

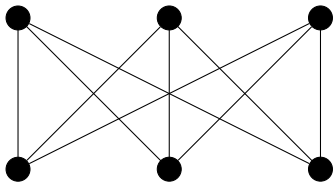
# List coloring the squares of sparse graphs

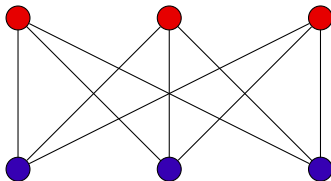
**Marthe Bonamy**  
Benjamin Lévêque  
Alexandre Pinlou

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Laboratoire  
Informatique  
Robotique  
Microélectronique  
Montpellier

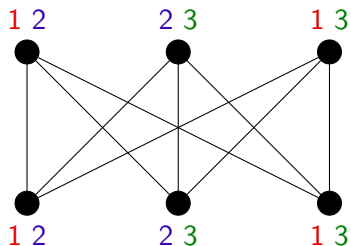




$\chi$ : Minimum number of colors to ensure that

$$\textcircled{a} \text{---} \textcircled{b} \Rightarrow a \neq b.$$

# Vertex coloring



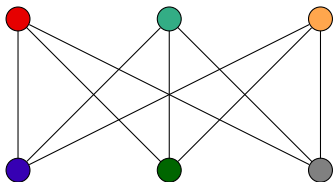
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$\chi_e$ : Minimum size of every  $L(v)$  such that

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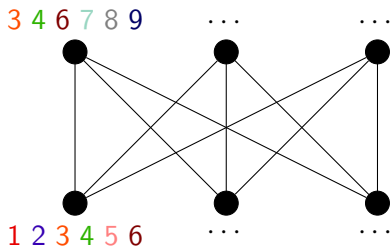
# Square coloring



$\chi^2$ : Minimum number of colors to ensure that

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# Square coloring



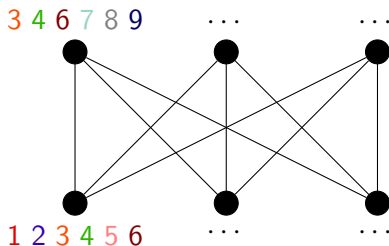
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$$\Delta + 1 \leq \chi^2 \leq \chi_\ell^2 \leq \Delta^2 + 1$$

# Two conjectures

Conjecture (Kostochka, Woodall '01)

$$\chi^2 = \chi_\ell^2.$$

**Planar** graph  $\Leftrightarrow$  can be drawn in a plane without crossing edges.

Conjecture (Wegner 1977)

For *planar* graphs,  $\chi^2 \leq \lfloor \frac{3\Delta}{2} \rfloor + 1$  if  $\Delta \geq 8$ .



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Recently disproved (Kim, Park '13)

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- For *planar* graphs,  $\exists d, g \geq 7, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1$ .
- $\exists (G_n)_{n \in \mathbb{N}}$ , planar,  $g = 6, \Delta \nearrow, \chi^2 > \Delta + 1$ .

# State of the art

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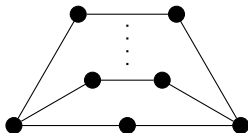
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$$g = 6.$$
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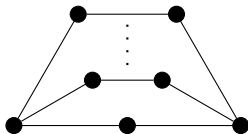
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$$g = 5.$$
$$\chi_l^2 = \Delta + 2 > \Delta + 1.$$



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For *planar* graphs:

- $g \geq 7 \Rightarrow \text{mad} < \frac{14}{5}$ .
- $g \geq 6 \Rightarrow \text{mad} < 3$ .

## Question

What is the maximal  $m \mid \exists d$ ,  
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Answer: True for any  $m < 3$ .

- $\forall \epsilon > 0$ ,  $\text{mad} < 3 - \epsilon$ ,  $\Delta \geq \mathcal{O}(\frac{1}{\epsilon^2}) \Rightarrow \chi_\ell^2 = \Delta + 1$ .
- $\exists (G_n)_{n \in \mathbb{N}}$ ,  $\text{mad} < 3$ ,  $\Delta \nearrow$ ,  $\chi_\ell^2 > \Delta + 1$ .

## Theorem

For a small enough  $\epsilon$ ,  $\text{mad} < 3 - \epsilon$  and  $\Delta \geq \frac{3}{\epsilon^2} \Rightarrow \chi_\ell^2 = \Delta + 1$ .

Take a minimal counter-example.



# Key idea in the proof

## Theorem

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## Theorem (Borodin, Kostochka, Woodall '97)

For any *bipartite multigraph*  $G$ , if  $L$  is an edge list assignment such that  $|L(u, v)| \geq d(u), d(v)$  for every  $(u, v)$ , then  $G$  is  *$L$ -edge-choosable*.

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$\Rightarrow$  Global discharging method.

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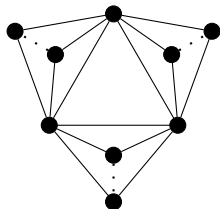
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$$\begin{aligned} mad &= \\ 4 - \frac{4}{\Delta}. \\ \chi_\ell^2 &= \frac{3\Delta}{2}. \end{aligned}$$

Optimal theorem  
 $M = 4$

# Conclusion

## Conjecture

For any  $C$  and  $\epsilon > 0$ ,

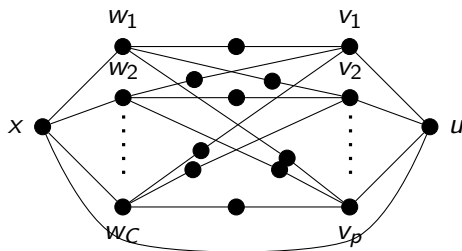
$$\text{mad} < \frac{4C+2}{C+1} - \epsilon \text{ and } \Delta \geq f(C, \epsilon) \Rightarrow \chi_\ell^2 \leq \Delta + C.$$

$$C = 1 \rightsquigarrow 3$$

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...

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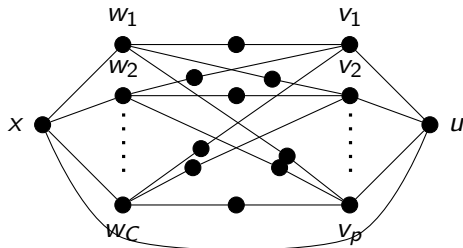
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Thanks for your attention.