Language, Automata and Logic for Finite Trees

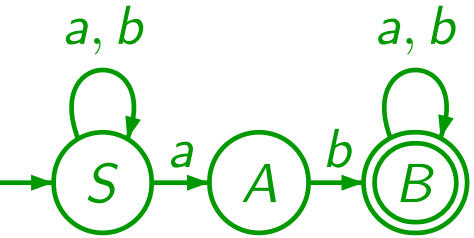
Olivier Gauwin

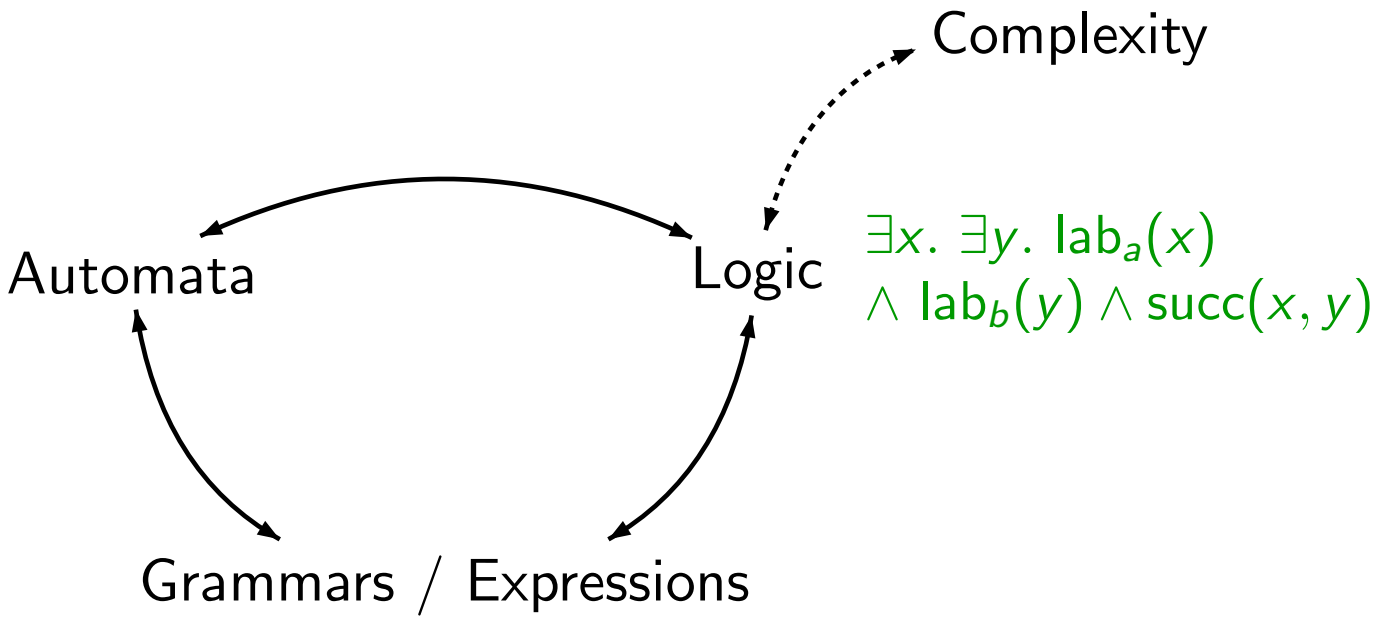
UMons

Feb/March 2010

Languages, Automata, Logic

Example for regular word languages: $\Sigma^*.a.b.\Sigma^*$





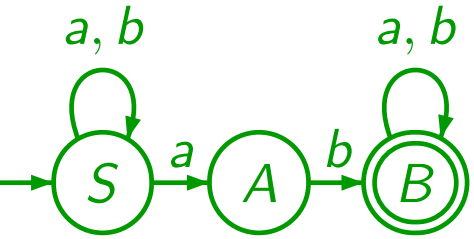
$\exists x.\ \exists y.\ \mathrm{lab}_a(x) \wedge \mathrm{lab}_b(y) \wedge \mathrm{succ}(x, y)$

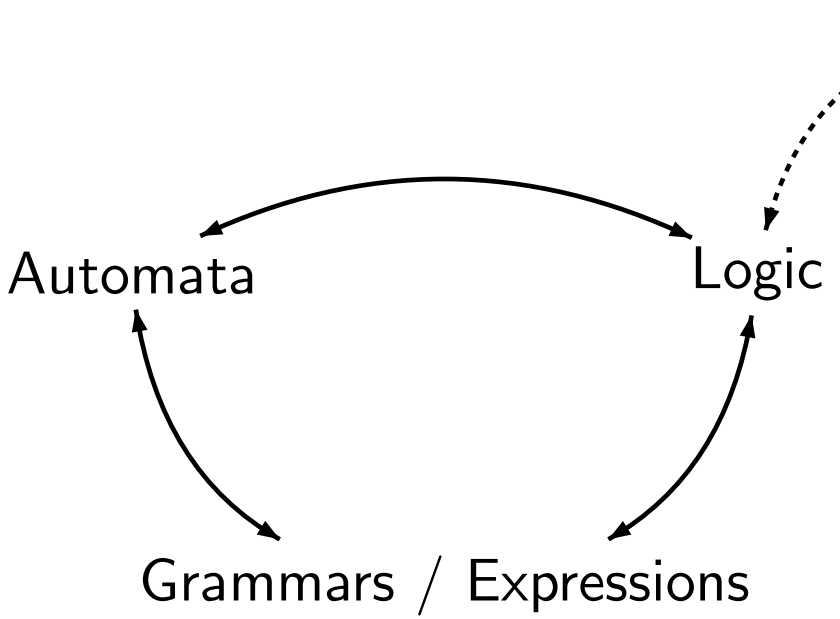
$$\begin{cases} S \to aS & S \to aB & X \to aX & X \to \epsilon \\ S \to bS & B \to bX & X \to bX & \end{cases}$$

Languages, Automata, Logic

Example for regular word languages: $\Sigma^*.a.b.\Sigma^*$

Example for regular tree languages: all trees with an a-node having a b-child





$\exists x.\ \exists y.\ \mathrm{lab}_a(x) \wedge \mathrm{lab}_b(y) \wedge \mathrm{succ}(x, y)$

$\exists x.\ \exists y.\ \mathrm{lab}_a(x) \wedge \mathrm{lab}_b(y) \wedge \mathrm{child}(x, y)$

$$\begin{cases} \_(S, X) \to S \\ \_(X, S) \to S \\ a(B, X) \to S \\ a(X, B) \to S \\ \dots \\ b \to B \\ \dots \end{cases}$$

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References

The main reference for this talk is the *TATA* book [CDG+07]:

Tree Automata, Techniques and Applications

by

Hubert Comon, Max Dauchet, Rémi Gilleron, Christof Löding, Florent Jacquemard, Denis Lugier, Sophie Tison, Marc Tommasi.

Other references will be mentionned progressively.

1. Ranked Trees
   - Trees on Ranked Alphabet
   - Tree Automata
   - Tree Grammars
   - Logic

2. Unranked Trees
   - Unranked Trees
   - Automata
   - Logic

Trees on Ranked Alphabet

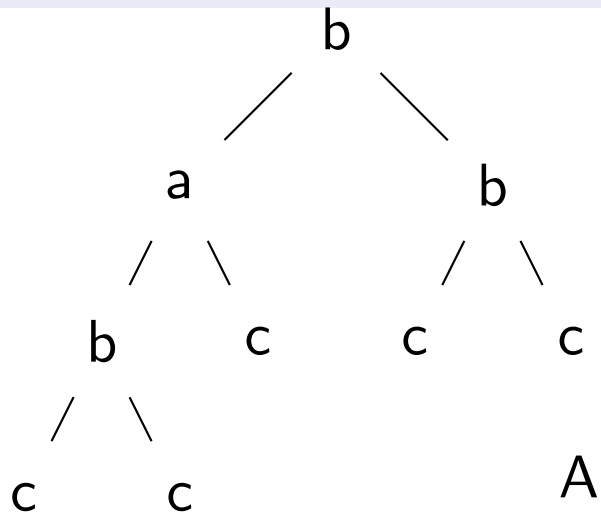
Ranked alphabet

Ranked alphabet Σ_r = finite alphabet $\Sigma = \{a, b, c\}$ + arity function $\mathrm{ar} : \Sigma \to \mathbb{N}$

$\mathrm{ar}(a) = 2$
$\mathrm{ar}(b) = 2$
$\mathrm{ar}(c) = 0$

Ranked trees over Σ_r

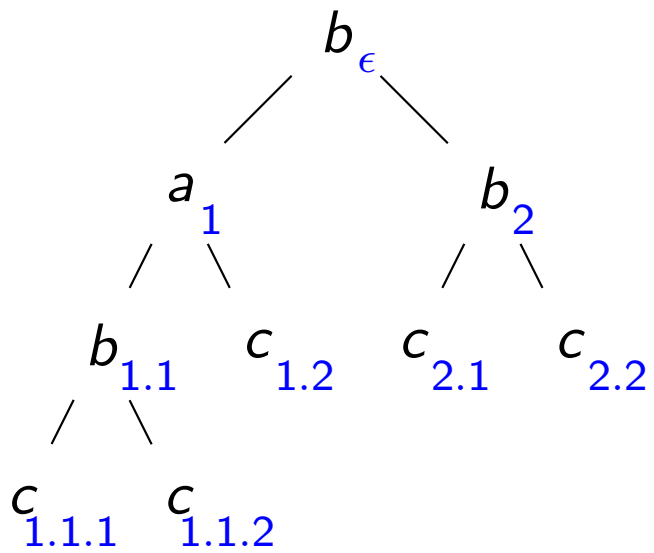
$\mathcal{T}_{\Sigma_r}$, the set of ranked trees, is the smallest set of terms $f(t_1, \ldots, t_k)$ such that: $f \in \Sigma_r$, $k = \mathrm{ar}(f)$, and $t_i \in \mathcal{T}_{\Sigma_r}$ for all $1 \leq i \leq k$.



A tree language T is a set of trees: $T \subseteq \mathcal{T}_{\Sigma_r}$.

Trees as relational structures

We will sometimes consider a ranked tree t as a relational structure $(\text{nodes}^t, \{\text{lab}_a^t, \text{lab}_b^t, \text{lab}_c^t, ch_1^t, ch_2^t\})$.



- $\text{lab}_a^t = \{1\}$
- $\text{lab}_b^t = \{\epsilon, 1.1, 2\}$
- $\text{lab}_c^t = \{1.1.1, 1.1.2, 1.2, 2.1, 2.2\}$
- $ch_1^t = \{(\epsilon, 1), (1, 1.1), (2, 2.1) \dots\}$
- $ch_2^t = \{(\epsilon, 2), (1, 1.2), (2, 2.2) \dots\}$

For convenience we write $\text{lab}(\pi)$ for the label of node π.

Tree Automata

over Ranked Trees

Definition

A tree automaton (TA) over $\Sigma_r = (\Sigma, \mathrm{ar})$ is a tuple $A = (Q, F, \Delta, \Sigma_r)$ where:

- Q is a finite set of states,
- $F \subseteq Q$ a set of final states,
- Δ are rules of type: $a(q_1, \ldots, q_k) \to q$

  with $a \in \Sigma$, $k = \mathrm{ar}(a)$ and $q, q_1, \ldots, q_k \in Q$

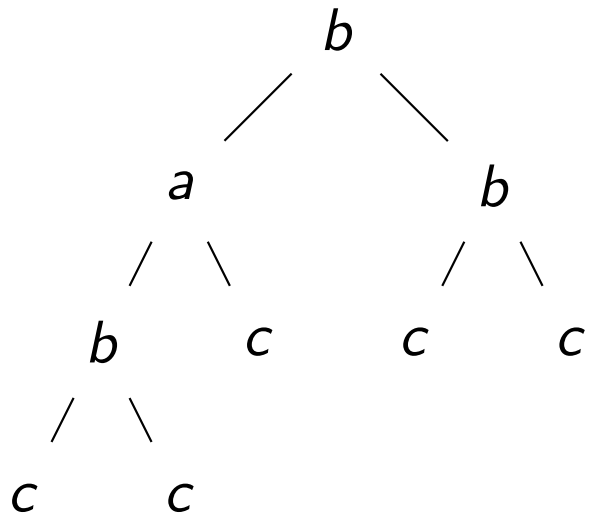
Runs

A run ρ of A on t is a function $\rho : \mathrm{nodes}^t \to Q$ such that:
if $\pi \in \mathrm{nodes}^t$ with children $\pi_1, \ldots, \pi_k$ and label a then
$a(\rho(\pi_1), \ldots, \rho(\pi_k)) \to \rho(\pi) \in \Delta$

Bottom-up vs top-down

Bottom-up view

$a(q_1, \ldots, q_k) \to q$ is a bottom-up point of view:



Rules:

$$\begin{cases} a(q_S, q_X) \to q_S \\ a(q_X, q_S) \to q_S \\ b(q_S, q_X) \to q_S \\ b(q_X, q_S) \to q_S \\ a(q_B, q_X) \to q_S \\ a(q_X, q_B) \to q_S \\ b(q_X, q_X) \to q_B \\ a(q_X, q_X) \to q_X \\ b(q_X, q_X) \to q_X \\ c \to q_X \end{cases}$$

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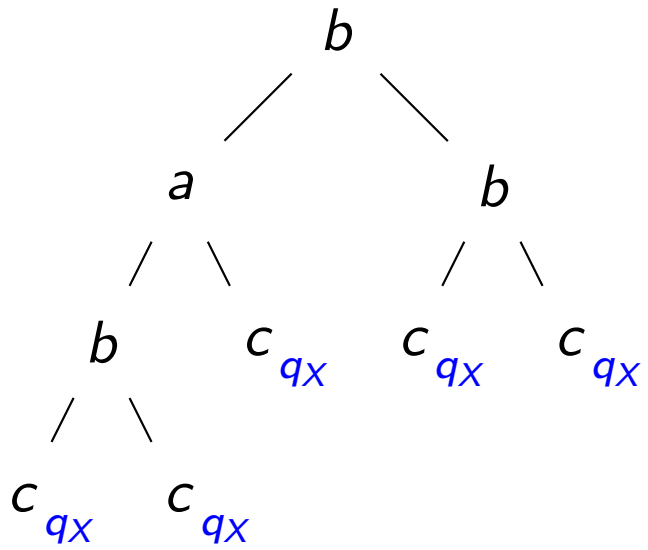

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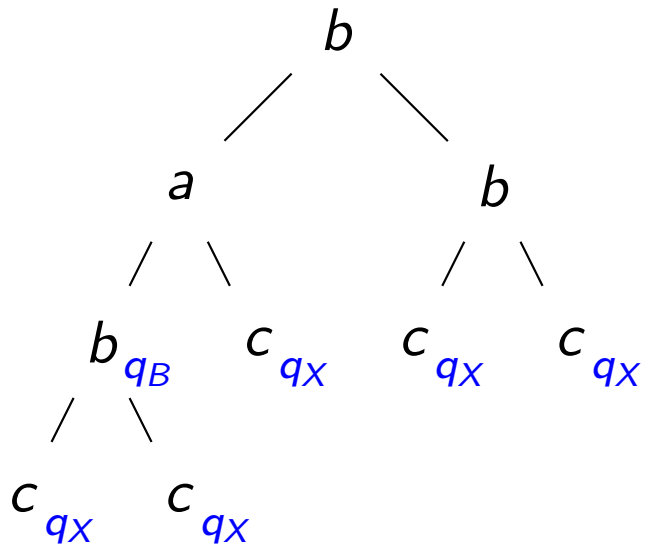

