

# Refined Cauchy and Littlewood identities, plane partitions and partition functions in the six-vertex model

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## Outline

- 1 Review: Cauchy and Littlewood identities, connection with plane partitions
- 2 More advanced examples:  $t$ -generalizations, and symplectic characters
- 3 Refined Cauchy and Littlewood identities, and refined plane partitions
- 4 Partition functions in the six-vertex model, connection with ASM symmetry classes

## Schur polynomials and SSYT

- The Schur polynomials  $s_\lambda(x_1, \dots, x_n)$  are the characters of irreducible representations of  $GL(n)$ . They are given by the Weyl formula:

$$s_\lambda(x_1, \dots, x_n) = \frac{\det \left( x_i^{\lambda_j - j + n} \right)_{1 \leq i, j \leq n}}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}$$

- A *semi-standard Young tableau* of shape  $\lambda$  is an assignment of one symbol  $\{1, \dots, n\}$  to each box of the Young diagram  $\lambda$ , such that
  - The symbols have the ordering  $1 < \dots < n$ .
  - The entries in  $\lambda$  increase weakly along each row and strictly down each column.
- The Schur polynomial  $s_\lambda(x_1, \dots, x_n)$  is also given by a weighted sum over semi-standard Young tableaux  $T$  of shape  $\lambda$ :

$$s_\lambda(x_1, \dots, x_n) = \sum_T \prod_{k=1}^n x_k^{\#(k)}$$

## SSYT and sequences of interlacing partitions

- Two partitions  $\lambda$  and  $\mu$  *interlace*, written  $\lambda \succ \mu$ , if

$$\lambda_i \geq \mu_i \geq \lambda_{i+1}$$

across all parts of the partitions. It is the same as saying  $\lambda/\mu$  is a *horizontal strip*.

- One can interpret a SSYT as a sequence of interlacing partitions:

$$T = \{\emptyset \equiv \lambda^{(0)} \prec \lambda^{(1)} \prec \dots \prec \lambda^{(n)} \equiv \lambda\}$$

- The correspondence works by “peeling away” partition  $\lambda^{(k)}$  from  $T$ , for all  $k$ :



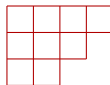
$T =$



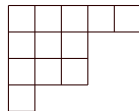
$\lambda^{(1)} \prec$



$\lambda^{(2)} \prec$



$\lambda^{(3)} \prec$



$\lambda^{(4)}$

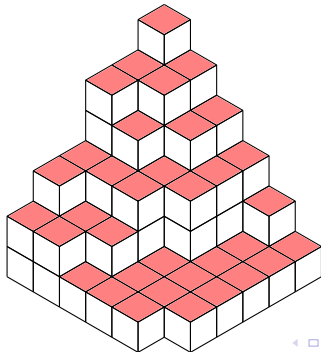
## Plane partitions

- A plane partition is a set of non-negative integers  $\pi(i, j)$  such that for all  $i, j \geq 1$

$$\pi(i, j) \geq \pi(i+1, j) \quad \pi(i, j) \geq \pi(i, j+1)$$

- Plane partitions can be viewed as an *increasing then decreasing* sequence of interlacing partitions. They are equivalent to *conjoined SSYT*.
- We define the set

$$\pi_{m,n} = \{\emptyset \equiv \lambda^{(0)} \prec \lambda^{(1)} \prec \dots \prec \lambda^{(m)} \equiv \mu^{(n)} \succ \dots \succ \mu^{(1)} \succ \mu^{(0)} \equiv \emptyset\}$$



## Cauchy identity and plane partitions

- The Cauchy identity for Schur polynomials,

$$\sum_{\lambda} s_{\lambda}(x_1, \dots, x_m) s_{\lambda}(y_1, \dots, y_n) = \prod_{i=1}^m \prod_{j=1}^n \frac{1}{1 - x_i y_j}$$

can thus be viewed as a generating series of plane partitions:

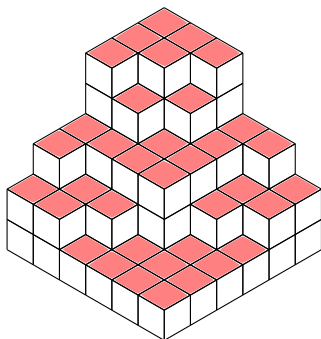
$$\sum_{\pi \in \pi_{m,n}} \prod_{i=1}^m x_i^{|\lambda^{(i)}| - |\lambda^{(i-1)}|} \prod_{j=1}^n y_j^{|\mu^{(j)}| - |\mu^{(j-1)}|} = \prod_{i=1}^m \prod_{j=1}^n \frac{1}{1 - x_i y_j}$$

