\bullet(l) \le \gamma_2 n^{1/2}\ln n)$$

$$\begin{aligned}
&\le \sum_{r=1}^{\gamma_2 n^{1/2}\ln n} \mathbb{P}\big(|c_k(l) - q_{k-1}R^\bullet(l)| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n, R^\bullet(l) = r\big)\\
&\le \sum_{r=1}^{\gamma_2 n^{1/2}\ln n} \mathbb{P}\big(|d_k(r) - q_{k-1}r| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n, R^\bullet(l) = r\big)\\
&\le \sum_{r=1}^{\gamma_2 n^{1/2}\ln n} \mathbb{P}\big(|d_k(r) - q_{k-1}r| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n\big). \qquad (24)
\end{aligned}$$

Hoeffding inequalities [16] p.78 allow to handle each term of the sum (24); if $b(m,p)$ is a binomial $B(m,p)$ r.v.:

$$\mathbb{P}(|b(m,p) - mp| > \sqrt{p}y) \le 2\exp(-2y^2/m).$$

One obtains that each term in the sum (24) is smaller than $2n^{-2\gamma^2/\gamma_2}$. Since the number of terms in the sum is smaller than $\gamma_2 n^{1/2}\ln n$, the sum (24) is $o(n^{-\beta})$ if $\gamma > \sqrt{(1/2+\beta)\gamma_2/2}$. □

Lemma 9 *Let β be a positive real number. For any $\gamma > \sqrt{(\beta+2)\sqrt{(\beta+3/2)g}/2}$,*

$$\sup_{k>0}\sup_{0\le l\le n} \mathbb{P}_n\left(|a_k(l) - q_{k-1}h(l)| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n\right) = o(n^{-\beta}) \qquad (25)$$

$$\sup_{k>0}\sup_{1\le j\le k}\sup_{0\le l\le n} \mathbb{P}_n\left(|a_{k,j}(l) - p_k h(l)| > \gamma\sqrt{p_k}n^{1/4}\ln n\right) = o(n^{-\beta}). \qquad (26)$$

Proof : Our proof relies on the conditioning argument. According to (10), (11), (13), (14),

$$\mathbb{P}_n\left(|a_k(l) - q_{k-1}h(l)| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n\right) = \mathbb{P}\left(|b_k(l) - q_{k-1}RM(l)| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n | W \in \mathcal{E}_W\right).$$

Using (16), (19), (20) and the conditioning argument one writes:

$$\begin{aligned}
\mathbb{P}_n\left(|a_k(l) - q_{k-1}h(l)| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n\right) &= O(n^{3/2})\mathbb{P}\left(|b_k(l) - q_{k-1}RM(l)| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n\right)\\
&= O(n^{3/2})\mathbb{P}\big(|c_k(l) - q_{k-1}R^\bullet(l)| > \gamma\sqrt{q_{k-1}}n^{1/4}\ln n\big).
\end{aligned}$$

Application of Lemma 8 (with β replaced by $(\beta+3/2)$) gives (25). The proof of (26) is similar. □.

Proof of Theorems 1 and 2: We just prove Theorem 1, the proof of Theorem 2 being similar. We write:

$$\mathbb{P}_n\Big(\sup_{u\in\zeta,\,k}\big|a_k(u) - kp_k h(u)\big| > \gamma n^{1/4}\ln n\Big) \le \sum_{u\in\zeta,\,k}\mathbb{P}_n\Big(\big|a_k(u) - kp_k h(u)\big| > \gamma n^{1/4}\ln n\Big).$$

For $k \ge n$, $|a_k(u) - kp_k h(u)\big| = kp_k h(u) \le kp_k n \le k^2 p_k$ which is obviously smaller than $\gamma n^{1/4}\ln n$ for n large enough. So, the sum can be reduced to at most n^2 terms each of them being (uniformally) $o(n^{-\beta-2})$ under the condition $\gamma > \sqrt{(\beta+4)\sqrt{(\beta+7/2)g}/2}$ (we use (25)).

For Theorem 2, there are n^3 terms, so we need $\gamma > \sqrt{(\beta+5)\sqrt{(\beta+9/2)g}/2}$. □

Proof of Proposition 1 We first establish that if $\gamma/m > \sqrt{(\beta+3)\sqrt{(\beta+5/2)g}/2}$, then for any (k,j)

$$\mathbb{P}_n\left(|A_{k,j} - p_k H_n| > \gamma\frac{\sqrt{p_k}}{m}n^{1/4}\ln n\right) = o(n^{-\beta}). \qquad (27)$$

