

On the injective chromatic number of graphs

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Definition (G. Hahn, J. Kratochvíl, J. Širáň, D. Sotteau '02)

Let $G = (V, E)$ be a graph,

- ▶ A vertex k -colouring is a function $c : V(G) \rightarrow \{0, 1, \dots, k - 1\}$. We say that a colouring of a graph is *injective* if its restriction to the neighbourhood of any vertex is injective.
- ▶ The *injective chromatic number* $\chi_i(G)$ of a graph G is the least k such that there is an injective k -colouring.

$$\Delta(G) \leq \chi_i(G) \leq |V(G)|$$



Example

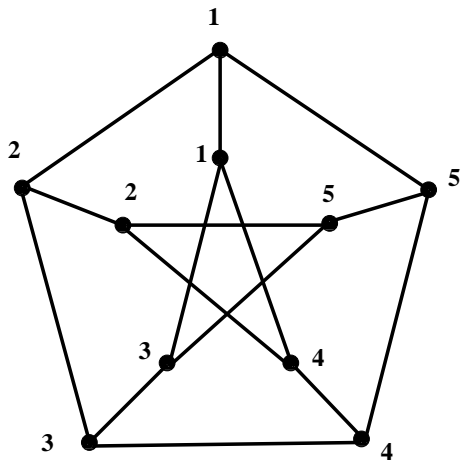


Figure: P_{10}



Instance: A graph $G = (V, E)$ and a natural number k

Question: Is there an injective k -coloring of G ?

Theorem (G. Hahn, J. Kratochvíl, J. Širáň, D. Sotteau '02)

The problem ICN is NP-complete



In the special case of the hypercube, we have :

Theorem (G. Hahn, J. Kratochvil, J. Širáň, D. Sotteau '02)

Let Q_n be the hypercube of dimension n .

1. $\chi_i(Q_n) = n$ if and only if n is a power of 2.
2. $\chi_i(Q_n) \leq 2n - 2$.
3. $\chi_i(Q_{2^m-j}) = 2^m$ for $0 \leq j \leq 3$.
4. $\chi_i(Q_{2n+1}) \leq 2\chi_i(Q_{n+1})$.

A part of these results was proved in 1997 by P. Wan (*Near Optical Conflict-Free Channel Set Assignments for an Optical Cluster-Based Hypercube Network*)



Definition

Let $G = (V, E)$ be a graph, the square of G is $G^2 = (V, E')$ s.t. $xy \in E'$ if and only if x and y are at distance at most 2 in G .



$$\chi(K_2) = 2$$



$$\chi_i(K_2) = 1$$

- ▶ If G is connected and not K_2 then $\chi(G) \leq \chi_i(G)$.
- ▶ $\chi(G) \leq \chi_i(G) \leq \chi(G^2)$
- ▶ $\chi_i(P_{10}) = 5$ and $\chi(P_{10}^2) = 10$



Conjecture (Wegner '77)

Let G be a planar graph. Then

$$\chi(G^2) \leq \begin{cases} \Delta(G) + 5 & \text{if } 4 \leq \Delta(G) \leq 7; \\ \lfloor 3\Delta(G)/2 \rfloor + 1 & \text{if } \Delta(G) \geq 8. \end{cases}$$



Theorem (C. Thomassen '01)

If G is planar and $\Delta(G) = 3$ then $\chi(G^2) \leq 7$

Theorem (O. Borodin, H.J. Broersma, A. Glebov, J. van den Heuvel '02)

If G is planar with $\Delta \geq 47$, $\chi(G^2) \leq \lceil 9\Delta(G)/5 \rceil + 1$

Theorem (M. Molloy, M.R. Salavatipour '05)

If G is planar, $\chi(G^2) \leq \lceil 5\Delta(G)/3 \rceil + 78$



The *maximum average degree* is a well used tool

Definition

$$\text{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} : H \text{ is a subgraph of } G\right\}.$$