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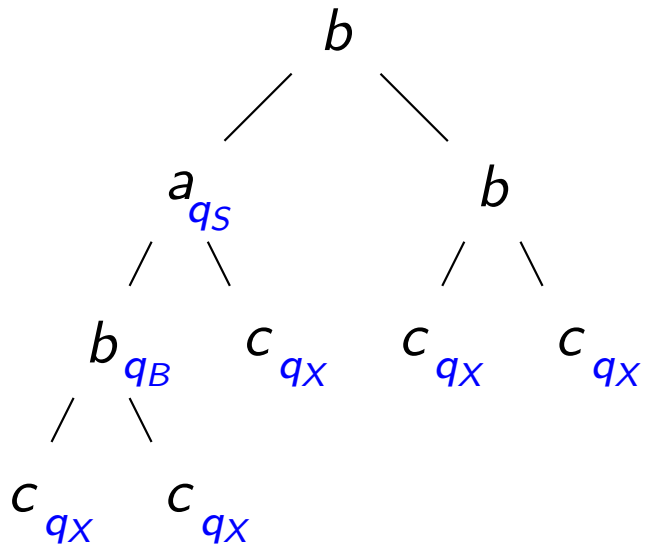

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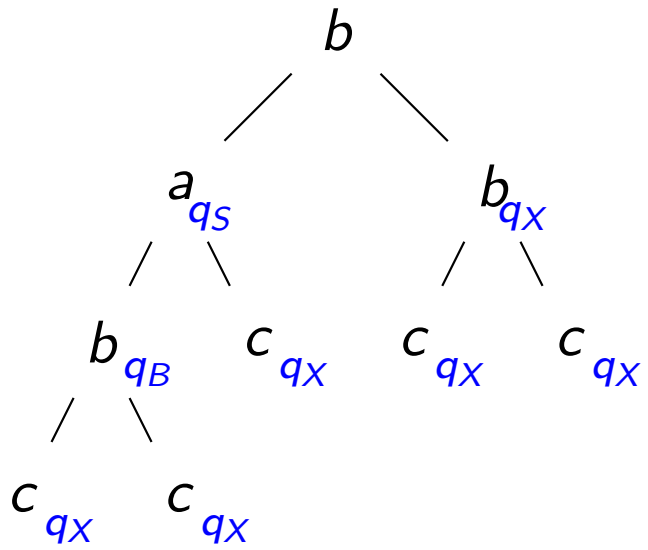

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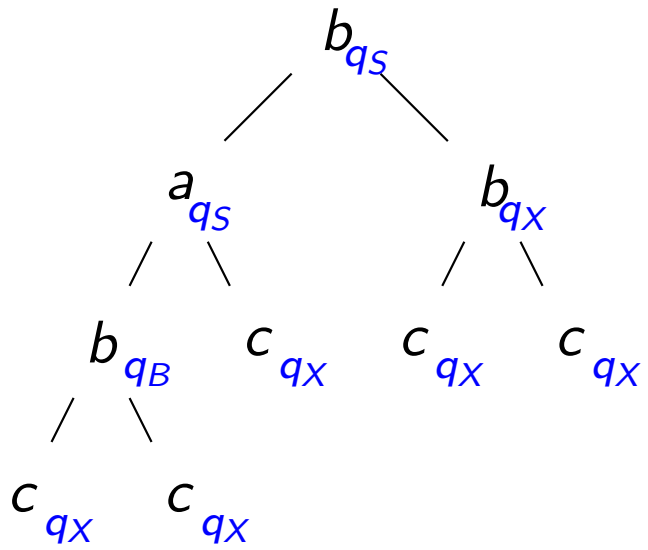

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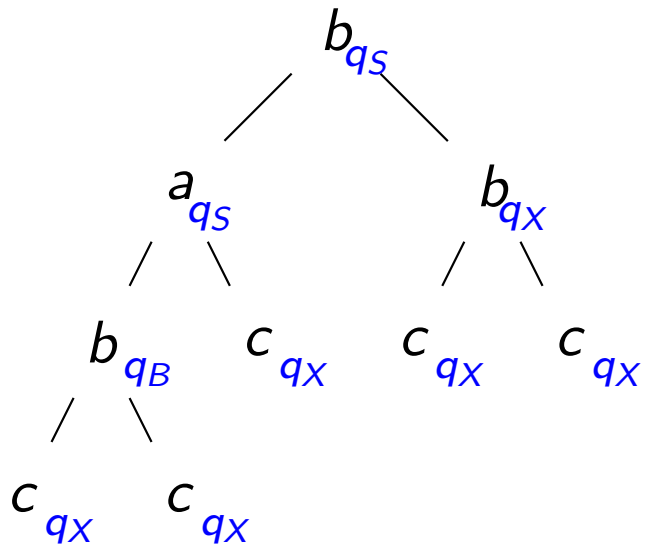

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A run of A on t is accepting if $\rho(\epsilon) \in F$.

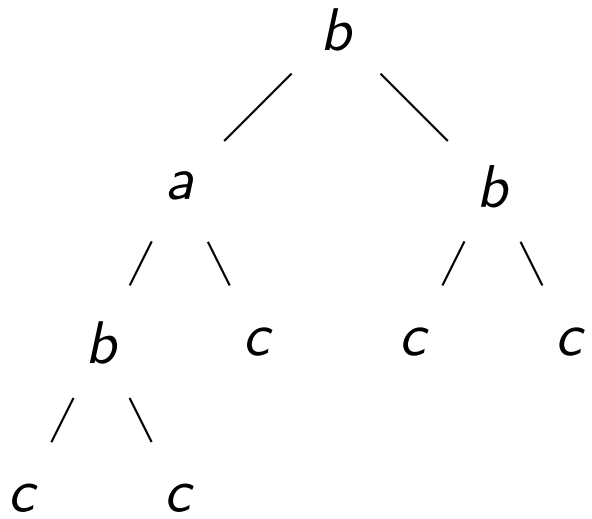
$\mathcal{L}(A) = \{t \mid \text{there exists an accepting run } \rho \text{ of } A \text{ on } t\}$

Bottom-up vs top-down

Top-down view

We could have written rules this way: $a(q) \to (q_1, \ldots, q_k)$, and name F the initial states.

This corresponds to a top-down definition:



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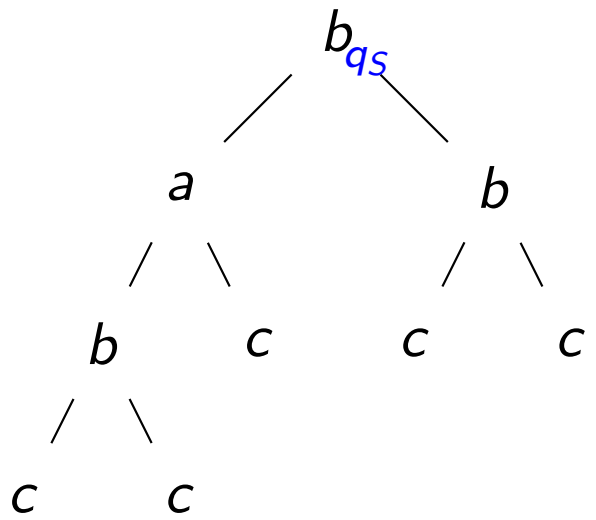

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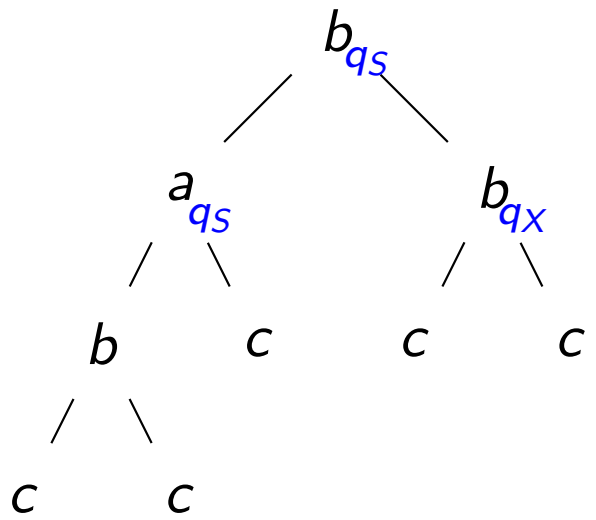

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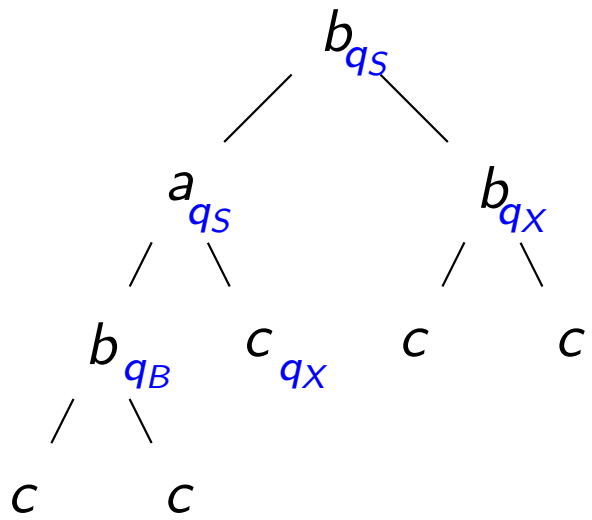

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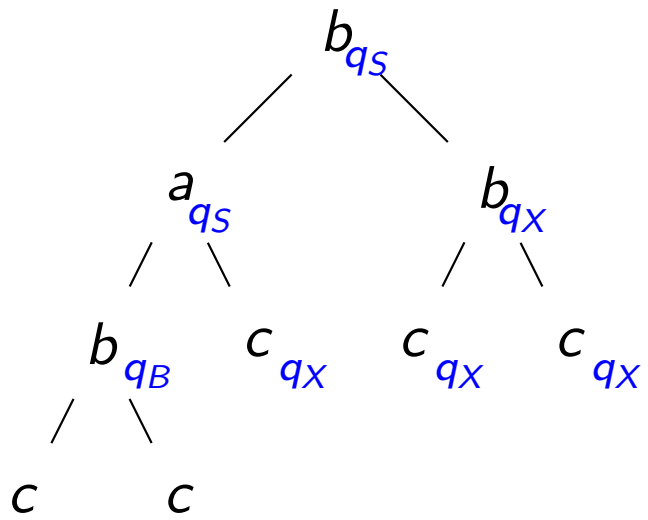

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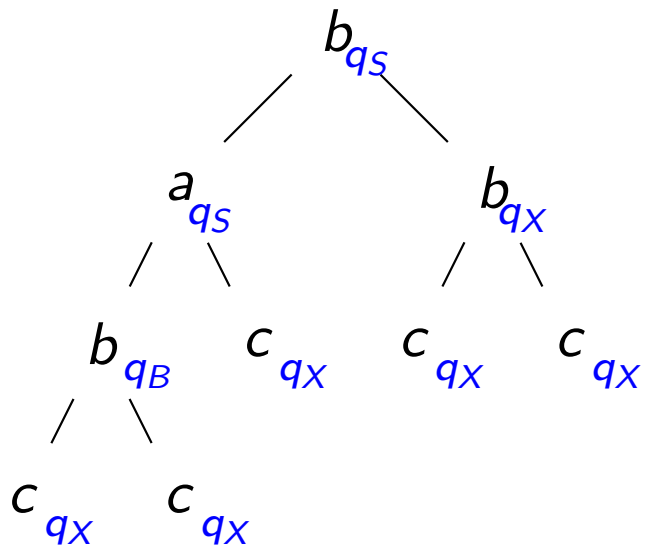

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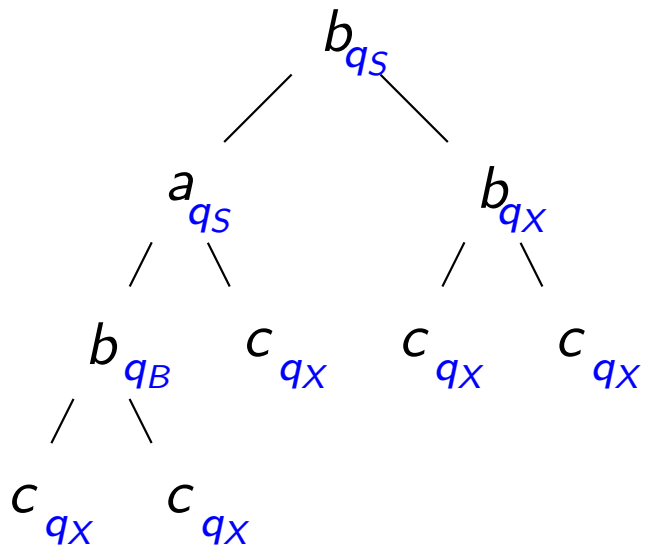

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Bottom-up vs top-down

Comparison

These definitions of runs coincide: a bottom-up run exists iff a top-down run exists, and they are strictly the same:

$$\text{bottom-up TA } (\uparrow\text{TA}) = \text{top-down TA } (\downarrow\text{TA}) = \text{“TA”}$$

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★★★

However, notions of determinism differ!

$$\text{det. } \downarrow\text{TA } (\text{d}\downarrow\text{TA }) \neq \text{det. } \uparrow\text{TA } (\text{d}\uparrow\text{TA })$$

For instance: $\{f(a,b), f(b,a)\}$ can be recognized by a det. bottom-up TA, but any det. top-down TA accepting $f(a,b)$ and $f(b,a)$ would recognize $f(a,a)$.

Determinization

of ↑TA

Proposition

For every ↑TA A, there exists a d↑TA A_d such that $\mathcal{L}(A) = \mathcal{L}(A_d)$.

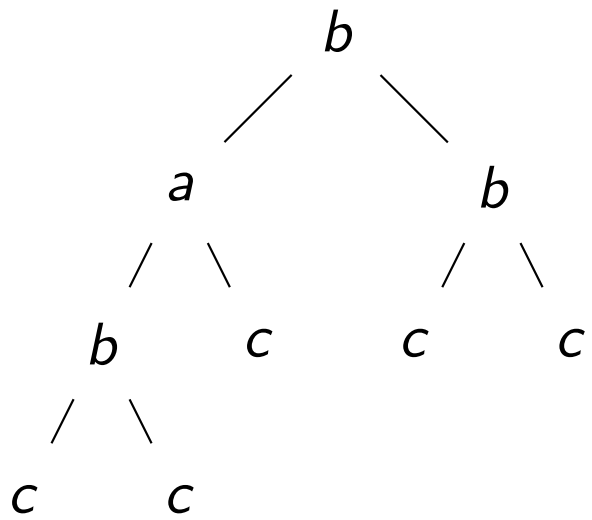
a subset construction, very similar to the determinization of NFAs:

- $Q_d = 2^Q$
- $\Delta_d = \{a(s_1, \ldots, s_n) \to s \mid$
  $s = \{ q \in Q \mid \exists q_1 \in s_1, \ldots, \exists q_n \in s_n,\ f(q_1, \ldots, q_n) \to q \in \Delta\}\}$
- $F_d = \{S \mid S \subseteq Q \text{ and } S \cap F \neq \emptyset\}$

This simulates all runs of A.

