

# Some Remarks on Logic and Topology

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# Why Topology?

- Topology seems to be at the right level of abstraction (eg. when compared to geometry).
- Basically, we have regions, their complements (intended to model “outside”), their interiors (intended to model “inside”) and frontiers.
- Rich and well-developed field of mathematics

# Why Topology?

- It allows us to model statements like:
  - “the frontier between France and Spain runs through the Pyrénées”
  - “France is the union of Continental France and Corsica, yet Continental France and Corsica are disconnected”



# Some Potential Problems

- Topology, being very general and very abstract, makes more distinctions than are useful when reasoning about real regions in space.
- Reasoning about topological spaces can be very complex.

# Goals

- Find some nice, well-behaved regions in topological space.
- Find a language which is expressive enough to make topologically interesting statements, yet has reasonable computational complexity.

# Reminder

## Basic Definitions

- A topology is a set  $X$  (the universe) and a collection  $\tau$  of subsets of  $X$  (the open sets) such that:
- $\tau$  contains  $X$  and  $\emptyset$
- The union of any collection of elements of  $\tau$  is in  $\tau$
- The intersection of any finite collection of elements of  $\tau$  is in  $\tau$

Sometimes it is useful to permit infinite intersections as well: in this case we will talk about an Alexandrov space or Alexandrov topology. An Alexandrov topology is isomorphic to specialization preorder on a set  $X$ , that is  $x \leq y$  if and only if all open sets which contain  $x$  contain  $y$  as well. This makes the link between preorders (reflexive and transitive relations) and  $S4$  (with a reflexive and transitive accessibility relation  $R$ ).



# Reminder

## Basic Definitions

- The complement of a subset  $A$  of  $X$ , which we will note by  $A'$  is defined as  $X \setminus A$ .
- The complement of an open set is closed.

Some people write  $A^c$  for the complement of a set. In order to avoid confusion with the closure of a set, I will write  $A'$  for the complement and  $A^-$  for the closure.

# Reminder

## Basic Definitions

- A set can be both closed and open (specifically, both  $X$  and  $\emptyset$  are both closed and open in any topology)
- a set can also be neither closed nor open.

Some people like the word "clopen" for a set which is both closed and open. I don't. It is very widely used, though.



# Reminder (Very) Basic Properties

- $(A \cup B)' = A' \cap B'$
- $X \setminus (A \cup B) = X \setminus A \cap X \setminus B$
- $\forall x \, x \in X \wedge x \notin (A \cup B) \Leftrightarrow$   
 $x \in X \wedge x \notin A \wedge x \notin B$

de Morgan law for union  
and complement

# Reminder (Very) Basic Properties

- $(A \cup B)' = A' \cap B'$
- $X \setminus (A \cup B) = X \setminus A \cap X \setminus B$
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# Reminder (Very) Basic Properties

- $(A \cap B)' = A' \cup B'$
- $A'' = A$

All of this is basic set theory  
We have, in essence, a boolean algebra.

# Reminder

## Basic Definitions

- Given a set  $A$ , the interior  $A^\circ$  of  $A$  is the union of all open sets  $O$  such that  $O \subseteq A$ .
- Given a set  $A$ , the closure  $A^-$  of  $A$  is the intersection of all closed sets  $C$  such that  $A \subseteq C$ .
- Evidently, we have  $A^\circ \subseteq A \subseteq A^-$



# Reminder

## Basic Definitions

- Evidently, we have  $A^\circ \subseteq A \subseteq A^-$ 
  - if  $A$  is open then  $A^\circ = A$
  - if  $A$  is closed then  $A^- = A$
- The boundary of  $A$ ,  $\delta A$  is defined as  $A^- \setminus A^\circ$

# Open and Closed Sets

General	$A^\circ$	$\subseteq$	$A$	$\subseteq$	$A^-$
Open	$A^\circ$	$=$	$A$	$\subseteq$	$A^-$
Closed	$A^\circ$	$\subseteq$	$A$	$=$	$A^-$
“Clopen”	$A^\circ$	$=$	$A$	$=$	$A^-$

# Reminder

## Example 0

Classically, first examples in topology are about intervals on the real line...

$$A = [0, 1>$$

$$A^{\circ} = <0, 1>$$

$$A^{-} = [0, 1]$$

$$\delta A = \{0, 1\}$$

$$A' = <-\infty, 1> \cup [0, \infty>$$



# Reminder

## Example 1

$$X = \{a, b, c, d, e\}$$

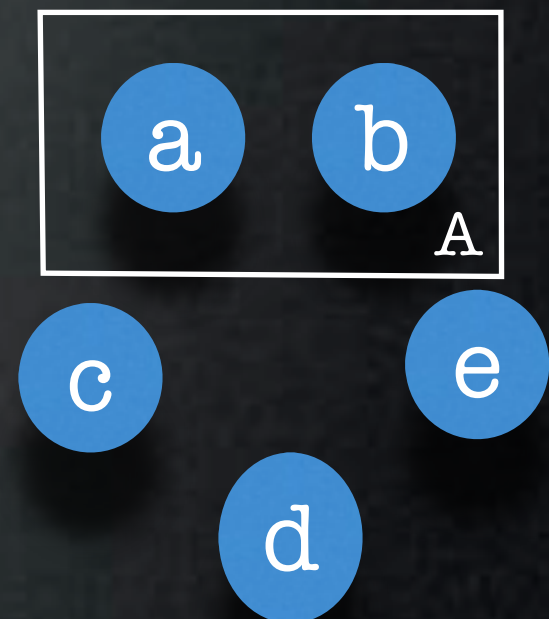
$$\tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

Open sets

$$\tau' = \{\emptyset, X, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}\}$$

Closed sets

$$A = \{a, b\}$$



# Reminder

## Example 1

$$X = \{a, b, c, d, e\}$$

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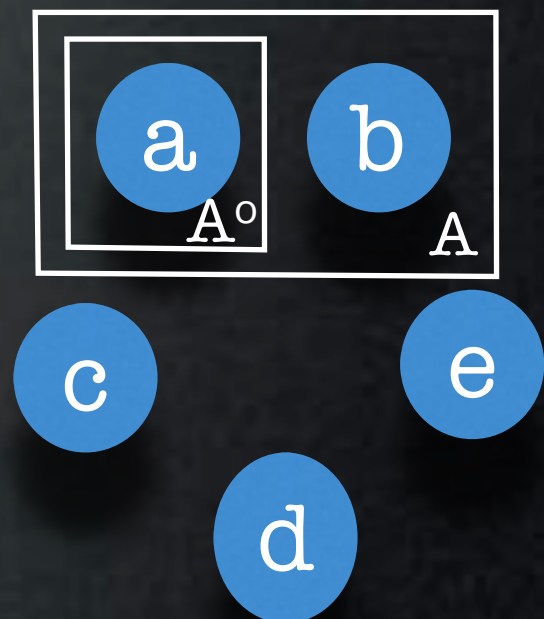
Open sets

$$\tau' = \{\emptyset, X, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}\}$$

Closed sets

$$A = \{a, b\}$$

$$A^\circ = \{a\} \cup \emptyset = \{a\}$$



# Reminder

## Example 1

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Open sets

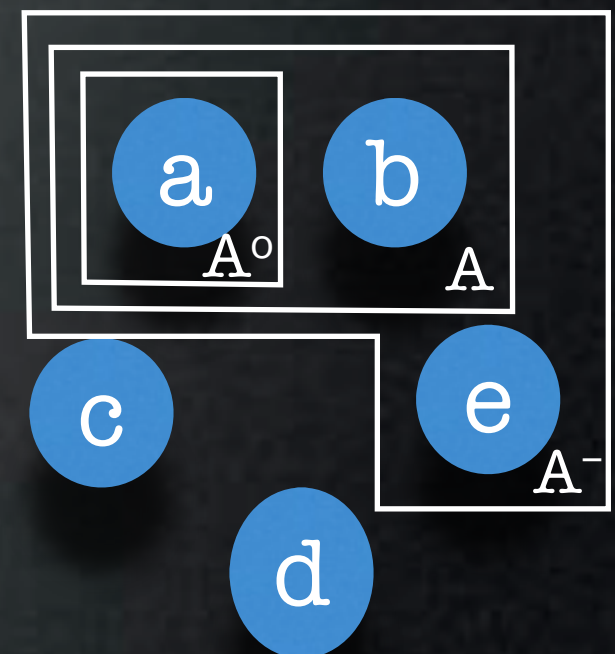
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Closed sets

$$A = \{a, b\}$$

$$A^\circ = \{a\} \cup \emptyset = \{a\}$$

$$A^- = X \cap \{a, b, e\} = \{a, b, e\}$$





# Reminder

## Example 1

$$X = \{a, b, c, d, e\}$$

$$\tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

Open sets

$$\tau' = \{\emptyset, X, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}\}$$

Closed sets

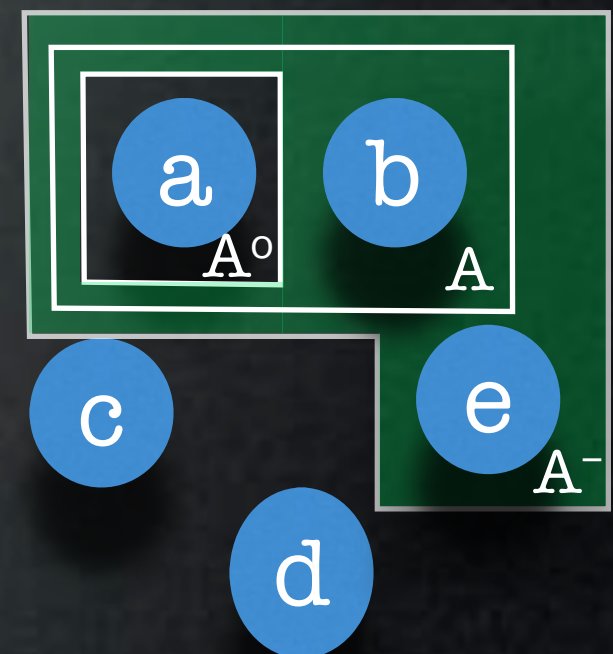
$$A = \{a, b\}$$

  $\delta A$

$$A^\circ = \{a\} \cup \emptyset = \{a\}$$

$$A^- = X \cap \{a, b, e\} = \{a, b, e\}$$

$$\delta A = \{a, b, e\} \setminus \{a\} = \{b, e\}$$



# Reminder

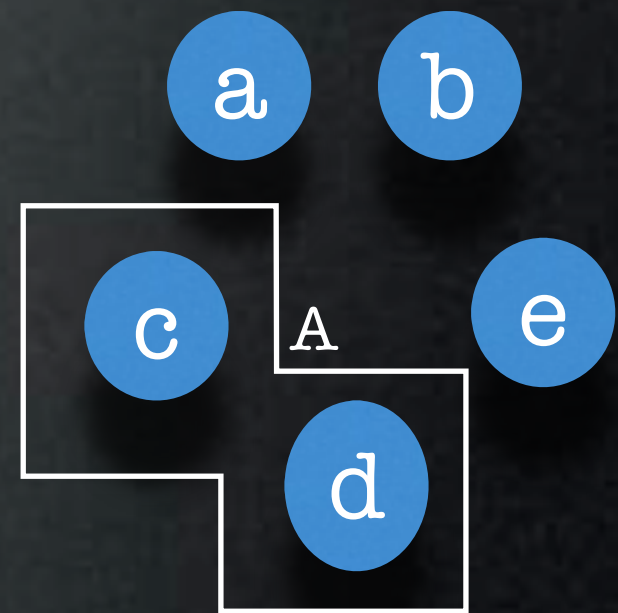
## Example 2

$$X = \{a, b, c, d, e\}$$

$$\tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

$$\tau' = \{\emptyset, X, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}\}$$

$$A = \{c, d\}$$



# Reminder

## Example 2

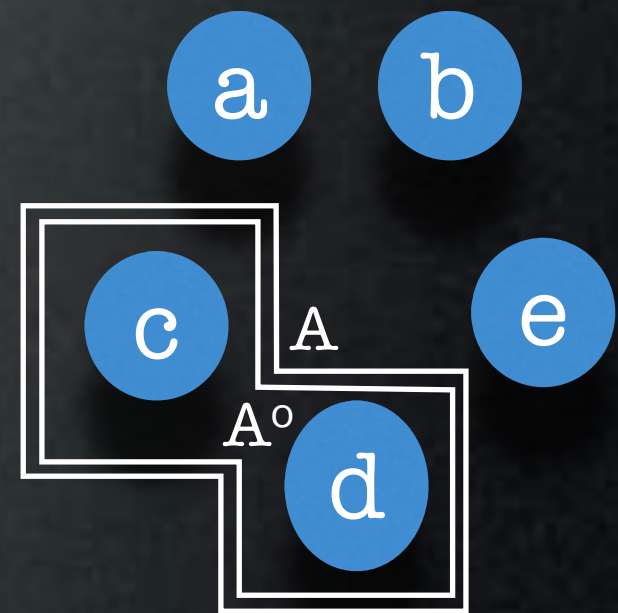
$$X = \{a, b, c, d, e\}$$

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$$\tau' = \{\emptyset, X, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}\}$$

$$A = \{c, d\}$$

$$A^\circ = \{c, d\} \cup \emptyset = \{c, d\}$$





# Reminder

## Example 2

$$X = \{a, b, c, d, e\}$$

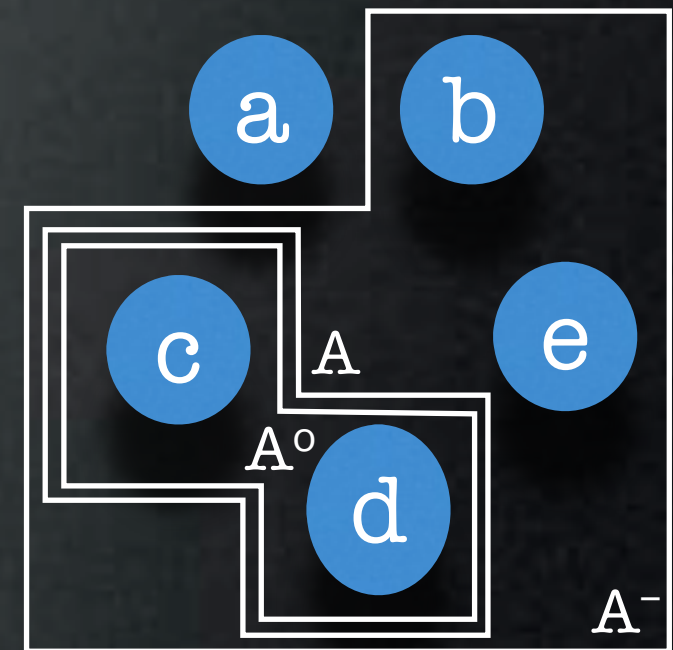
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$$A = \{c, d\}$$

$$A^{\circ} = \{c, d\} \cup \emptyset = \{c, d\}$$

$$A^{-} = X \cap \{b, c, d, e\} = \{b, c, d, e\}$$



# Reminder

## Example 2

$$X = \{a, b, c, d, e\}$$

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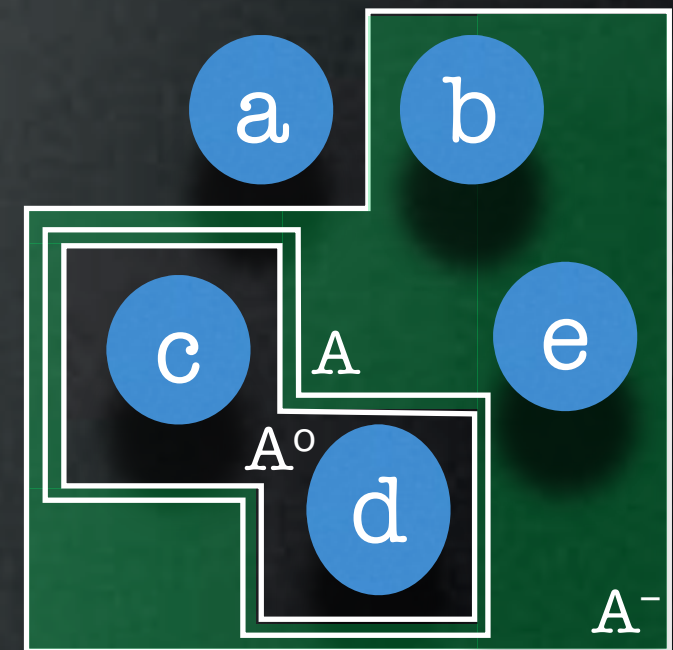
$$A = \{c, d\}$$

