

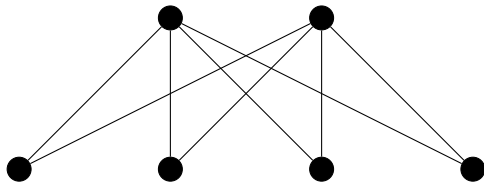
# Bounded spectrum list coloring

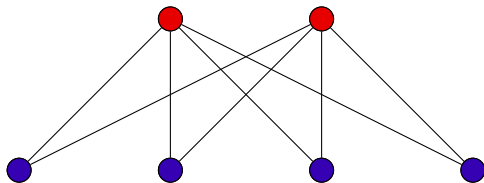
Marthe Bonamy

19/09/2013

Joint work with Ross Kang



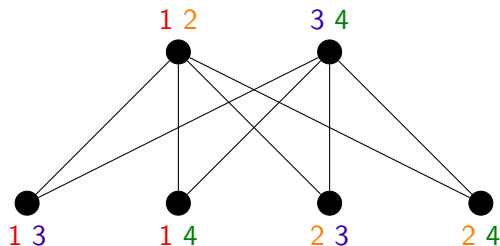




Graph  $k$ -colorable if:

$$\textcircled{a} - \textcircled{b} \Rightarrow \begin{cases} a \neq b \\ a, b \in [1, \dots, k] \end{cases}$$

# Vertex coloring



Graph  $k$ -colorable if:

$$\textcircled{a} - \textcircled{b} \Rightarrow \begin{cases} a \neq b \\ a, b \in [1, \dots, k] \end{cases}$$

Graph  $k$ -choosable if:

For every  $L$  with  $|L(v)| \geq k$ ,

$$\begin{array}{c} \textcircled{a} - \textcircled{b} \\ x \quad y \end{array} \Rightarrow \begin{cases} a \neq b \\ a \in L(x) \\ b \in L(y) \end{cases}$$

# Bounded spectrum list coloring

Graph  $(k, p)$ -choosable if:

For every  $L$  with  $|L(v)| \geq k$  and  $|\cup L(v)| \leq p$ ,

$$\begin{array}{c} \textcircled{a} \\ x \end{array} \text{---} \begin{array}{c} \textcircled{b} \\ y \end{array} \Rightarrow \begin{cases} a \neq b \\ a \in L(x) \\ b \in L(y) \end{cases}$$

# Bounded spectrum list coloring

Graph  $(k, p)$ -choosable if:

For every  $L$  with  $|L(v)| \geq k$  and  $\overbrace{|\cup L(v)|}^{\text{Spectrum}} \leq p$ ,

$$\begin{array}{c} \textcircled{a} \\ x \end{array} \text{---} \begin{array}{c} \textcircled{b} \\ y \end{array} \Rightarrow \begin{cases} a \neq b \\ a \in L(x) \\ b \in L(y) \end{cases}$$

# Bounded spectrum list coloring

Graph  $(k, p)$ -choosable if:

For every  $L$  with  $|L(v)| \geq k$  and  $\overbrace{|\cup L(v)|}^{\text{Spectrum}} \leq p$ ,

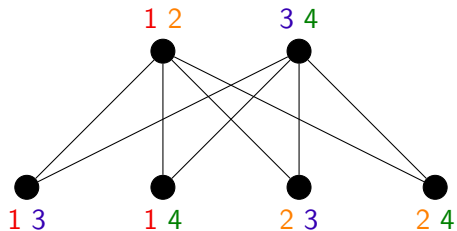
$$\begin{array}{c} \textcircled{a} \\ x \end{array} \text{---} \begin{array}{c} \textcircled{b} \\ y \end{array} \Rightarrow \begin{cases} a \neq b \\ a \in L(x) \\ b \in L(y) \end{cases}$$

$(k, k)$ -choosable  $\Leftrightarrow k$ -colorable.

$(k, p + 1)$ -choosable  $\Rightarrow (k, p)$ -choosable.

$(k, \infty)$ -choosable  $\Leftrightarrow k$ -choosable.

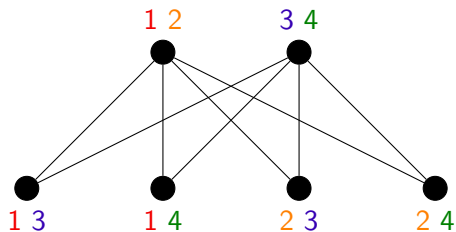
# Bipartite graphs



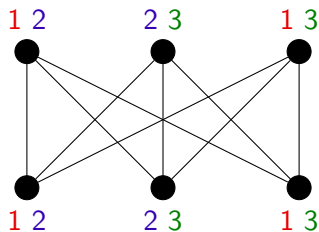
Not  $(2, 4)$ -choosable, but  $(2, 3)$ -choosable.



