Tree-width of hypergraphs and surface duality

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Abstract. In Graph Minor III, Robertson and Seymour conjecture that the difference between the tree-width of a planar graph and that of its dual is at most one. We prove that given a hypergraph $H$ on a surface of Euler genus $k$, the tree-width of $H^*$ is at most the maximum of $\text{tw}(H) + 1 + k$ and the maximum size of a hyperedge of $H^*$.

1 Preliminaries

A surface is a connected compact 2-manifold without boundaries. A surface $\Sigma$ can be obtained, up to homeomorphism, by adding $k(\Sigma)$ “crosscaps” to the sphere. $k(\Sigma)$ is the Euler genus or just genus of the surface.

Let $\Sigma$ be a surface. A graph $G = (V,E)$ on $\Sigma$ is a drawing of a graph in $\Sigma$, i.e. each vertex $v$ is an element of $\Sigma$, each edge $e$ is an open curve between two vertices, and edges are pairwise disjoint. We only consider graphs up to homomorphism. A face of $G$ is a connected component of $\Sigma \setminus G$. We denote by $V(G)$, $E(G)$ and $F(G)$ the vertex, edge and face sets of $G$. We only consider 2-cell graphs, i.e. graph whose faces are homeomorphic to open discs. The Euler formula links the number of vertices, edges and faces of a graph $G$ to the genus of the surface $|V(G)| - |E(G)| + |F(G)| = 2 - k(G)$.

The set $A(G) = V(G) \cup E(G) \cup F(G)$ of atoms of $G$ is a partition of $\Sigma$. Two Atom $x$ and $y$ of $G$ are incident if $x \cap \bar{y}$ or $y \cap \bar{x}$ is non empty, $\bar{z}$ being the closure of $z$. A cutting edge in a graph $G$ on $\Sigma$ is an edge $e$ separates $G$, i.e. $G$ intersects at least two connected components of $\Sigma \setminus \bar{e}$. As an example, if a planar graph $G$ has a cut-vertex $u$, any loop on $u$ that goes “around” a connected component of $G \setminus \{u\}$ is a cutting edge.

Let $G = (V \cup V_E, E)$ be a bipartite graph on $\Sigma$. The graph $G$ can be seen as the incidence graph of a hypergraph. For each $v_e \in V_E$, merge $v_e$ and its incident edges into a hyperedge $e$, and call $v_e$ its center. Let $E$ be the set of all hyperedges. A hypergraph on $\Sigma$ is any such pair $H = (V, E)$. For brevity, we also say edges for hyperedges. We extend the notions of cutting edges, 2-cell graphs, atoms and incidence to hypergraphs. Moreover, since they naturally correspond to abstract

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graphs and hypergraphs, graph and hypergraph on surface inherit terminology from them. For example, we denote \(|e|\) the number of vertices incident to a hyperedge \(e\), and we denote \(\alpha(H)\) the maximum size of an edge of \(H\). Note that a graph on \(\Sigma\) is also a hypergraph on \(\Sigma\).

The dual of a hypergraph \(H = (V, E)\) on \(\Sigma\) is obtained by choosing a vertex \(v_f\) for every face \(f\) of \(H\). For every edge \(e\) of center \(v_e\), we pick up an edge \(e^*\) as follows: choose a local orientation of the surface around \(v_e\). This local orientation induces a cyclic order \(v_1, f_1, v_2, f_2, \ldots, v_d, f_d\) of the ends of \(e\) and of the faces incident with \(e\) (possibly with repetition). The edge \(e^*\) is the edge obtained by “rotating” \(e\) and whose ends are \(v_{f_1}, \ldots, v_{f_d}\).

A tree-decomposition of a hypergraph \(H\) on \(\Sigma\) is a pair \(T = (T, (X_v)_{v \in V(T)})\) with \(T\) a tree and \((X_v)_{v \in V(T)}\) a family of bags such that:

i. \(\bigcup_{v \in V(T)} X_v = H\);

ii. \(\forall x, y, z \in V(T)\) with \(y\) on the path from \(x\) to \(z\), \(X_x \cap X_z \subseteq X_y\).

The width of \(T\) is \(\text{tw}(T) = \max(|V(X_t)| - 1 : t \in V(T))\) and the tree-width \(\text{tw}(H)\) of \(H\) is the minimum width of one of its tree-decompositions.

Tree-width was introduced by Robertson and Seymour in connection with graph minors. In [RS84], they conjectured that for a planar graph \(G\), \(\text{tw}(G)\) and \(\text{tw}(G^*)\) differ by at most one. In an unpublished paper, Lapoire [Lap96] proves a more general result: for any hypergraph \(H\) in an orientable surface \(\Sigma\), \(\text{tw}(H^*) \leq \max(\text{tw}(H) + 1 + k(\Sigma), \alpha(H^*) - 1)\). Nevertheless, his proof is rather long and technical. Later, Bouchitté et al. [BMT03] gave an easier proof for planar graphs. Here we generalises Lapoire’s result to arbitrary surfaces while being less technical.

