

Kempe Equivalence of Colourings of Graphs

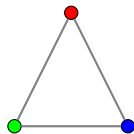
Carl Feghali
Durham University

7 June 2016,

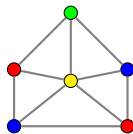
SIAM conference on Discrete Mathematics 2016

Graph Colouring

- A *k*-colouring of a graph $G = (V, E)$ is an assignment of colours to the vertices in V from the set $\{1, \dots, k\}$ such that no two-adjacent vertices receive the same colour.



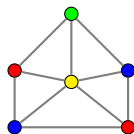
3-colouring



4-colouring

Kempe chains

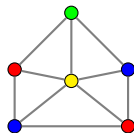
- $G(a, b)$ denotes the subgraph of G induced by colours a, b .



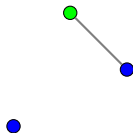
G

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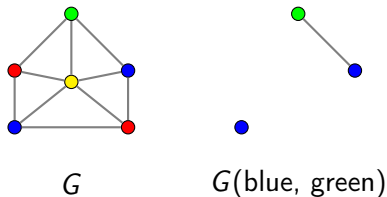
G



$G(\text{blue, green})$

Kempe chains

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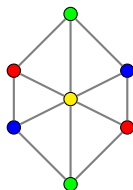
- A *Kempe chain* is a connected component of $G(a, b)$.

Kempe changes

- A *Kempe change* is the operation of **exchanging** the colours of a Kempe chain.

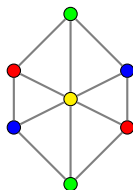
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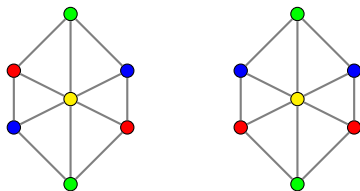
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Exchange colours in the left component of G (blue, red)

Kempe changes

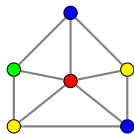
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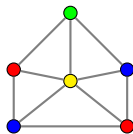
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Kempe equivalence

- Two colourings are **Kempe equivalent** if one can be obtained from the other by a sequence of **Kempe changes**.



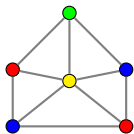
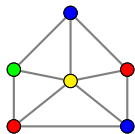
α



β

Kempe equivalence

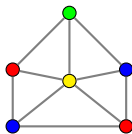
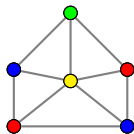
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β

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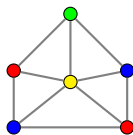
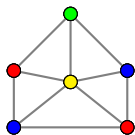
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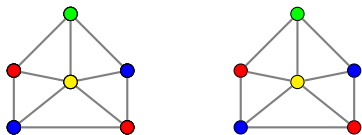
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β

Are there graphs where all k -colourings are Kempe equivalent?

Planar graphs

Theorem (Meyniel, 1978)

5-colourings of planar graphs are Kempe equivalent.

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By showing that 4-colourings of 3-colourable planar graphs are Kempe equivalent, Mohar extended the first theorem:

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If $k > \chi(G)$, all k -colourings of a planar graph G are Kempe equivalent.

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A graph G is 4-critical if it is not 3-colourable but $G - e$ is 3-colourable for any edge e of G . Mohar posed the following problem.

Problem

Suppose G is a 4-critical planar graph. Is it possible that G has two 4-colourings that are not Kempe equivalent to each other?

K_n -Minor-free graphs

Meyniel's result was later extended to K_5 -minor free graphs:

Theorem (Las Vergnas, Meyniel, 1981)

5-colourings of K_5 -minor free graphs are Kempe equivalent.

They went on to assert the following conjecture.

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If $k \geq 6$, then the k -colourings of K_k -minor free graphs are Kempe equivalent.

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A graph is apex if there is a vertex whose removal makes the graph planar. Apex graphs are K_6 -minor free.

Proposition (F, unpublished)

The 6-colourings of apex graphs are Kempe equivalent.

Perfect graphs

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There is a perfect graph G and a $(\chi(G) + 1)$ -colouring of G that is not Kempe equivalent to a $\chi(G)$ -colouring of G .

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Problem

Are the $(\chi(G) + 2)$ -colourings of a perfect graph G Kempe equivalent?

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A graph G is k -regular if every vertex $v \in V(G)$ has degree k .

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Let $k \geq 3$. If G is a k -regular graph that is not complete, then all k -colourings of G are Kempe equivalent.

