

Shortest reconfiguration paths in the solution space of Boolean formulas

Amer Mouawad

Department of Informatics
University of Bergen

Joint work with Naomi Nishimura, Vinayak Pathak, and Venkatesh Raman

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Problem statement

Input: A Boolean formula ϕ (e.g. $x_1 \vee x_2$) and two satisfying assignments \mathbf{s} and \mathbf{t} (e.g. 01 and 10).

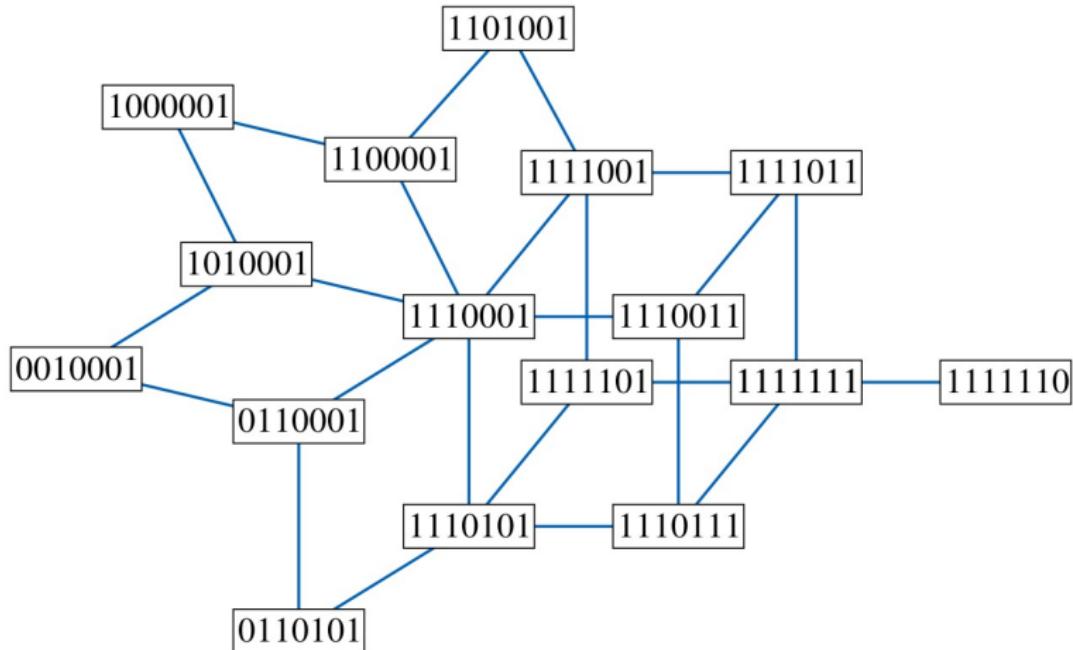
Output: Smallest number of flips to transform \mathbf{s} to \mathbf{t} , where a flip negates a single variable so that the resulting assignment is also satisfying.

Example: $\phi = x_1 \vee x_2$, $\mathbf{s} = 01$, and $\mathbf{t} = 10$.

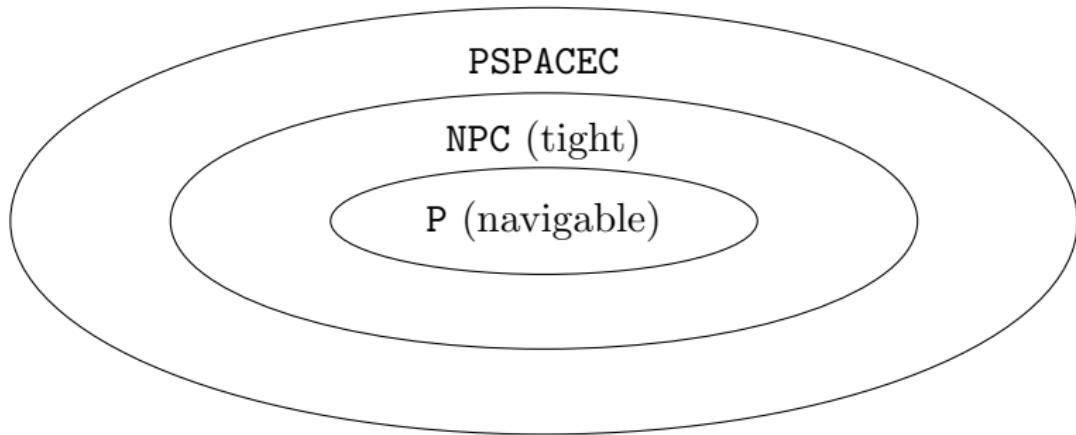
$01 \rightarrow 11 \rightarrow 10$ is valid.