- Taking the  $q$ -specialization  $x_i = q^{m-i+1/2}$  and  $y_j = q^{n-j+1/2}$ , we recover volume-weighted plane partitions:

$$\sum_{\pi \in \pi_{m,n}} q^{|\pi|} = \prod_{i=1}^m \prod_{j=1}^n \frac{1}{1 - q^{m+n-i-j+1}} = \prod_{i=1}^m \prod_{j=1}^n \frac{1}{1 - q^{i+j-1}}$$

## Symmetric plane partitions

- *Symmetric* plane partitions satisfy the condition that  $\pi(i, j) = \pi(j, i)$  for all  $i, j \geq 1$ .
- A symmetric plane partition is determined by an increasing sequence of interlacing partitions. (The decreasing part is obtained from the symmetry.)
- They are in one-to-one correspondence with SSYT.



## Littlewood identities and symmetric plane partitions

- The three (simplest) Littlewood identities for Schur polynomials

$$\sum_{\lambda} s_{\lambda}(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j} \prod_{i=1}^n \frac{1}{1 - x_i}$$

$$\sum_{\lambda \text{ even}} s_{\lambda}(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j} \prod_{i=1}^n \frac{1}{1 - x_i^2}$$

$$\sum_{\lambda' \text{ even}} s_{\lambda}(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j}$$

can each be viewed as generating series for symmetric plane partitions, with a (possible) constraint on the partition forming the main diagonal.



## Hall–Littlewood polynomials

- Hall–Littlewood polynomials are  $t$ -generalizations of Schur polynomials. They can be defined as a sum over the symmetric group:

$$P_\lambda(x_1, \dots, x_n; t) = \frac{1}{v_\lambda(t)} \sum_{\sigma \in S_n} \sigma \left( \prod_{i=1}^n x_i^{\lambda_i} \prod_{1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j} \right)$$

- Alternatively, the Hall–Littlewood polynomial  $P_\lambda(x_1, \dots, x_n; t)$  is given by a weighted sum over semi-standard Young tableaux  $T$  of shape  $\lambda$ :

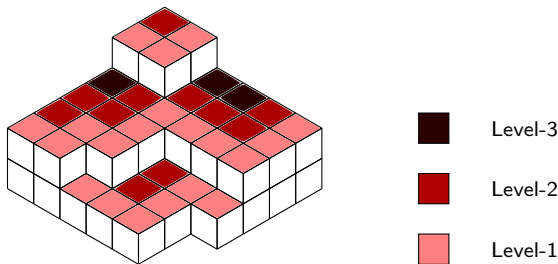
$$P_\lambda(x_1, \dots, x_n; t) = \sum_T \prod_{k=1}^n \left( x_k^{\#(k)} \psi_{\lambda^{(k)}/\lambda^{(k-1)}}(t) \right)$$

where the function  $\psi_{\lambda/\mu}(t)$  is given by

$$\psi_{\lambda/\mu}(t) = \prod_{\substack{i \geq 1 \\ m_i(\mu) = m_i(\lambda) + 1}} (1 - t^{m_i(\mu)})$$

## Path-weighted plane partitions

- As Vuletić discovered, the effect of the  $t$ -weighting in tableaux has a nice combinatorial interpretation on plane partitions.
- The refinement is that all *paths* at level  $k$  receive a weight of  $1 - t^k$ .
- Example of a plane partition with weight  $(1 - t)^3(1 - t^2)^4(1 - t^3)^2$  shown below:



## Hall–Littlewood Cauchy identity and path-weighted plane partitions

- The Cauchy identity for Hall–Littlewood polynomials,

$$\sum_{\lambda} \prod_{i=1}^{\infty} \prod_{j=1}^{m_i(\lambda)} (1-t^j) P_{\lambda}(x_1, \dots, x_m; t) P_{\lambda}(y_1, \dots, y_n; t) = \prod_{i=1}^m \prod_{j=1}^n \frac{1-tx_i y_j}{1-x_i y_j}$$

is thus a generating series of (path-weighted) plane partitions:

$$\sum_{\pi \in \pi_{m,n}} \prod_{i \geq 1} (1-t^i)^{P_i(\pi)} \prod_{i=1}^m x_i^{|\lambda^{(i)}| - |\lambda^{(i-1)}|} \prod_{j=1}^n y_j^{|\mu^{(j)}| - |\mu^{(j-1)}|} = \prod_{i=1}^m \prod_{j=1}^n \frac{1-tx_i y_j}{1-x_i y_j}$$