  $\delta A$

$$A^\circ = \{c, d\} \cup \emptyset = \{c, d\}$$

$$A^- = X \cap \{b, c, d, e\} = \{b, c, d, e\}$$

$$\delta A = \{b, c, d, e\} \setminus \{c, d\} = \{b, e\}$$

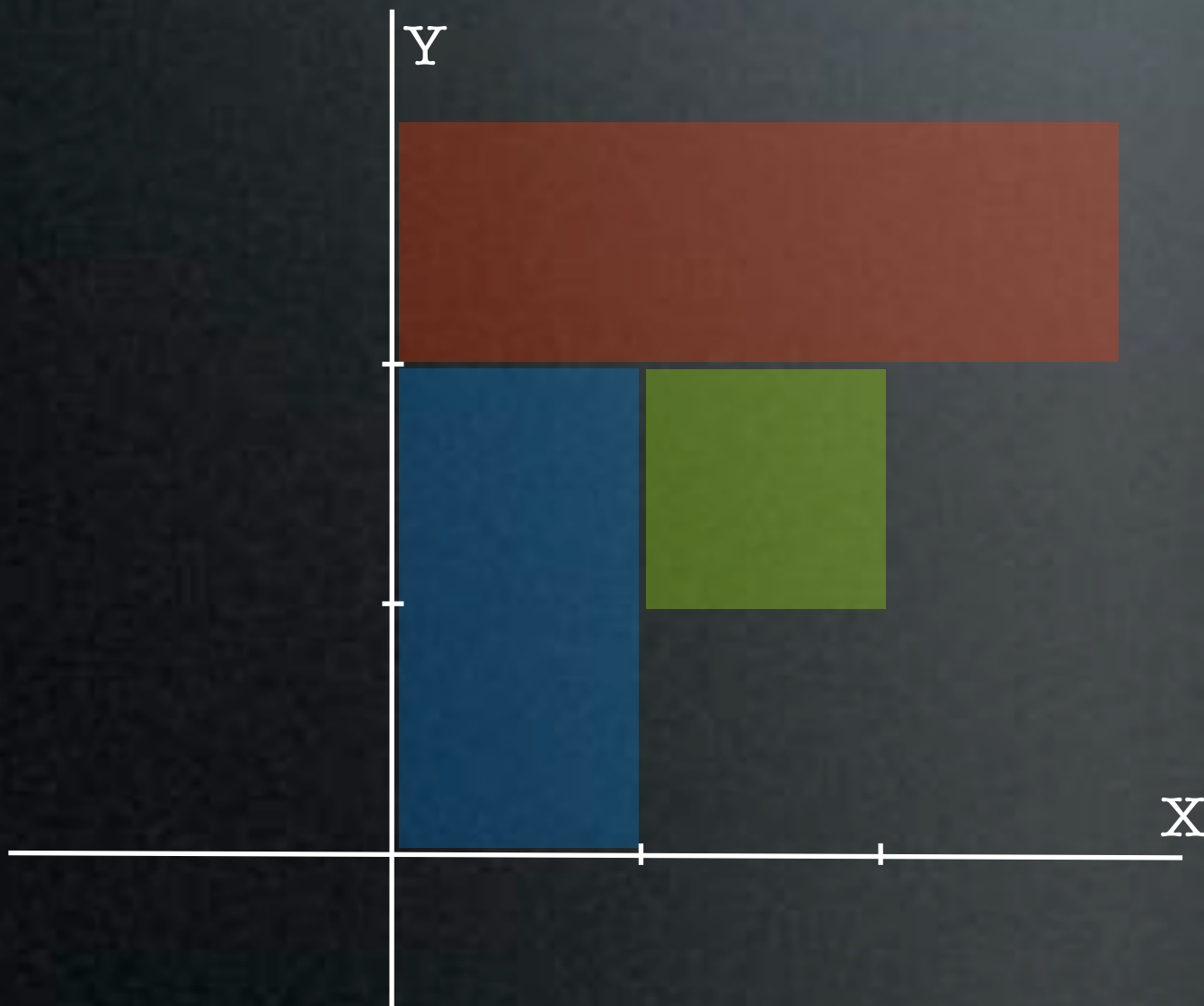


# Reminder

## Example 3

$$X = \mathbb{R}^2$$

$$\tau = \{\emptyset, \mathbb{R}^2, \langle 0,0 \rangle - \langle 1,2 \rangle, \langle 0,2 \rangle - \langle 3,3 \rangle, \langle 1,1 \rangle - \langle 2,2 \rangle, \dots\}$$



- Open rectangles generate one of the standard topologies on  $\mathbb{R}^2$
- As an example, the following open rectangles are part of this standard topology.



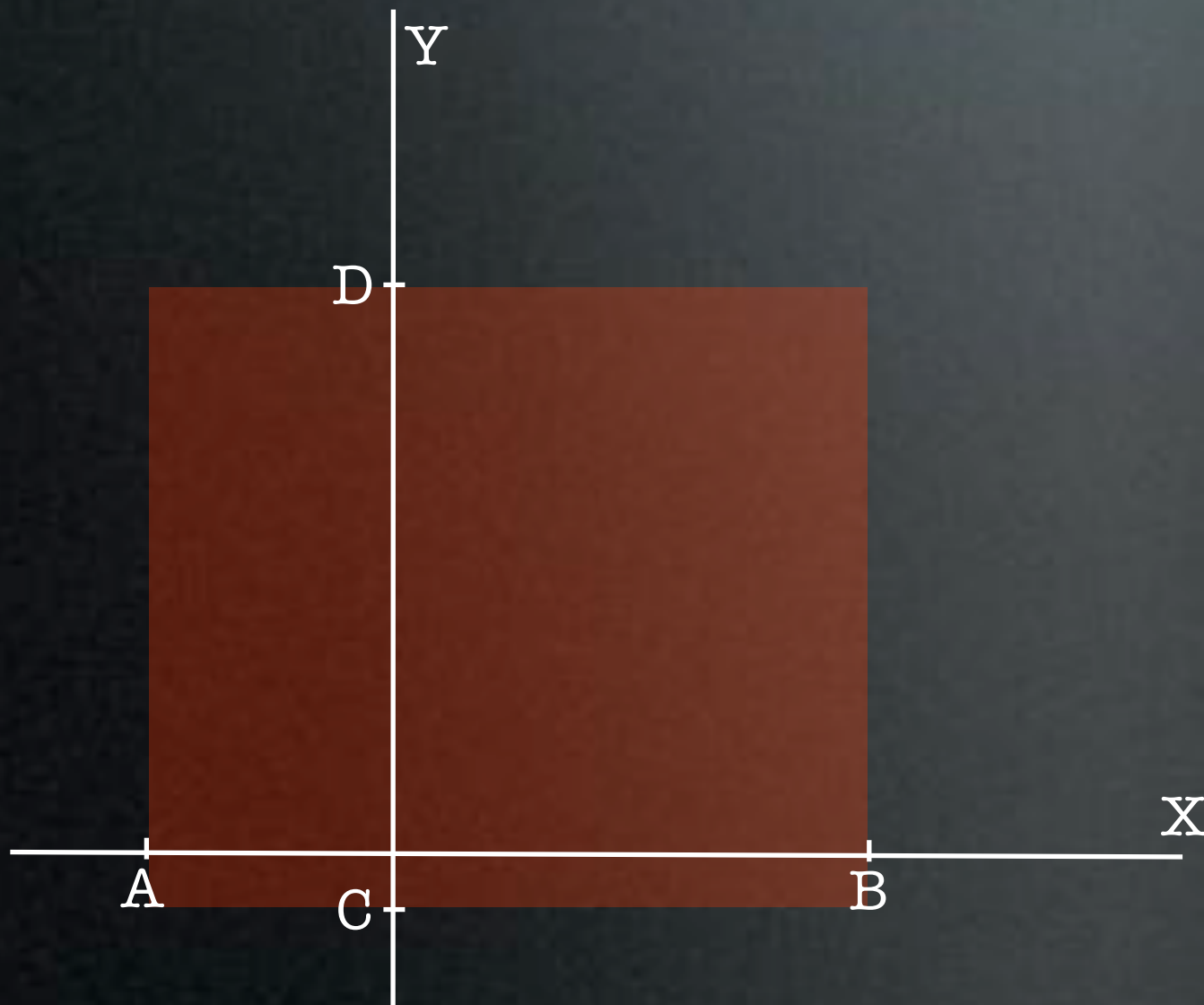
# Reminder

## Example 3

$$X = \mathbb{R}^2$$

$$\tau = \{\emptyset, \mathbb{R}^2, \langle A, B \rangle - \langle C, D \rangle\}$$

$$A < C, B < D$$



- Open rectangles generate one of the standard topologies on  $\mathbb{R}^2$
- However, even the general definition of open sets listed above does not form a topology (why?)

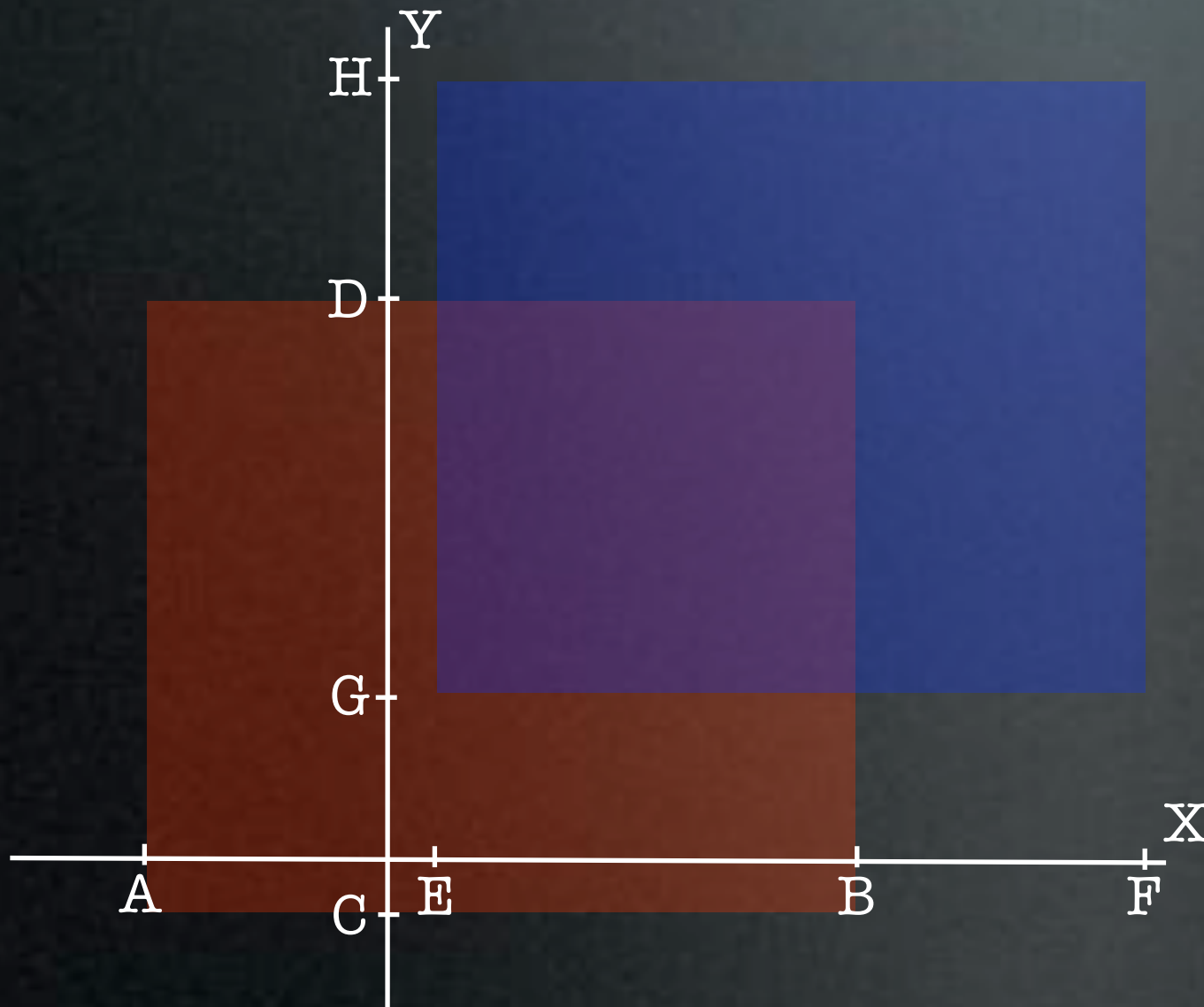
# Reminder

## Example 3

$$X = \mathbb{R}^2$$

$$\tau = \{\emptyset, \mathbb{R}^2, \langle A, B \rangle - \langle C, D \rangle\}$$

$$A < B, C < D$$



- When we have two open rectangles, their intersection is either an open

So while the purple part of the figure to the right is a part of the defined topology, the union (that is, the union of the blue, purple and red part) is not a rectangle and therefore not a part of the topology as defined above.

or empty.

