

The Erdős-Hajnal conjecture for paths and cycles.

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Two complementary invariants



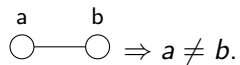
largest clique.



largest independent set.

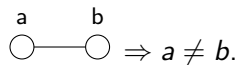
...And two related chromatic numbers

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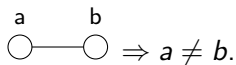
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$$w : S(G) \rightarrow \{0; 1\}.$$

$$\forall v \in V, \sum_{S|v \in S} w(S) = 1.$$

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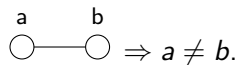
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w' optimal solution \Leftrightarrow

$$\sum_S w'(S) = \chi(G).$$

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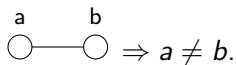
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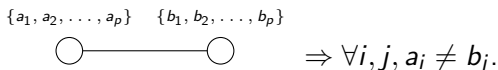
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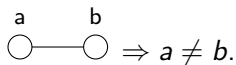
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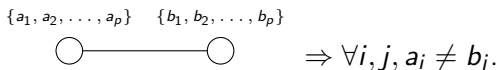
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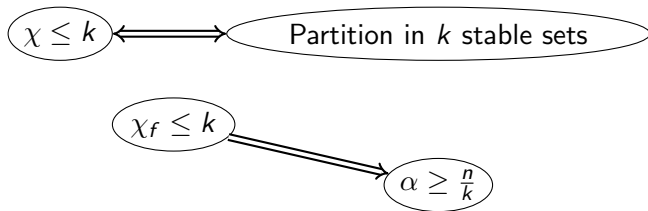
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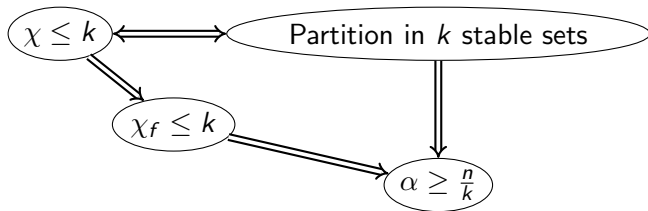
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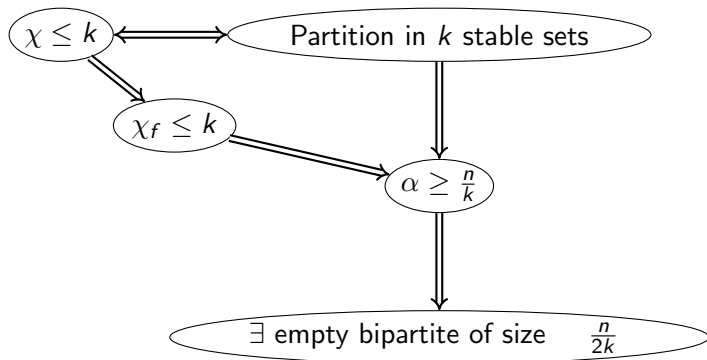


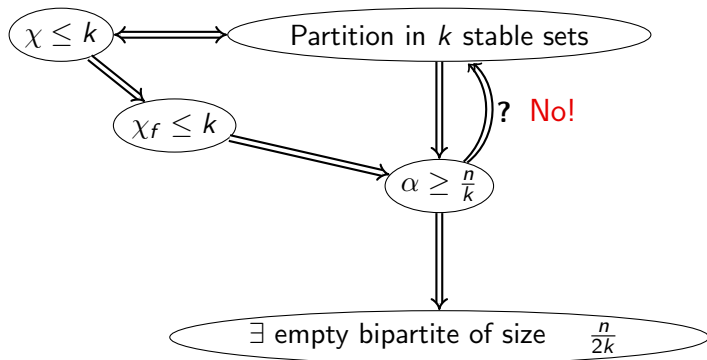
$$\chi_f(G) = \inf_p \frac{\chi_p(G)}{p}$$