- Taking the same  $q$ -specialization as earlier, we obtain

$$\sum_{\pi \in \pi_{m,n}} \prod_{i \geq 1} (1-t^i)^{P_i(\pi)} q^{|\pi|} = \prod_{i=1}^m \prod_{j=1}^n \frac{1-tq^{i+j-1}}{1-q^{i+j-1}}$$

## Littlewood identities for Hall–Littlewood polynomials

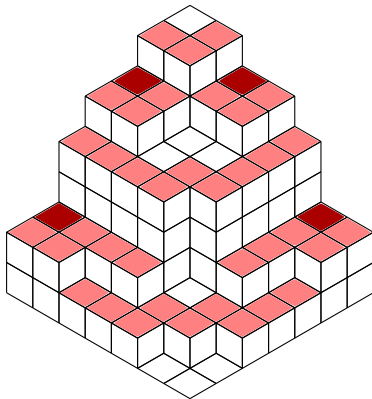
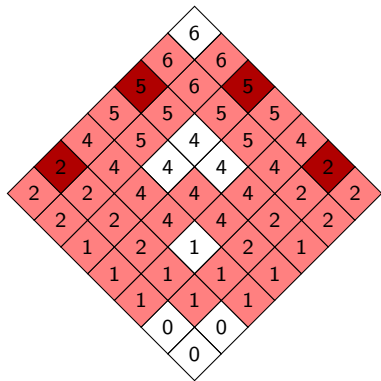
- The  $t$ -analogues of the previously stated Littlewood identities are

$$\sum_{\lambda} P_{\lambda}(x_1, \dots, x_n; t) = \prod_{1 \leq i < j \leq n} \frac{1 - tx_i x_j}{1 - x_i x_j} \prod_{i=1}^n \frac{1}{1 - x_i}$$

$$\sum_{\lambda \text{ even}} P_{\lambda}(x_1, \dots, x_n; t) = \prod_{1 \leq i < j \leq n} \frac{1 - tx_i x_j}{1 - x_i x_j} \prod_{i=1}^n \frac{1}{1 - x_i^2}$$

$$\sum_{\lambda' \text{ even}} \prod_{i=1}^{\infty} \prod_{j \text{ even}}^{m_i(\lambda)} (1 - t^{j-1}) P_{\lambda}(x_1, \dots, x_n; t) = \prod_{1 \leq i < j \leq n} \frac{1 - tx_i x_j}{1 - x_i x_j}$$

- These can be regarded as generating series for path-weighted symmetric plane partitions.
- Warning!** Paths which intersect the main diagonal might not have a  $t$ -weight!

$t$ -weighting of symmetric plane partitions

$$\sum_{\lambda' \text{ even}} \prod_{i=1}^{\infty} \prod_{j \text{ even}}^{m_i(\lambda)} (1 - t^{j-1}) P_{\lambda}(x_1, \dots, x_n; t)$$

## Symplectic Schur polynomials

- The symplectic Schur polynomials  $sp_\lambda(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)$  are the characters of irreducible representations of  $Sp(2n)$ . In this case, the Weyl formula gives

$$sp_\lambda(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n) = \frac{\det \left( x_i^{\lambda_j - j + n + 1} - \bar{x}_i^{\lambda_j - j + n + 1} \right)_{1 \leq i, j \leq n}}{\prod_{i=1}^n (x_i - \bar{x}_i) \prod_{1 \leq i < j \leq n} (x_i - x_j)(1 - \bar{x}_i \bar{x}_j)}$$

- A *symplectic tableau* of shape  $\lambda$  is an assignment of one symbol  $\{1, \bar{1}, \dots, n, \bar{n}\}$  to each box of the Young diagram  $\lambda$ , such that

- 1 The symbols have the ordering  $1 < \bar{1} < \dots < n < \bar{n}$ .
- 2 The entries in  $\lambda$  increase weakly along each row and strictly down each column.
- 3 All entries in row  $k$  of  $\lambda$  are at least  $k$ .