their  
not an open

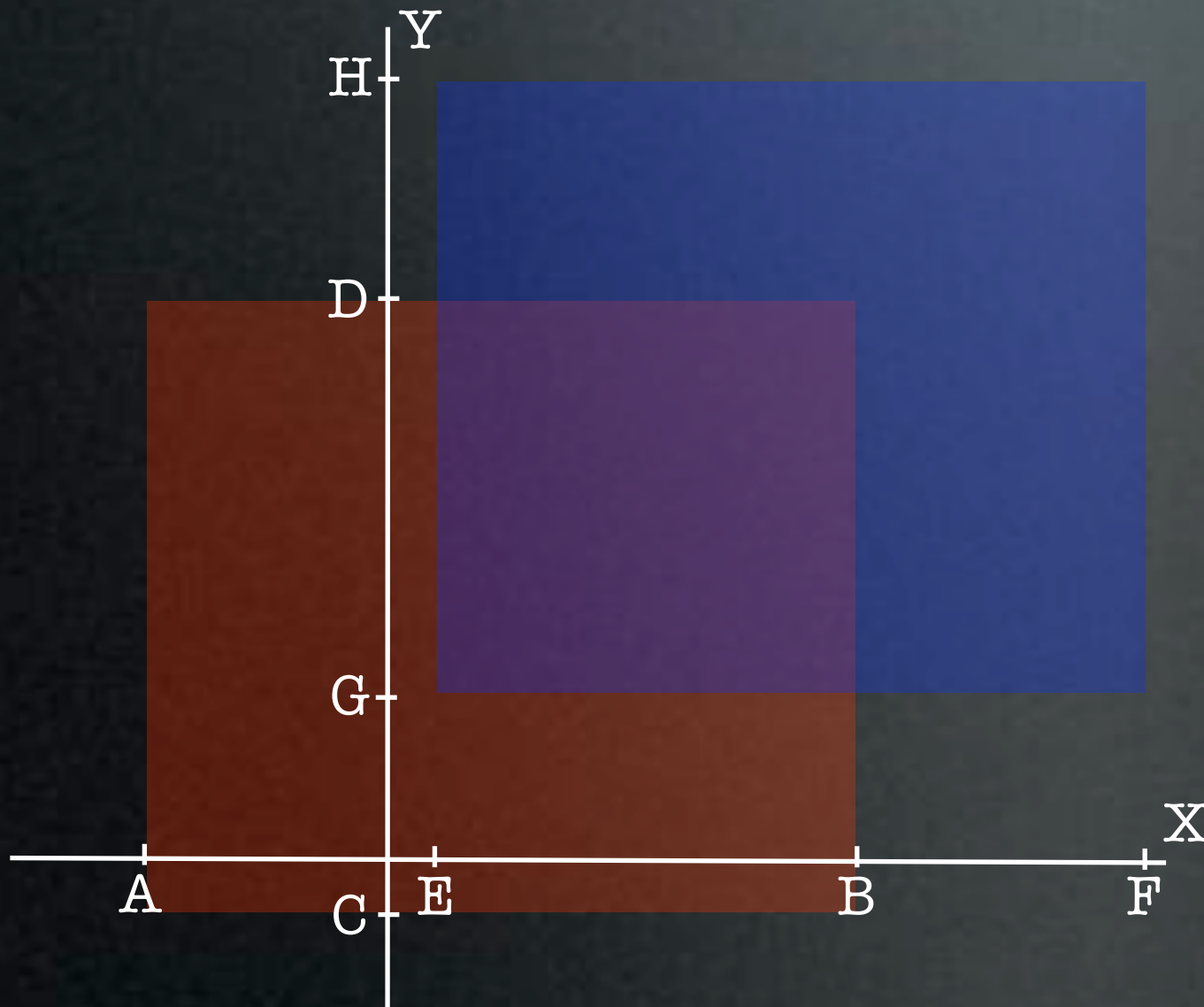
# Reminder

## Example 3

$$X = \mathbb{R}^2$$

$$B = \{\emptyset, \mathbb{R}^2, \langle A, B \rangle - \langle C, D \rangle\}$$

$$A < B, C < D$$



- We say a collection of open sets  $B$  is a base (basis) for a topology if any element of  $\tau$  is the union of elements of  $B$



# Reminder

## Bases

- Bases allow us to give a compact way to describe topologies.
- Some topological properties can also be verified directly on the base elements.

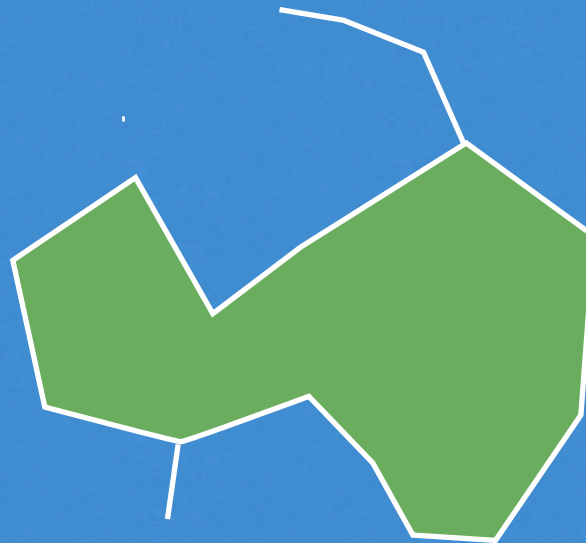
# Regular Closed Sets

Regular closed



A

Not reg. closed



B

- A is regular closed iff  $A = A^{0-}$
- roughly speaking, this means no “loose points” and no “hanging lines”

# Regular Closed Sets

Regular closed



$A^0$

Not reg. closed



$B^0$

- $A$  is regular closed iff  $A = A^{0-}$
- roughly speaking, this means no “loose points” and no “hanging lines”



# Regular Closed Sets

Regular closed



$$\begin{aligned} A^{o-} \\ = \underline{A} \end{aligned}$$

Not reg. closed



$$\begin{aligned} B^{o-} \\ \neq \underline{B} \end{aligned}$$

- A is regular closed iff  $A = A^{o-}$
- roughly speaking, this means no “loose points” and no “hanging lines”

Note, this does not show that  $A^{o-}$  is necessarily regular closed, I will come to that in a few slides.

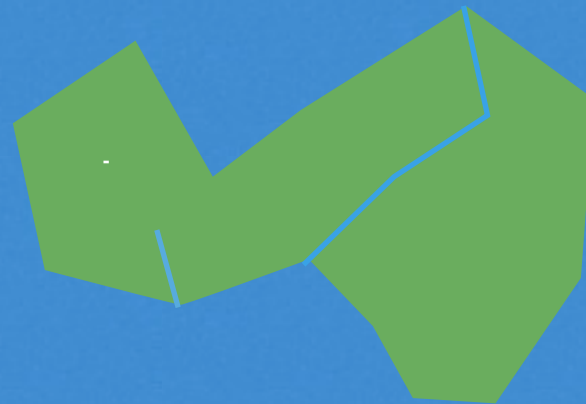
# Regular Open Sets

Regular open



A

Not reg. open



B

- A is regular open iff  $A = A^{-\circ}$
- roughly speaking, this means no “holes” and no “cracks”

# Regular Open Sets

Regular open



$A^-$

Not reg. open



$B^-$

- $A$  is regular open iff  $A = A^{-\circ}$
- roughly speaking, this means no “holes” and no “cracks”



# Regular Open Sets

Regular open



$$\begin{aligned} A^{-\circ} \\ = \underline{A} \end{aligned}$$

Not reg. open



$$\begin{aligned} B^{-\circ} \\ \neq \underline{B} \end{aligned}$$

- A is regular open iff  $A = A^{-\circ}$
- roughly speaking, this means no “holes” and no “cracks”

Again, this does not suffice to show that  $A^{-\circ}$  is regular open.

# Regular Sets

- Suppose  $A$  is a regular open set, that is  $A = A^{-\circ}$ .
- Thus we have, for its complement  $A'$  that  $A' = A^{-\circ'} = A^{-'\circ} = A'^{\circ-}$
- In addition, we have  $A^- = A^{-\circ-}$
- In other words, both  $A'$  and  $A^-$  are regular closed.

Similar remarks hold for regular closed sets. Given a regular closed set  $A$ , both its complement and its interior are regular open.

# Combinations

- Given a region variable  $A$ , how many (potentially) different regions can we construct using the closure, interior and complement relations?
- Our possibilities are limited by the fact that a double complement is the identity function and that the closure and interior operations are idempotent.



# Combinations

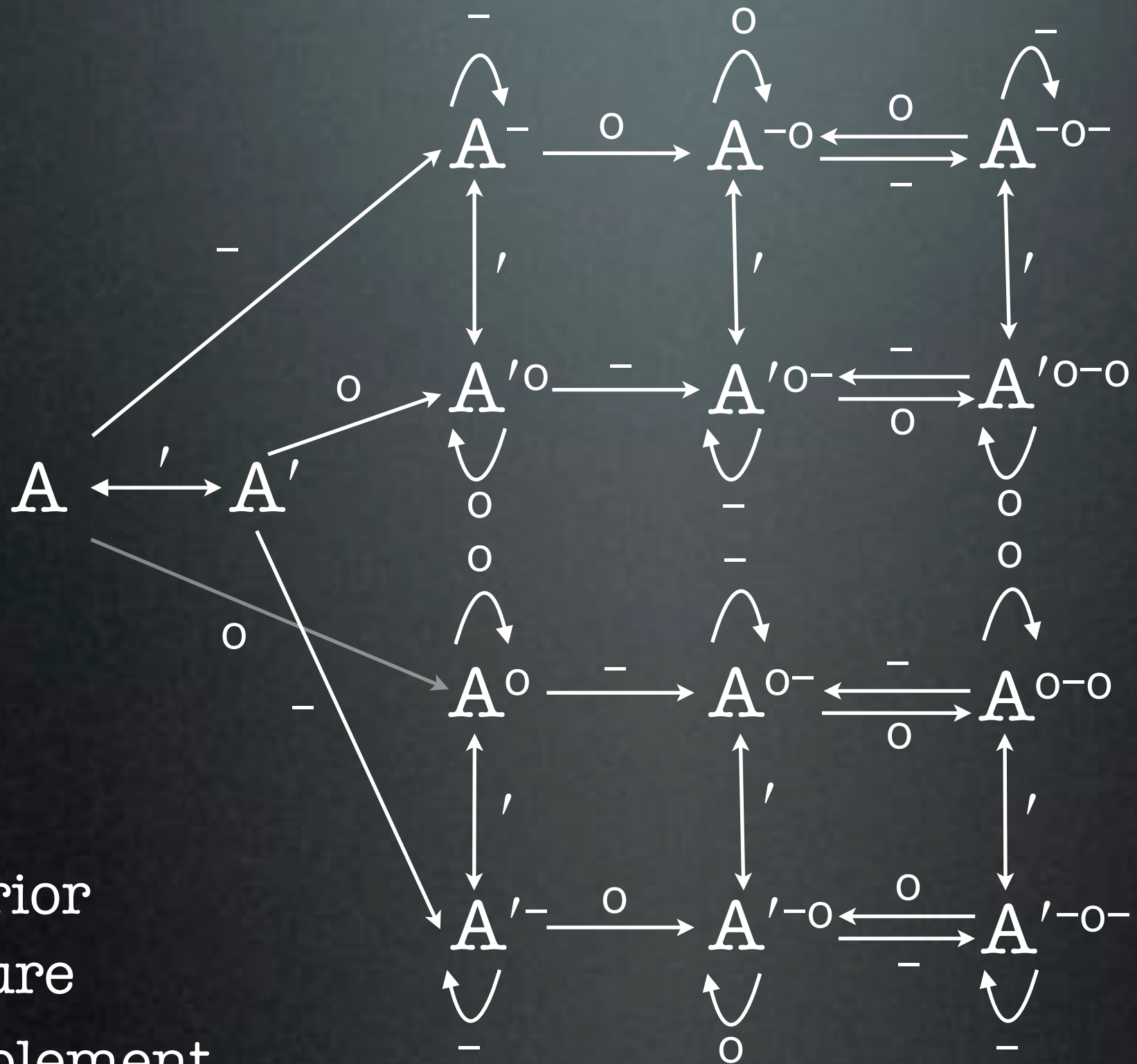
- Equivalences

- $A'' = A$
- $A^{oo} = A^o$
- $A^{--} = A^-$
- $A^{o-o-} = A^{o-}$
- $A^{-o-o} = A^{-o}$

- Non-equivalent

- $A$
- $A^-, A^o$
- $A^{-o}, A^{o-}$
- $A^{-o-}, A^{o-o}$
- and their negations

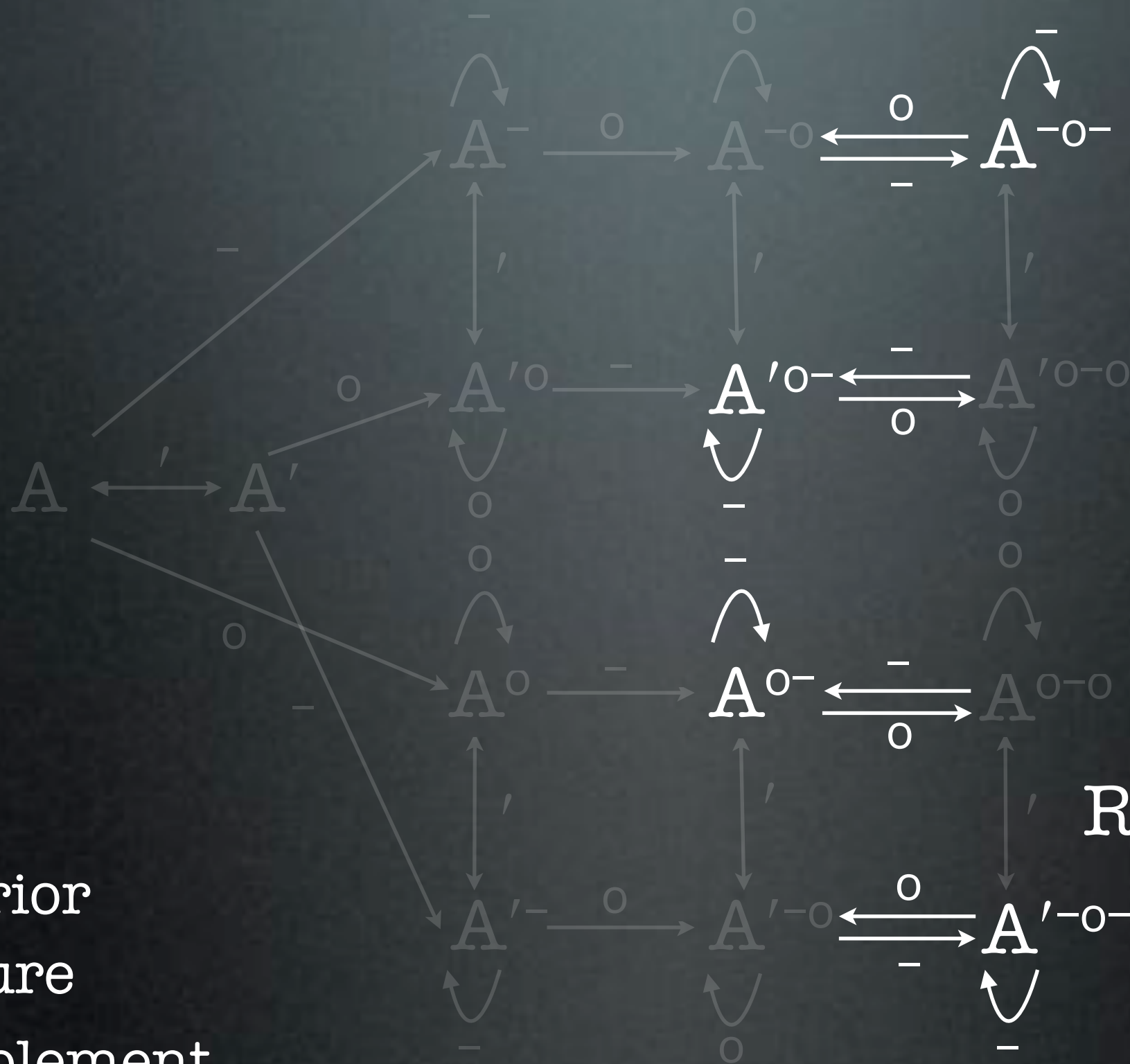
# Combinations



$o$  : interior  
 $-$  : closure  
 $'$  : complement

Note: there are examples where all these 14 regions are different.