As a matter of fact,

$$\begin{aligned}\mathbb{P}_n\left(|A_{k,j}-p_kH_n|>\gamma\frac{\sqrt{p_k}}{m}n^{1/4}\ln n\right) &= \mathbb{P}_n\left(|\sup_{u\in\zeta}a_{k,j}(u)-p_k\sup_{u'\in\zeta}h(u')|>\gamma\frac{\sqrt{p_k}}{m}n^{1/4}\ln n\right)\\ &\le \mathbb{P}_n\left(\sup_{u\in\zeta}|a_{k,j}(u)-p_kh(u)|>\gamma\frac{\sqrt{p_k}}{m}n^{1/4}\ln n\right)\\ &\le \sum_{u\in\zeta}\mathbb{P}_n\left(|a_{k,j}(u)-p_kh(u)|>\gamma\frac{\sqrt{p_k}}{m}n^{1/4}\ln n\right).\end{aligned}$$

Each of the terms of the sum is smaller than $o(n^{-(\beta+1)})$ under the condition $\gamma/m>\sqrt{(\beta+3)\sqrt{(\beta+5/2)g}/2}$. So (27) is proved. Now, we write

$$\mathbb{P}_n\Big(\big|H_n-\sum_{k,j}A_{k,j}\big|>\gamma n^{1/4}\ln n\Big)\le\sum_{k\ge1}\sum_{j=1}^{k}\mathbb{P}_n\left(|A_{k,j}-p_kH_n|>\gamma\frac{\sqrt{p_k}}{m}n^{1/4}\ln n\right).$$

Each of the (at most n^2) terms are $o(n^{-\beta-2})$ under the condition $\gamma/m>\sqrt{(\beta+5)\sqrt{(\beta+9/2)g}/2}$. Hence Proposition 1 is proved. □

2.7 Sketch of the proof of Proposition 3

The proof of Proposition 3 just relies on the deviations obtained on the random variables $a_{k,j}$. We present below a sketch of the proof.

We begin by a simple remark. If we know the values of $a_{k,j}(u)$ for all k and j, we know how many copies of $X_j^{(k)}$ have been summed to obtain $W_u(h(u))$. Moreover, the $X_j^{(k)}$s involved are independent (since two nodes cannot be ancestors of u and brothers). We express $W_u(h(u))$ and $\Lambda h(u)$ as sums and we introduce a r.v. $B(u)$:

$$W_u(h(u))=\sum_j\sum_{i=1}^{j}\sum_{k=1}^{a_{j,i}(u)}X_i^{(j),k},\ B(u)=\sum_j\sum_{i=1}^{j}a_{j,i}(u)\mathbb{E}(X_i^{(j)}),\ \Lambda h(u)=\sum_j\sum_{i=1}^{j}p_jh(u)\mathbb{E}(X_i^{(j)}),$$

where the $(X_i^{(j),k})_k$ are independent copies of $X_i^{(j)}$.

Since the number of nodes is n, Proposition 3 is proved by checking that for any $\beta>0$ there exists a constant $\gamma_0>0$ such that for any $\gamma>\gamma_0$,

$$\sup_{u\in\zeta}\mathbb{P}_n(|W_u(h(u))-\Lambda h(u)|>\gamma n^{1/4}\ln n)=o(n^{-\beta-1}).\tag{28}$$

Proposition 3 is then a consequence of lemmas 10 and 11 below. □

Lemma 10 *Let β be a positive real number. There exists a constant $\gamma_0>0$ such that for any $\gamma>\gamma_0$,*

$$\sup_{u\in\zeta}\mathbb{P}_n\Big(|W_u(h(u))-B(u)|>\gamma\frac{n^{1/4}\ln n}{2}\Big)=o(n^{-\beta-1}).$$

Proof : Standard computations show that assumption $(8,i)$ ensures the existence of two real numbers $a>0$ and $\mu>0$ such that, for each i and j,

$$\mathbb{E}\Big(\exp(t(X_i^j-\mathbb{E}(X_i^j))\Big)\leq \exp(\mu\frac{t^2}{2}) \text{ if } |t|\leq a. \tag{29}$$

$W_u(h(u))-B(u)$ is in fact a sum involving $h(u)$ independent r.v. $X_i^{(j)}-\mathbb{E}(X_i^{(j)})$ that all satisfy (29). We then split $\mathbb{P}_n(|W_u(h(u))-B(u)|>\frac{\gamma n^{1/4}\ln n}{2})$ into two parts.

$$\begin{aligned}\mathbb{P}_n(|W_u(h(u))-B(u)|>\frac{\gamma n^{1/4}\ln n}{2}) &\leq \mathbb{P}_n(h(u)\geq \gamma_2 n^{1/2}\ln n)\\ &+ \mathbb{P}_n(|W_u(h(u))-B(u)|>\frac{\gamma n^{1/4}\ln n}{2},\ h(u)\leq \gamma_2 n^{1/2}\ln n).\end{aligned}$$

The first part (in the right hand side) is bounded thanks to Lemma 7. Petrov's Lemma 6 applies for the second term, which is smaller than:

$$\sum_{k=1}^{\gamma_2 n^{1/2}\ln n}\mathbb{P}_n(|W_u(h(u))-B(u)|>\frac{\gamma n^{1/4}\ln n}{2}\,|\,h(u)=k) \leq \gamma_2 n^{1/2}\ln(n)\exp(-\gamma^2\ln n/(8\gamma_2)).$$

for n large enough. Hence, one can choose constants γ and γ_2 in order to bound this second part as required. □

