

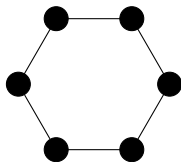
# Coloring the squares of graphs

Marthe Bonamy

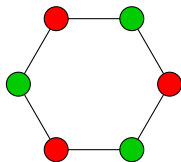
Bratislava

November 28, 2013





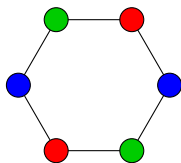
# Square coloring



$\chi$ : Minimum number of colors to ensure that

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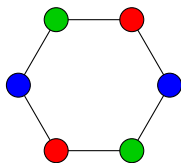


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$\Delta$ : Maximum degree of the graph.

$$\Delta^2 + 1 \geq \chi^2 \geq \Delta + 1.$$

A graph is **planar** if it can be drawn in a plane without crossing edges.

## Conjecture (Wegner 1977)

If  $G$  is a *planar* graph, then:

- $\chi^2 \leq 7$  if  $\Delta = 3$
- $\chi^2 \leq \Delta + 5$  if  $4 \leq \Delta \leq 7$
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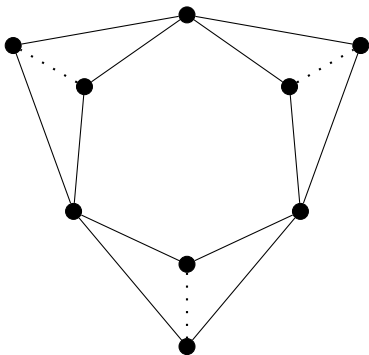


# Sparse planar graphs

$g$  : girth = minimum length of a cycle.

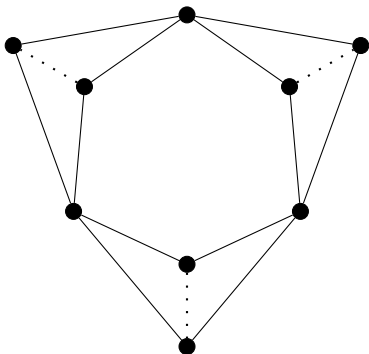
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Theorem (Bu, Zhu '11)

Wegner's conjecture is *true* for planar graphs with  $g \geq 6$ .

## Sparse planar graphs (2)

Question (Wang, Lih '01)

$$g \geq k, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1.$$

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 $\exists? k_0, \forall k \geq k_0, \exists d,$   
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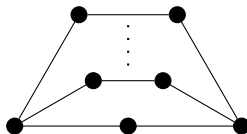
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Theorem (Borodin, Glebov, Ivanova, Neustroeva, Tashkinov '04)

*False* for  $k \leq 6$ . *True* for  $k \geq 7$ .



Theorem (Borodin, Ivanova, Neustroeva '08 & Ivanova '11)

For any *planar* graph,  $\chi^2 = \Delta + 1$  in each of the following cases :

- $g \geq 24$  and  $\Delta \geq 3$
- $g \geq 15$  and  $\Delta \geq 4$
- $g \geq 12$  and  $\Delta \geq 5$
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$$|E| - |V| - |F| = -2$$

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$$\sum_{f \in F} \underbrace{(d(f) - g)}_{\geq 0} + c \cdot \sum_{v \in V} \left( d(v) - \frac{2g}{g-2} \right) < 0$$

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ad: average degree =  $\frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|}$ .

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$M$  maximal |  $\forall m < M, \exists d,$   
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For *any* graph,  $\chi^2 = \Delta + 1$  in each of the following cases :

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•  $\underbrace{mad < \frac{18}{7}}_{g \geq 9}$  and  $\Delta \geq 8$

•  $\underbrace{mad < \frac{70}{29}}_{g \geq 12}$  and  $\Delta \geq 5$

•  $\underbrace{mad < \frac{8}{3}}_{g \geq 8}$  and  $\Delta \geq 14 > 10$

•  $\underbrace{mad < \frac{5}{2}}_{g \geq 10}$  and  $\Delta \geq 6$

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What about  $mad < 3$ ?

# First results on non-planar sparse graphs (2)

Theorem (B., Lévêque, Pinlou '13)

For a small enough  $\epsilon$ ,  $\text{mad} < 3 - \epsilon$  and  $\Delta \geq \frac{3}{\epsilon^2} \Rightarrow \chi^2 = \Delta + 1$ .

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For any *bipartite multigraph*  $G$ , if  $L$  is an edge list assignment such that  $|L(u, v)| \geq d(u), d(v)$  for every  $(u, v)$ , then  $G$  is *L-edge-choosable*.

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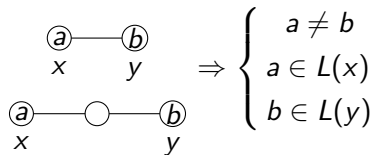
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$\Rightarrow$  Global discharging method.

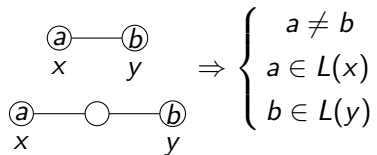
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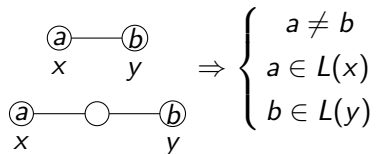
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What changes in the previous theorems?

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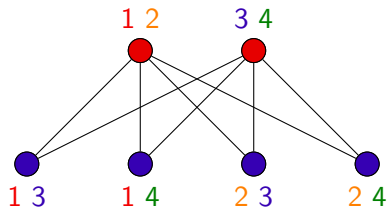
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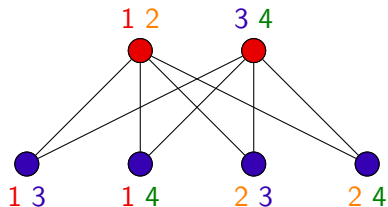
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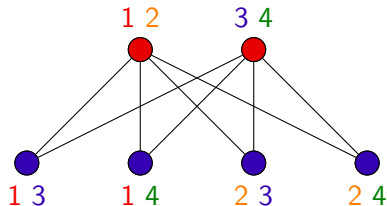
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Theorem (Kim, Kwon, Park '13, Kosar, Petrickova, Reiniger, Yeager '13)

For any  $k$ ,  $\chi_\ell^k - \chi^k$  can be *arbitrarily large*.

$$\Delta^2 + 1 \geq \chi_\ell^2 \geq \chi^2 \geq \Delta + 1.$$

Conjecture (Cranston, Kim '08)

If  $\Delta \geq 3$ , *no* graph satisfies  $\chi_\ell^2 = \Delta^2$ .

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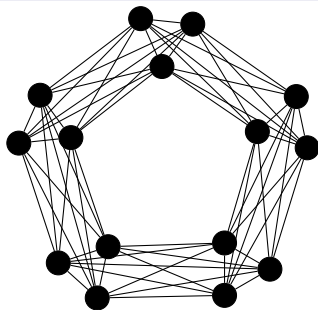
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## Conjecture

For any  $k \geq 3$ , and  $\Delta \geq 3$ , *no* graph satisfies

$$\chi_l^k \geq D(k, \Delta) - k + 2.$$

# Conclusion

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For any  $k \geq 3$ , and  $\Delta \geq 3$ , no graph satisfies

$$\chi_k \geq D(k, \Delta) - k + 2.$$

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For any  $C$  and  $\epsilon > 0$ ,

$$\text{mad} < \frac{4C+2}{C+1} - \epsilon \text{ and } \Delta \geq f(C, \epsilon) \Rightarrow \chi_k^2 \leq \Delta + C.$$