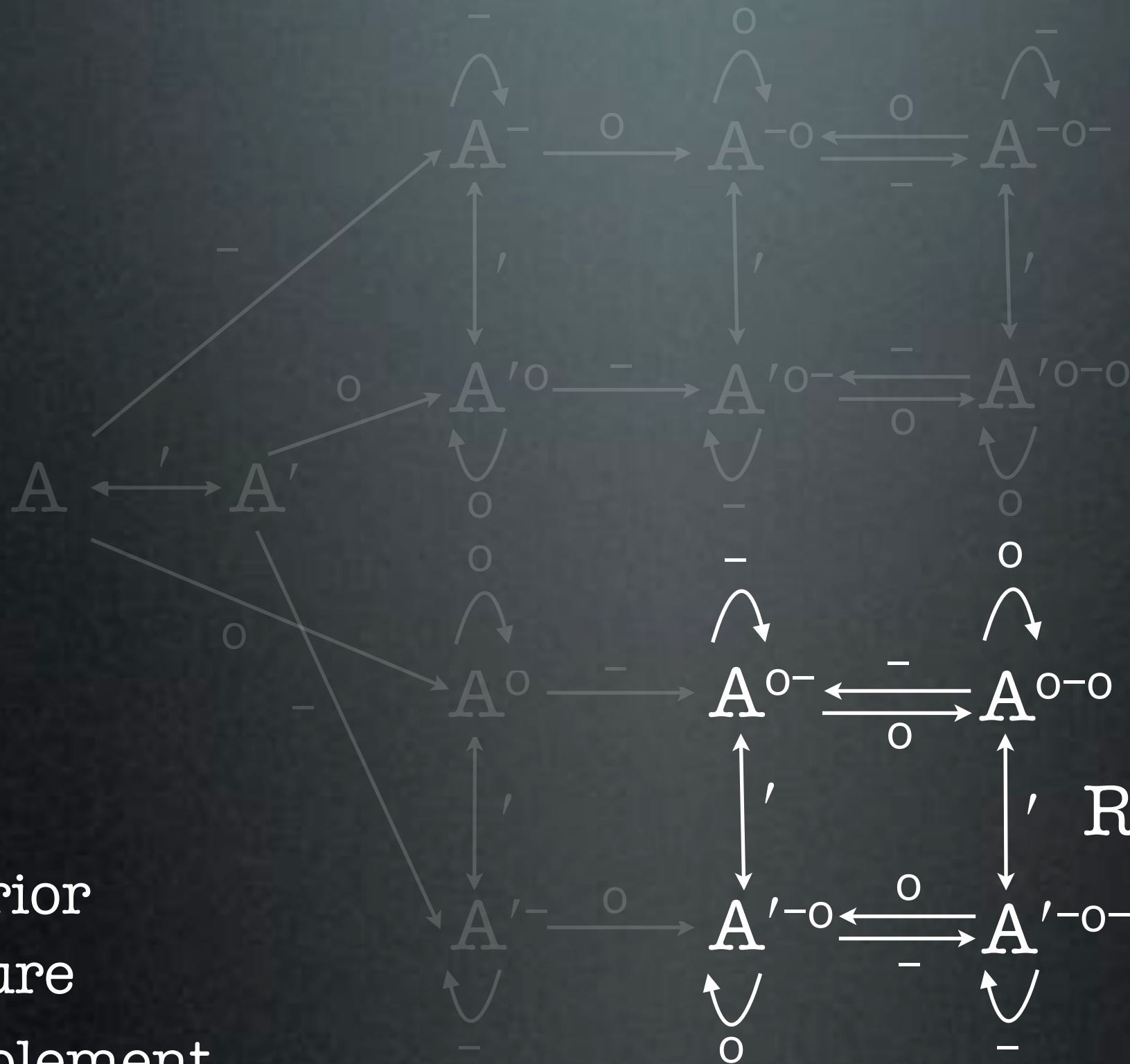
# Combinations



Regular Closed



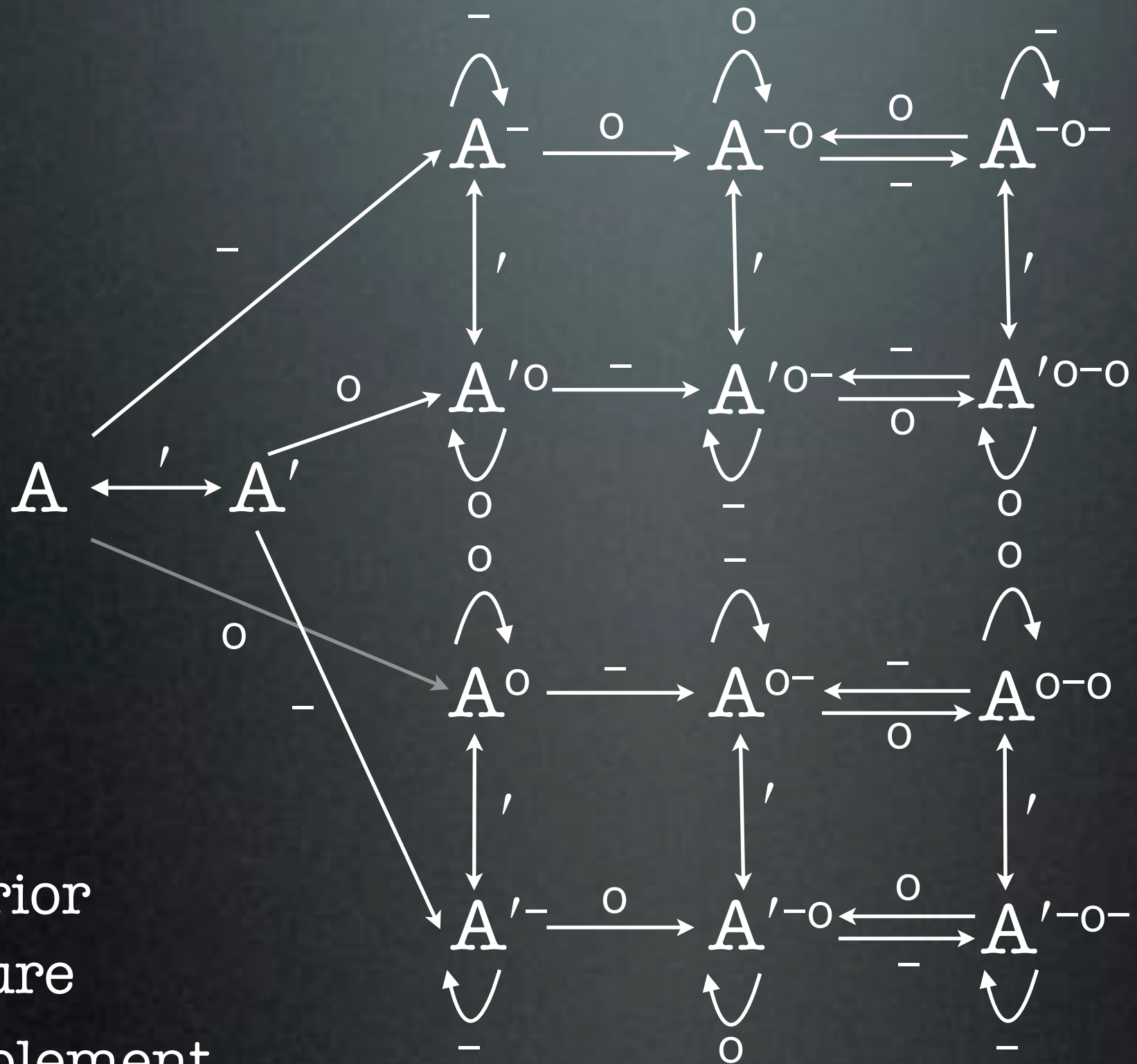
# Combinations



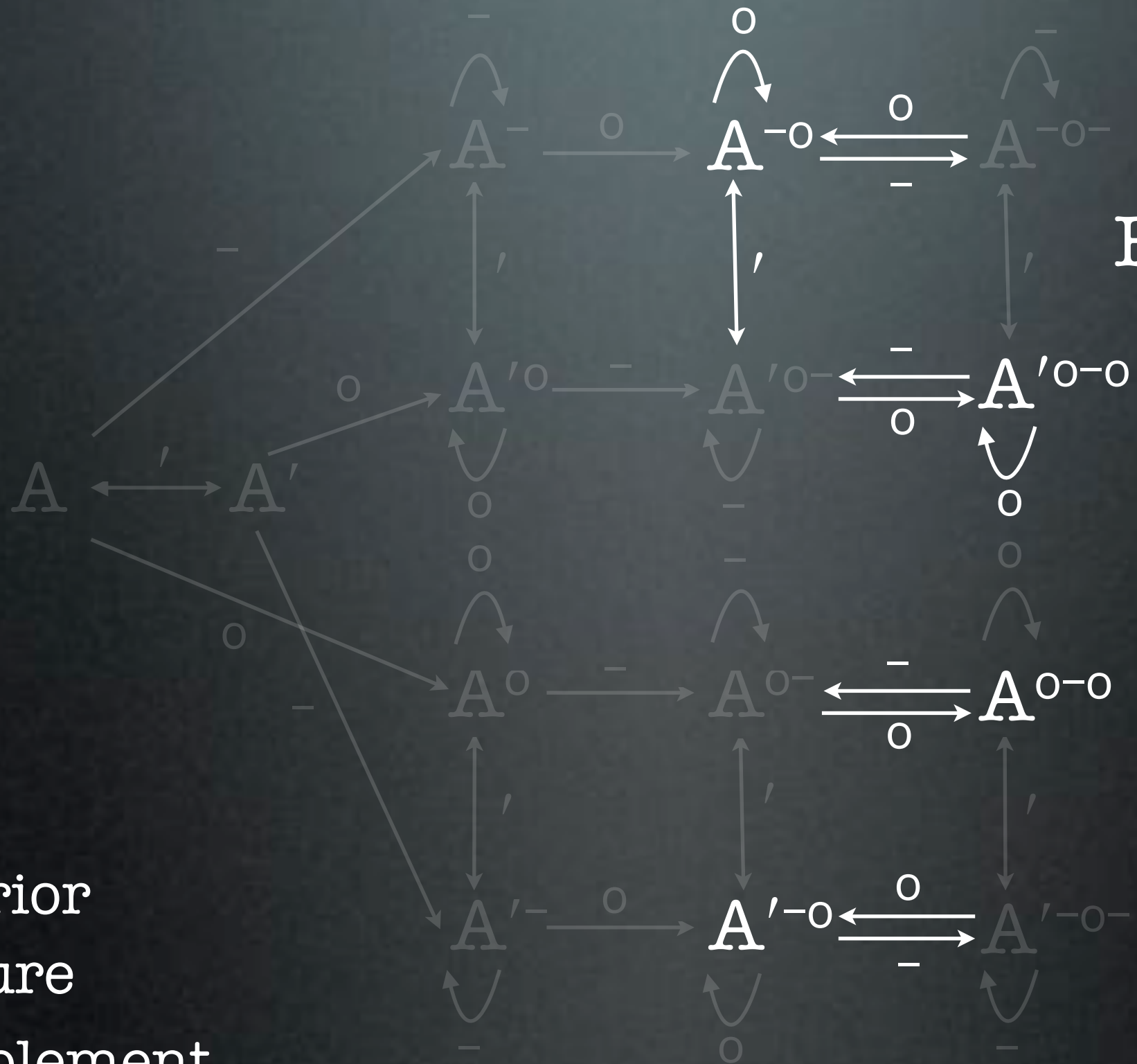
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Regular Closed

