

Planar graphs with $\Delta \geq 8$ are $(\Delta + 1)$ -edge-choosable.*

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Abstract

We consider the problem of *list edge coloring* for planar graphs. Edge coloring is the problem of coloring the edges while ensuring that two edges that are incident receive different colors. A graph is k -edge-choosable if for any assignment of k colors to every edge, there is an edge coloring such that the color of every edge belongs to its color assignment. Vizing conjectured in 1965 that every graph is $(\Delta + 1)$ -edge-choosable. In 1990, Borodin solved the conjecture for planar graphs with maximum degree $\Delta \geq 9$, and asked whether the bound could be lowered to 8. We prove here that planar graphs with $\Delta \geq 8$ are $(\Delta + 1)$ -edge-choosable.

1 Introduction

We consider simple graphs. A k -edge-coloring of a graph G is a coloring of the edges of G with k colors such that two edges that are incident receive distinct colors. We denote by $\chi'(G)$ the smallest k such that G admits a k -edge-coloring. Let $\Delta(G)$ be the maximum degree of G . Since incident edges have to receive distinct colors in an edge coloring, every graph G satisfies $\chi'(G) \geq \Delta(G)$. A trivial upper-bound on $\chi'(G)$ is $2\Delta(G) - 1$, which can be greatly improved, as follows.

Theorem 1 (Vizing [12]). *Every graph G satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

Vizing [13] proved that $\chi'(G) = \Delta(G)$ for every planar graph G with $\Delta(G) \geq 8$. He gave examples of planar graphs with $\Delta(G) = 4, 5$ that are not $\Delta(G)$ -edge-colorable, and conjectured that no such graph exists for $\Delta(G) = 6, 7$. This remains open for $\Delta(G) = 6$, but the case $\Delta(G) = 7$ was solved by Sanders and Zhao [10], as follows.

Theorem 2 (Sanders and Zhao [10]). *Every planar graph G with $\Delta(G) \geq 7$ satisfies $\chi'(G) = \Delta(G)$.*

An extension of the problem of edge coloring is the *list edge coloring* problem, defined as follows. For any $L : E \rightarrow \mathcal{P}(\mathbb{N})$ list assignment of colors to the edges of a graph $G = (V, E)$, the graph G is L -edge-colorable if there exists an edge coloring of G such that the color of every edge $e \in E$ belongs to $L(e)$. A graph $G = (V, E)$ is said to be *list k -edge-colorable* (or *k -edge-choosable*) if G is L -edge-colorable for any list assignment L such that $|L(e)| \geq k$ for any edge $e \in E$. We denote by $\chi'_\ell(G)$ the smallest k such that G is k -edge-choosable.

One can note that edge coloring is a special case of list edge coloring, where all the lists are equal. Thus $\chi'(G) \leq \chi'_\ell(G)$. This inequality is in fact conjectured to be an equality (see [8] for more information).

Conjecture 1 (List Coloring Conjecture). *Every graph G satisfies $\chi'(G) = \chi'_\ell(G)$.*

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The conjecture is still widely open. Some partial results were however obtained in the special case of planar graphs: for example, the conjecture is true for planar graphs of maximum degree at least 12, as follows.

Theorem 3 (Borodin et al. [6]). *Every planar graph G with $\Delta(G) \geq 12$ satisfies $\chi'_\ell(G) = \Delta(G)$.*

There is still a large gap with the lower bound of 7 that should hold by Theorem 2 if Conjecture 1 were true.

Using Vizing's theorem, the List Coloring Conjecture can be weakened into Conjecture 2.

Conjecture 2 (Vizing [14]). *Every graph G satisfies $\chi'_\ell(G) \leq \Delta(G) + 1$.*

Conjecture 2 has been actively studied in the case of planar graphs with some restrictions on cycles (see for example [11, 15, 16]), and was settled by Borodin [4] for planar graphs of maximum degree at least 9 (a simpler proof was later found by Cohen and Havet [7]).

Theorem 4 (Borodin [4]). *Every planar graph G with $\Delta(G) \geq 9$ satisfies $\chi'_\ell(G) \leq \Delta(G) + 1$.*

Here we prove the following theorem.

Theorem 5. *Every planar graph G with $\Delta(G) \leq 8$ satisfies $\chi'_\ell(G) \leq 9$.*

This improves Theorem 4 and settles Conjecture 2 for planar graphs of maximum degree 8.

Corollary 1. *Every planar graph G with $\Delta(G) \geq 8$ satisfies $\chi'_\ell(G) \leq \Delta(G) + 1$.*

This answers Problem 5.9 in a survey by Borodin [5]. For small values of Δ , Theorem 5 implies that every planar graph G with $5 \leq \Delta(G) \leq 7$ is also 9-edge-choosable. To our knowledge, this was not known. It is however known that planar graphs with $\Delta(G) \leq 4$ are $(\Delta(G) + 1)$ -edge-choosable [9, 14].

2 Method

The discharging method was introduced in the beginning of the 20th century. It has been used to prove the celebrated Four Color Theorem ([1] and [2]).

We prove Theorem 5 using a discharging method, as follows. A graph is *minimal* for a property if it satisfies this property but none of its proper subgraphs does. The first step is to consider a minimal counter-example G (i.e. a graph G such that $\Delta(G) \leq 8$ and $\chi'_\ell(G) > 9$, whose every proper subgraph is 9-edge-choosable), and prove it cannot contain some configurations. We assume by contradiction that G contains one of the configurations. We consider a particular subgraph H of G . For any list assignment L on the edges of G , with $|L(e)| \geq 9$ for every edge e , we L -edge-color H by minimality. We show how to extend the L -edge-coloring of H to G , a contradiction.

The second step is to prove that a connected planar graph on at least two vertices with $\Delta \leq 8$ that does not contain any of these configurations does not satisfy Euler's Formula. To that purpose, we consider a planar embedding of the graph. We assign to each vertex its degree minus six as a weight, and to each face two times its degree minus six. We apply discharging rules to redistribute weights along the graph with conservation of the total weight. As some configurations are forbidden, we can prove that after application of the discharging rules, every vertex and every face has a non-negative final weight. This implies that $\sum_v (d(v) - 6) + \sum_f (2d(f) - 6) = 2 \times |E(G)| - 6 \times |V(G)| + 4 \times |E(G)| - 6 \times |F(G)| \geq 0$, a contradiction with Euler's Formula that $|E| - |V| - |F| = -2$. Hence a minimal counter-example cannot exist.

The complete proof was omitted due to space limitations. The proof requires eleven forbidden configurations and eleven discharging rules, which we do not present here because they rely on additional definitions. The full proof can be found in [3].

3 Conclusion

The key idea in the proof lies in some recoloring arguments using directed graphs. It allowed us to deal with configurations that would not yield under usual techniques, and thus to improve Theorem 4. Though this simple argument does not seem to be enough to prove Conjecture 2 for $\Delta = 7$, it might be interesting to try to improve similarly Theorem 3.

Note that the proof could easily be adapted to prove that planar graphs with $\Delta \geq 8$ are $(\Delta + 1)$ -edge-choosable. This would however be of little interest considering the simple proof for $\Delta \geq 9$ presented in [7].

Conjecture 2 remains open for $\Delta = 5, 6$ and 7 . It might be interesting to weaken the conjecture and ask whether all planar graphs are $(\Delta + 2)$ -edge-choosable. This is true for planar graphs with $\Delta \geq 7$ by Theorems 4 and 5. What about planar graphs with $\Delta = 6$?

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