To avoid technicalities, we suppose that \(H\) is connected, contains at least two edges, and has no cutting edge.

2 P-trees and duality

From now on, \(H = (V, E)\) is a hypergraph on a surface \(\Sigma\). The border of a partition \(\mu\) of \(E\) is the set of vertices \(\delta(\mu)\) that are incident with edges in at least two parts of \(\mu\), and the border of \(X \subseteq E\) is the border of the partition \(\{X, E \setminus X\}\). A partition \(\mu = \{X_1, \ldots, X_p\}\) of \(E\) is connected if the \(A(H) \setminus \delta(\mu)\) can be partitioned in \(V_1, X_1, F_1, \ldots, V_p, X_p, F_p\) so that each \(V_i \cup X_i \cup F_i\) is an arc-wise connected subset of \(\Sigma\). Note that \(V_i\) is the set of vertices only incident to edges in \(X_i\).

A p-tree of \(H\) is a tree \(T\) whose internal nodes have degree three and whose leaves are identified with the edges of \(H\) in a bijective way. Removing an internal node \(v\) of \(T\) results in a partition \(\{A_v, B_v, C_v\}\) of \(E\). By identifying the internal nodes of \(T\) with the border of its corresponding partition, it is easy to see that \(T\) is a tree-decomposition. The tree-width of a p-tree is its tree-width, seen as a tree-decomposition. A p-tree is connected if all its nodes partitions are connected.

If \(\{A, B\}\) is a bipartition of an abstract hypergraph \(\hat{H}\), \(\hat{H} / A\) is obtained by replacing the edges in \(A\) by a single hyperedge \(\delta(\{A, B\})\). We want to define
a similar operation for hypergraphs on surfaces that is “compatible” with the surface. To do so we may only consider connected bipartitions. More precisely, let \( \{A, B\} \) be a connected bipartition of \( H \). Choose a partition of \( A(H) \setminus \delta(\{A, B\}) \) so that \( V_A \cup A \cup F_A \) and \( V_B \cup A \cup F_B \) are connected. We consider the incidence graph \( G_H(V \cup V_E, L) \) of \( H \), and we identify the edges in \( A \) with their centers. By adding edges through faces in \( F_A \), we can make \( G_H[A \cup V_A] \) connected. We then contract \( A \cup V_A \) into a new edge center \( v_A \). To make the resulting graph bipartite, we remove all \( v_A \)-loops. When removing a loop \( e \) incident to only one face \( F \), the new face \( F \cup e \) is not a disc but a crosscap. Since its border is a loop, we can “cut” \( \Sigma \) along this loop and replace \( F \cup e \) by an open disc while decreasing the genus of the surface. The obtained graph is the bipartite graph of \( H/A \).

We need the following folklore lemma:

**Lemma 1.** For any connected bipartition \( \{A, B\} \) of \( H \), \( \text{tw}(H) \leq \max(\text{tw}(H/A), \text{tw}(H/B)) \). If \( \delta(\{A, B\}) \) belongs to a bag of an optimal tree-decomposition, then \( \text{tw}(H) = \max(\text{tw}(H/A), \text{tw}(H/B)) \).

Let \( S \) be a set of vertices of \( H \). An \( S \)-bridge is a minimal subset \( X \) of \( E \) with the property that \( \delta(X) \subseteq S \). There are two kind of \( S \)-bridges: singletons containing an edge whose ends all belong to \( S \) and sets \( E_C \) containing all the edges incident to at least one vertex in \( C \), a connected component of \( G \setminus S \). The \( S \)-bridges partition \( E \). We define the abstract graph \( G_S \) whose vertices are the \( S \)-bridges and in which \( \{X, Y\} \) is an edge if there is a face incident with both an edge in \( X \) and an edge in \( Y \). A key fact is that any bipartition \( \{A, B\} \) of \( V(G_S) \) such that \( G_S[A] \) and \( G_S[B] \) is connected corresponds to the connected bipartition \( \{\cup A, \cup B\} \).

**Proposition 1.** There exists a connected p-tree \( T \) of \( H \) with \( \text{tw}(T) = \text{tw}(H) \).

**Proof.** By induction on \( |E| \), if \( |E| \leq 3 \), since \( H \) has no cutting edge, the only p-tree is connected and optimal. We can suppose that \( |E| \geq 4 \).