# Combinations



# Combinations

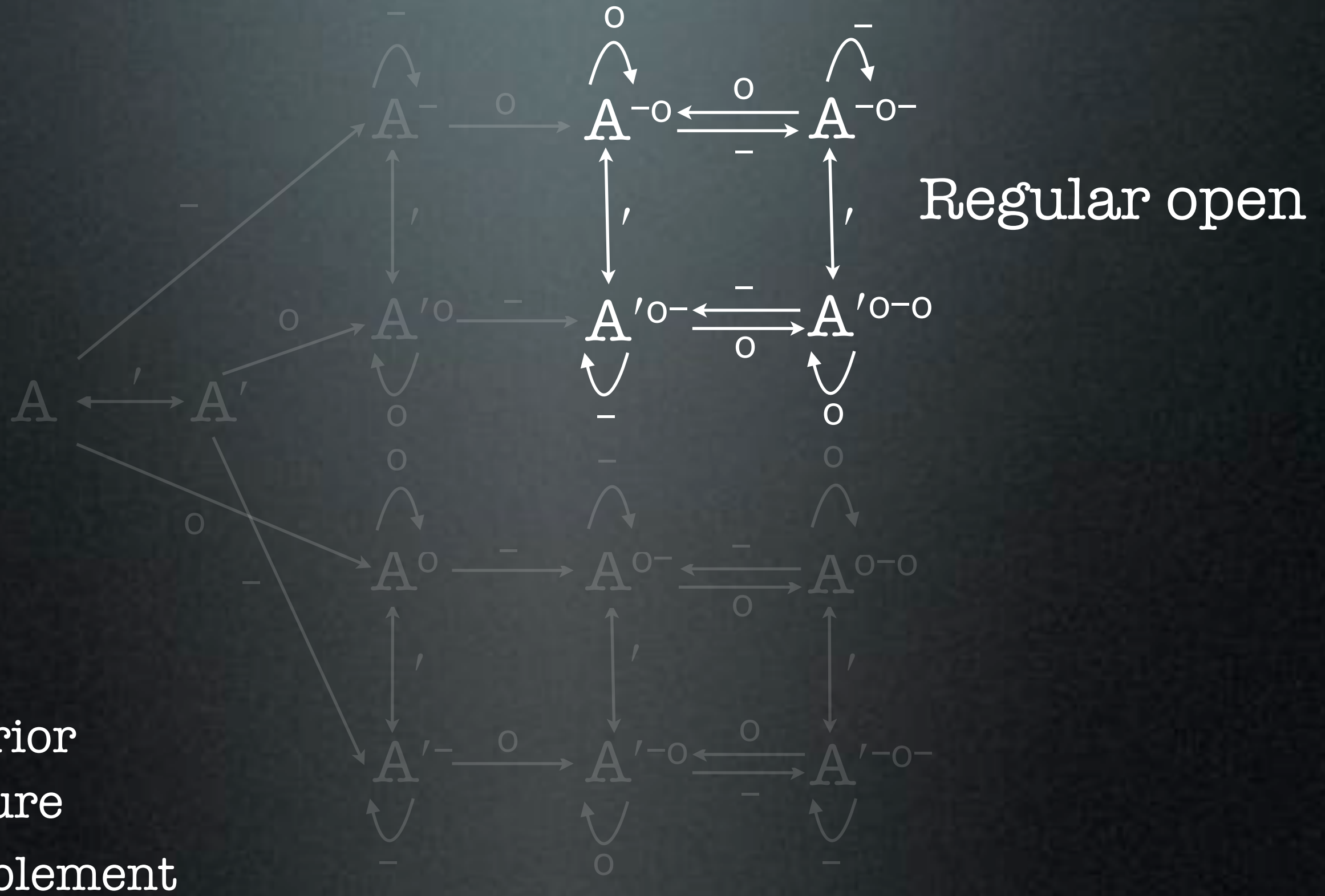


# Regular open

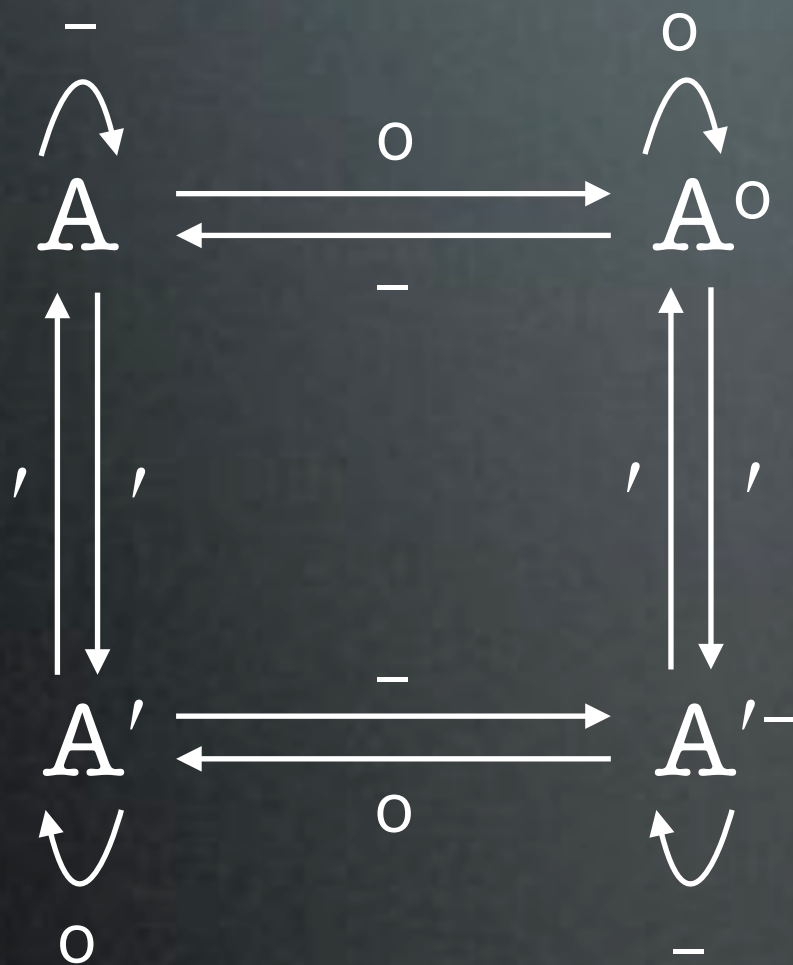
o : interior  
 - : closure  
 ' : complement



# Combinations



# Regular closed regions



A regular closed  $\stackrel{\text{def}}{=} A = A^{\circ\overline{\phantom{x}}}$

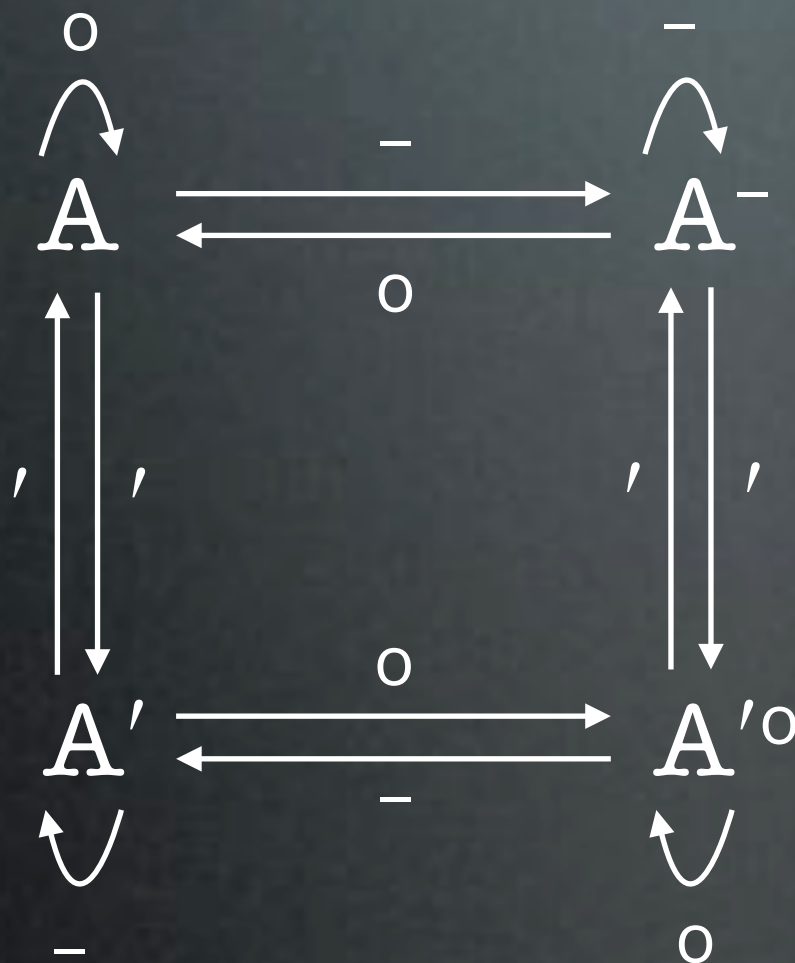
$$A = A^{\circ\overline{\phantom{x}}}$$

$\overline{\phantom{x}}$  : closure

$\circ$  : interior

$'$  : complement

# Regular open regions



A regular open  $\stackrel{\text{def}}{=}$

$$A = A^{-o}$$

$-$  : closure

$o$  : interior

$'$  : complement



# Combinations

- We have seen that the regular closed and regular open sets reduce the combinatorics of the closure, interior and complement relations on a single region.
- What we have not looked at yet is how the regular closed and open sets combine with unions and intersections.

# Intersection



- A is the union of the closed rectangles  $[0,0]-[1-2]$  and  $[0-1][1-2]$
- B is the union of the closed rectangles  $[0-0]-[1-1]$  and  $[0-1]-[2-2]$

# Intersection

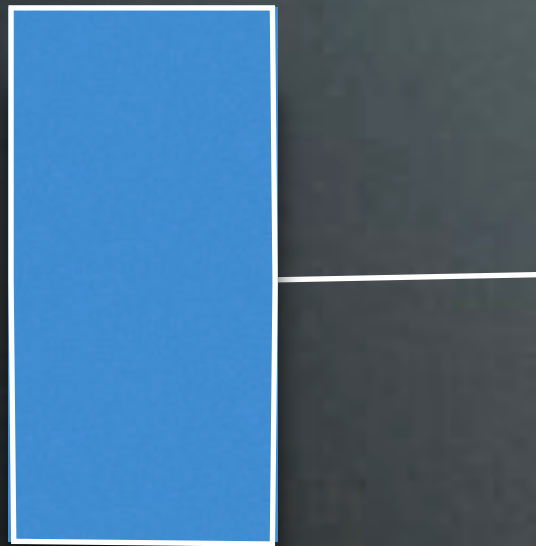
A,B	B
A,B	A

- A is the union of the closed rectangles  $[0,0]-[1-2]$  and  $[0-1][1-2]$
- B is the union of the closed rectangles  $[0-0]-[1-1]$  and  $[0-1]-[2-2]$



# Intersection

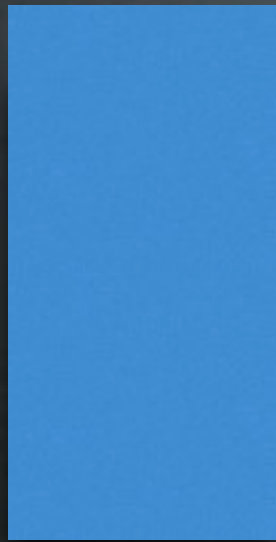
$A \cap B$



- The intersection of two regular closed sets (though closed) is not necessarily regular closed.
- Therefore we define the intersection of two regular closed sets as  $(A \cap B)^{0-}$

# Intersection

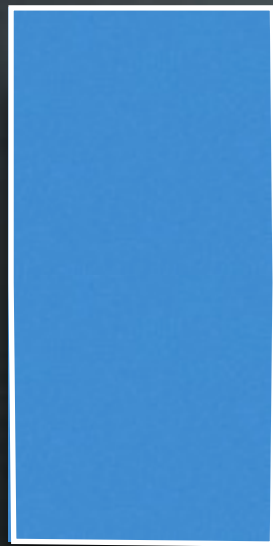
$$A \cap B^{\circ}$$



- The intersection of two regular closed sets (though closed) is not necessarily regular closed.
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# Intersection

$$A \cap B \text{ } ^{0-}$$

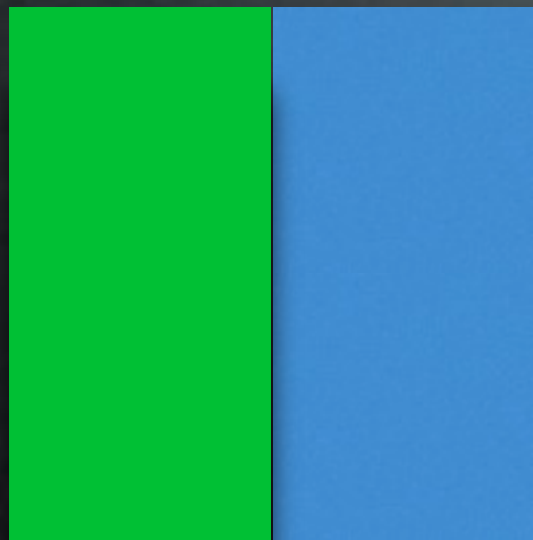


- The intersection of two regular closed sets (though closed) is not necessarily regular closed.
- Therefore we define the intersection of two regular closed sets as  $(A \cap B)^{0-}$



# Union

$A \cup B$

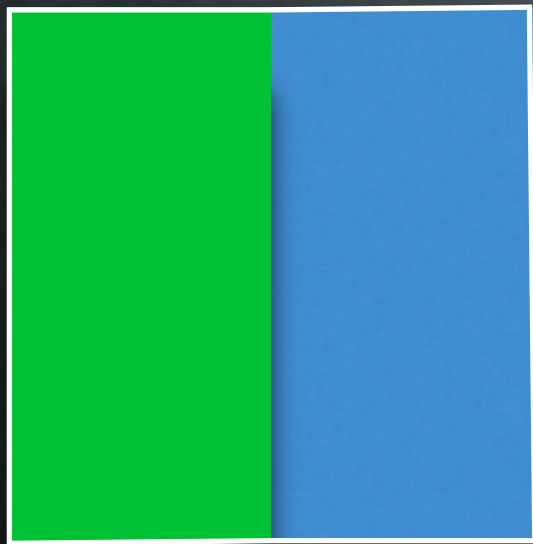


$$A \cup B = \langle 0, 0 \rangle - \langle 1, 2 \rangle \cup \langle 1, 0 \rangle - \langle 2, 2 \rangle$$

- Likewise, the union of two regular open sets (though open) is not necessarily regular open.
- Therefore we define the union of two regular open sets as  $(A \cup B)^{-\circ}$

# Union

$$(A \cup B)^-$$

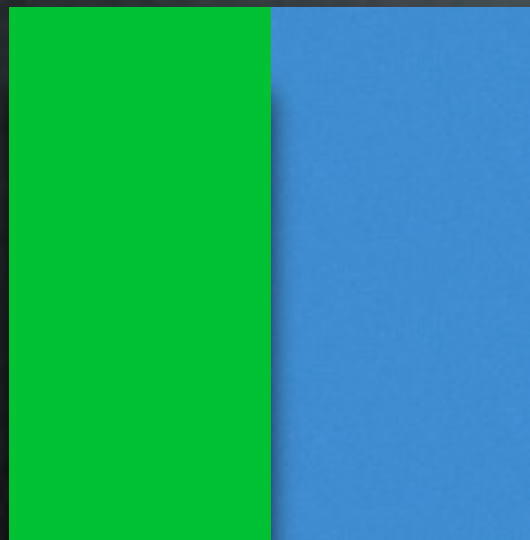


$$(A \cup B)^- = [0,0] - [2,2]$$

- Likewise, the union of two regular open sets (though open) is not necessarily regular open.
- Therefore we define the union of two regular open sets as  $(A \cup B)^{\circ}$

# Union

$$(A \cup B)^{-\circ}$$



$$(A \cup B)^{-\circ} = \langle 0, 0 \rangle - \langle 2, 2 \rangle$$

- Likewise, the union of two regular open sets (though open) is not necessarily regular open.
- Therefore we define the union of two regular open sets as  $(A \cup B)^{-\circ}$



# Boolean operations

## Regular Open Regions      Regular Closed Regions

- $RO(p) = p^{-0}$
- $RO(\neg A) = X \setminus RO(A)^-$
- $RO(A \wedge B) = RO(A) \cap RO(B)$
- $RO(A \vee B) = (RO(A) \cup RO(B))^{-0}$
- $RC(p) = p^{0-}$
- $RC(\neg A) = X \setminus RC(A)^0$
- $RC(A \wedge B) = (RC(A) \cap RC(B))^{0-}$
- $RC(A \vee B) = RC(A) \cup RC(B)$

The combinations listed here give us a way to produce boolean combination of regular closed regions. They are in fact a sublanguage of  $S4u$  describing regions terms. This will be useful later, when we will use it to move from  $RCC8$  to  $BRCC8$  by changing from regular close atoms to formulas of  $RC$ .