$01 \rightarrow 00 \rightarrow 10$ is **NOT** valid.

Walking in the solution space



The result



Theorem

For any “class” of Boolean formulas computing the shortest reconfiguration path is either in P, NP-complete, or PSPACE-complete.

A bit of history

Schaefer's Theorem (1978)

2-SAT, Horn-SAT, Dual-Horn-SAT, and Affine-SAT are the only (non-trivial) classes in P. Every other class is NP-complete.

Ladner's Theorem (1975)

If $P \neq NP$ then there are an infinite number of non-equivalent SAT classes in NP that are neither in P nor NP-complete.

Catch: Schaefer's definition of "class" is restrictive.

Rest of the talk

- Schaefer's framework
- **st**-connectivity (Gopalan et al., 2009)
- **st**-shortest path

Schaefer's framework

- ▶ Let \mathcal{S} be a finite set of Boolean relations of **constant** arity.
- ▶ A Boolean formula ϕ belongs to the class $\text{CNF}(\mathcal{S})$ if each clause in ϕ is “built” from a relation $R \in \mathcal{S}$.

$$R^0 = \{0, 1\}^3 \setminus \{000\}$$

$$\mathcal{S} = \{R^0, R^1, R^2, R^3\}$$

$$R^1 = \{0, 1\}^3 \setminus \{100\}$$

$$\text{CNF}(\mathcal{S}) \equiv 3\text{CNF}$$

$$R^2 = \{0, 1\}^3 \setminus \{110\}$$

$$R^3 = \{0, 1\}^3 \setminus \{111\}$$

$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \equiv R^0(x_1, x_2, x_3) \wedge R^2(x_1, x_3, x_2)$$

The reconfiguration graph

- ▶ For any relation R , we let $G_R = (V, E)$ be the graph consisting of one node for each element $\mathbf{v} \in R$ and two nodes \mathbf{u} and \mathbf{v} are connected whenever $Ham(\mathbf{u}, \mathbf{v}) = 1$.
- ▶ For a formula ϕ , we let R_ϕ denote the relation containing all satisfying assignments for ϕ and we let $G_\phi = G_{R_\phi}$.

The reconfiguration graph

- ▶ **st-connectivity:** check if s and t belong to the same connected component of G_ϕ .
- ▶ **st-shortest path:** compute the length of a shortest path from s to t in G_ϕ .

st-connectivity

Theorem (Folklore)

st-connectivity is in P for 2CNF formulas.

Proof.

We only need to flip variables that are assigned different values in **s** and **t**. So just keep flipping such variables as long as possible. If we get stuck then there is no path from **s** to **t**. Otherwise, we will eventually find a (shortest) path.

st-connectivity

Theorem (Gopalan et al., 2009)

st-connectivity is in P for any formula ϕ where each connected component of G_ϕ can be “written” as a 2CNF formula, also called component-wise bijunctive formulas.

Proof.

Generalizes the 2CNF case (also produces shortest paths).

st-connectivity

Definition

A k -ary relation R is called NAND-free (OR-free) if no $k - 2$ of its variables can be assigned values such that the relation induced on the remaining two variables is a NAND (OR), i.e. $\neg x \vee \neg y$ (i.e. $x \vee y$).