We claim that there exists a connected bipartition \( \{A, B\} \) of \( E \) whose border is contained in a bag of an optimal tree-decomposition of \( H \). Suppose that the trivial one vertex tree-decomposition whose bag is \( H \) is optimal. Consider the graph \( G_V \).

Otherwise, let \( X \) be a bag of any optimal tree-decomposition of \( H \). By Lemma 2, there exists a connected non trivial bipartition \( \{A, B\} \) with \( \delta(\{A, B\}) \in X \). Since \( \{A, B\} \) is connected, \( e_A \) and \( e_B \) are respectively not cutting edges in \( H/A \) and \( H/B \). By induction, there exists connected p-trees \( T_A \) and \( T_B \) of optimal width of \( H/A \) and \( H/B \). By removing the leaves labelled \( e_A \) and \( e_B \) and adding an edge between their respective neighbour, we obtain from \( T_A \cup T_B \) a p-tree of \( H \) which is connected. Its width is \( \max(\text{tw}(T/A), \text{tw}(T/B)) \) which is equal, by Lemma 1 to \( \text{tw}(H) \). \( \square \)

Because of the natural bijection between \( E(H) \) and \( E(H^*) \), a p-tree \( T \) of \( H \) also corresponds to a p-tree \( T^* \) of \( H^* \).
Proposition 2. For any connected p-tree T of H,
\[ \text{tw}(T^\ast) \leq \max(\text{tw}(T) + 1 + k(\Sigma), \alpha(H^\ast) - 1). \]

Proof. Let v be a vertex of T labelled is X_v in T and X_v^\ast in T^\ast. Suppose that v is a leaf. Then X_v^\ast = \{v\} and |X_v^\ast| - 1 \leq \max(\text{tw}(T) + 1 + k(\Sigma), \alpha(H^\ast) - 1).

Suppose that v is an internal node, and \{A, B, C\} is the associated E-partition. X_v = \delta(\{A, B, C\}), and X_v^\ast is the set of faces incident with the center of edges in at least two parts among A, B and C.

As for the proof of Proposition 1, since \{A, B, C\} is connected, we may contract A (and B and C). But since we now care about the faces of H, we have to be more careful. Since we want an upper bound on |X_v^\ast|, we may add but not remove faces to X_v^\ast. So adding edges to make \( G_H[V \cup A] \) connected is OK, but if a loop e on say v_A is incident with two faces in X_v^\ast, we cannot remove e.

Instead, we cut \Sigma along e and fill the holes with open discs. While doing so, we removed e, we cut v_A in two siblings, and we decreased the genus of \Sigma.

After contracting A, B and C, we obtain a bipartite graph G_v on \Sigma' that has |X_v| + 3 + s vertices with s the number of siblings, at least |X_v^\ast| faces and with \( k(\Sigma') \leq k(\Sigma) - s \). Since \( G_v \) is bipartite and faces in \( X_v^\ast \) are incident with at least 4 edges, \( 2|E(G_v)| = 4|F_1| + 6|F_2| + \cdots \geq 4|F(G_v)| \) with \( F_{2k} \) the set of 2k-gones faces of \( G_v \), and thus \( |E(G_v)| \geq 2|F(G_v)| \). If we apply Euler's formula to \( G_v \) on \Sigma', we obtain:
\[ |X_v| + 3 + s - |E(G_v)| + |F(G_v)| = 2 - k(\Sigma') \geq 2 - k(\Sigma) + s. \]
Adding this to \( |E(G_v)| \geq 2|F(G_v)| \), we get \( |X_v| + 1 + k(\Sigma) \geq |F(G_v)| \geq |X_v^\ast| \) which proves that \( |X_v^\ast| - 1 \leq \max(\text{tw}(T) + 1 + k(\Sigma), \alpha(H^\ast) - 1) \), and thus \[ \text{tw}(T^\ast) \leq \max(\text{tw}(T) + 1 + k(\Sigma), \alpha(H^\ast) - 1). \]

Let us now prove the main theorem.

Theorem 1. For any hypergraph H on a surface \Sigma,
\[ \text{tw}(H^\ast) \leq \max(\text{tw}(H) + 1 + k(\Sigma), \alpha(H^\ast) - 1). \]

Proof. By Proposition 1, let T be a connected p-tree of H such that \( \text{tw}(T) = \text{tw}(H) \). By Proposition 2, \( \text{tw}(T^\ast) \leq \max(\text{tw}(T) + 1 + k(\Sigma), \alpha(H^\ast) - 1) \). Since \( \text{tw}(H^\ast) \leq \text{tw}(T^\ast) \), we deduce, \( \text{tw}(H^\ast) \leq \max(\text{tw}(H) + 1 + k(\Sigma), \alpha(H^\ast) - 1). \)

References