Determinization

Example run



$$F = \{q_S\}$$

Rules of $\uparrow$TA:

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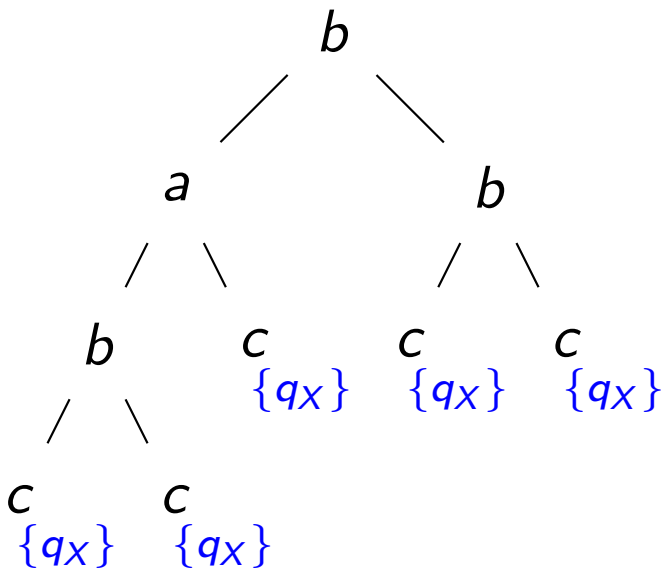
Determinization

Example run



$F = \{q_S\}$

Rules of $\uparrow$TA:

$$\begin{cases} a(q_S, q_X) \to q_S \\ a(q_X, q_S) \to q_S \\ b(q_S, q_X) \to q_S \\ b(q_X, q_S) \to q_S \\ a(q_B, q_X) \to q_S \\ a(q_X, q_B) \to q_S \\ b(q_X, q_X) \to q_B \\ a(q_X, q_X) \to q_X \\ b(q_X, q_X) \to q_X \\ c \to q_X \end{cases}$$

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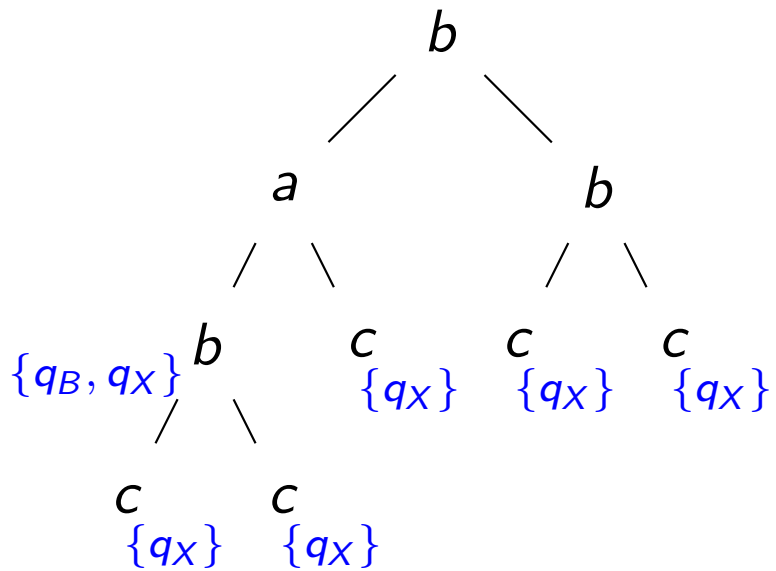

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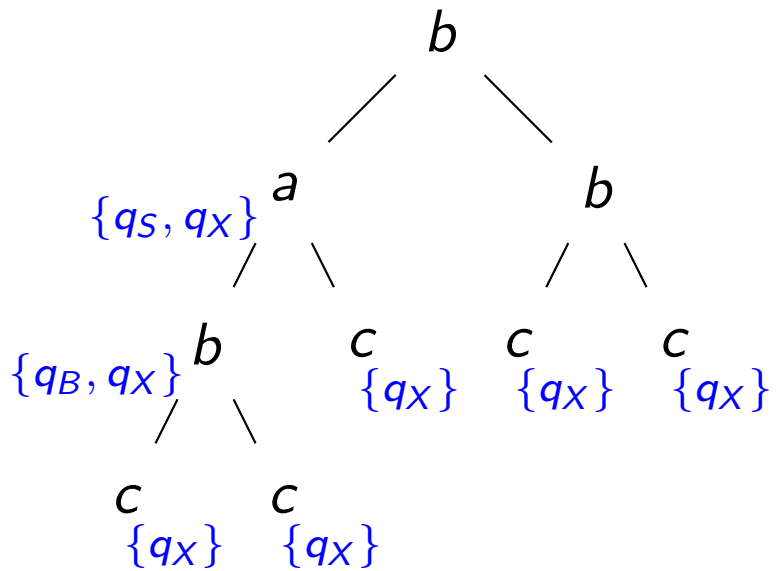

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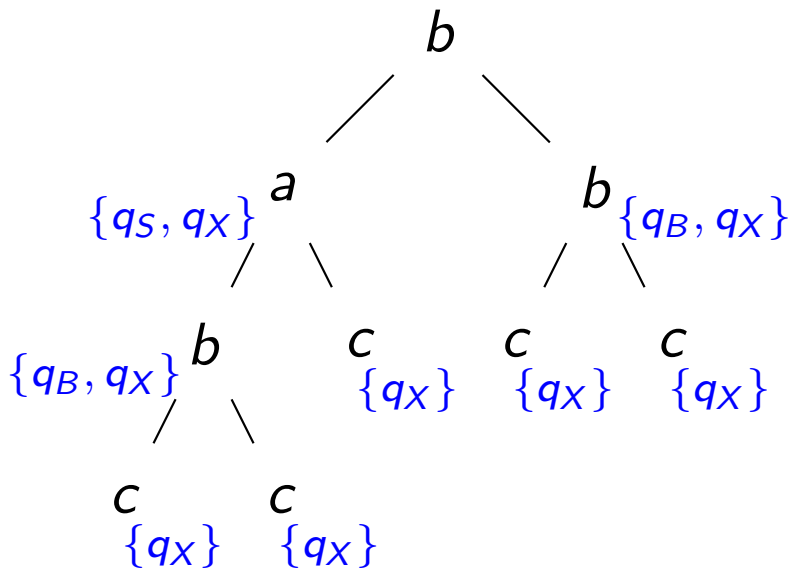

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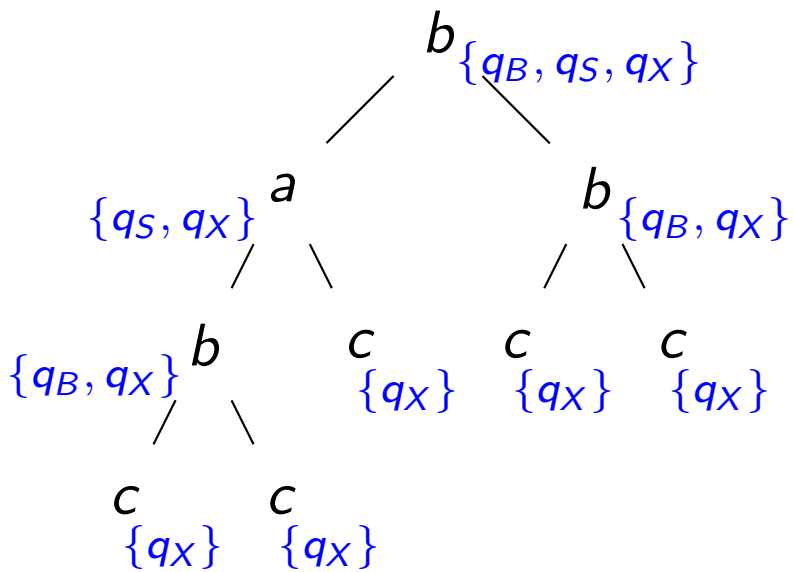

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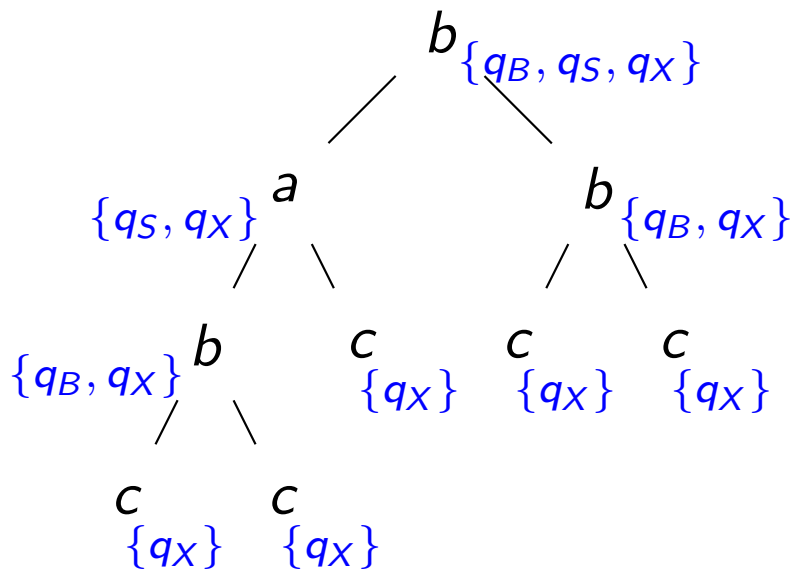

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$\{q_B, q_S, q_X\} \cap F \neq \emptyset$ so this tree is accepted by A_d

TA classes

$$\mathsf{d}{\uparrow}\mathsf{TA} = {\uparrow}\mathsf{TA}$$

TA classes

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TA classes

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Definition

A ranked tree language $L \subseteq \mathcal{T}_{\Sigma_r}$ is recognizable if there is a $\uparrow$TA recognizing L.

Closure Properties

of recognizable tree languages

If L_1 and L_2 are recognizable tree languages, then:

$\overline{L_1}$ is recognizable

if $A = (Q, F, \Delta, \Sigma_r)$ is a complete $\uparrow$TA, then $A' = (Q, Q \setminus F, \Delta, \Sigma_r)$ recognizes $\overline{\mathcal{L}(A)}$.
Completing is easy, by adding a sink state.

$L_1 \cup L_2$ is recognizable

$L_1 \cap L_2$ is recognizable

Closure Properties

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Let $\begin{cases} A_1 = (Q_1, F_1, \Delta_1, \Sigma_r) \text{ be a complete } \uparrow\text{TA recognizing } L_1 \\ A_2 = (Q_2, F_2, \Delta_2, \Sigma_r) \text{ be a complete } \uparrow\text{TA recognizing } L_2 \end{cases}$

We build the product automaton

$A_1 \times A_2 = (Q_1 \times Q_2, F_1 \times Q_2 \cup Q_1 \times F_2, \Delta', \Sigma_r)$ with

$$\frac{a(q_1, \ldots, q_n) \to q \in \Delta_1 \qquad a(q'_1, \ldots, q'_n) \to q' \in \Delta_2}{a((q_1, q'_1), \ldots, (q_n, q'_n)) \to (q, q') \in \Delta'}$$

Then $\mathcal{L}(A_1 \times A_2) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$.

$L_1 \cap L_2$ is recognizable

Closure Properties

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If L_1 and L_2 are recognizable tree languages, then:

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$L_1 \cap L_2$ is recognizable

$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$ (but more efficient using another product construction)

Pumping Lemma

Context $=$ tree over $\Sigma_r \uplus \{\Box\}$ where $\begin{cases} \mathrm{ar}(\Box) = 0 \text{ and} \\ \Box \text{ appears exactly once.} \end{cases}$

Intuitively, *context = tree with a hole*.

(context C) $+$ (tree t) $=$ a new tree $C[t]$, where t replaces $\Box$ in C.

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Pumping lemma

Let L be a *recognizable* tree language. Then there exists $k > 0$ such that for every tree $t \in L$ of depth $> k$, there exist contexts C_1, C_2 (with $C_2 \neq \Box$) and a tree t' such that $\begin{cases} t = C_1[C_2[t']] \\ \forall n \geq 0.\ C_1[C_2^n[t']] \in L \end{cases}$

idea: for A s.t. $\mathcal{L}(A) = L$, take $k = |Q|$ and consider a branch of length k.

Tree homomorphisms

$$\text{Tree homomorphism} = \begin{cases} \text{transformation } h : \mathcal{T}_\Sigma \to \mathcal{T}_{\Sigma'} \\ \text{based on a mapping } m : \Sigma \to \mathcal{T}_{\Sigma' \cup \mathcal{X}_{\text{ar}(m)}} \end{cases}$$

Theorem

$$\left.\begin{array}{l} L\text{: recognizable tree language in } \mathcal{T}_\Sigma \\ h\text{: linear tree homom. } h : \mathcal{T}_\Sigma \to \mathcal{T}_{\Sigma'} \end{array}\right\} \Rightarrow h(L) \text{ is recognizable (over } \mathcal{T}_{\Sigma'})$$

Linear means that each variable appears at most once in $m(a)$.

Simple counter-example with $m(a) = a(x_1, x_1)$.

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Theorem

$$\left. \begin{array}{l} L\text{: recognizable tree language in } \mathcal{T}_{\Sigma'} \\ h\text{: any tree homom. } h : \mathcal{T}_\Sigma \to \mathcal{T}_{\Sigma'} \end{array} \right\} \Rightarrow h^{-1}(L) \text{ is recognizable (over } \mathcal{T}_\Sigma)$$

Minimization

of Tree Automata

Let $\equiv$ be an equivalence relation. It is:

- a congruence if $\forall a \in \mathcal{T}_{\Sigma_r}$,

  $$\text{if} \quad t_i \equiv t'_i \text{ for all } 1 \leq i \leq n \quad \text{then} \quad a(t_1, \ldots, t_n) \equiv a(t'_1, \ldots, t'_n)$$

- of finite index if there are only finitely many $\equiv$-classes

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★★★

Given a tree language L, we define the congruence $\equiv_L$:

$$t \equiv_L t' \qquad \text{if for all contexts } C \text{ over } \Sigma_r, C[t] \in L \Leftrightarrow C[t'] \in L$$

Minimization

of Tree Automata

Def: $t \equiv_L t'$ if $\forall C.\ C[t] \in L \Leftrightarrow C[t'] \in L$

Myhill-Nerode Theorem

L is a recognizable tree language iff $\equiv_L$ is of finite index.

- ($\Rightarrow$) Let A be a complete d$\uparrow$TA recognizing L. Let $\equiv_A$ defined by: $t \equiv_A t'$ iff $\Delta(t) = \Delta(t')$ (state of A). It is of finite index ($\leq |Q_A|$), and $L = \bigcup_{t \mid \Delta(t) \in F_A} \mathit{class}_{\equiv_A}(t)$.

Minimization

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  As $\equiv_A$ is of finite index, it suffices to prove that $t \equiv_A t' \Rightarrow t \equiv_L t'$.
  Assume $t \equiv_A t'$. By an easy induction, $C[t] \equiv_A C[t']$ for all C. As L is a union of classes of $\equiv_A$, $C[t] \in L \Leftrightarrow C[t'] \in L$, and thus $t \equiv_L t'$.

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- ($\Leftarrow$) $A_{min} = (Q_{min}, F_{min}, \Delta_{min}, \Sigma_r)$ with:
  - $Q_{min} = class_{\equiv_L}$
  - $F_{min} = \{class_{\equiv_L}(t) \mid t \in L\}$
  - Δ_{min} contains rules:
    $f(class_{\equiv_L}(t_1), \ldots, class_{\equiv_L}(t_n)) \to class_{\equiv_L}(f(t_1, \ldots, t_n))$

Minimization

of Tree Automata

Consequence of Myhill-Nerode

The minimum d↑TA recognizing L is unique, up to a renaming of states.

if A recognizes L, $\equiv_A$ is a refinement of $\equiv_L$, so $|Q_A| \geq |Q_{A_{min}}|$.