001101100

010~~0~~10010

010~~1~~10010

0100~~1~~1010

001101100

001101100

010~~1~~11010

010~~1~~10010

0100~~0~~1010

001101100

A formula ϕ is NAND-free (OR-free) if it consists entirely of NAND-free (OR-free) relations.

st-connectivity

Theorem (Gopalan et al., 2009)

For a NAND-free formula ϕ , any sequence of flips can be reordered so that all $0 \rightarrow 1$ flips occur before all $1 \rightarrow 0$ flips.
The reverse is true for OR-free relations.

Proof. (NAND-free case)

$..0..1.. \rightarrow ..0..0.. \rightarrow ..1..0..$ becomes $..0..1.. \rightarrow ..1..1.. \rightarrow ..1..0..$ If not possible then some relation is not NAND-free.

st-connectivity

Corollary

For a NAND-free (OR-free) formula ϕ , any connected component of G_ϕ contains a unique local maximum (minimum) with respect to hamming weight (polynomial diameter if G_ϕ is connected).

st-connectivity

Definition (Gopalan et al., 2006)

\mathcal{S} is *tight* if at least one of the following conditions holds:

- (1) Every relation in \mathcal{S} is component-wise bijunctive.
- (2) Every relation in \mathcal{S} is OR-free.
- (3) Every relation in \mathcal{S} is NAND-free.

st-connectivity

Theorem (Gopalan et al., 2006)

For a $\text{CNF}(\mathcal{S})$ formula ϕ , st-connectivity is in P if \mathcal{S} is tight and PSPACE-complete otherwise.

Proof. (NAND-free case)

Try to reach the same assignment by doing $0 \rightarrow 1$ flips from both s and t (i.e. try to reach the local maximum).

Finding shortest paths

- If not tight \rightarrow PSPACE-complete
- 2CNF and component-wise bijunctive $\in P$ (symmetric difference)

Is there a tight formula for which finding the shortest path is NP-complete?

NP-completeness

- Let $\phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$ (NAND-free).
- Let $\mathbf{s} = 0000$ and $\mathbf{t} = 1100$.
- If $x_3 = x_4 = 0$, then neither x_1 nor x_2 can be flipped.
- Hence we need at least one of them to be 1.
- Reduction from VERTEX COVER (two clauses per edge).
- Since the formula is tight G_ϕ has polynomial diameter.

A false (but useful) conjecture

- Hardness crucially relied on deciding which common variables should be flipped.
- Conjecture: whenever we have to flip common variables the problem becomes hard.

A false (but useful) conjecture

- Let $P = \{000, 100, 101, 111, 011\}$. So G_P is a path:
 $000 \leftrightarrow 100 \leftrightarrow 101 \leftrightarrow 111 \leftrightarrow 011$.
- There exists formulas in $\text{CNF}(\{P\})$ where any path between two assignments needs to flip common variables.
- However computing the shortest path between any two assignments is in P for any formula in $\text{CNF}(\{P\})$.
 - Each clause specifies a **unique order** on flips.
 - We just need to check if two clauses are in conflict.

The trichotomy

Definition

\mathcal{S} is *navigable* if at least one of the following conditions holds:

- (1) Every relation in \mathcal{S} is component-wise bijunctive.
- (2) Every relation in \mathcal{S} is OR-free and **Horn-free**.
- (3) Every relation in \mathcal{S} is NAND-free and **dual-Horn-free**.

Theorem

For a $\text{CNF}(\mathcal{S})$ formula ϕ , **st**-shortest path is in **P** if \mathcal{S} is navigable, **NP**-complete if \mathcal{S} is tight but not navigable, and **PSPACE**-complete otherwise.

Proof ideas

- (1) Component-wise bijunctive: Use the symmetric difference.
- (2) OR-free and Horn-free: Unique local minimum +
partial order.
- (3) NAND-free and dual-Horn-free: Unique local maximum +
partial order.

Open problems

- ▶ Generalize the dichotomy/trichotomy beyond the Boolean domain.
- ▶ How about weighted SAT and/or CSPs?
- ▶ Study the variations in the (parameterized) complexity landscape if we consider simultaneous flips.

Thank you