- The symplectic Schur polynomial  $sp_\lambda(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n)$  is given by a weighted sum over symplectic tableaux  $\bar{T}$  of shape  $\lambda$ :

$$sp_\lambda(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n) = \sum_{\bar{T}} \prod_{k=1}^n x_k^{\#(k) - \#(\bar{k})}$$

## Symplectic tableaux as restricted interlacing sequences

- We can interpret a symplectic tableau  $\bar{T}$  as a sequence of interlacing partitions, subject to an extra constraint:

$$\bar{T} = \{\emptyset \equiv \bar{\lambda}^{(0)} \prec \lambda^{(1)} \prec \bar{\lambda}^{(1)} \prec \dots \prec \lambda^{(n)} \prec \bar{\lambda}^{(n)} \equiv \lambda \mid \ell(\bar{\lambda}^{(i)}) \leq i\}.$$

1	1	$\bar{1}$	$\bar{2}$	3
2	$\bar{2}$	3		
$\bar{3}$	$\bar{3}$	4		
4				

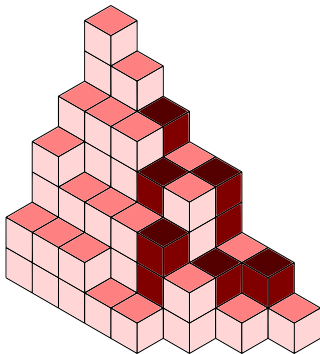
- The symplectic Schur polynomial can now be expressed as

$$\begin{aligned} sp_{\lambda}(x_1, \bar{x}_1, \dots, x_n, \bar{x}_n) &= \sum_{\bar{T}} \prod_{i=1}^n x_i^{|\lambda^{(i)}| - |\bar{\lambda}^{(i-1)}|} \prod_{j=1}^n x_j^{|\lambda^{(j)}| - |\bar{\lambda}^{(j)}|} \\ &= \sum_{\bar{T}} \prod_{i=1}^n x_i^{2|\lambda^{(i)}| - |\bar{\lambda}^{(i)}| - |\bar{\lambda}^{(i-1)}|} \end{aligned}$$

## A restricted class of plane partitions

- We consider the class of plane partitions formed by a conjoined SSYT and symplectic tableau.
- We define the set

$$\overline{\pi}_{m,2n} = \{\emptyset \equiv \lambda^{(0)} \prec \lambda^{(1)} \prec \dots \prec \lambda^{(m)} \equiv \bar{\mu}^{(n)} \succ \mu^{(n)} \succ \dots \succ \bar{\mu}^{(1)} \succ \mu^{(1)} \succ \bar{\mu}^{(0)} \equiv \emptyset\}$$





## Symplectic Cauchy identity and associated plane partitions

- The Cauchy identity for symplectic Schur polynomials,

$$\sum_{\lambda} s_{\lambda}(x_1, \dots, x_m) sp_{\lambda}(y_1, \bar{y}_1, \dots, y_n, \bar{y}_n) = \frac{\prod_{1 \leq i < j \leq m} (1 - x_i x_j)}{\prod_{i=1}^m \prod_{j=1}^n (1 - x_i y_j)(1 - x_i \bar{y}_j)}$$

can now be regarded as a generating series for the plane partitions defined:

$$\sum_{\pi \in \overline{\pi}_{m, 2n}} \prod_{i=1}^m x_i^{|\lambda^{(i)}| - |\lambda^{(i-1)}|} \prod_{j=1}^n y_j^{2|\mu^{(j)}| - |\bar{\mu}^{(j)}| - |\bar{\mu}^{(j-1)}|} = \frac{\prod_{1 \leq i < j \leq m} (1 - x_i x_j)}{\prod_{i=1}^m \prod_{j=1}^n (1 - x_i y_j)(1 - x_i \bar{y}_j)}$$

- One can experiment with the parameters to find a “good”  $q$ -specialization. We choose  $x_i = q^{m-i+3/2}$  and  $y_j = q^{1/2}$ , giving

$$\sum_{\pi \in \overline{\pi}_{m, 2n}} q^{|\pi_{\leq}^o| - |\pi_{>}^e|} = \frac{\prod_{1 \leq i < j \leq m} (1 - q^{i+j+1})}{\prod_{i=1}^m (1 - q^i)^n (1 - q^{i+1})^n}$$

## Example 1(a): Refined Cauchy identity for Schur polynomials

### Theorem

$$\sum_{\lambda} \prod_{i=1}^n (1 - t^{\lambda_i - i + n + 1}) s_{\lambda}(x_1, \dots, x_n) s_{\lambda}(y_1, \dots, y_n) \\ = \frac{1}{\Delta(x)_n \Delta(y)_n} \det \left\{ \frac{(1-t)}{(1 - tx_i y_j)(1 - x_i y_j)} \right\}_{1 \leq i, j \leq n}$$

### Proof.

Expand the entries of the determinant as formal power series, and use Cauchy–Binet:

$$\det \left\{ \frac{(1-t)}{(1 - tx_i y_j)(1 - x_i y_j)} \right\}_{1 \leq i, j \leq n} = \det \left\{ \sum_{k=0}^{\infty} (1 - t^{k+1}) x_i^k y_j^k \right\}_{1 \leq i, j \leq n} \\ = \sum_{k_1 > \dots > k_n \geq 0} \prod_{i=1}^n (1 - t^{k_i + 1}) \det \left\{ x_i^{k_j} \right\}_{1 \leq i, j \leq n} \det \left\{ y_j^{k_i} \right\}_{1 \leq i, j \leq n}$$

The proof follows after the change of indices  $k_i = \lambda_i - i + n$ .