Minimization algorithm (sketch)

- input: complete and reduced d↑TA A
- start from $\equiv_A$ and build $\equiv_L$,
- by merging q and q' if $\forall a \in \Sigma_r$, $\forall i$, $\forall q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n \in Q_A$, $\Delta(a(q_1, \ldots, q_{i-1}, q, q_{i+1}, \ldots, q_n)) \equiv \Delta(a(q_1, \ldots, q_{i-1}, q', q_{i+1}, \ldots, q_n))$
- until fixed point

Complexity of some decision problems

Name	Input	Output	Complexity
Membership	$\uparrow$TA A, tree t	$t \in \mathcal{L}(A)$?	PTIME
Emptiness	$\uparrow$TA A	$\mathcal{L}(A) = \emptyset$?	PTIME
Intersection non-emptiness	set S of $\uparrow$TAs	$\bigcap_{A \in S} \mathcal{L}(A) = \emptyset$?	EXPTIME-compl.
	set S of d$\uparrow$TAs		EXPTIME-compl.
Universality	$\uparrow$TA A	$\mathcal{L}(A) = T_{\Sigma_r}$?	EXPTIME-compl.
	d$\uparrow$TA A		PTIME
Equivalence	$\uparrow$TAs A_1, A_2	$\mathcal{L}(A_1) = \mathcal{L}(A_2)$?	EXPTIME-compl.
	d$\uparrow$TAs A_1, A_2		PTIME

Tree Grammars

Let $\mathcal{X}$ be a set of variables.

Tree grammar

A tree grammar is a tuple $G = (S, N, F, R)$ where:

- N is a set of non-terminal symbols,
- $S \in N$ an axiom,
- F a set of terminal symbols ($F \cap N = \emptyset$),
- R a set of production rules $\alpha \to \beta$ with $\alpha, \beta \in \mathcal{T}_{N \cup F \cup \mathcal{X}}$ and α contains at least one non-terminal.

Each element of $N \cup F$ has a fixed arity, $\mathrm{ar}(S) = 0$, and $\mathrm{ar}(x) = 0$, for $x \in \mathcal{X}$.

$a(b(x, d(N_1(c)))) \to a(d(x), b(N_2, c))$

Regular Tree Grammars

Definition

Regular tree grammars

A tree grammar G is regular if $\mathrm{ar}(N_i) = 0$ for all $N_i \in N$, and production rules are of the form $N_i \to \beta$ with $N_i \in N$ and $\beta \in \mathcal{T}_{N \cup F}$.

$\mathcal{L}(G)$ is the set of trees obtained by applying a series of rules, starting from S.
$L \subseteq \mathcal{T}_{\Sigma_r}$ is said regular if $L = \mathcal{L}(G)$ for some regular tree grammar G.

Example

$$R = \left\{ \begin{array}{lllll} S \to a(S,X) & S \to b(S,X) & S \to a(B,X) & B \to b(X,X) & X \to b(X,X) \\ S \to a(X,S) & S \to b(X,S) & S \to a(X,B) & X \to a(X,X) & X \to c \end{array} \right.$$

$a(b(c,c),c) \in \mathcal{L}(G)$ because:
$S \to_G a(B,X) \to_G a(b(X,X),X) \to_G a(b(c,X),X) \to_G a(b(c,c),X) \to_G a(b(c,c),c)$

Regular tree grammars can be normalized, so that rules have the form $N_0 \to a(N_1, \ldots, N_n)$.

Regular Tree Grammars

Expressiveness

Proposition

regular tree languages $=$ recognizable tree languages

Normalized regular tree grammars are $\downarrow$TAs...

Regular Tree Grammars

Expressiveness

Proposition

regular tree languages = recognizable tree languages

Normalized regular tree grammars are $\downarrow$TAs...

Regular Expressions

There also exists a notion of regular expressions for trees, with the same expressiveness as regular tree grammar (Kleene's theorem).

skipped in this talk...

1. Ranked Trees
   - Trees on Ranked Alphabet
   - Tree Automata
   - Tree Grammars
   - Logic

2. Unranked Trees
   - Unranked Trees
   - Automata
   - Logic

Monadic Second-Order (MSO) Logic

Syntax

- Σ_r a ranked signature
- $\mathcal{X}$ contains first-order $(x, y \ldots)$ and second-order $(X, Y \ldots)$ variables.
- $\Omega = \{\mathrm{lab}_a \mid a \in \Sigma_r\} \cup \{ch_i \mid \exists a \in \Sigma_r.\ \mathrm{ar}(a) \geq i\}$
- predicates lab_a are unary, while ch_i are binary

$MSO[\Omega]$: Syntax

$$\phi ::= \mathrm{lab}_a(x) \mid ch_i(x, y) \mid \phi \wedge \phi \mid \neg\phi \mid \exists x.\ \phi \mid \exists X.\ \phi \mid x \in X$$

where $a \in \Sigma$, $\exists a \in \Sigma_r.\ \mathrm{ar}(a) \geq i$, and $x, x_1, \ldots, x_k, X \in \mathcal{X}$.

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$MSO[\Omega]$ is also known as WSkS, the *Weak Second-order logic with k Successors*. "Weak" means "interpreted over finite structures" (here, terms).

Monadic Second-Order (MSO) Logic

Example

For instance: all trees having an even number of a-labeled nodes.

hint: define X as the set of nodes having an even number of a-descendants

Monadic Second-Order (MSO) Logic

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hint: define X as the set of nodes having an even number of a-descendants

$$even_a = \exists X\ \forall x\ \mathsf{lab}_a(x) \Rightarrow \begin{cases} \exists y_1\ ch_1(x, y_1) \wedge \\ \exists y_2\ ch_2(x, y_2) \wedge \\ x \in X \Leftrightarrow (y_1 \in X \oplus y_2 \in X) \end{cases}$$

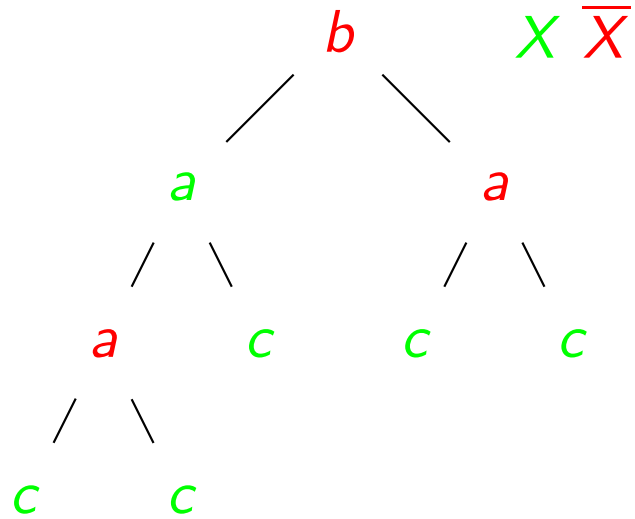
$$\bigwedge_{\alpha \neq a} \mathsf{lab}_\alpha(x) \Rightarrow \begin{cases} \bigwedge_{1 \leq i \leq \mathsf{ar}(\alpha)} \exists y_i\ ch_i(x, y_i) \wedge \\ x \notin X \Leftrightarrow (y_1 \notin X \oplus \ \ldots\ \oplus y_{\mathsf{ar}(\alpha)} \notin X) \end{cases}$$

$$\wedge \exists x\ root(x)\ \wedge\ x \in X$$

where:

$\phi \oplus \phi' = (\phi\ \wedge \neg\phi')\ \vee (\neg\phi\ \wedge\ \phi')$

$root(x) = \neg\exists y.\ ch_1(y, x)$



Monadic Second-Order (MSO) Logic

Semantics

convention: $\phi(\overline{x}, \overline{X})$ means that ϕ has free FO variables $\overline{x}$ and free SO variables $\overline{X}$.

A formula $\phi(\overline{x}, \overline{X}) \in MSO[\Omega]$ is interpreted over a tree t under an assignment $\mu : \overline{x} \cup \overline{X} \to \text{nodes}(t)$.

Satisfiability $t, \mu \models \phi$ is defined inductively by:

$$
\begin{array}{ll}
t, \mu \models \text{lab}_a(x) & \text{iff } \text{lab}_a^t(\mu(x)) \\
t, \mu \models ch_i(x, y) & \text{iff } ch_i(\mu(x), \mu(y)) \\
t, \mu \models \phi \ \wedge \ \phi' & \text{iff } t, \mu \models \phi \text{ and } t, \mu \models \phi' \\
t, \mu \models \neg\phi & \text{iff } t, \mu \not\models \phi \\
t, \mu \models \exists x \phi & \text{iff there exists } \pi \in \text{nodes}(t) \text{ s.t. } t, \mu[x \leftarrow \pi] \models \phi \\
t, \mu \models \exists X \phi & \text{iff there exists } S \subseteq \text{nodes}(t) \text{ s.t. } t, \mu[X \leftarrow S] \models \phi \\
t, \mu \models x \in X & \text{iff } \mu(x) \in \mu(X)
\end{array}
$$

MSO and Tree Languages

Expressiveness of MSO vs ↑TA?

For closed formulas ϕ, define $\mathcal{L}_\phi = \{t \mid t \models \phi\} \subseteq \mathcal{T}_{\Sigma_r}$.

What about $\phi(\overline{x}, \overline{X})$?

$$\phi(\overline{x}, \overline{X}) \xrightarrow{\text{semantics}} \{(t, \mu) \mid t, \mu \models \phi\} \xrightarrow{\text{tree language}} \mathcal{L}_\phi = \{t * \mu \mid t, \mu \models \phi\}$$

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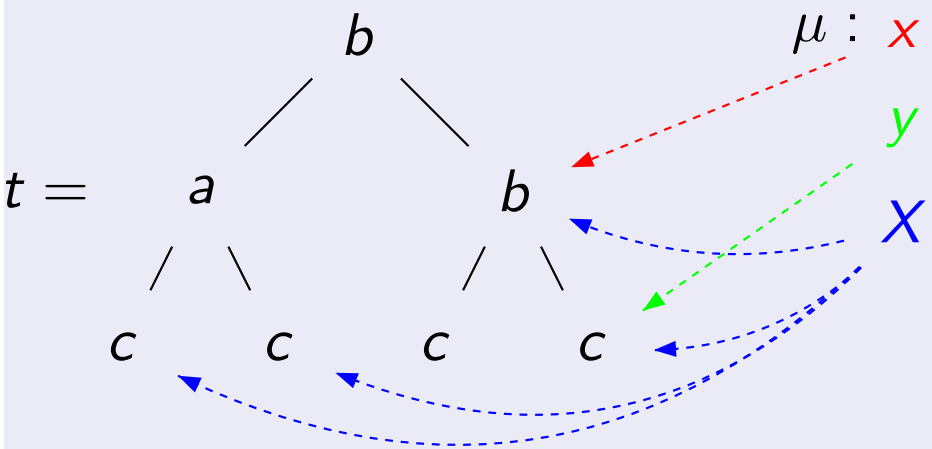
$t * \mu \in \mathcal{T}_{\Sigma_r \times \mathbb{B}^n}$



where $n = |\overline{x}| + |\overline{X}|$ and $\mathbb{B} = \{0, 1\}$.

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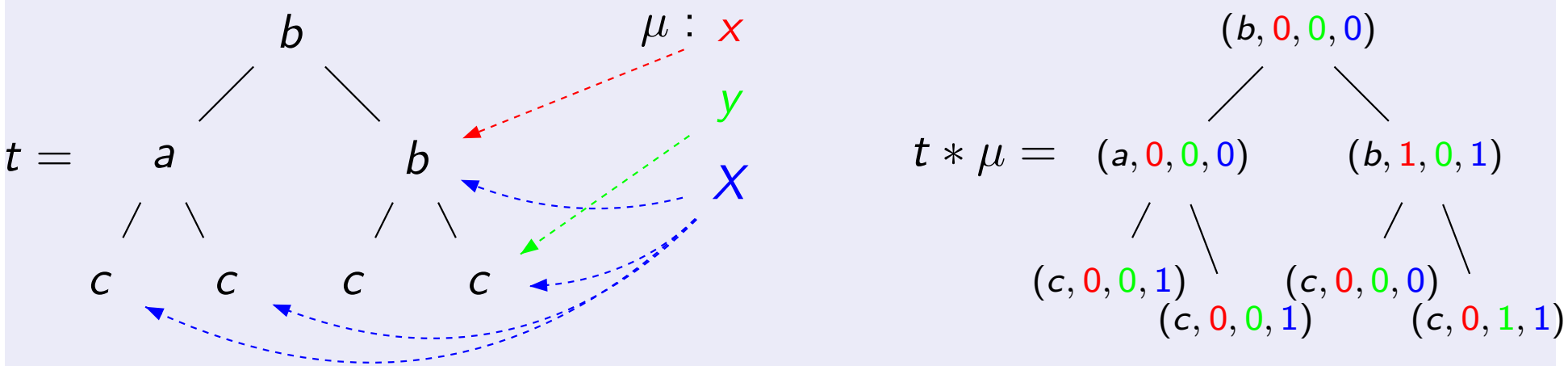
For closed formulas ϕ, define $\mathcal{L}_\phi = \{t \mid t \models \phi\} \subseteq \mathcal{T}_{\Sigma_r}$.

What about $\phi(\overline{x}, \overline{X})$?

$$\phi(\overline{x}, \overline{X}) \xrightarrow{\text{semantics}} \{(t, \mu) \mid t, \mu \models \phi\} \xrightarrow{\text{tree language}} \mathcal{L}_\phi = \{t * \mu \mid t, \mu \models \phi\}$$

$t * \mu \in \mathcal{T}_{\Sigma_r \times \mathbb{B}^n}$



where $n = |\overline{x}| + |\overline{X}|$ and $\mathbb{B} = \{0, 1\}$.