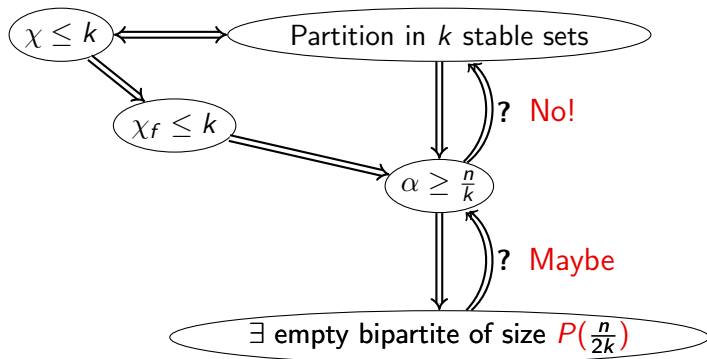












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Definition

\mathcal{C} is χ -bounded $\Leftrightarrow \exists f, \chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{C}$.

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For which \mathcal{C} do we have a polynomial f ?

Conjecture (Gyarfás '87)

Graphs with *no induced path of length k* satisfy $\chi \leq P(k, \omega)$.

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

Conjecture (Erdős Hajnal '89)

Graphs with *no induced path of length k* contain *a clique or a stable set of size $n^{\epsilon k}$* .

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

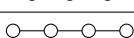
H-free graphs contain a clique or a stable set of size n^{ϵ_H} .

H	α, ω
	$\alpha = n$
	$\alpha \geq \sqrt{n \log n}$
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Theorem (Grimmet, McDiarmid '75)

The H -free hypothesis is *necessary*.

(Random graphs satisfy w.h.p. $\alpha, \omega = \mathcal{O}(\log n)$)

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Theorem (Alon, Pach, Solymosi '01)

Looking at *prime* H is enough.

Interesting graphs on 4 vertices: $\circ - \circ - \circ - \circ$ ✓.

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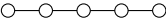
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
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Interesting graphs on 5 vertices:

  ✓ (Chudnovsky Safra '08).

A more tractable rewording

Conjecture (Erdős Hajnal)

Graphs in a non-trivial class \mathcal{C} contain a clique or a stable set of size $n^{\epsilon_{\mathcal{C}}}$.

Closed under induced subgraphs

True for any polynomially χ -bounded \mathcal{C} .

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True for graphs with no induced P_5 or $\overline{P_7}$.

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For any k , true for graphs with no induced P_k or $\overline{P_k}$.

Theorem (B. Bousquet Thomassé '13)

For any k , true for graphs with no induced $C_{\geq k}$ or $\overline{C_{\geq k}}$.

Sketch of the proof for paths

- 1 Extract a linear sparse or dense subgraph.
(sparse = every vertex of degree $\leq \epsilon n$,
dense = every vertex of degree $\geq (1 - \epsilon)n$)
- 2 Extract a linear empty or complete bipartite graph.
- 3 Conclude.

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Theorem (Rödl '86)

Every graph satisfies one of the following:

- *It contains all graphs on k vertices.*
- *It has a linear subset with average degree $\leq \epsilon n$.*
- *It has a linear subset with average degree $\geq (1 - \epsilon)n$.*

Theorem (Alon et al. '05, Fox and Pach '08)

*Every graph with an empty or complete linear bipartite graph contains a **polynomial cograph**.*

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Assume wlog there is a sparse linear subgraph.

Lemma

Every graph G contains a dominating tree T such that:

- *there is a linear empty bipartite graph, or*
- *there is a path of T that dominates $\frac{1}{4}$ of the vertices.*

Lemma

A sparse $C_{\geq k}$ -free graph that admits a dominating path contains a linear empty bipartite graph.

E-H for C_5 or P_5 ?

χ -boundedness of graphs with no long cycle? (Gyarfás)

Existence of a graph with

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Thanks!