## Example 1(b): Refined Cauchy identity for Hall–Littlewood polynomials

### Theorem (Kirillov–Noumi, Warnaar)

$$\sum_{\lambda} \prod_{i=0}^{\infty} m_i(\lambda) \prod_{j=1}^{\infty} (1-t^j) P_{\lambda}(x_1, \dots, x_n; t) P_{\lambda}(y_1, \dots, y_n; t)$$

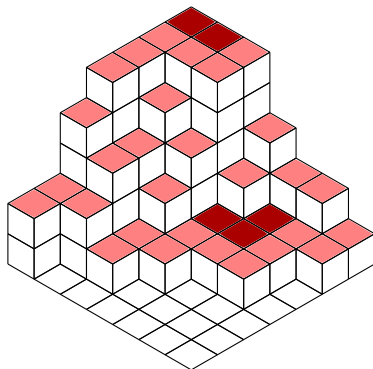
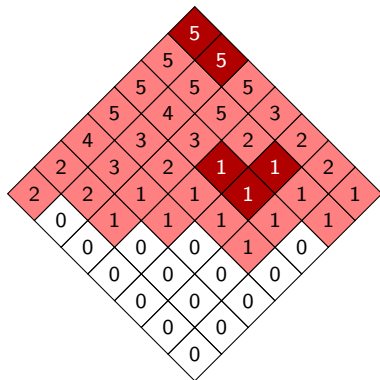
$$= \frac{\prod_{i,j=1}^n (1-tx_i y_j)}{\Delta(x)_n \Delta(y)_n} \det \left\{ \frac{(1-t)}{(1-tx_i y_j)(1-x_i y_j)} \right\}_{1 \leq i, j \leq n}$$

### Proof.

The key idea is to act on the Hall–Littlewood Cauchy identity with a generating series of Macdonald's difference operators. The left hand side follows immediately. The right hand side follows after acting on the Cauchy kernel, and performing some manipulation.  $\square$

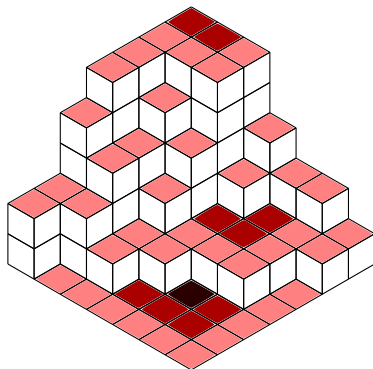
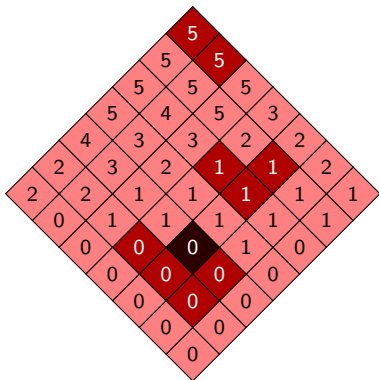
## Example 1(b): Refined Cauchy identity for Hall–Littlewood polynomials

- Question: What does the refinement do at the level of plane partitions?



## Example 1(b): Refined Cauchy identity for Hall–Littlewood polynomials

- Answer: The zero-height entries are treated like the rest!



## Example 2(a): Refined Littlewood identity for Schur polynomials

### Theorem (DB,MW)

$$\sum_{\lambda' \text{ even}} \prod_{i \text{ even}} (1 - t^{\lambda_i - i + 2n + 1}) s_{\lambda}(x_1, \dots, x_{2n})$$

$$= \prod_{1 \leq i < j \leq 2n} \frac{1}{(x_i - x_j)} \operatorname{Pf} \left\{ \frac{(1-t)(x_i - x_j)}{(1 - tx_i x_j)(1 - x_i x_j)} \right\}_{1 \leq i < j \leq 2n}$$