# Kuratowski Axioms

## Interior

- $A^{\circ} \leq A$
- $A^{\circ} = A^{\circ\circ}$
- $\top = \top^{\circ}$
- $A^{\circ} \cap B^{\circ} = (A \cap B)^{\circ}$

These rules should  
remind us of something...

## Closure

- $A \leq A^{-}$
- $A^{--} = A^{-}$
- $\perp^{-} = \perp$
- $(A \cup B)^{-} = A^{-} \cup B^{-}$

# Kuratowski Axioms

## Interior

- $A^{\circ} \leq A$
- $A^{\circ} = A^{\circ\circ}$
- $\top = \top^{\circ}$
- $A^{\circ} \cap B^{\circ} = (A \cap B)^{\circ}$

## S4

- $\Box A \vdash A$
- $\Box A \vdash \Box \Box A$
- $\top \vdash \Box \top$
- $\Box A \wedge \Box B \vdash \Box (A \wedge B)$
- $\Box (A \wedge B) \vdash \Box A \wedge \Box B$

You will note in the following that I sometimes switch rather freely from the “modal logic” notation with diamonds and boxes to the “topology notation” with interiors and closures. I trust this will not lead to confusion.



# Kuratowski Axioms

## Closure

- $A \leq A^-$
- $A^{--} = A^-$
- $\perp^- = \perp$
- $(A \cup B)^- = A^- \cup B^-$

## S4

- $A \vdash \Diamond A$
- $\Diamond \Diamond A \vdash \Diamond A$
- $\Diamond \perp \vdash \perp$
- $\Diamond(A \vee B) \vdash \Diamond A \vee \Diamond B$
- $\Diamond A \vee \Diamond B \vdash \Diamond(A \vee B)$

# Kuratowski Axioms

## Interior

- $A^0 \leq A$
- $A^0 = A^{00}$
- $\top = \top^0$
- $A^0 \cap B^0 = (A \cap B)^0$

## S4

- $!A \vdash A$
- $!A \vdash !!A$
- $\top \vdash !\top$
- $!A \wedge !B \vdash !(A \wedge B)$
- $!(A \wedge B) \vdash !A \wedge !B$

Of course, the logical rules for the exponentials of linear logic are just the rules for S4 as well!

# Logic and Topology

- The relation between S4 and topology was first noted by Tarski (1938) McKinsey & Tarski (1944)
- However, its application to spatial reasoning is fairly new (see references: the earliest I found are from the mid 90s)



# Modeling

- Let's turn to some possible applications.
- How would we model a statement like “the interior of region A and the interior of region B have a non-empty intersection”?
- First try:  $\neg(\Box A \wedge \Box B \leftrightarrow \perp)$

# Modeling

- First try:  $\neg(\Box A \wedge \Box B \leftrightarrow \perp)$
- This formula is equivalent to  $\Box A \wedge \Box B$
- Let's construct a model of this formula.

Of course this will fail, but it will be instructive to see why (and to see what form the models of these kinds of formulas take)

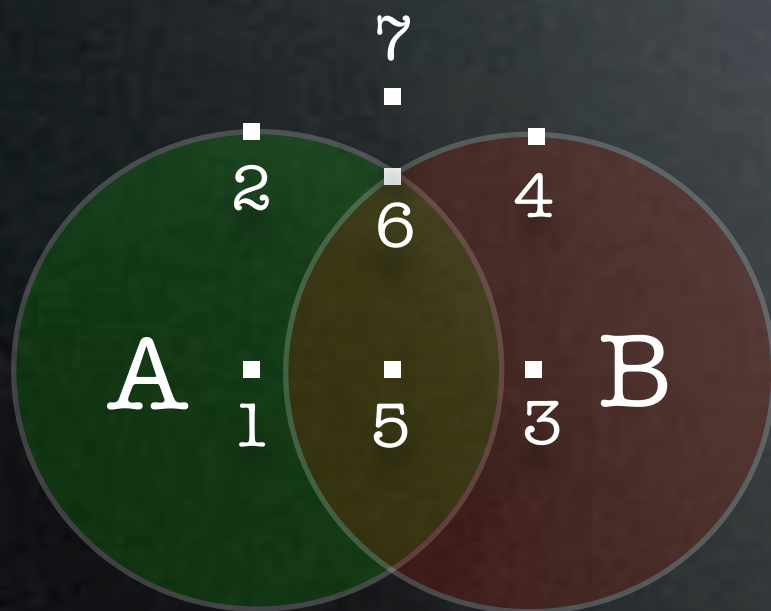
# A model of $A^0 \wedge B^0$

Of course, this is just one of many possible models! In particular, this is a model where  $A \cup B$  is not equal to the universe

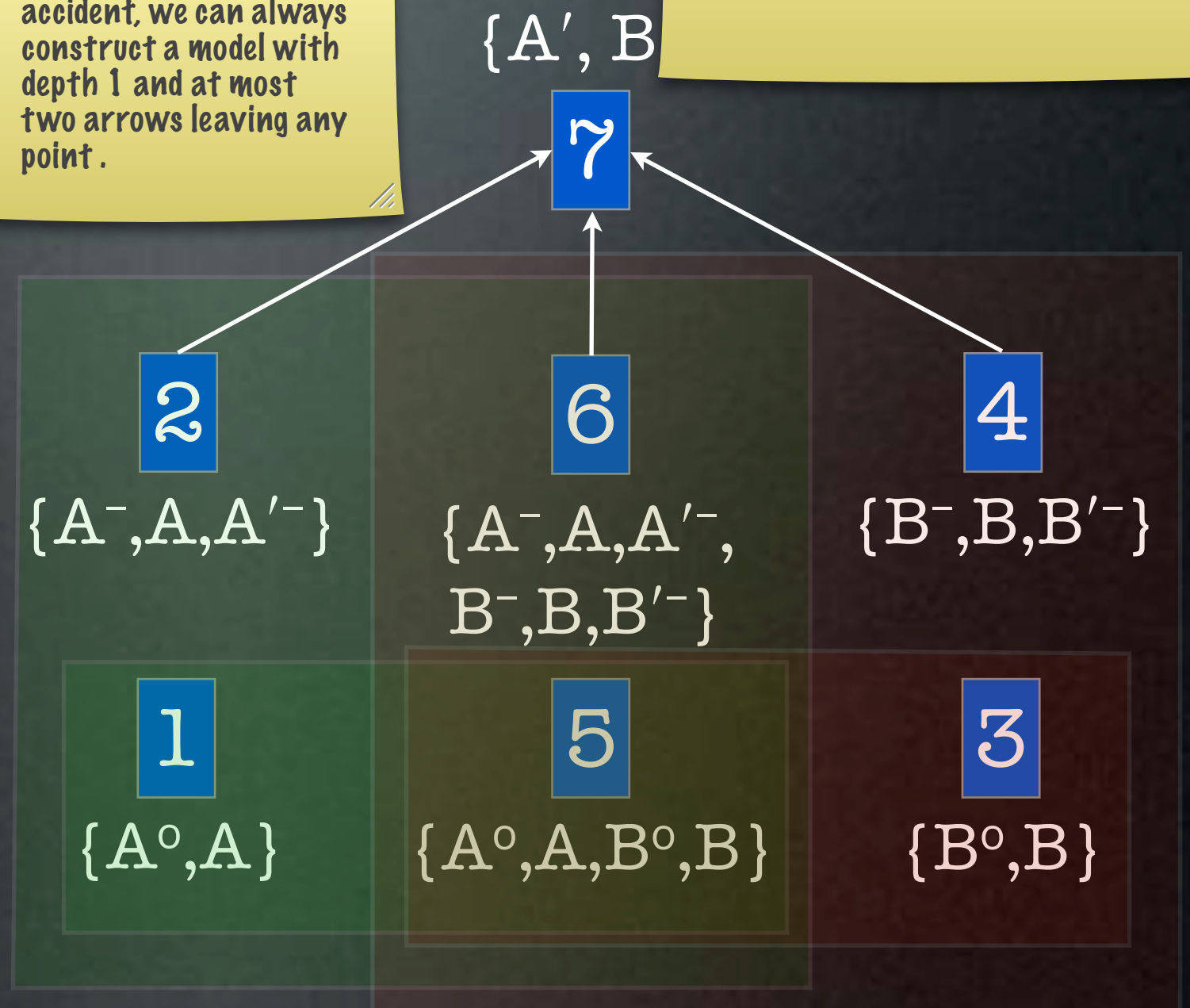
Remark: none of the models are disjoint

The model seems very simple. This is no accident, we can always construct a model with depth 1 and at most two arrows leaving any point.

Question: can we add arrow from 1 to 2?  
Answer: No! If we add an arrow from 1 to 2, transitivity would force us to add an arrow from 1 to 7 as well and then  $A$  would no longer hold at all states accessible from 1.



Reminder:  $A^-$  means there is an arrow to a state where  $A$  holds.  
 $A^0$  means all arrows lead to a state where  $A$  holds.



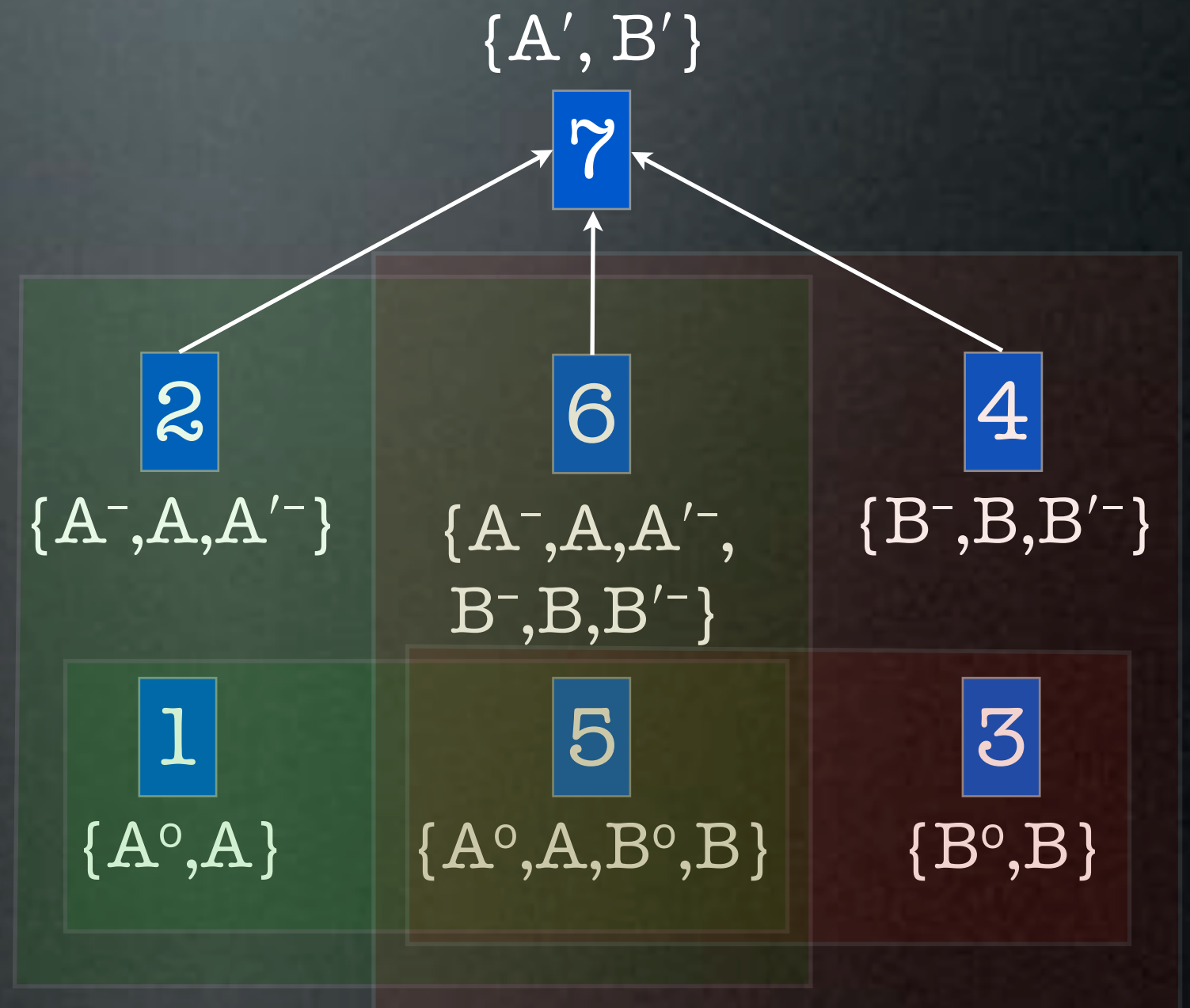


# A model of $A^0 \wedge B^0$

Now, in order for the formula  $A^0 \wedge B^0$  to hold, it has to be true at every point.

This means that for every point and every point we can reach from this point by following an arrow both A and B must hold.

Let's look at point 7.