# Bipartite graphs



Not (2, 4)-choosable, but (2, 3)-choosable.



Not (2, 3)-choosable.

## Lemma (Easy)

Any *bipartite* graph is  $(k + 1, 2k)$ -choosable.

# Bipartite graphs (2)

## Lemma (Easy)

Any *bipartite* graph is  $(k + 1, 2k)$ -choosable.

## Lemma (Easy)

Some *bipartite* graphs are *not*  $(k + 1, 2k + 1)$ -choosable.

# Bipartite graphs (2)

## Lemma (Easy)

Any *bipartite* graph is  $(k + 1, 2k)$ -choosable.

## Lemma (Easy)

Some *bipartite* graphs are *not*  $(k + 1, 2k + 1)$ -choosable.

## Lemma (Easy)

$(2, 3)$ -choosable  $\not\Rightarrow$  2-choosable.

$(2, 4)$ -choosable  $\Rightarrow$  2-choosable.

# Bipartite graphs (2)

## Lemma (Easy)

Any  $\chi$ -colorable graph is  $(k(\chi - 1) + 1, k\chi)$ -choosable.

## Lemma (Easy)

Some bipartite graphs are *not*  $(k + 1, 2k + 1)$ -choosable.

## Lemma (Easy)

$(2, 3)$ -choosable  $\not\Rightarrow$  2-choosable.

$(2, 4)$ -choosable  $\Rightarrow$  2-choosable.

# Bipartite graphs (2)

## Lemma (Easy)

Any  $\chi$ -colorable graph is  $(k(\chi - 1) + 1, k\chi)$ -choosable.

## Lemma (Easy)

Some  $\chi$ -colorable graphs are *not*  $(k(\chi - 1) + 1, k\chi + 1)$ -choosable.

## Lemma (Easy)

$(2, 3)$ -choosable  $\not\Rightarrow$  2-choosable.

$(2, 4)$ -choosable  $\Rightarrow$  2-choosable.

## Question

$\exists? C_k, (k, C_k)$ -choosable  $\Rightarrow$   $k$ -choosable.

## Question

$\exists? C_k, (k, C_k)$ -choosable  $\Rightarrow$   $k$ -choosable.

$k = 1$  ✓

$k = 2$  ✓



## Question

$\exists?$   $C_k, (k, C_k)$ -choosable  $\Rightarrow$   $k$ -choosable.

$k = 1$  ✓

$k = 2$  ✓

## Theorem (Kráľ' Sgall 2005)

For  $k \geq 3$  and any  $p \geq k$ ,  $(k, p)$ -choosable  $\not\Rightarrow$   $k$ -choosable.

## Question

$\exists?$   $C_k$ ,  $(k, C_k)$ -choosable  $\Rightarrow$   $k$ -choosable.

$k = 1$  ✓

$k = 2$  ✓

## Theorem (Kráľ' Sgall 2005)

For  $k \geq 3$  and any  $p \geq k$ , there is a graph  $(k, p)$ -choosable and **not**  $(k, p + 1)$ -choosable.

## Question

$\exists? C_k, (k, C_k)$ -choosable  $\Rightarrow$   $k$ -choosable.

$k = 1$  ✓

$k = 2$  ✓

## Theorem (Kráľ' Sgall 2005)

For  $k \geq 3$  and any  $p \geq k$ , there is a graph  $(k, p)$ -choosable and **not**  $(k, p + 1)$ -choosable.

## Question (Kráľ' Sgall 2005)

$\exists? D_k, (k, D_k)$ -choosable  $\Rightarrow$   $(k + 1)$ -choosable.

## Question

$\exists? C_k, (k, C_k)$ -choosable  $\Rightarrow$   $k$ -choosable.

$k = 1$  ✓

$k = 2$  ✓

## Theorem (Kráľ' Sgall 2005)

For  $k \geq 3$  and any  $p \geq k$ , there is a graph  $(k, p)$ -choosable and **not**  $(k, p + 1)$ -choosable.

## Question (Kráľ' Sgall 2005)

$\exists? D_k, (k, D_k)$ -choosable  $\Rightarrow$   $(k + 1)$ -choosable.

True for  $k = 2$  (then  $D_2 = 3$ ).

## Questions (2)

### Question

$\exists?$   $C_{k,p}$ ,  $(k, p)$ -choosable  $\Rightarrow C_{k,p}$ -choosable.

## Questions (2)

### Question

$\exists? C_{k,p}, (k, p)$ -choosable  $\Rightarrow C_{k,p}$ -choosable.

$p \leq 2k - 2$  : **NO**

## Questions (2)

### Question

$\exists? C_{k,p}, (k,p)$ -choosable  $\Rightarrow C_{k,p}$ -choosable.