### Proof.

Expand the entries of the Pfaffian and use a Pfaffian analogue of Cauchy–Binet:

$$\operatorname{Pf} \{ \dots \}_{1 \leq i < j \leq 2n} = \operatorname{Pf} \left\{ \sum_{0 \leq k < l} \delta_{l, k+1} (1 - t^{k+1}) (x_i^l x_j^k - x_i^k x_j^l) \right\}_{1 \leq i < j \leq 2n}$$

$$= \sum_{1 \leq s_1 < \dots < s_{2n}} (-1)^n \operatorname{Pf} \left\{ \delta_{s_i + 1, s_j} (1 - t^{s_i}) \right\}_{1 \leq i < j \leq 2n} \det \left\{ x_i^{s_j - 1} \right\}_{1 \leq i, j \leq 2n}$$

The Pfaffian in the sum factorizes, to produce the correct (blue) factor and the restriction on the summation. □

## Example 2(b): Refined Littlewood identity for Hall–Littlewood polynomials

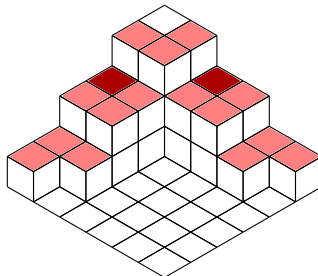
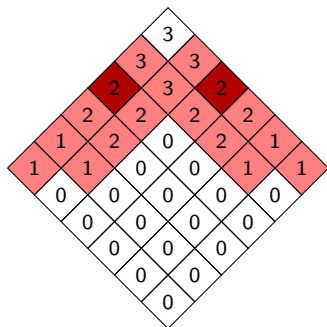
Conjecture (DB,MW)

$$\sum_{\lambda' \text{ even}} \prod_{i=0}^{\infty} \prod_{j \text{ even}}^{m_i(\lambda)} (1 - t^{j-1}) P_{\lambda}(x_1, \dots, x_{2n}; t)$$

$$= \prod_{1 \leq i < j \leq 2n} \frac{(1 - tx_i x_j)}{(x_i - x_j)} \text{Pf} \left\{ \frac{(1-t)(x_i - x_j)}{(1 - tx_i x_j)(1 - x_i x_j)} \right\}_{1 \leq i < j \leq 2n}$$

## Example 2(b): Refined Littlewood identity for Hall–Littlewood polynomials

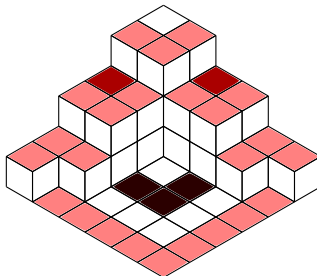
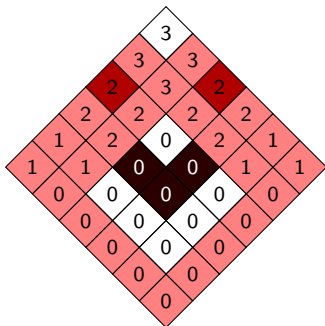
- At the level of plane partitions, this is (again) a very simple refinement!





## Example 2(b): Refined Littlewood identity for Hall–Littlewood polynomials

- At the level of plane partitions, this is (again) a very simple refinement!



## Example 3(a): Refined Cauchy identity for symplectic Schur polynomials

Theorem (DB,MW)

$$\sum_{\lambda} \prod_{i=1}^n (1 - t^{\lambda_i - i + n + 1}) s_{\lambda}(x_1, \dots, x_n) sp_{\lambda}(y_1, \bar{y}_1, \dots, y_n, \bar{y}_n) = \frac{\prod_{i=1}^n (1 - tx_i^2)}{\Delta(x)_n \Delta(y)_n \prod_{i < j} (1 - \bar{y}_i \bar{y}_j)} \det \left\{ \frac{(1-t)}{(1 - tx_i y_j)(1 - tx_i \bar{y}_j)(1 - x_i y_j)(1 - x_i \bar{y}_j)} \right\}_{1 \leq i, j \leq n}$$

Proof.

Similar to the ordinary Schur case:

$$\prod_{i=1}^n (1 - tx_i^2)(y_i - \bar{y}_i) \det \{ \dots \}_{1 \leq i, j \leq n} = \det \left\{ \sum_{k=0}^{\infty} (1 - t^{k+1}) x_i^k (y_j^{k+1} - \bar{y}_j^{k+1}) \right\}_{1 \leq i, j \leq n}$$

$$= \sum_{k_1 > \dots > k_n \geq 0} \prod_{i=1}^n (1 - t^{k_i+1}) \det \{ x_i^{k_j} \}_{1 \leq i, j \leq n} \det \{ y_j^{k_i+1} - \bar{y}_j^{k_i+1} \}_{1 \leq i, j \leq n}$$

The proof follows after the change of indices  $k_i = \lambda_i - i + n$ .