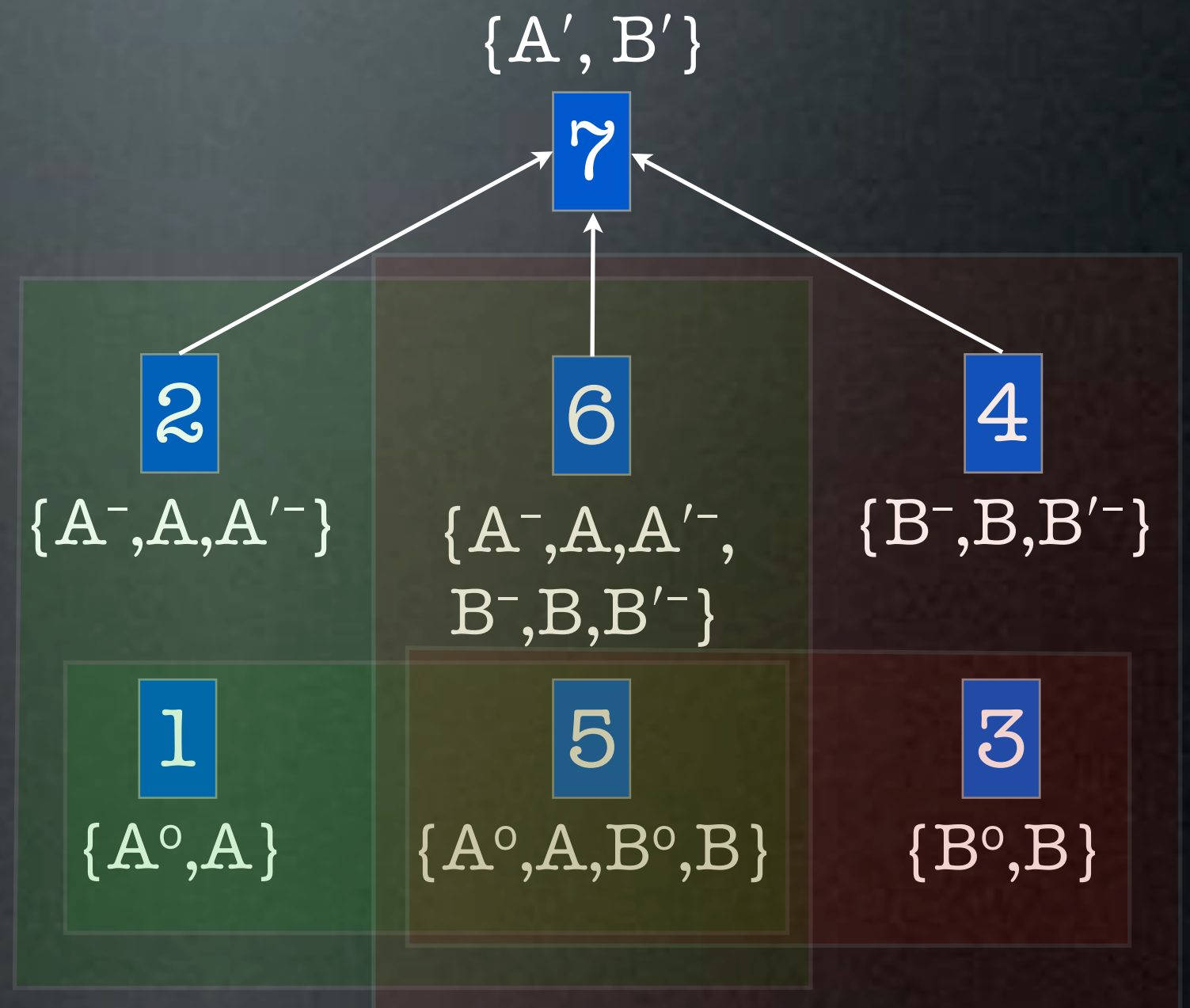
# A model of $A^0 \wedge B^0$

The only point we can reach from point 7 is point 7 itself.

Neither A nor B holds at point 7.

Therefore, this model is a countermodel to  $A^0 \wedge B^0$

What went wrong?



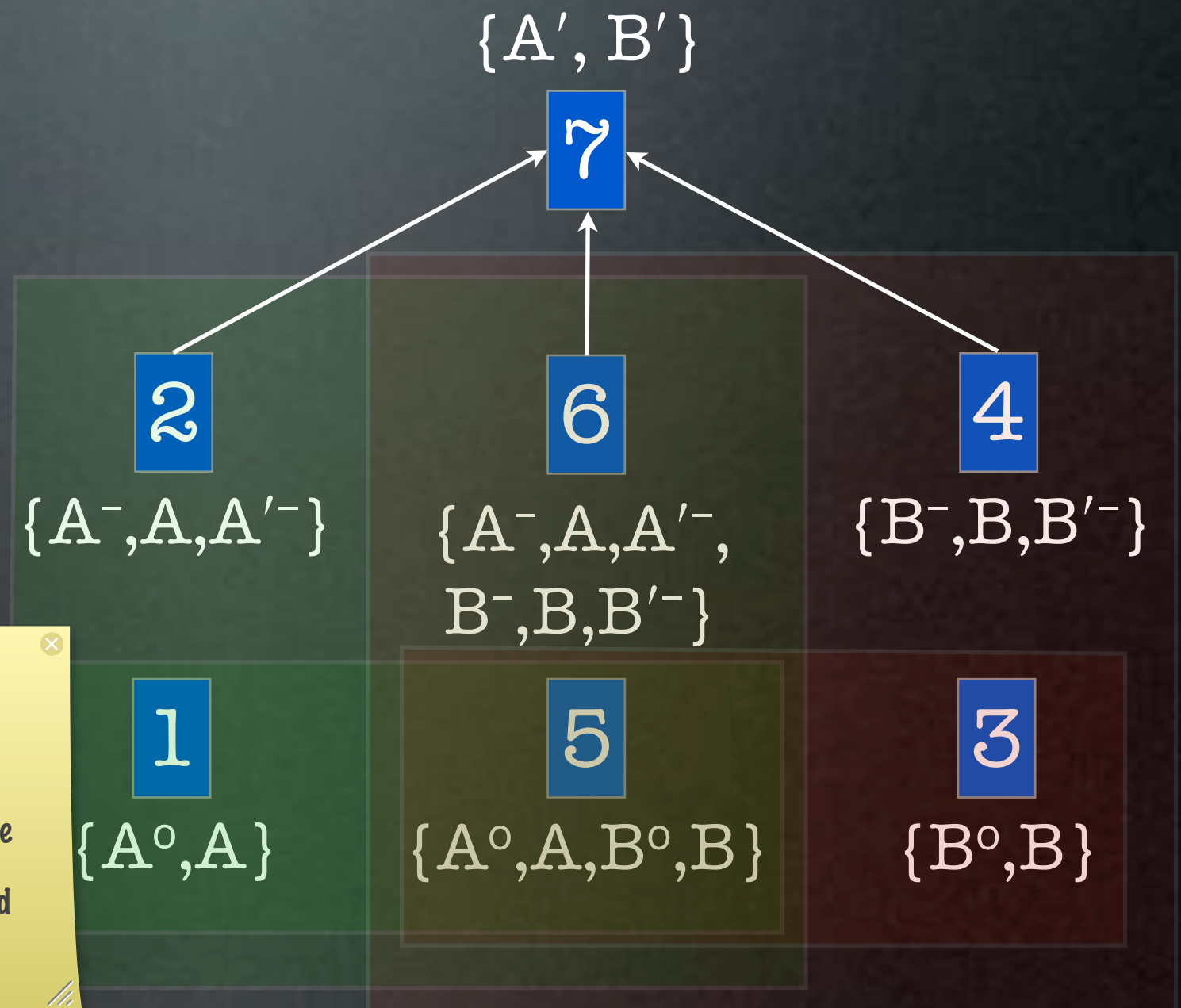
# A model of $A^{\circ} \wedge B^{\circ}$

We want to state that  $A^{\circ} \wedge B^{\circ}$  is true at at least one point in the model.

This would correctly model the fact that this intersection is not empty

However in standard cannot express this.

At least not if we want to keep the current interpretation of the connectives, with its advantage of being very close to the topological operators. We want to preserve this and add as little as possible.



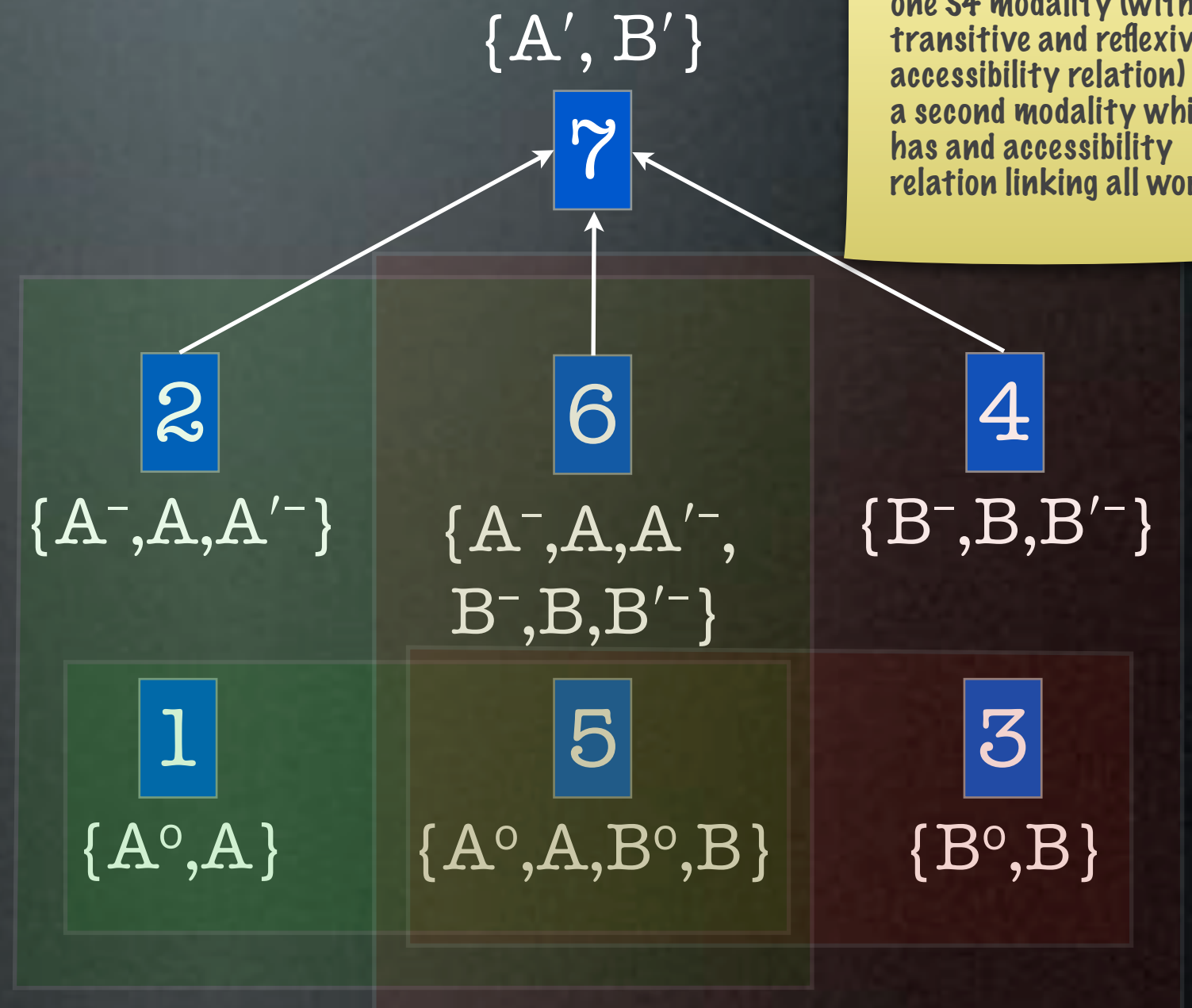


# A model of $A^0 \wedge B^0$

A solution is to add a universal modality to S4, giving the system  $S4_u$

A formula  $\forall F$  is true if for all points in the model the formula  $F$  is true.

A formula  $\exists F$  is true if  $F$  holds at at least one point.



One way to see the system  $S4_u$  is that it is a multimodal system with one S4 modality (with a transitive and reflexive accessibility relation) and a second modality which has an accessibility relation linking all worlds.

# S4<sub>u</sub>

So we have “true everywhere” for the universal quantifier and “true somewhere” for the existential quantifier. Their negations are “false somewhere” (not forall) and “false everywhere” (not exists)

- In other words,  $\forall F$  will mean  $|F| = X$  and  $\exists F$  will mean  $|F| \neq \emptyset$
- The negated forms are interpreted as expected:  $\neg \forall F$  will mean  $|F| \neq X$  and  $\neg \exists F$  will mean  $|F| = \emptyset$
- In the following, I will often use formulas containing “ $F \neq \emptyset$ ”, “ $F \neq$ ”, “ $F = \emptyset$ ”, “ $F = X$ ”.

This is a slight abuse of notation, but, in my opinion, it makes the formulas a lot easier to read!

# S4<sub>u</sub> - Language

- We can restrict ourselves without loss of generality to formulas without nested occurrences of the universal/existential modality (see eg. Aiello & van Benthem).
- This is because a formula  $\exists F$  or  $\forall F$  is true globally.
- That is to say, we can replace a formula  $F[\exists G]$  by the equivalent formula  $(\exists G \wedge F[\top]) \vee \neg(\exists G \wedge F[\perp])$



# S4<sub>u</sub> - Language

- In other words, we can say that the formulas of S4<sub>u</sub> are of the following form:

$$\begin{array}{l} F = F \wedge F \\ | F \vee F \\ | \neg F \\ | \forall S \\ | \exists S \\ | S \end{array}$$

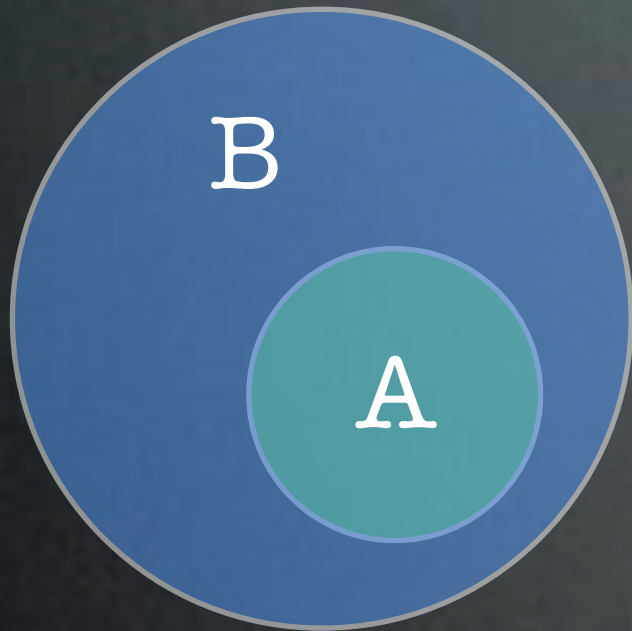
- Where S is a “normal” S4 formula.

