

# Adjacent vertex-distinguishing edge coloring of graphs with bounded maximum degree

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## EXTENDED ABSTRACT

In the following we only consider simple graphs with no isolated edges. A proper edge coloring of a graph  $G = (V, E)$  is an assignment of colors to the edges of the graph such that two adjacent edges do not use the same color. An *adjacent vertex-distinguishing  $k$ -edge coloring*, or  *$k$ -avd-coloring* for short, of a graph  $G$  is a proper edge coloring of  $G$  using at most  $k$  colors such that, for every pair of adjacent vertices  $u, v$ , the set of colors of the edges incident to  $u$  differs from the set of colors of the edges incident to  $v$ . Adjacent vertex-distinguishing colorings are also known as *adjacent strong edge coloring* and *1-strong edge coloring*. We denote by  $\chi'_{avd}(G)$  the *avd-chromatic number* of  $G$ , which is the smallest integer  $k$  such that  $G$  can be  $k$ -avd-colored.

Zhang *et al.* completely determined the avd-chromatic number for paths, cycles, trees, complete graphs, and complete bipartite graphs [5]. Moreover, they proposed the following conjecture [5], where  $\Delta(G)$  denotes the maximum degree of  $G$ :

**Conjecture 1** *If  $G$  is a connected graph with at least 6 vertices, then  $\chi'_{avd}(G) \leq \Delta(G) + 2$ .*

Balister *et al.* [1] proved Conjecture 1 for graphs with maximum degree three and for bipartite graphs.

Let  $\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \subseteq G \right\}$  be the maximum average degree of the graph  $G$ , where  $V(H)$  and  $E(H)$  are the sets of vertices and edges of  $H$ , respectively.

Wang and Wang [3] linked maximum average degree and avd-colorings, and their results were further improved by Hocquard and Montassier [2]:

**Theorem 2** [3, 2] *Let  $G$  be a graph with maximum degree  $\Delta(G)$ .*

1. *If  $\Delta(G) \geq 3$  and  $\text{mad}(G) < 3$ , then  $\chi'_{avd}(G) \leq \Delta(G) + 2$ .*
2. *If  $\Delta(G) \geq 3$  and  $\text{mad}(G) < 3 - \frac{2}{\Delta(G)}$ , then  $\chi'_{avd}(G) \leq \Delta(G) + 1$ .*

Two main questions arise from these partial results: can this threshold of 3 as an upper-bound on  $\text{mad}(G)$  be reached with a sufficiently large lower-bound on  $\Delta(G)$  in the case of Theorem 2.2, and broken in the case of Theorem 2.1?

We answer positively to these questions with Theorem 3.1, and prove more precisely that there is no threshold in the case of Theorem 2.2 (and thus in the case of Theorem 2.1). Theorem 3.2-3 are refinements of it for small upper-bounds on  $\text{mad}(G)$ .

**Theorem 3** *Let  $G$  be a graph with maximum degree  $\Delta(G)$  and maximum average degree  $\text{mad}(G)$ .*

1. *For all  $m \geq \frac{5}{2}$ , if  $\Delta(G) \geq 16m^2$  and  $\text{mad}(G) < m$ , then  $\chi'_{avd}(G) \leq \Delta(G) + 1$ .*
2. *If  $\Delta(G) \geq 7$  and  $\text{mad}(G) < 3$ , then  $\chi'_{avd}(G) \leq \Delta(G) + 1$ .*
3. *If  $\Delta(G) \geq 3$  and  $\text{mad}(G) < 3 - \frac{1}{\Delta(G)-1}$ , then  $\chi'_{avd}(G) \leq \Delta(G) + 1$ .*

Theorem 3 is proved using discharging methods. The proof of Theorem 3.1 is heavily inspired from a beautiful proof by Woodall [4] for a similar statement in the case of list edge coloring with  $\Delta$  colors. Note that list edge coloring is similarly conjectured to be always possible with  $\Delta + 1$  colors. The main idea in the proof of Theorem 3.2 is that a minimal (minimal in the lexico-graphic order) counter-example contains no cycle in which every edge is incident to at least one vertex of degree 2. In the proof of Theorem 3.3, we consider a tree-like configuration of unbounded size that contains all the locally problematic cases, and we prove that it cannot be problematic on the whole graph if the upper-bound on the maximum average degree is respected.

## References

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