## Example 3(b): Identity involving $BC_n$ Hall–Littlewood functions

- Macdonald extended his theory of symmetric functions to other root systems. We will use the type  $BC_n$  Hall–Littlewood functions:

$$K_\lambda(y_1, \bar{y}_1, \dots, y_n, \bar{y}_n; t) = \frac{1}{v_\lambda(t)} \sum_{\omega} \omega \left( \prod_{i=1}^n \frac{y_i^{\lambda_i}}{(1 - \bar{y}_i^2)} \prod_{1 \leq i < j \leq n} \frac{(y_i - ty_j)(1 - t\bar{y}_i\bar{y}_j)}{(y_i - y_j)(1 - \bar{y}_i\bar{y}_j)} \right)$$

where the sum is taken over the hyperoctahedral group,  $\omega \in S_n \times \mathbb{Z}_2^n$ .

### Conjecture (DB,MW)

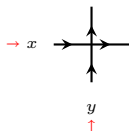
$$\sum_{\lambda} \prod_{i=0}^{\infty} \prod_{j=1}^{m_i(\lambda)} (1 - t^j) P_{\lambda}(x_1, \dots, x_n; t) K_{\lambda}(y_1, \bar{y}_1, \dots, y_n, \bar{y}_n; t) =$$

$$\frac{\prod_{i,j=1}^n (1 - tx_i y_j)(1 - tx_i \bar{y}_j)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)(y_i - y_j)(1 - tx_i x_j)(1 - \bar{y}_i \bar{y}_j)}$$

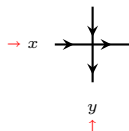
$$\times \det \left\{ \frac{(1 - t)}{(1 - tx_i y_j)(1 - tx_i \bar{y}_j)(1 - x_i y_j)(1 - x_i \bar{y}_j)} \right\}_{1 \leq i, j \leq n}$$

## The six-vertex model

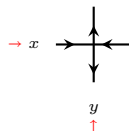
- The vertices of the six-vertex model are



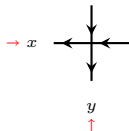
$a_+(x, y)$



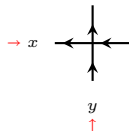
$b_+(x, y)$



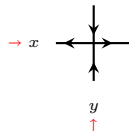
$c_+(x, y)$



$a_-(x, y)$



$b_-(x, y)$



$c_-(x, y)$

## The six-vertex model

- The Boltzmann weights are given by

$$a_+(x, y) = \frac{1 - tx/y}{1 - x/y}$$

$$a_-(x, y) = \frac{1 - tx/y}{1 - x/y}$$

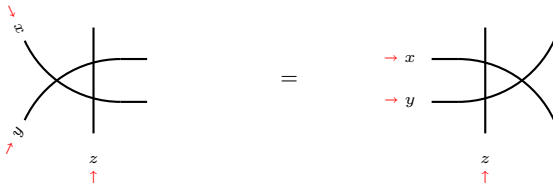
$$b_+(x, y) = 1$$

$$b_-(x, y) = t$$

$$c_+(x, y) = \frac{(1-t)}{1 - x/y}$$

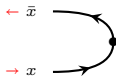
$$c_-(x, y) = \frac{(1-t)x/y}{1 - x/y}$$

- The parameter  $t$  from Hall–Littlewood is now the crossing parameter of the model.
- The Boltzmann weights obey the *Yang–Baxter* equations (the  $\mathcal{U}_q(\widehat{sl}_2)$  solution):

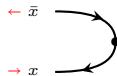


## Boundary vertices

- In addition to the bulk vertices, we introduce U-turn vertices

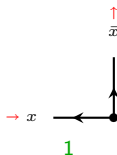
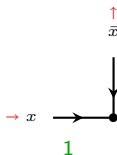


$$1/(1-x^2)$$



$$1/(1-x^2)$$

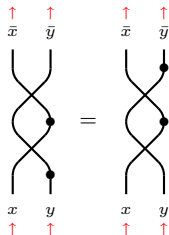
which depend on a single spectral parameter and are spin-conserving, and corner vertices



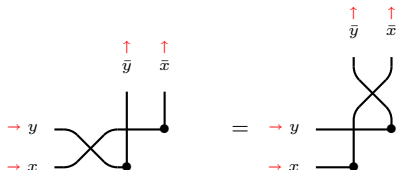
which *do not* depend on a spectral parameter and behave like sources/sinks.