Recognizable languages = MSO-definable languages

Theorem [TW68, Don70]

$L \subseteq \mathcal{T}_{\Sigma_r}$ is recognizable iff there exists $\phi \in MSO[\Omega]$ such that $\mathcal{L}_\phi = L$.

consequence: $MSO[\Omega]$ is decidable (convert to $\uparrow$TA, test emptiness)

($\Rightarrow$) *idea: encode a run of A in MSO.*
$A = (Q, F, \Delta, \Sigma_r)$ a complete d$\uparrow$TA recognizing L. Let $\{q_1, \ldots, q_n\} = Q$.

$$\phi_A = \exists X_{q_1} \ldots \exists X_{q_n}\ partition(X_{q_1}, \ldots, X_{q_n}) \wedge$$
$$\bigwedge_{a \to q \in \Delta} leaf(x) \wedge \mathrm{lab}_a(x) \Rightarrow x \in X_q$$
$$\bigwedge_{a(q_{i_1}, \ldots, q_{i_k}) \to q_i \in \Delta} \forall x\ \forall y_1 \ldots \forall y_k \left\{ \begin{array}{l} \mathrm{lab}_a(x) \wedge \\ ch_1(x, y_1) \wedge \ldots \wedge ch(x, y_k) \wedge \\ y_1 \in X_{q_{i_1}} \wedge \ldots \wedge y_k \in X_{q_{i_k}} \end{array} \right\} \Rightarrow x \in X_{q_i}$$
$$\wedge\ \exists x. root(x) \wedge \bigvee_{q \in F} x \in X_q$$

with: $\left\{ \begin{array}{l} leaf(x) = \nexists y.\ ch_1(x, y) \\ partition(X_{q_1}, \ldots, X_{q_n}) = \ldots \end{array} \right.$

($\Leftarrow$) *idea: equivalence betw. logical connectives and automata operators.*

- $\mathcal{L}_{\mathsf{lab}_a} = \left\{ \ldots, \begin{array}{ccc} & (a,1) & \\ \swarrow & & \searrow \\ (a,0) & & (b,0) \end{array}, \begin{array}{ccc} & (a,0) & \\ \swarrow & & \searrow \\ (a,1) & & (b,0) \end{array}, \ldots \right\}$ recognizable

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- $\mathcal{L}_{\exists X \phi}$: same idea
- $\mathcal{L}_{x \in X}$: easily recognizable (a 1 on x-component implies a 1 on X-component)

$$\begin{array}{ccc} & (a,1,1) & \\ / & & \backslash \\ (a,0,1) & & (b,0,0) \end{array} \in \mathcal{L}_{x \in X} \quad \text{but} \quad \begin{array}{ccc} & (a,0,1) & \\ / & & \backslash \\ (a,1,0) & & (b,0,1) \end{array} \notin \mathcal{L}_{x \in X}$$

Bonus: Recognizable tree relations

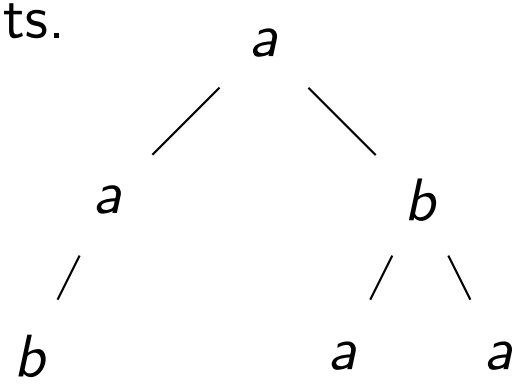
see [CDG+07] Chapter 3, and also [BLN07, Gau09]

1. Ranked Trees
   - Trees on Ranked Alphabet
   - Tree Automata
   - Tree Grammars
   - Logic

2. Unranked Trees
   - Unranked Trees
   - Automata
   - Logic

Unranked Trees

An unranked tree over Σ is a tree, labeled by elements of Σ, without arity constraints.



We write $\mathcal{T}_\Sigma$ for the set of unranked trees over Σ.

Which notion of automata can we use?

For ranked trees, rules were $a(q_1, \ldots, q_n) \to q$ but here n is not bounded.
Several approaches:

- horizontal languages: use a string language on the states of children
  - hedge automata
  - DTDs
- binary encodings: encode unranked trees into binary trees
  - stepwise tree automata...
- linearization: serialize trees and use a pushdown system
  - nested word automata
  - visibly pushdown automata

Horizontal Languages: Hedge Automata

idea: use a regular language on the states of children

"hedge" = finite series of trees

Hedge automata [BKWM01]

A hedge automaton over Σ is a tuple $A = (Q, F, \Delta, \Sigma)$ where:

- Q is a finite set of states
- $F \subseteq Q$ is the set of final states
- Δ is a set of rules $a(L) \to q$ where $a \in \Sigma$, $q \in Q$ and L is a regular string language over Q.

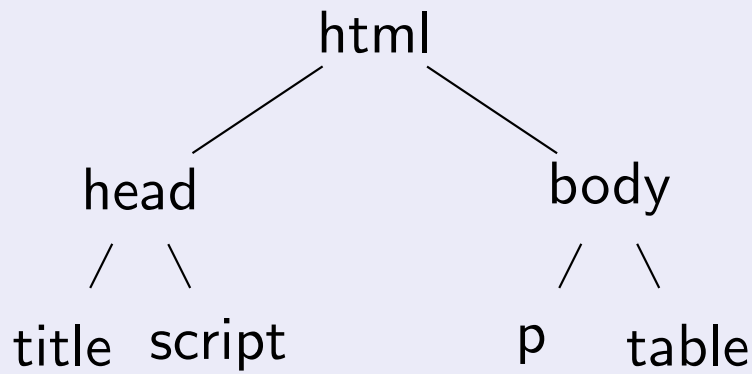
A run of A on t is a function $\rho : \text{nodes}(t) \to Q$ such that for all nodes π of t with children $\pi_1, \ldots, \pi_n$ and label a, there is a rule $a(L) \to \rho(\pi) \in \Delta$ with $\rho(\pi_1) \ldots \rho(\pi_n) \in L$.

Horizontal Languages: DTDs

- Document Type Definitions [BPSM+08]
- W3C standard for specifying valid XML documents
- hedge automata, where horizontal languages are specified by regexp.

Example: HTML DTD

$$
\begin{array}{lcl}
\mathit{html} & \rightarrow & \mathit{head}.\mathit{body} \\
\mathit{head} & \rightarrow & \mathit{title}.\mathit{meta}?.\mathit{style}?.\mathit{script}?... \\
\mathit{body} & \rightarrow & (p|\mathit{div}|\mathit{table}|h1|...)^* \\
... & &
\end{array}
$$



Binary Encodings

- define a bijection between unranked trees and binary trees
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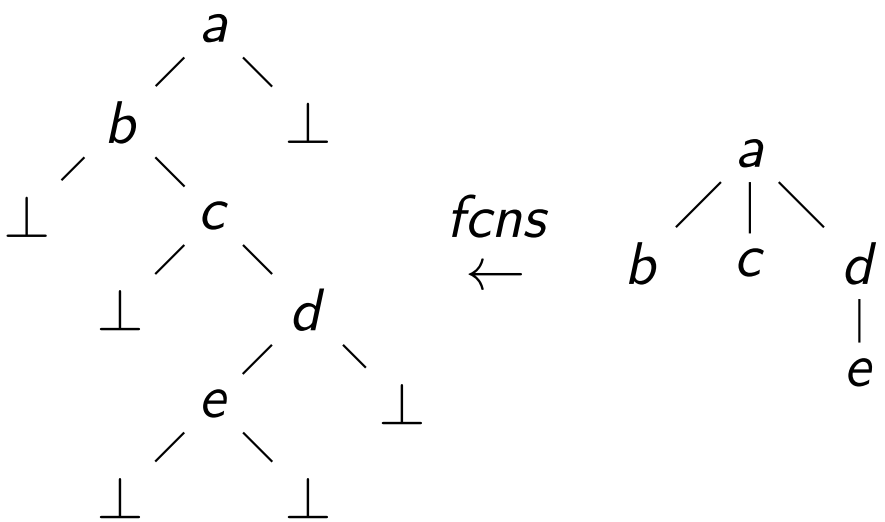
first-child next-sibling [Rab69, Koc03]

$fcns : \mathcal{T}_\Sigma \to \mathcal{T}_{\Sigma_f}$ where $\Sigma_f = (\Sigma \uplus \{\bot\}, \mathrm{ar})$ with $\mathrm{ar}(a)=2$ for $a \in \Sigma$ and $\mathrm{ar}(\bot)=0$



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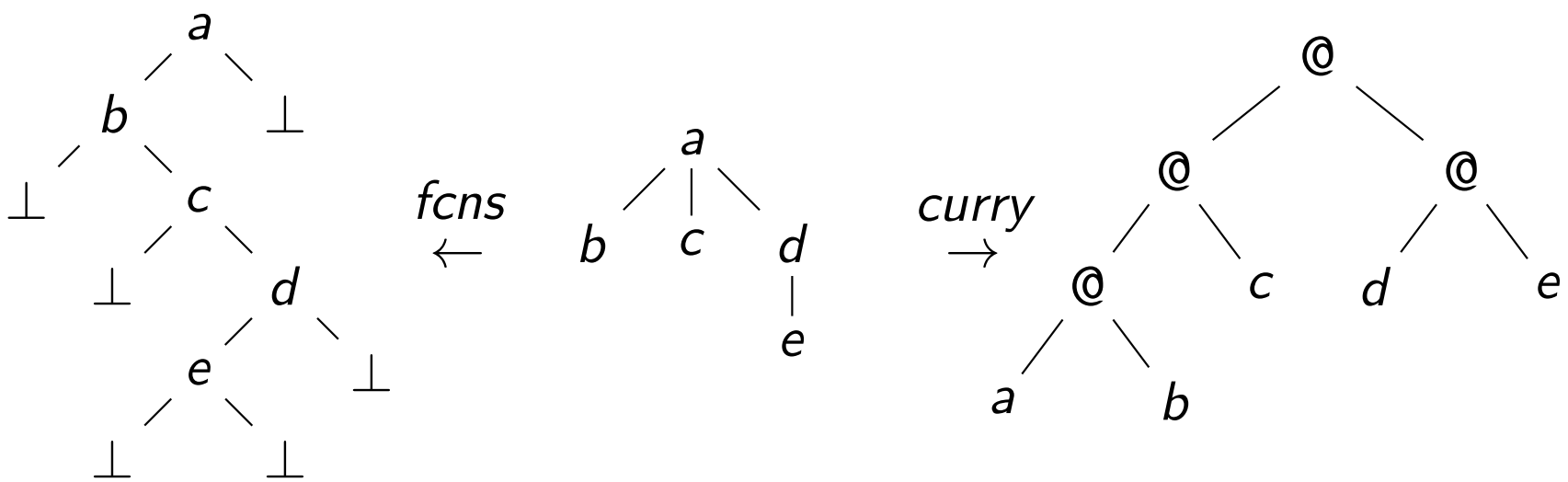
Curryfication

$curry : \mathcal{T}_\Sigma \to \mathcal{T}_{\Sigma_c}$ where $\Sigma_c = (\Sigma \uplus \{@\}, \mathrm{ar})$ with $\mathrm{ar}(a)=0$ for $a \in \Sigma$ and $\mathrm{ar}(@)=2$
$t_1 @ t_2 =$ "add t_2 as last child of the root of t_1"



Binary Encodings

Automata

We can use tree automata over ranked languages on encoded trees:

	fcns	*curry*
$\uparrow$TA	C_1	C_2
$\downarrow$TA	C_3	C_4

These 4 classes are equally expressive.

C_2 corresponds to stepwise tree automata [CNT04].

Binary Encodings

Automata

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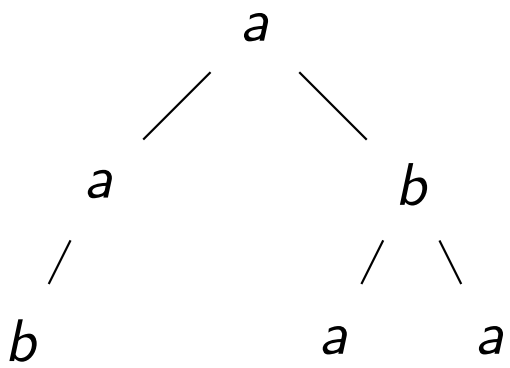
d$\downarrow$TAs $\circ \{fcns, curry\}$ define two other classes, that have different expressiveness. d$\downarrow$TAs $\circ fcns$ is the determinism of DTDs.

Linearization

Let $\overline{\Sigma} = \{\overline{a} \mid a \in \Sigma\}$.
$lin(t)$, the linearization of t, is the word over $\Sigma \cup \overline{\Sigma}$ produced by the pre-order traversal of t:



$$lin(t) = a.a.b.\overline{b}.\overline{a}.b.a.\overline{a}.a.\overline{a}.\overline{b}.\overline{a}$$

This corresponds to the XML serialization: <a><a>...

Visibly Pushdown Automata [AM04]

- a pushdown automaton over $\Sigma \uplus \overline{\Sigma}$
- visible means that 1 action (push/pop) is performed by each letter:
  - rules using $a \in \Sigma$ only push
  - rules using $\overline{a} \in \overline{\Sigma}$ only pop

Visibly Pushdown Automaton (VPA)

A VPA is a tuple $(Q, I, F, \Gamma, \Delta, \Sigma \uplus \overline{\Sigma})$ with $I, F \subseteq Q$ and Δ is a set of rules with the form:

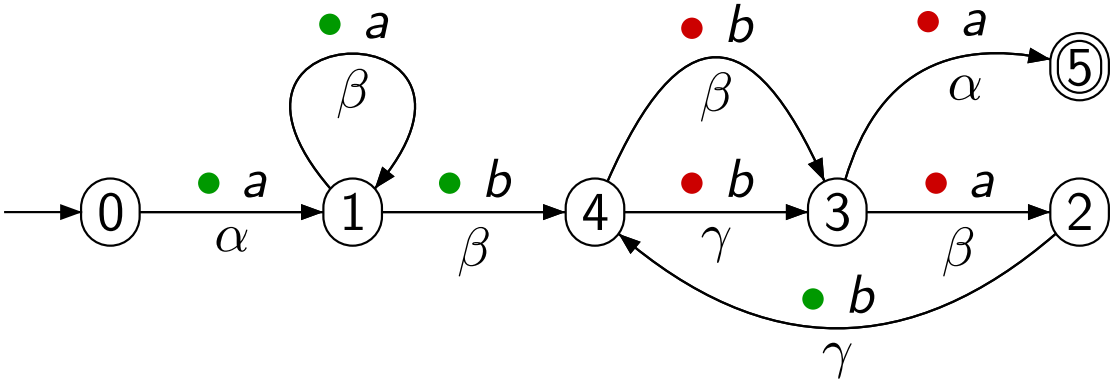
- $q_1, a \to \gamma, q_2$
- $q_1, \overline{a}, \gamma \to q_2$

with $q_1, q_2 \in Q$, $a \in \Sigma$, $\overline{a} \in \overline{\Sigma}$, $\gamma \in \Gamma$.

The semantics is defined as for usual pushdown automata (ends on final states, not on empty stack).