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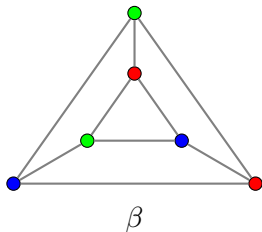
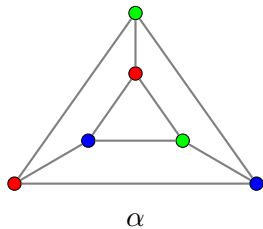
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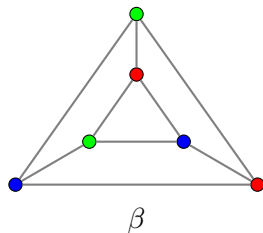
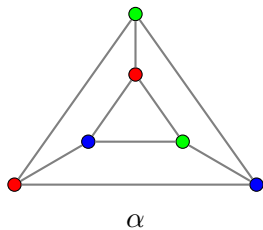
Let $k \geq 3$. If G is a k -regular graph that is not complete, then all k -colourings of G are Kempe equivalent.

van den Heuvel found a counterexample: a 3-regular graph with 3-colourings that are not Kempe equivalent ...

The triangular prism

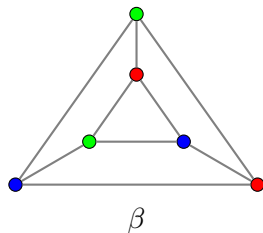
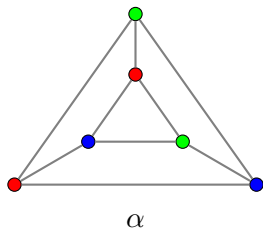


The triangular prism



α and β are not Kempe equivalent:

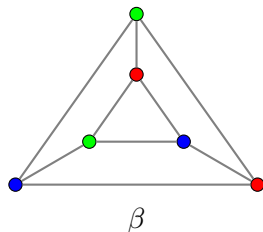
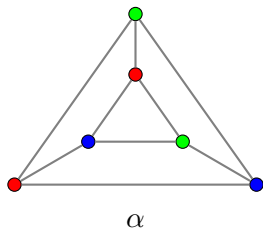
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α and β are not Kempe equivalent:

- $G(\text{blue, red})$, $G(\text{blue, green})$, $G(\text{green, red})$ are connected.

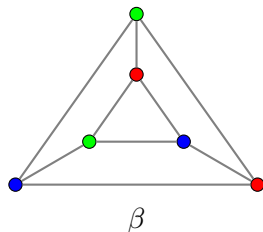
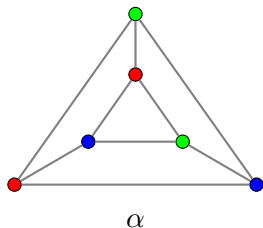
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- So two 3-colourings are Kempe equivalent if and only if the partitions of the vertex set into colour classes are the same.

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- $G(\text{blue, red})$, $G(\text{blue, green})$, $G(\text{green, red})$ are connected.
- So two 3-colourings are Kempe equivalent if and only if the partitions of the vertex set into colour classes are the same.
- The colour classes of α and β induce different partitions.

Regular graphs

Mohar's conjecture has been recently settled by combining the following two theorems.

Theorem (F, Johnson, Paulusma; 2015)

If G is a connected cubic graph that is neither complete nor the triangular prism, then all 3-colourings of G are Kempe equivalent.

Theorem (Bonamy, Bousquet, F, Johnson; 2015)

Let $k \geq 4$. The k -colourings of non-complete k -regular graphs are Kempe equivalent.

- Proof techniques

Lemma

Let G be a graph. If the colourings of the graph G' obtained by identifying x and y into a new vertex are Kempe equivalent, then the colourings of G that colour x and y alike are Kempe equivalent.

- This enables to use an argument by induction on $|V(G)|$ by first describing a sequence of Kempe changes that colours some fixed pair of vertices alike.

Lemma (Mohar, 2006)

Let H be a supergraph of a graph G , and let α and β be colourings of H . If α and β are Kempe equivalent, then the restriction of α and β to G are also Kempe equivalent.

- A graph G is *edge maximal* w.r.t. property P if the graph obtained from G by joining any pair vertices does not satisfy P .
- Edge maximal graphs have nice structural properties.
- One may adopt an argument by reverse induction on $|E(G)|$ by first describing a sequence of Kempe changes that assigns distinct colours to some fixed pair of non-adjacent vertices.

Lemma (Las Vergnas, Meyniel; 1981)

Let G_1 and G_2 be two graphs such that $G_1 \cap G_2$ is a complete graph. If the colourings of G_1 are Kempe equivalent and the colourings of G_2 are Kempe equivalent, then the colourings of $G_1 \cup G_2$ are Kempe equivalent.

- Let G be a graph with minimum vertex cut S . Let F_1 and F_2 be two graphs of $G - S$ such that $F_1 \cup F_2 \cup S = G$.
- Suppose the colourings of $G_i = F_i \cup S$, $i = 1, 2$, are Kempe equivalent.
- One may perform the earlier techniques (edge addition and vertex identification) to S so that the resulting vertex cut is a complete graph.

Thank You

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