Lemma 11 *Let β be a positive real number. There exists a constant $\gamma_0>0$ such that for any $\gamma>\gamma_0$,*

$$\sup_{u\in\zeta}\mathbb{P}_n(|B(u)-\Lambda h(u)|>\frac{\gamma n^{1/4}\ln n}{2})=o(n^{-\beta-1}).$$

Proof :

$$\begin{aligned}\mathbb{P}_n(|B(u)-\Lambda h(u)|>\frac{\gamma n^{1/4}\ln n}{2}) &\leq \mathbb{P}_n(\exists v,\, j,\, i, |a_{j,i}(v)-p_j h(v)|>\sqrt{p_j}n^{1/4}\sqrt{\ln n})\\ &+\mathbb{P}_n(|B(u)-\Lambda h(u)|>\frac{\gamma n^{1/4}\ln n}{2}\cap \forall v,\, j,\, i, |a_{j,i}(v)-p_j h(v)|\leq\sqrt{p_j}n^{1/4}\sqrt{\ln n}).\end{aligned}$$

The first term in the right hand side is bounded thanks to (26); for the second one, note that:

$$\begin{aligned}&\Big\{|B(u)-\Lambda h(u)|>\frac{\gamma n^{1/4}\ln n}{2}\Big\}\cap\Big\{\forall v,\, j,\, i, |a_{j,i}(v)-p_j h(v)|\leq\sqrt{p_j}n^{1/4}\sqrt{\ln n}\Big\}\\ \subset&\Big\{\sum_j\sum_{i=1}^{j}|a_{j,i}(v)-p_j h(v)|.|\mathbb{E}(X_i^{(j)})|\geq\frac{\gamma n^{1/4}\ln n}{2}\cap\forall v,\, j,\, i, |a_{j,i}(v)-p_j h(v)|\leq\sqrt{p_j}n^{1/4}\sqrt{\ln n}\Big\}\\ \subset&\Big\{\sum_j\sum_{i=1}^{j}\sqrt{p_j}n^{1/4}\sqrt{\ln n}|\mathbb{E}(X_i^{(j)})|\geq\frac{\gamma n^{1/4}\ln n}{2}\}.\end{aligned}$$

This set is empty for n large enough according to $(8,ii)$. □

Acknowledgment : We would like to thank a referee for his very helpful comments and suggestions.

References

[1] D. Aldous, (1991). *The continuum random tree. II: An overview.*, Stochastic analysis, Proc. Symp., Durham/UK 1990, Lond. Math. Soc. Lect. Note Ser. 167, 23-70.

[2] J. Bennies, G. Kersting, (2000). *A random walk approach to Galton-Watson trees.*, J. Theor. Probab. 13, No.3, 777-803.

[3] W. Feller, *An Introduction to Probability Theory and its Applications. Vol II. 2nd ed.*, Wiley Series in Prob. and Math. Stats. New York etc.

[4] P. Flajolet, A. Odlyzko, (1982). *The average height of binary trees and other simple trees*, J. Comput. System Sci. **25**, no. 2, 171–213.

[5] T.E. Harris (1952). *First passage and recurrence distributions*, Trans. Amer. Math. Soc. 73, 471-486.

[6] D.G. Kendall, (1951). *Some problems in the theory of queues*, J. Royal Stat. Soc., B 13, 151-185.

[7] D.P. Kennedy, (1976). *The distribution of the maximum Brownian excursion*, J. Appl. Probab. **13**, 371-376.

[8] V.F. Kolchin, (1986). *Random Mappings*, Translation Series in Mathematics and Engineering.

[9] J.F. Le Gall, Y. Le Jan, (1998). *Branching processes in Lévy processes: The exploration process.* Ann. Probab. 26, No.1, 213-252.

[10] J.F. Le Gall, (2000). *Lévy Processes and Branching Processes*, Course Notes.

[11] T. Duquesne, J.F. Le Gall, (2001). *Random trees, Lévy Processes and Spatial Branching Processes*, monograph, Preliminary version.

[12] R. Lyons, Y. Peres, (2002). *Probability on Trees and Network*, http://www.stat.berkeley.edu/∼peres/ chap.10

[13] J.F. Marckert, A. Mokkadem, (2003). *The depth first processes of Galton-Watson trees converge to the same Brownian excursion*, Ann. Proba., Vol. 31, No. 3.

[14] A. Meir, J.W. Moon, (1978). *On the altitude of nodes in random trees*, Canadian Journal of Math 30, 997-1015.

[15] R. Otter, (1949). *The multiplicative process.*, Ann. Math. Statist., Baltimore Md. 20, 206-224.

[16] V.V. Petrov,(1995). *Limit Theorems of Probability Theory*, Oxford.

[17] S.C. Port, (1994). *Theoretical Probability for Applications,* Wiley & Sons.