$p \leq 2k - 2$  : **NO**

### Theorem (Kráľ' Sgall 2005)

$C_{k,2k-1} = \mathcal{O}(k \cdot \ln k \cdot 2^{4k})$ .

Sketch of the proof (blackboard).

## Questions (2)

### Question

$\exists? C_{k,p}, (k,p)$ -choosable  $\Rightarrow C_{k,p}$ -choosable.

$p \leq 2k - 2$  : **NO**

### Theorem (Kráľ' Sgall 2005)

$C_{k,2k-1} = \mathcal{O}(k \cdot \ln k \cdot 2^{4k})$ .

Sketch of the proof (blackboard).

### Question (Kráľ' Sgall 2005)

Must  $C_{k,2k-1}$  grow *exponentially* in  $k$ ?



## Questions (2)

### Question

$\exists? C_{k,p}$ ,  $(k, p)$ -choosable  $\Rightarrow C_{k,p}$ -choosable.

$p \leq 2k - 2$  : **NO**

### Theorem (Kráľ' Sgall 2005)

$C_{k,2k-1} = \mathcal{O}(k \cdot \ln k \cdot 2^{4k})$ .

Sketch of the proof (blackboard).

### Question (Kráľ' Sgall 2005)

Must  $C_{k,2k-1}$  grow *exponentially* in  $k$ ?

### Theorem (B. Kang 2013)

**Yes.**

Sketch of the proof (blackboard).

Theorem (B. Kang 2013)

For any  $p \geq 2k - 1$ ,  $C_{k,p} > \exp\left(\frac{(k-1)^2}{p}\right)$ .

Only interesting if  $p = o(k^2)$ .

# Properties B and K

Theorem (B. Kang 2013)

For any  $p \geq 2k - 1$ ,  $C_{k,p} > \exp\left(\frac{(k-1)^2}{p}\right)$ .

Only interesting if  $p = o(k^2)$ .

Definition (Berstein 1908)

A family of sets  $\mathcal{F} \subseteq \binom{[p]}{k}$  has **property B** if there is an **intersecting set**  $S$  of  $\mathcal{F}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

# Properties B and K

Theorem (B. Kang 2013)

For any  $p \geq 2k - 1$ ,  $C_{k,p} > \exp\left(\frac{(k-1)^2}{p}\right)$ .

Only interesting if  $p = o(k^2)$ .

Definition (Berstein 1908)

A family of sets  $\mathcal{F} \subseteq \binom{[p]}{k}$  has **property B** if there is an **intersecting set**  $S$  of  $\mathcal{F}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

(2-colorable  $k$ -uniform  $p$ -hypergraphs).

# Properties B and K

## Theorem (B. Kang 2013)

For any  $p \geq 2k - 1$ ,  $C_{k,p} > \exp\left(\frac{(k-1)^2}{p}\right)$ .

Only interesting if  $p = o(k^2)$ .

## Definition (Berstein 1908)

A family of sets  $\mathcal{F} \subseteq \binom{[p]}{k}$  has **property B** if there is an **intersecting set**  $S$  of  $\mathcal{F}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

(2-colorable  $k$ -uniform  $p$ -hypergraphs).

## Definition (B. Kang 2013)

A family of sets  $\mathcal{F} \subseteq \binom{[p]}{k}$  has **property K** if there is a set  $S \in \binom{[p]}{p-k+1}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

# Properties B and K

Theorem (B. Kang 2013)

For any  $p \geq 2k - 1$ ,  $C_{k,p} > \exp\left(\frac{(k-1)^2}{p}\right)$ .

Only interesting if  $p = o(k^2)$ .

Definition (Berstein 1908)

A family of sets  $\mathcal{F} \subseteq \binom{[p]}{k}$  has **property B** if there is an **intersecting set**  $S$  of  $\mathcal{F}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

(2-colorable  $k$ -uniform  $p$ -hypergraphs).

Definition (B. Kang 2013)

A family of sets  $\mathcal{F} \subseteq \binom{[p]}{k}$  has **property K** if there is a set  $S \in \binom{[p]}{p-k+1}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

Hypergraphs?

# Property B

## Definition (Berstein 1908)

A family of sets  $\mathcal{F}$  has **property B** if there is an **intersecting set**  $S$  of  $\mathcal{F}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

## Definition (Erdős 1963)

$M_{k,p}$  = size of the smallest  $\mathcal{F} \subseteq \binom{[p]}{k}$  that does **not** have **property B** (i.e. every **intersecting set** of  $\mathcal{F}$  **belongs to**  $\mathcal{F}$ ).