## Boundary vertices

- The U-turn vertices satisfy Sklyanin's reflection equation:

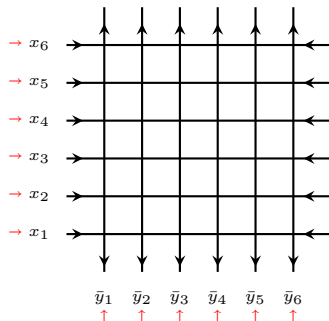


- The corner vertices satisfy their own variant of the reflection equation:



## Domain wall boundary conditions

- The six-vertex model on a lattice with *domain wall* boundary conditions:



- This partition function is of fundamental importance in periodic quantum spin chains based on  $\mathcal{Y}(sl_2)$  and  $\mathcal{U}_q(\widehat{sl_2})$ .



## Domain wall boundary conditions

- Configurations on this lattice are in one-to-one correspondence with alternating sign matrices:

$$\begin{pmatrix} 0 & 0 & + & 0 & 0 & 0 \\ 0 & + & - & 0 & + & 0 \\ + & - & 0 & + & 0 & 0 \\ 0 & + & 0 & - & 0 & + \\ 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 \end{pmatrix}$$

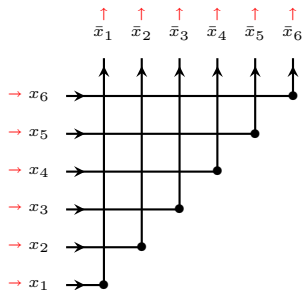
- The domain wall partition function was evaluated in determinant form by Izergin:

$$Z_{ASM}(x_1, \dots, x_n; y_1, \dots, y_n; t) = \frac{\prod_{i,j=1}^n (1 - tx_i y_j)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)(y_i - y_j)} \det \left[ \frac{(1-t)}{(1-tx_i y_j)(1-x_i y_j)} \right]_{1 \leq i, j \leq n}$$

- This is precisely what appears on the right hand side of Example 1(b).

## Off-diagonally symmetric boundary conditions

- *Off-diagonally symmetric* boundary conditions were introduced by Kuperberg. They involve the corner vertices:



## Off-diagonally symmetric boundary conditions

- Configurations on this lattice are in one-to-one correspondence with off-diagonally symmetric ASMs (OSASMs):

$$\begin{pmatrix} 0 & 0 & + & 0 & 0 & 0 \\ + & 0 & - & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ 0 & + & 0 & 0 & - & + \\ 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \end{pmatrix}$$

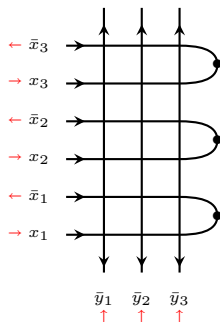
- Kuperberg evaluated this partition function as a Pfaffian:

$$Z_{\text{OSASM}}(x_1, \dots, x_{2n}; t) = \prod_{1 \leq i < j \leq 2n} \frac{(1 - tx_i x_j)}{(x_i - x_j)} \text{Pf} \left[ \frac{(1-t)(x_i - x_j)}{(1 - tx_i x_j)(1 - x_i x_j)} \right]_{1 \leq i < j \leq 2n}$$

- This is precisely what appears on the right hand side of Example 2(b).

## Reflecting domain wall boundary conditions

- The six-vertex model has also been studied on a lattice with *reflecting* boundary conditions:



- This quantity is important in quantum spin-chains with open boundary conditions.

## Reflecting domain wall boundary conditions

- Configurations on this lattice are in one-to-one correspondence with U-turn ASMs (UASMs):

$$\begin{pmatrix} 0 & + & 0 \\ + & - & 0 \\ 0 & 0 & + \\ 0 & + & - \\ 0 & 0 & 0 \\ 0 & 0 & + \end{pmatrix}$$

- The partition function was evaluated in determinant form by Tsuchiya:

$$\begin{aligned} Z_{\text{UASM}}(x_1, \dots, x_n; y_1, \bar{y}_1, \dots, y_n, \bar{y}_n; t) = & \\ & \frac{\prod_{i,j=1}^n (1 - tx_i y_j)(1 - tx_i \bar{y}_j)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)(y_i - y_j)(1 - tx_i x_j)(1 - \bar{y}_i \bar{y}_j)} \\ & \times \det \left[ \frac{(1-t)}{(1 - tx_i y_j)(1 - tx_i \bar{y}_j)(1 - x_i y_j)(1 - x_i \bar{y}_j)} \right]_{1 \leq i, j \leq n} \end{aligned}$$

- This matches what appears on the right hand side of Example 3(b).