VPAs as Tree Automata

- a run of a VPA on a tree consists in:
  - when opening node π: update the current state, and assign $\gamma \in \Gamma$ to π
  - when closing node π: update the current state according to γ



A: VPA on $\mathcal{T}_\Sigma$ with $\Sigma = \{a, b\}$
and $Q = \{0, 1, 2, 3, 4, 5\}$
and $\Gamma = \{\alpha, \beta, \gamma\}$

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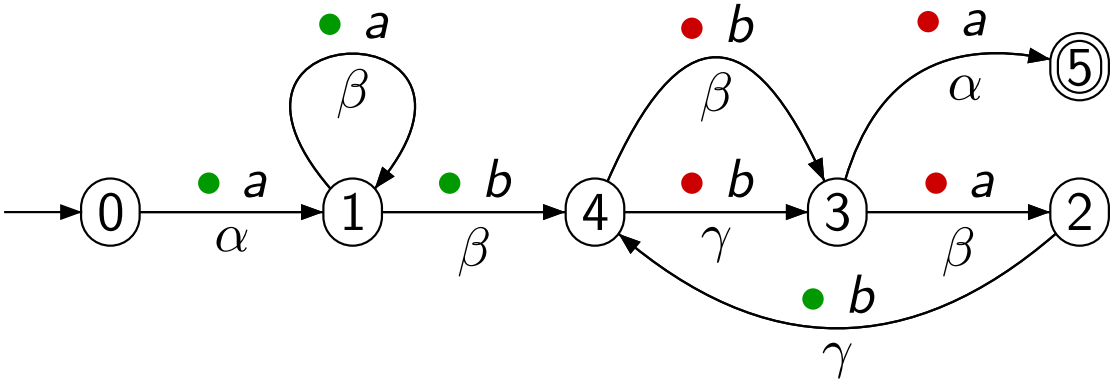

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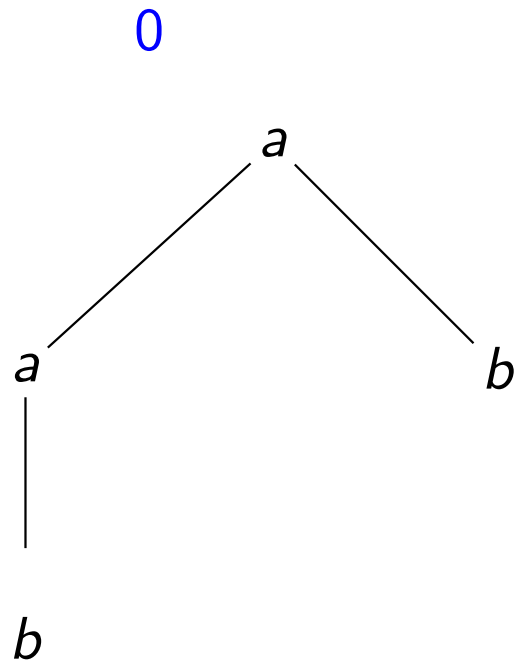
$(0, \emptyset)$



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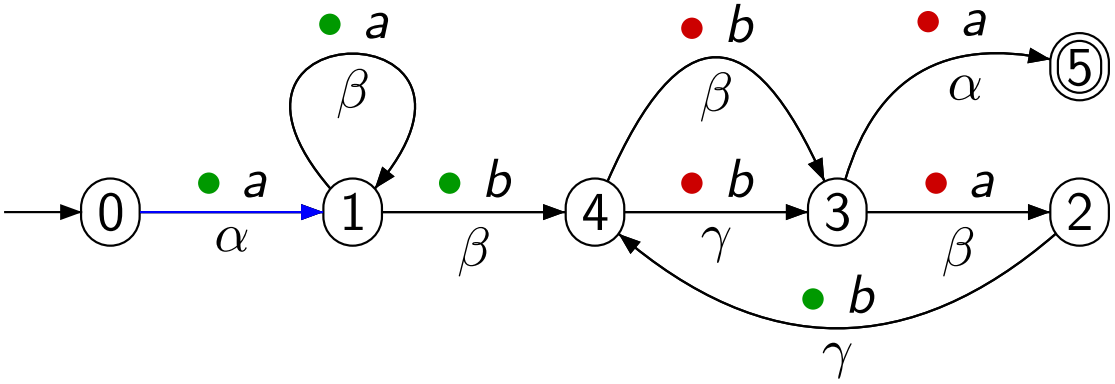

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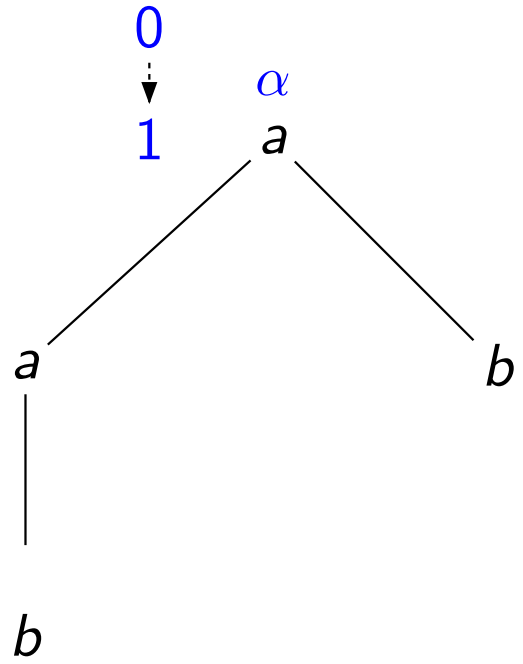
$(0, \emptyset) \xrightarrow{\bullet a} (1, \alpha)$



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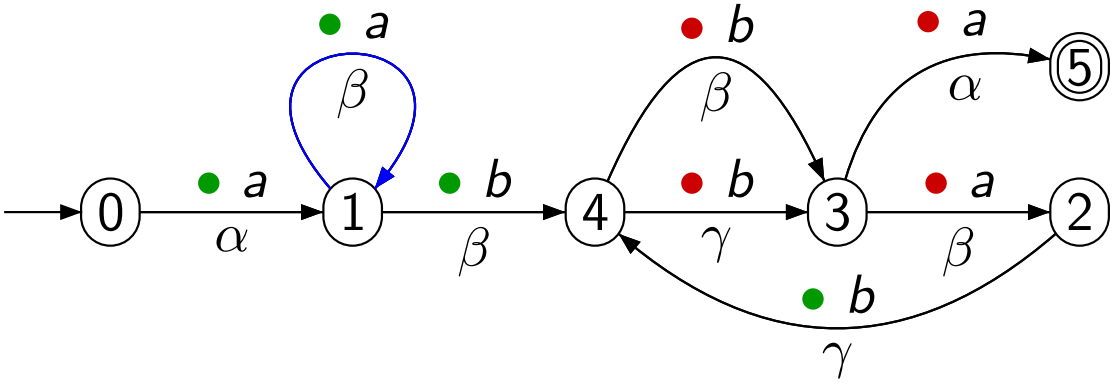

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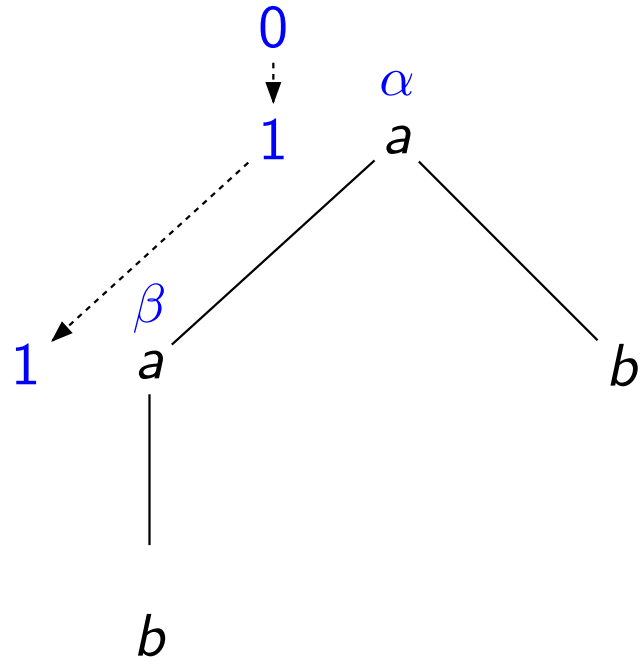

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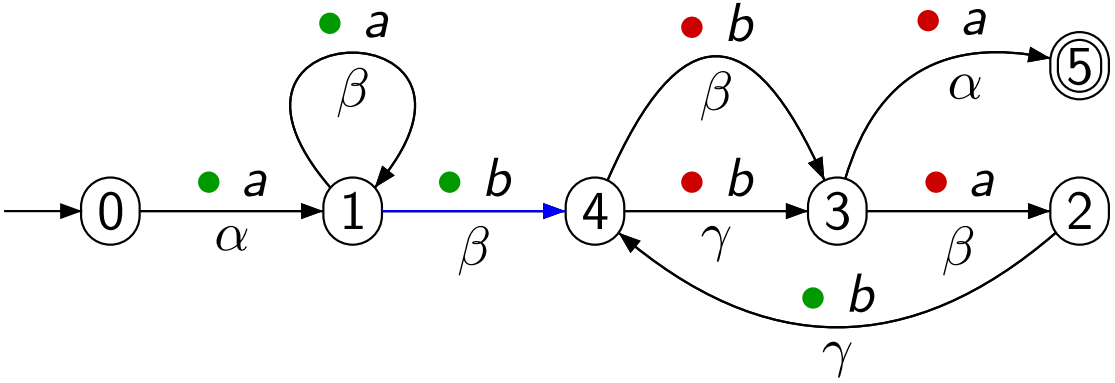

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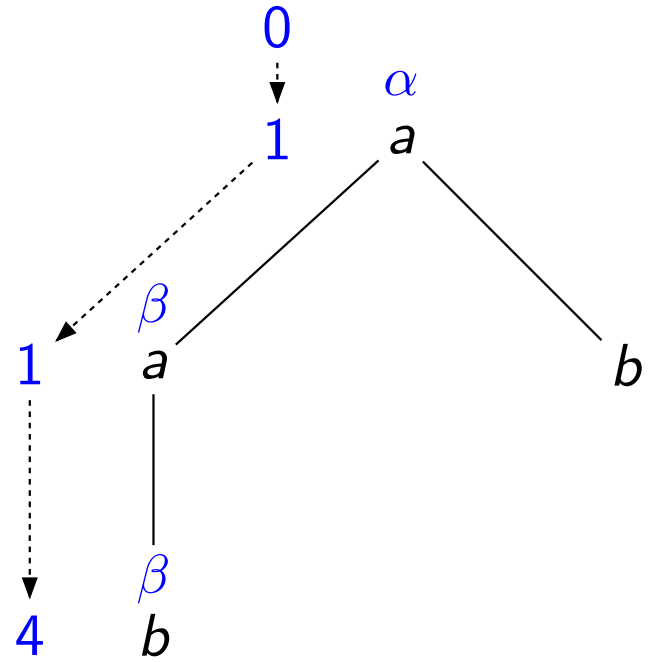
$$(0, \emptyset) \xrightarrow{\bullet a} (1, \alpha) \xrightarrow{\bullet a} (1, \alpha.\beta) \xrightarrow{\bullet b} (4, \alpha.\beta.\beta)$$



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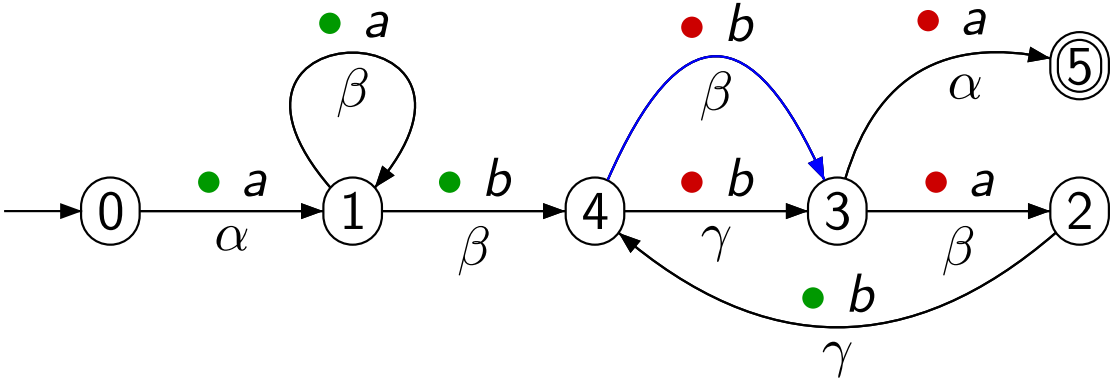

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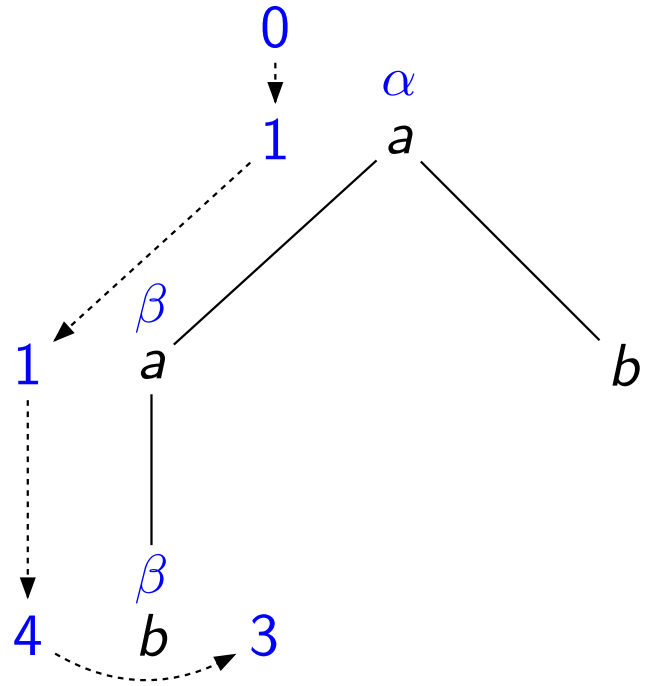
$(0, \emptyset) \xrightarrow{\bullet a} (1, \alpha) \xrightarrow{\bullet a} (1, \alpha.\beta) \xrightarrow{\bullet b} (4, \alpha.\beta.\beta)$
$\xrightarrow{\bullet b} (3, \alpha.\beta)$



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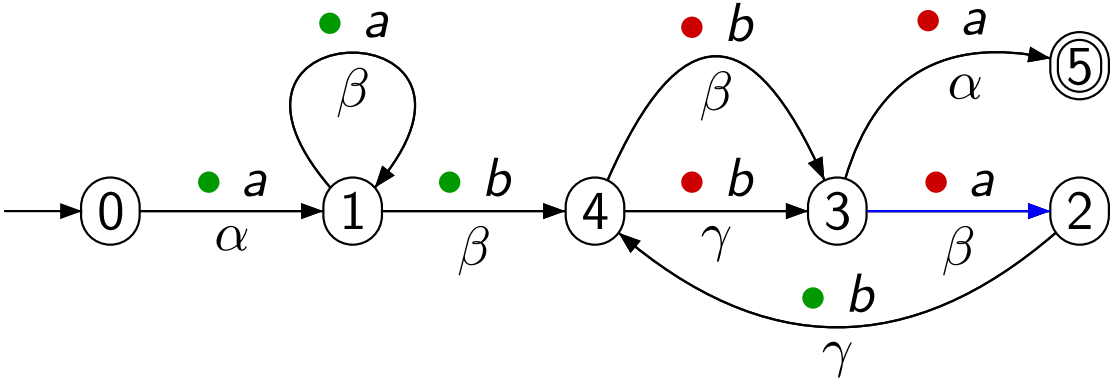
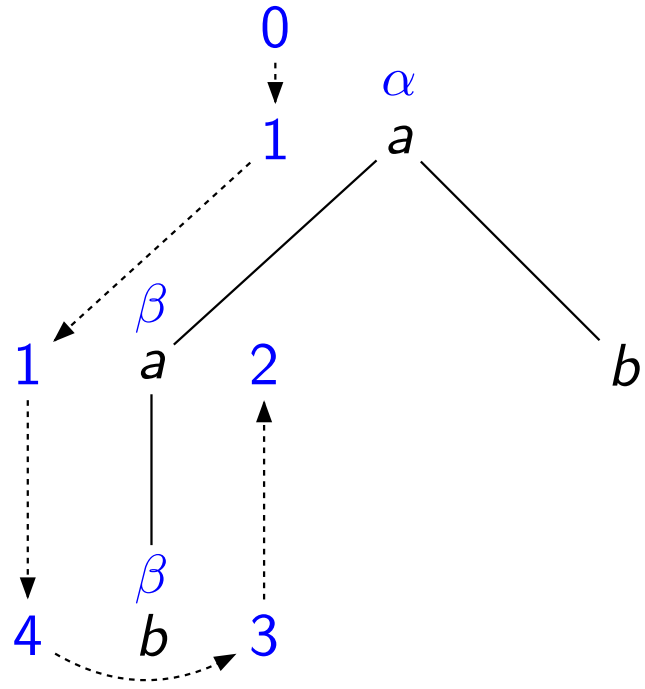