# Property B

## Definition (Berstein 1908)

A family of sets  $\mathcal{F}$  has **property B** if there is an **intersecting set**  $S$  of  $\mathcal{F}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

## Definition (Erdős 1963)

$M_{k,p}$  = size of the smallest  $\mathcal{F} \subseteq \binom{[p]}{k}$  that does **not** have **property B** (i.e. every **intersecting set** of  $\mathcal{F}$  **belongs to**  $\mathcal{F}$ ).

$$M_{k,2k-1} = \binom{2k-1}{k}$$



# Property B

## Definition (Berstein 1908)

A family of sets  $\mathcal{F}$  has **property B** if there is an **intersecting set**  $S$  of  $\mathcal{F}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

## Definition (Erdős 1963)

$M_{k,p}$  = size of the smallest  $\mathcal{F} \subseteq \binom{[p]}{k}$  that does **not** have **property B** (i.e. every **intersecting set** of  $\mathcal{F}$  **belongs to**  $\mathcal{F}$ ).

$$M_{k,2k-1} = \binom{2k-1}{k}$$

## Theorem (Kráľ' Sgall 2005)

$\delta \geq 110 \cdot M(k, p)^2 \cdot \ln(M(k, p)) \Rightarrow$  **not**  $(k, p)$ -**choosable**.

# Property B

## Definition (Berstein 1908)

A family of sets  $\mathcal{F}$  has **property B** if there is an **intersecting set**  $S$  of  $\mathcal{F}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

## Definition (Erdős 1963)

$M_{k,p}$  = size of the smallest  $\mathcal{F} \subseteq \binom{[p]}{k}$  that does **not** have **property B** (i.e. every **intersecting set** of  $\mathcal{F}$  **belongs to**  $\mathcal{F}$ ).

$$M_{k,2k-1} = \binom{2k-1}{k}$$

## Theorem (Kral' Sgall 2005)

$\delta \geq 110 \cdot M(k,p)^2 \cdot \ln(M(k,p)) \Rightarrow$  **not**  $(k,p)$ -**choosable**.

Kang 2012: generalization to improper choosability.

# Property K

## Definition (B. Kang 2013)

A family of sets  $\mathcal{F} \subseteq \binom{[p]}{k}$  has **property K** if there is a set  $S \in \binom{[p]}{p-k+1}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

## Definition

$R_{k,p}$  = size of the smallest  $\mathcal{F} \subseteq \binom{[p]}{k}$  that does **not** have **property K** (i.e. every set  $S \in \binom{[p]}{p-k+1}$  contains an element of  $\mathcal{F}$ ).

## Definition (B. Kang 2013)

A family of sets  $\mathcal{F} \subseteq \binom{[p]}{k}$  has **property K** if there is a set  $S \in \binom{[p]}{p-k+1}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

$$R_{k,2k-1} = \binom{2k-1}{k}.$$

# Property K

## Definition (B. Kang 2013)

A family of sets  $\mathcal{F} \subseteq \binom{[p]}{k}$  has **property K** if there is a set  $S \in \binom{[p]}{p-k+1}$  with  $F \not\subseteq S \quad \forall F \in \mathcal{F}$ .

$$R_{k,2k-1} = \binom{2k-1}{k}.$$

## Lemma

Any *bipartite* graph  $(A, B)$  is  $(k, p)$ -choosable if  $|A| < R(k, p)$ .

## Theorem (B. Kang 2013)

$$R_{k,p} \geq \exp\left(\frac{(k-1)^2}{p}\right).$$

Sketch of the proof (blackboard).

$$R_{k,\geq k^2} = k.$$

Question (Kráľ' Sgall 2005)

$\exists? D_k, (k, D_k)$ -choosable  $\Rightarrow (k + 1)$ -choosable.

- $D_k \stackrel{?}{=} k^2$ .

## Question (Kráľ' Sgall 2005)

$\exists? D_k, (k, D_k)$ -choosable  $\Rightarrow (k + 1)$ -choosable.

- $D_k \stackrel{?}{=} k^2$ .
- $\delta \geq f(R_{k,p}) \stackrel{?}{\Rightarrow}$  not  $(k, p)$ -choosable.

## Question (Kráľ' Sgall 2005)

$\exists? D_k, (k, D_k)$ -choosable  $\Rightarrow (k + 1)$ -choosable.

- $D_k \stackrel{?}{=} k^2$ .
- $\delta \geq f(R_{k,p}) \stackrel{?}{\Rightarrow}$  not  $(k, p)$ -choosable.
- $(k, p)$ -choosability of complete bipartite graphs?



## Question (Kráľ' Sgall 2005)

$\exists? D_k, (k, D_k)$ -choosable  $\Rightarrow (k + 1)$ -choosable.

- $D_k \stackrel{?}{=} k^2$ .
- $\delta \geq f(R_{k,p}) \stackrel{?}{\Rightarrow}$  not  $(k, p)$ -choosable.
- $(k, p)$ -choosability of complete bipartite graphs?

# Thanks!