# S4<sub>u</sub> - Examples

- $\exists a$
- $\exists \Box a$
- $\exists (\neg \Box a \wedge \Diamond a)$
- $\neg \forall a$

- a is not empty  
- a has a non-empty interior  
- the frontier of a is not empty  
- a is not the universe (there is a point outside of a)

# S4<sub>u</sub> expressivity

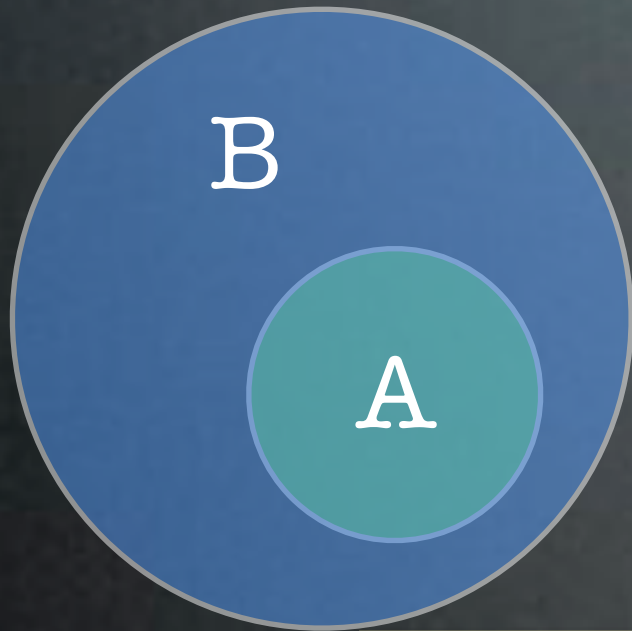


- Define  $A \subseteq B$  as  
 $\forall(\neg A \vee B)$  or  
 $\neg A \vee B = X$  or  
 $A \rightarrow B = X$
- Define  $A \not\subseteq B$  as  
 $\neg \forall(\neg A \vee B)$  or  
 $\neg A \vee B \neq X$  or  
 $A \rightarrow B \neq X$
- Define  $A \subset B$  as  
 $A \subseteq B \wedge B \not\subseteq A$



# S4<sub>u</sub> expressivity

- The subset relation gives us a fairly crude way of partitioning the different possible relations between a region A and a region B



These are the distinctions we can make in mereology. By adding some topologically interesting information, we obtain a system of mereotopology.

$$1. A \subseteq B \wedge B \subseteq A$$

$$2. A \subseteq B \wedge B \not\subseteq A$$

$$3. A \not\subseteq B \wedge B \subseteq A$$

$$4. A \not\subseteq B \wedge B \not\subseteq A$$

# S4<sub>u</sub> expressivity

$$A = a^{0-} \text{ and } B = b^{0-}$$



$$A \wedge B = \emptyset$$

A → B	B → A	
0	0	DC, EC, PO
0	1	TPP <sup>-1</sup> NTPP <sup>-1</sup>
1	0	TPP NTPP
1	1	EQ

- Define DC(A,B) as  $\neg \exists (A \wedge B)$ , which is equivalent to  $A \wedge B = \emptyset$



$$A^0 \wedge B^0 = \emptyset \wedge A \wedge B \neq \emptyset$$

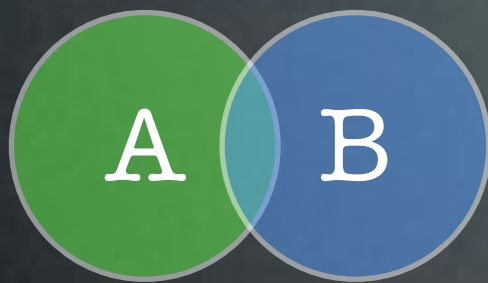
Note that this definition presupposes that A and B are closed: given that we are working with regular closed regions only, this condition is satisfied by construction.

Exercise: give the correct definitions of the RCC8 relations for regular open sets.

- Define EC(A,B) as  $\neg \exists (A^0 \wedge B^0) \wedge \exists (A \wedge B)$ , or  $A^0 \wedge B^0 = \emptyset \wedge A \wedge B \neq \emptyset$

# S4<sub>u</sub> expressivity

$$A = a^{0-} \text{ and } B = b^{0-}$$



The interiors share a point but neither  $A \rightarrow B$  nor  $B \rightarrow A$

$$A^0 \wedge B^0 \neq \emptyset \wedge A \rightarrow B \neq X \wedge B \rightarrow A \neq X$$

$$A^0 \wedge B^0 \neq \emptyset \wedge A \not\subseteq B \wedge B \not\subseteq A$$

Both  $A \rightarrow B$  and  $B \rightarrow A$ . The remaining cases are therefore  $A \rightarrow B$  and not  $B \rightarrow A$  ( $\neg B / \wedge A$ ) and  $B \rightarrow A$  and not  $A \rightarrow B$  ( $\neg A / \wedge B$ )



$$A \leftrightarrow B = X$$

$$A \subseteq B \wedge B \subseteq A$$

- Define  $PO(A, B)$  as  $\exists(A^0 \wedge B^0) \wedge \neg(A \subseteq B) \wedge \neg(B \subseteq A)$  or  $A^0 \wedge B^0 \neq \emptyset \wedge \neg(A \subseteq B) \wedge \neg(B \subseteq A)$

- Define  $EQ(A, B)$  as  $A \subseteq B \wedge B \subseteq A$ .



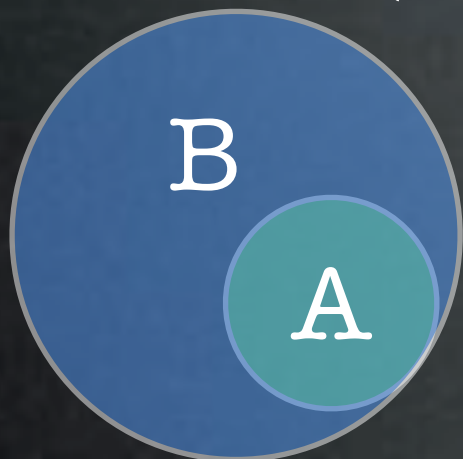
# S4<sub>u</sub> expressivity

$A = a^{0-}$  and  $B = b^{0-}$



$$\neg A \vee B^0 = X \wedge \neg A \wedge B \neq \emptyset$$

$$A \subseteq B^0 \wedge B \not\subseteq A$$



$$\neg A \vee B = X \wedge \neg A \wedge B \neq \emptyset \wedge A \wedge \neg(B^0) \neq \emptyset$$

$$A \subseteq B \wedge A \not\subseteq B^0 \wedge B \not\subseteq A$$

- Define NTPP(A,B) as

$$\neg A \vee B^0 = X \wedge$$

$$\neg A \wedge B \neq \emptyset$$

- Define TPP(A,B)

$$\text{as } \neg A \vee B = X \wedge$$

$$A \wedge \neg(B^0) \neq \emptyset \wedge$$

$$\neg A \wedge B \neq \emptyset$$

$A \rightarrow B$  and  $\neg(B \rightarrow A)$  and  
 $\neg(A \rightarrow B^0)$

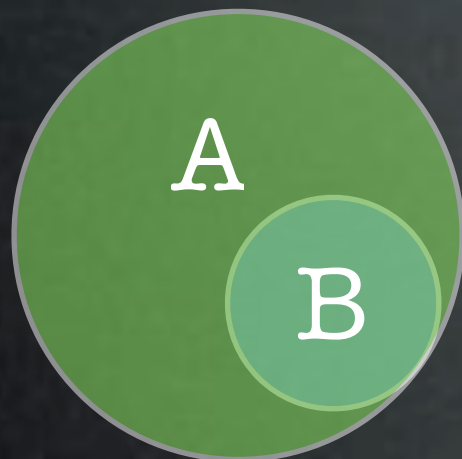
# S4<sub>u</sub> expressivity

$A = a^{0-}$  and  $B = b^{0-}$



$$\neg B \vee A^0 = X \wedge \neg B \wedge A \neq \emptyset$$

$$B \subseteq A^0 \wedge A \not\subseteq B$$



$$\neg B \vee A = X \wedge \neg B \wedge A \neq \emptyset \wedge B \wedge \neg(A^0) \neq \emptyset$$

$$B \subset A \wedge B \not\subseteq A^0$$

- Define  $\text{NTPP}^{-1}(A, B)$  as  $\text{NTPP}(B, A)$
- Define  $\text{TPP}^{-1}(A, B)$  as  $\text{TPP}(B, A)$

# $S4_u$ expressivity

- We have shown that there are formulas defining the RCC8 relations in  $S4_u$
- A natural question is: are there any useful things we can express in  $S4_u$  which are not expressible in RCC8?



# S4<sub>u</sub> expressivity

- In RCC8, we apply the 8 relations only to region variables, which we assume to be regular closed, ie.  $A = a^{0-}$  and  $B = b^{0-}$  for  $a$  and  $b$  atomic.
- If instead we allow regular closed formulas, which use boolean combinations of regions, then we can express more.
- The resulting calculus is sometimes called BRCC8 (B for Boolean)

# S4u expressivity

- $EQ(\text{UnionEuropéenne}, \text{PaysBays} \vee \text{Belgique} \vee \text{France} \vee \dots)$
- $EQ(\text{Aquitaine}, \text{Dordogne} \vee \text{Gironde} \vee \text{Landes} \vee \text{LotEtGaronne} \vee \text{PyrénéesAtlantiques})$
- $TPP(\text{Pyrénées}, \text{France} \vee \text{Espagne} \vee \text{A})$
- $EC(\text{France} \wedge \text{Pyrénées}, \text{Espagne} \wedge \text{Pyr})$

We can use equality statements to specify that a region is exactly the union of a number of other regions. In RCC8 we can only specify that each of the different regions is a part of (tangential or not) a super-region but not the inverse.

These two statements capture the fact that France and Spain are connected by means of the Pyrénées. This is stronger than the RCC8 statements  $PO(\text{France}, \text{Pyrénées})$ ,  $PO(\text{Espagne}, \text{Pyrénées})$ , and  $EC(\text{France}, \text{Espagne})$ .

# S4u expressivity

- $EC(\text{Andorre}, \text{France} \wedge \text{Pyrénées})$
- $EC(\text{Andorre}, \text{Espagne} \wedge \text{Pyrénées})$
- $NTPP(\text{Andorre}, \text{Pyrénées})$
- $EQ(\text{France}, \text{FranceContinental} \vee \text{Corse})$
- $DC(\text{FranceContinental}, \text{Corse})$

We can state that a region denoted by a certain variable is discontinuous. This is impossible in RCC8



# S4<sub>u</sub> expressivity

- Besides BRCC8, there is another way to model regions and their interactions in S4<sub>u</sub>.
- To see how, remember that the formula  $\neg \Box A \wedge \Diamond A$  denotes the frontier of A,  $\Box A$  denotes the interior of A and  $\neg \Diamond A$  denotes the exterior of A (that is, the complement of the closure)

# S4<sub>u</sub> expressivity

- To see how, remember that the formula  $\neg\Box A \wedge \Diamond A$  denotes the frontier of  $A$ ,  $\Box A$  denotes the interior of  $A$  and  $\neg\Diamond A$  denotes the exterior of  $A$  (that is, the complement of the closure)
- In addition  $\neg\exists A$  means region  $A$  is empty, whereas  $\exists A$  means it is not empty.

# S4<sub>u</sub> expressivity

- Egenhofer and colleagues, have created a taxonomy of topological possibilities (under certain hypotheses about the regions under consideration) specifying whether each of the 3 divisions of the topological space (interior, frontier, exterior) introduced by two different regions is empty or not.
- In the case of regions “without holes” this gives the same 8 relations as RCC8; without this restriction, the 9-intersection model makes more distinctions.



# S4<sub>u</sub> expressivity

	$A^\circ$	$\delta A$	$A'$
$B^\circ$	$\neg \emptyset$	$\neg \emptyset$	$\neg \emptyset$
$\delta B$	$\emptyset$	$\emptyset$	$\neg \emptyset$
$B'$	$\emptyset$	$\emptyset$	$\neg \emptyset$



The NTPP relation (non-tangential proper part) of RCC8 shown as a 9-

# S4<sub>u</sub> expressivity

The same matrix translated into a S4<sub>u</sub> formula, which is slightly big!

	$\Box A$	$\neg \Box A \wedge \Diamond A$	$\neg \Diamond A$
$\Box B$	$\neg \emptyset$	$\neg \emptyset$	$\neg \emptyset$
$\neg \Box B \wedge \Diamond B$	$\emptyset$	$\emptyset$	$\neg \emptyset$
$\neg \Diamond B$	$\emptyset$	$\emptyset$	$\neg \emptyset$

$$\begin{aligned}
 & \exists(\Box A \wedge \Box B) \wedge \exists(\neg \Box A \wedge \Diamond A \wedge \Box B) \wedge \exists(\neg \Diamond A \wedge \Box B) \wedge \\
 & \neg \exists(\Box A \wedge \neg \Box B \wedge \Diamond B) \wedge \neg \exists(\neg \Box A \wedge \Diamond A \wedge \neg \Box B \wedge \Diamond B) \wedge \exists(\neg \Diamond A \wedge \neg \Box B \wedge \Diamond B) \wedge \\
 & \neg \exists(\Box A \wedge \neg \Diamond B) \wedge \neg \exists(\neg \Box A \wedge \Diamond A \wedge \neg \Diamond B) \wedge \exists(\neg \Diamond A \wedge \neg \Diamond B)
 \end{aligned}$$

# S4<sub>u</sub> expressivity

The partial overlap relation (PO) from RCC8

	$\Box A$	$\neg \Box A \wedge \Diamond A$	$\neg \Diamond A$
$\Box B$	$\neg \emptyset$	$\neg \emptyset$	$\neg \emptyset$
$\neg \Box B \wedge \Diamond B$	$\neg \emptyset$	$\neg \emptyset$	$\neg \emptyset$
$\neg \Diamond B$	$\neg \emptyset$	$\neg \emptyset$	$\neg \emptyset$

$$\begin{aligned}
 &\exists(\Box A \wedge \Box B) \wedge \exists(\neg \Box A \wedge \Diamond A \wedge \Box B) \wedge \exists(\neg \Diamond A \wedge \Box B) \wedge \\
 &\exists(\Box A \wedge \neg \Box B \wedge \Diamond B) \wedge \exists(\neg \Box A \wedge \Diamond A \wedge \neg \Box B \wedge \Diamond B) \wedge \exists(\neg \Diamond A \wedge \neg \Box B \wedge \Diamond B) \wedge \\
 &\exists(\Box A \wedge \neg \Diamond B) \wedge \exists(\neg \Box A \wedge \Diamond A \wedge \neg \Diamond B) \wedge \exists(\neg \Diamond A \wedge \neg \Diamond B)
 \end{aligned}$$



# S4<sub>u</sub> expressivity

- $\forall((\text{NorthKorea} \wedge \text{SouthKorea}) \rightarrow \text{DemilitarizedZone})$
- $\exists p \wedge \forall(p \rightarrow \Diamond \neg p) \wedge \forall(\neg p \rightarrow \Diamond p)$

A tiny model for the above formula

$$\neg p \longleftrightarrow p$$

We can state the following fact: the border between North and South Korea is a subset of the Demilitarized Zone.

The second is very strange:  $p$  is not empty and all points of  $p$  are boundary points.

Note however, that  $p$  is not regular closed ( $p$ 's interior is the empty set)

Note how the second conjunct makes it impossible for  $p$  to hold in a point from which only states where " $p$ " holds are accessible: all world where " $p$ " holds have an accessible world where " $\neg p$ " holds. The third conjunct expresses the same for " $\neg p$ "

# S4<sub>u</sub> complexity

- The complexity of S4<sub>u</sub> is the same as the complexity of S4: it is PSPACE complete.
- Many interesting fragments (all have have seen: RCC8, BRCC8 and 9-I, at least for atomic formulas) have simpler models than S4 and are NP complete (see eg. Gabelaia e.a.).

# Going Further

- It seems natural to interpret our regions in an Euclidian space: this would give us a natural distance metric.
- Convex hulls seem useful for many purposes.
- Integration with temporal logic has been investigated (but there is surely more to be done!)
- Is there a good way to strike a balance between expressivity and computational complexity?



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