A: VPA on $\mathcal{T}_\Sigma$ with $\Sigma = \{a, b\}$
and $Q = \{0, 1, 2, 3, 4, 5\}$
and $\Gamma = \{\alpha, \beta, \gamma\}$

$(0, \emptyset) \xrightarrow{\bullet a} (1, \alpha) \xrightarrow{\bullet a} (1, \alpha.\beta) \xrightarrow{\bullet b} (4, \alpha.\beta.\beta)$
$\xrightarrow{\bullet b} (3, \alpha.\beta) \xrightarrow{\bullet a} (2, \alpha)$



VPAs as Tree Automata

- a run of a VPA on a tree consists in:
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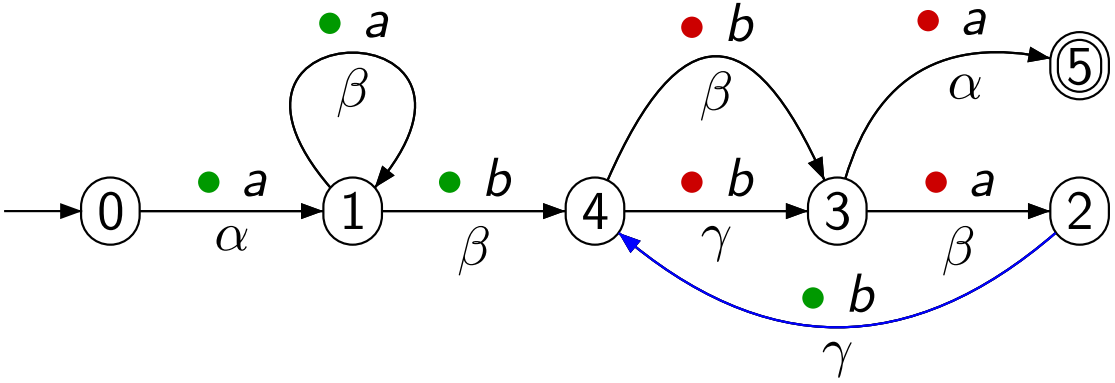

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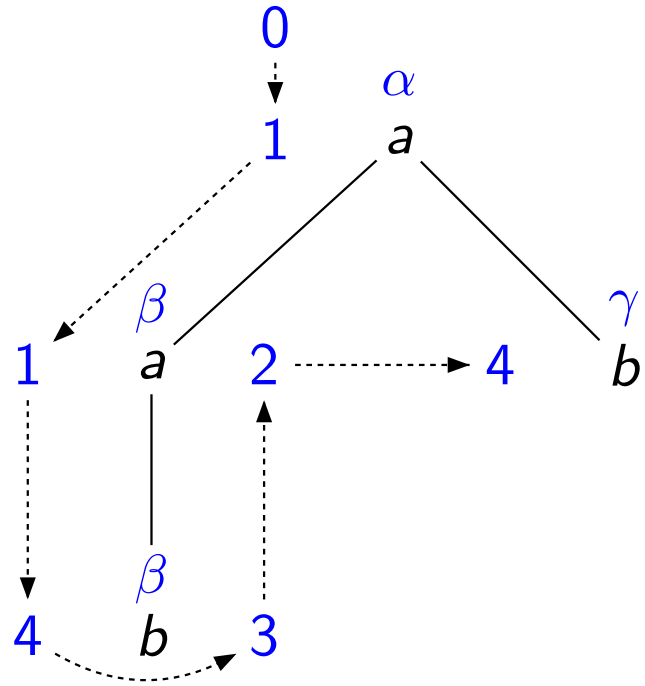

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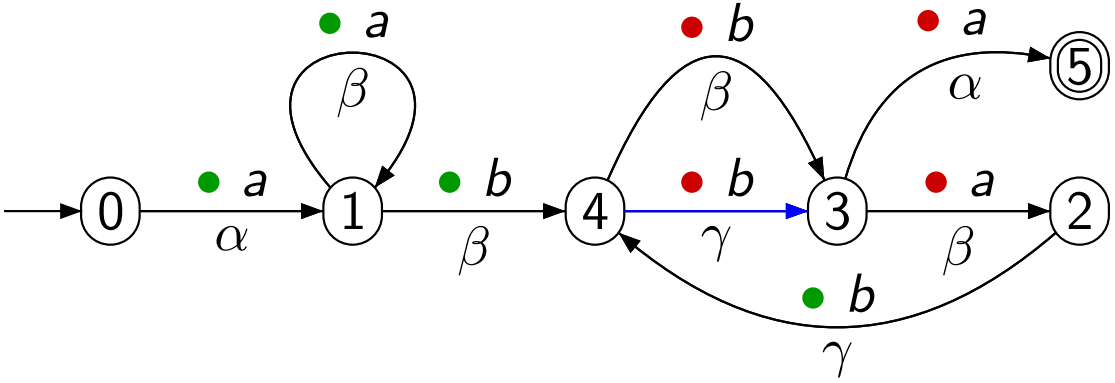

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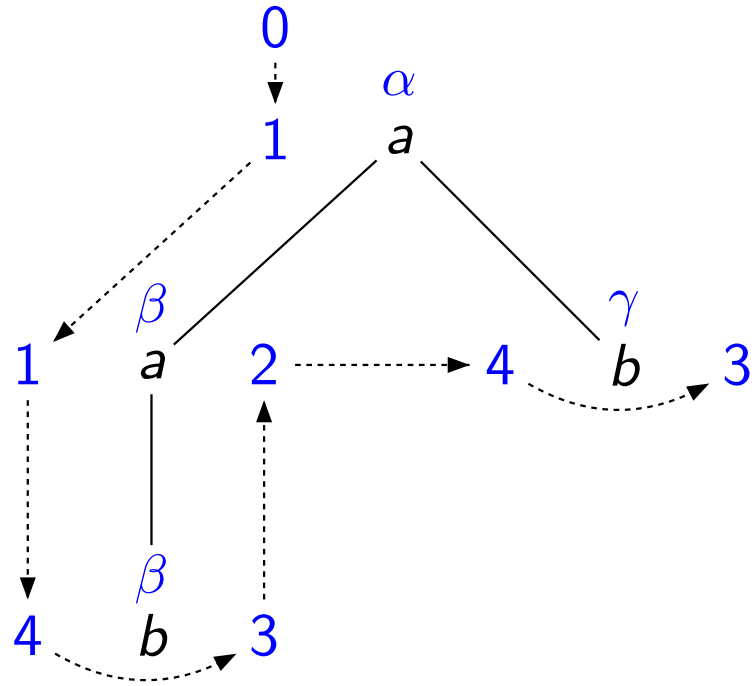
$\xrightarrow{\bullet b} (3, \alpha)$



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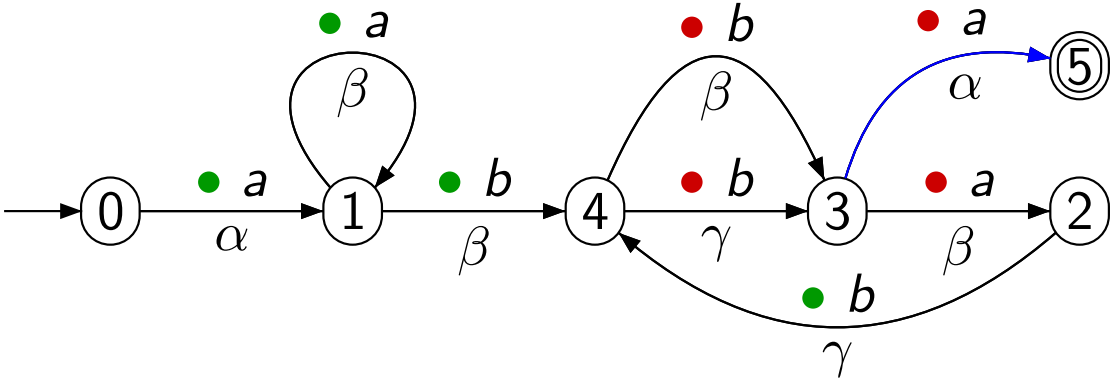

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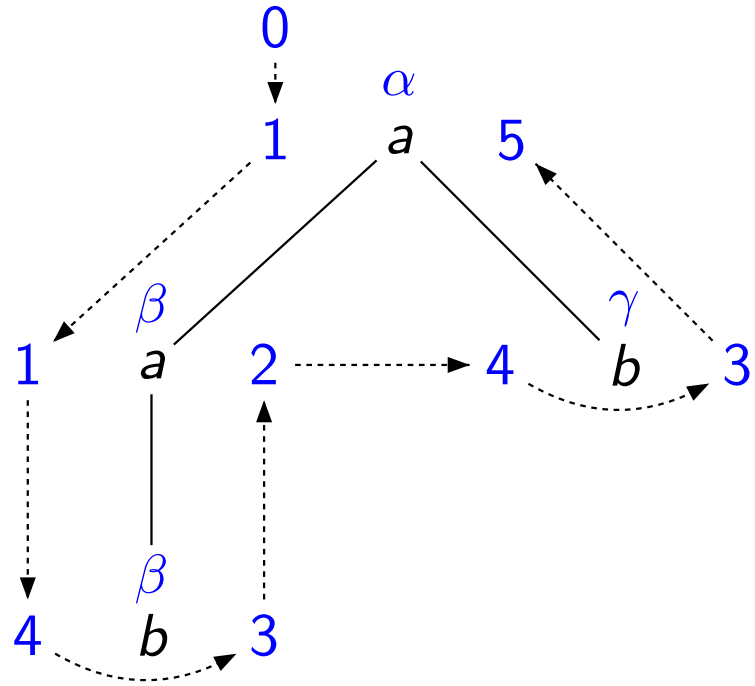
$\xrightarrow{\bullet b} (3, \alpha) \xrightarrow{\bullet a} (5, \emptyset)$



VPAs: properties & related models

From now on, we will consider VPAs as tree automata.

Translations between VPAs and $\uparrow$TA $\circ$*curry* exist [Gau09], so:

- VPAs are as expressive as the 4 classes using $\{\uparrow \text{TA}, \downarrow \text{TA}\} + \{\mathit{fcns}, \mathit{curry}\}$
- hedge automata also have this expressiveness
- we will call an unranked tree language recognizable if it belongs to this class
- other equivalent models (in expressiveness):
  - nested word automata [Alu07]: reformulation of VPAs
  - pushdown forest automata [NS98]: VPAs on forests (i.e. hedges)
  - streaming tree automata [GNR08, Gau09]: VPAs on trees

VPAs: determinism

the class of deterministic VPAs (dVPAs) has numerous interesting properties:

- as expressive as VPAs
  - so, as string acceptors: $NFAs \subsetneq VPAs = dVPAs \subsetneq dPAs \subsetneq PAs$
- determinization procedure in $O(2^{|Q|^2})$
- corresponds to streaming XML deterministically
  - yardstick class for streamability

VPAs: determinization

VPA $A = (Q, I, F, \Gamma, \Delta, \Sigma \uplus \overline{\Sigma})$ recognizing encodings of trees
(see [AM04] otherwise)

For an hedge h, let
$acc_A(h) = \{(q, q') \in Q^2 \mid \text{there is a run of } A \text{ on } h \text{ from } q \text{ to } q'\}$

The determinization procedure computes $acc_A(h)$ for all hedges h of the tree t. More precisely, the current state at node π is $acc(h)$ where h is the hedge of left siblings of π:

- when opening a node,
  - the previous state is pushed on the stack
  - the current state is set to the identity of Q^2 (h=empty hedge)
- when closing a node, the current state is updated from:
  - the top of the stack, i.e. hedge accessibility before traversing π
  - the previous state, i.e. hedge accessibility through $t.\pi$

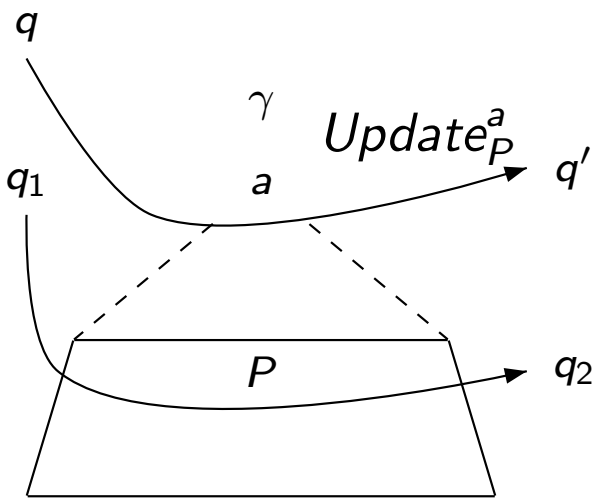
Determinization

$$Q^{A'} = 2^{Q^A \times Q^A}$$
$$I^{A'} = id_{I^A}$$
$$F^{A'} = \{P \mid \pi_2(P) \cap F^A \neq \emptyset\}$$

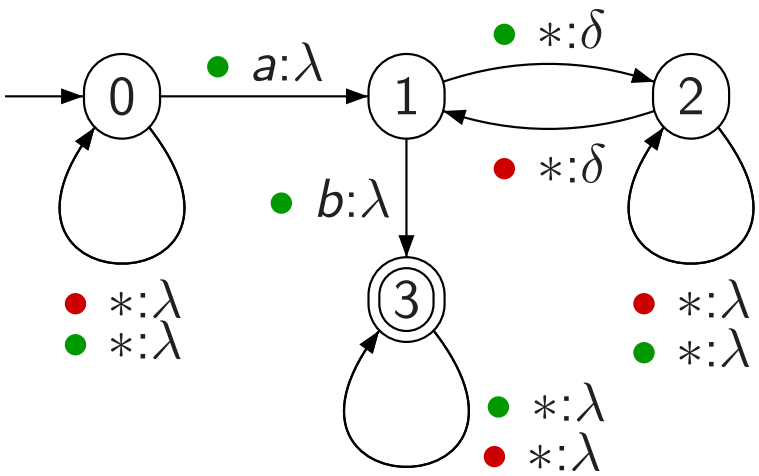
$$\frac{a \in \Sigma \qquad P \in Q^{A'}}{P \xrightarrow{\bullet\ a:P} id_{Q^A} \in \Delta^{A'}} \qquad \frac{a \in \Sigma \qquad P, P' \subseteq Q^A}{P \xrightarrow{\bullet\ a:P'} P' \circ Update^a_P \in \Delta^{A'}}$$

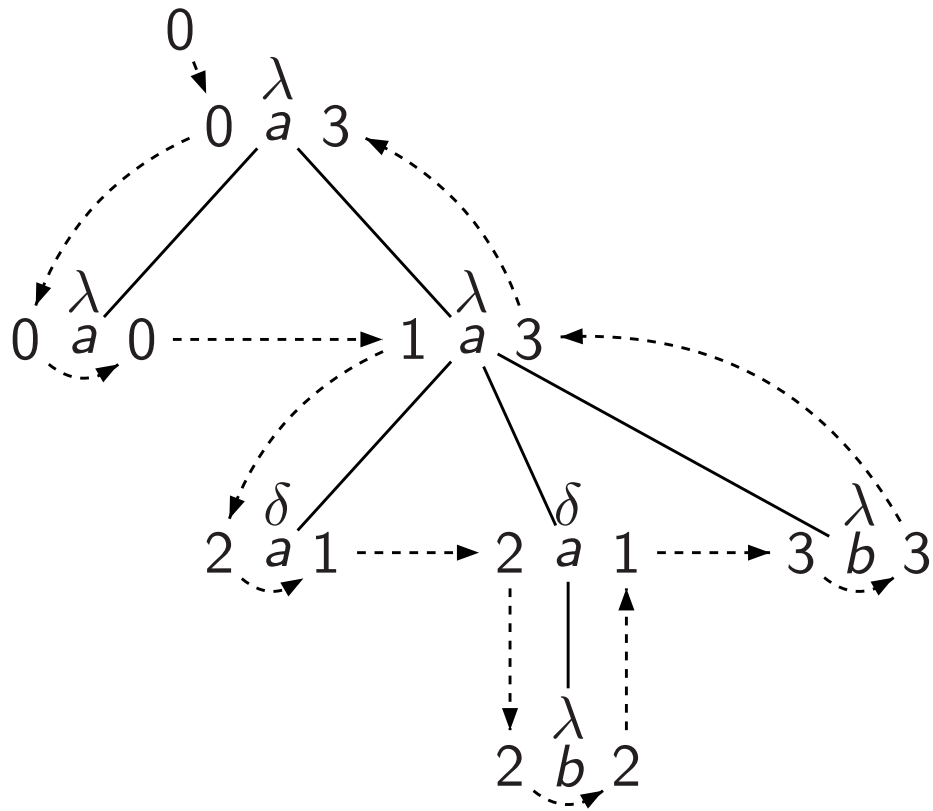
with

$$Update^a_P = \{(q, q') \mid \exists (q_1, q_2) \in P.\ \exists \gamma.\ q \xrightarrow{\bullet\ a:\gamma} q_1 \in \Delta^A \ \&\ q_2 \xrightarrow{\bullet\ a:\gamma} q' \in \Delta^A\}$$



Determinization: example





exercise: run of $det(A)$ on t

Linearization

Place of recognizable languages

$$NFAs \subsetneq VPAs = dVPAs \subsetneq dPAs \subsetneq PAs = CFLs$$

	closed by			det.	decidable		
	$\cap$	$\cup$	compl.		$\mathcal{L}(A)=\emptyset$	$\mathcal{L}(A)=\Sigma^*_{wf}$ $\mathcal{L}(A)=\mathcal{L}(B)$	$\mathcal{L}(A)\subseteq\mathcal{L}(B)$
NFAs	✓	✓	✓	✓	✓	✓	✓
VPAs	✓	✓	✓	✓	✓	✓	✓
dPAs	x	x	✓	✓	✓	✓	x
PAs	x	✓	x	x	✓	x	x

Minimization

of unranked tree automata

	unique minimal automaton?	procedure cost	ref.
det. hedge automata using DFAs for horiz. lang.	x	not PTIME unless PTIME=NP	[MN07]
$d \uparrow \mathrm{TA} \circ \mathit{curry}$ = stepwise tree automata	✓	PTIME	[MN07]
$d \uparrow \mathrm{TA} \circ \mathit{fcns}$	✓	PTIME	[MN07]
dVPAs	x	*open?*	[AKMV05] [CW07]

Congruence of a language $L \subseteq \hat{\Sigma}^*$ with $\hat{\Sigma} = \Sigma_{push} \uplus \Sigma_{pop}$

For well-matched words w and w',

$$w \equiv_L w' \quad \Leftrightarrow \quad \forall u, v \in \hat{\Sigma}^*,\ uwv \in L \text{ iff } uw'v \in L$$

A well-matched language $L \subseteq \hat{\Sigma}^*$ is VPA-recognizable iff $\equiv_A$ is of finite index.
This permits to define canonical VPAs, but not minimal.

1. Ranked Trees
   - Trees on Ranked Alphabet
   - Tree Automata
   - Tree Grammars
   - Logic

2. Unranked Trees
   - Unranked Trees
   - Automata
   - Logic

MSO

Equivalence with recognizable tree languages

for unranked trees, use first-child/next-sibling predicates:
$\Omega_u = \{\mathsf{lab}_a \mid a \in \Sigma\} \cup \{fc, ns\}$

Equivalence with automata

A tree language $L \subseteq \mathcal{T}_\Sigma$ is recognizable iff $\exists \phi \in MSO[\Omega_u]$ s.t. $L = \mathcal{L}_\phi$.

($\Rightarrow$) Similar to the ranked case: define a formula recognizing runs.

($\Leftarrow$) From ϕ, define ϕ' recognizing $fcns(\mathcal{L}_\phi)$. Then use equivalence for ranked trees.

XPath

XPath $\rightarrow$ det. automata: in $O(2^{2^{|e|}})$

★★★

For the fragment k-Downward XPath, it is in PTIME.

k-Downward XPath

Syntax

axis	d	$::=$	$self \mid ch \mid ch^*$
steps	S	$::=$	$d::a \mid d::* \quad$ (where $a \in \Sigma$)
paths	P	$::=$	$S \mid P[F] \mid P_1/P_2$
filters	F	$::=$	$P \mid \neg F \mid F_1 \wedge F_2$
rooted paths	R	$::=$	$/P$

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Semantic

$$[\![d::*]\!]_{path}(t) = d^t$$
$$[\![d::a]\!]_{path}(t) = \{(\pi, \pi') \in d^t \mid \mathsf{lab}_a^t(\pi')\}$$
$$[\![P_1/P_2]\!]_{path}(t) = [\![P_1]\!]_{path}(t) \circ [\![P_2]\!]_{path}(t)$$
$$[\![P[F]]\!]_{path}(t) = \{(\pi, \pi') \in [\![P]\!]_{path}(t) \mid \pi' \in [\![F]\!]_{filter}(t)\}$$

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$$[\![P]\!]_{filter}(t) = \{\pi \mid \exists \pi'.\ (\pi, \pi') \in [\![P]\!]_{path}(t)\}$$
$$[\![\neg F]\!]_{filter}(t) = \mathsf{nodes} \setminus [\![F]\!]_{filter}(t)$$
$$[\![F_1 \wedge F_2]\!]_{filter}(t) = [\![F_1]\!]_{filter}(t) \cap [\![F_2]\!]_{filter}(t)$$
$$[\![/P]\!]_{filter}(t) = \{\pi \mid (root, \pi) \in [\![P]\!]_{path}(t)\}$$

k-Downward XPath

Restrictions:

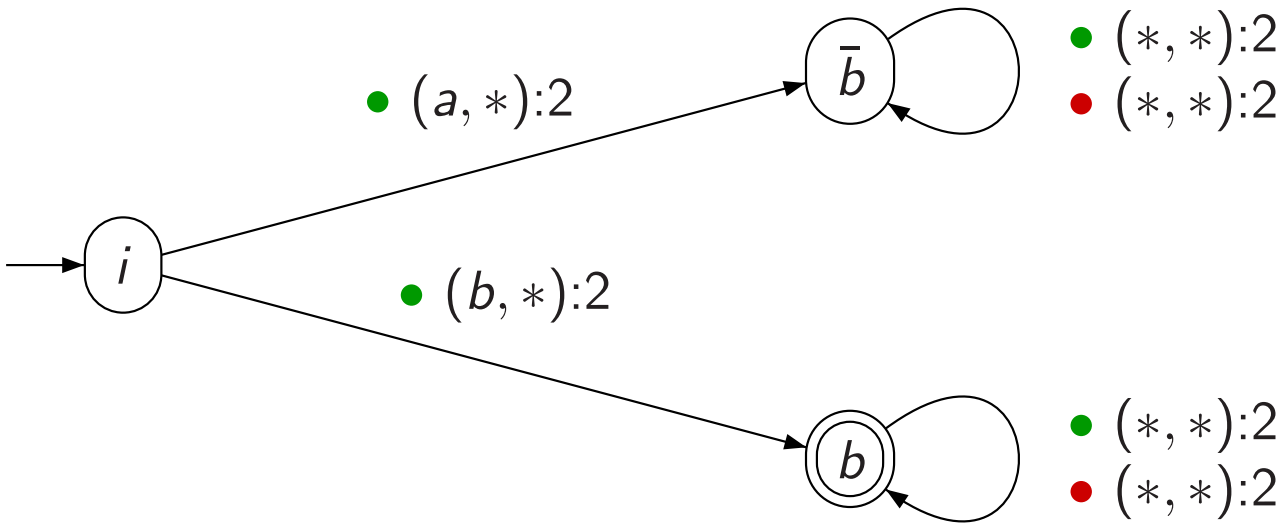
- $|\text{conjunctions} + \text{filters}| \leq k$
- if $ch^*::a$ appears, then there are no 2 a-nodes on the same branch

k-Downward XPath $\rightarrow$ dVPA

- inductive construction
- at each step, automata are deterministic and pseudo-complete:
  $\rightarrow$ for every tree t, there is exactly one run on t.

Example: $[ch::b]$

First step: A_b checks whether the root is labeled by b



Example: $[ch::b]$

Second step: $A_{ch::b}$ runs A_b on every child of the root

Procedure:

- add 3 states: `start`, 0 and 1
- add the rules inferred from what follows
  - this builds rules for $F = [ch[F']]$

Example: $[ch::b]$

Second step: $A_{ch::b}$ runs A_b on every child of the root

Procedure:

- add 3 states: `start`, 0 and 1
- add the rules inferred from what follows
  - this builds rules for $F = [ch[F']]$

$$\frac{a \in \Sigma \qquad V \in \{0,1\}}{\texttt{start} \xrightarrow{\bullet\ (a,V):0} 0}$$

opening the root: move to 0

$$\frac{q_1 \xrightarrow{\bullet\ (a,V):\gamma} q_2 \in \Delta^{A'} \qquad q_1 \in I^{A'} \qquad \flat \in \{0,1\}}{\flat \xrightarrow{\bullet\ (a,V):\ \flat} q_2}$$

opening a child: start testing F'

$$\frac{q_1 \xrightarrow{\alpha\ (a,V):\gamma} q_2 \in \Delta^{A'}}{q_1 \xrightarrow{\alpha\ (a,V):\gamma} q_2}$$

run test of F'

Example: $[ch::b]$

$$\frac{q_1 \xrightarrow{\bullet\ (a,V):\gamma} q_2 \in \Delta^{A'} \qquad q_2 \notin F^{A'} \qquad q_1' \xrightarrow{\bullet\ (a,V):\gamma} q_2' \in \Delta^{A'} \qquad q_1' \in I^{A'} \qquad \flat \in \{0,1\}}{q_1 \xrightarrow{\bullet\ (a,V):\ \flat} \flat}$$

failure of F': no new match

$$\frac{q_1 \xrightarrow{\bullet\ (a,V):\gamma} q_2 \in \Delta^{A'} \qquad q_2 \in F^{A'} \qquad q_1' \xrightarrow{\bullet\ (a,V):\gamma} q_2' \in \Delta^{A'} \qquad q_1' \in I^{A'} \qquad \flat \in \{0,1\}}{q_1 \xrightarrow{\bullet\ (a,V):\ \flat} 1}$$

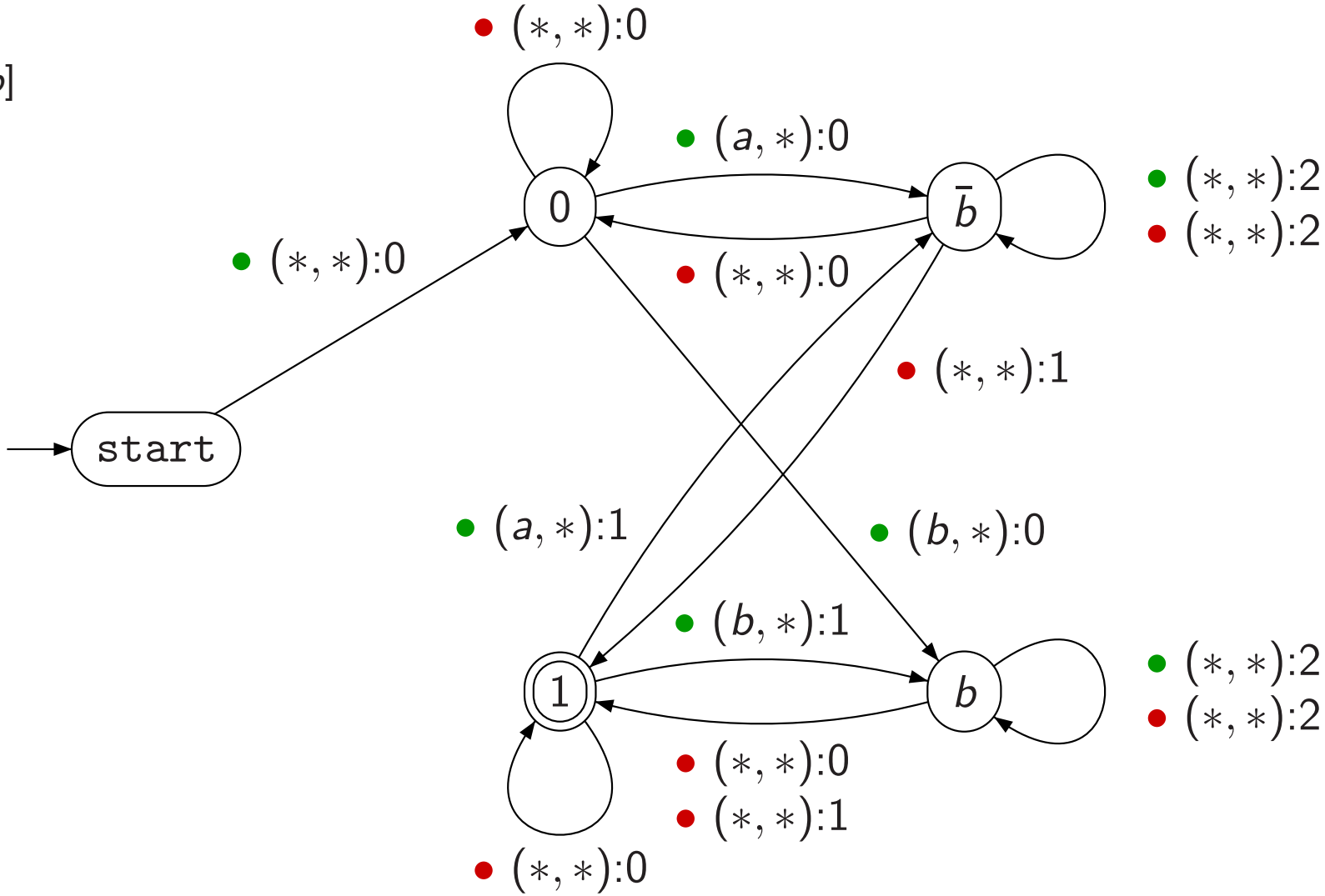
success of F': move to 1

$$\frac{a \in \Sigma \qquad V \in \{0,1\} \qquad \flat \in \{0,1\}}{\flat \xrightarrow{\bullet\ (a,V):0} \flat}$$

closing the root

Example: $[ch::b]$

$A_{[ch::b]}$



k-Downward XPath $\rightarrow$ dVPAs

other steps

- ch^* is similar to ch
- P_1/P_2 is transformed into filter (w/ variables): $ch^*::a/ch::b$ becomes $[ch^*::a[ch::b[x]]]$
- $A_{F_1 \wedge F_2} = A_{F_1} \wedge A_{F_2}$ (product construction)
- $A_{\neg F} = \neg A_F$

Other logics

- Conditional XPath, Regular XPath
- other modal logics: Tree TL, CTL*, PDL$_{tree}$
- μ-calculus

Language, Automata and Logic for Finite Trees

Olivier Gauwin

UMons

Feb/March 2010

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