The degree of a vertex u will be denoted by $\deg(u)$

Proposition

If G is a planar graph with girth at least g then

$$\text{mad}(G) < 2 + \frac{4}{g-2}$$


Theorem (A. Doyon, G. Hahn, A.R. '05)

Let G be a graph

1. *If $\text{mad}(G) < \frac{14}{5}$ then $\chi_i(G) \leq \Delta(G) + 3$.*
2. *If $\text{mad}(G) < 3$ then $\chi_i(G) \leq \Delta(G) + 4$.*
3. *If $\text{mad}(G) < \frac{10}{3}$ then $\chi_i(G) \leq \Delta(G) + 8$.*



$mad(G) < \frac{14}{5}$ then $\chi_i(G) \leq \Delta(G) + 3$

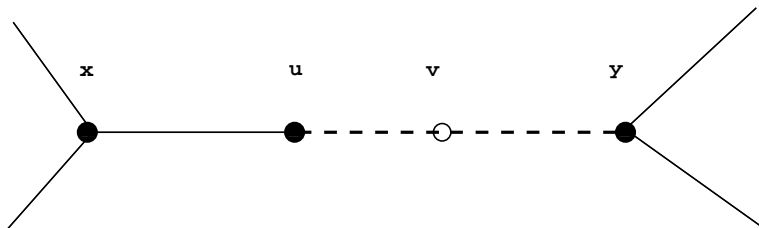
Let H be a minimum counterexample.

1. Both neighbours of a vertex of degree 2 are of degree at least 3.
2. If uvz is a simple path in G with $\deg(u) = 2$, $\deg(v) = 3$ then $\deg(z) > 3$.
3. A vertex of degree 4 has at most two neighbours of degree 2.
4. A vertex of degree 4 cannot have two neighbours of degree 2 and two neighbours of degree 3



Both neighbours of a vertex of degree 2 are of degree at least 3.

If there is two vertices u and v of degree 2 adjacent in H . We color $H \setminus v$.

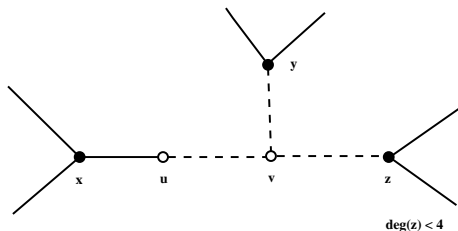


1. $Forb(u) \leq \Delta - 1 + 1 = \Delta$
2. $Forb(v) \leq \Delta - 1 + 1 = \Delta$



If uvz is a simple path in G with $\deg(u) = 2$, $\deg(v) = 3$ then $\deg(z) > 3$

If $\deg(z) \leq 3$ we color $H \setminus uv$.

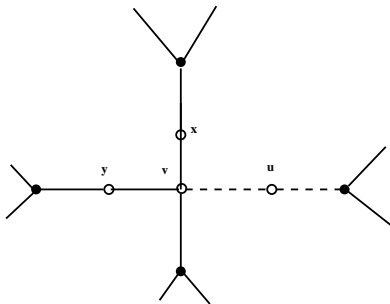


1. $Forb(u) \leq \Delta - 1 + 2 = \Delta + 1$
2. $Forb(v) \leq \Delta - 1 + 2 + 1 = \Delta + 2$



A vertex of degree 4 has at most two neighbours of degree 2

If $\deg(v) = 4$ and v has 3 neighbors of degree 2. We color $H \setminus u$

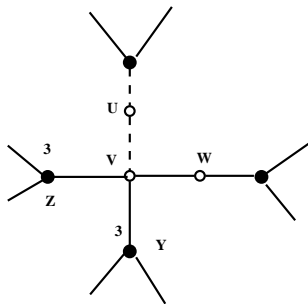


1. $Forb(v) \leq \Delta - 1 + 3 = \Delta + 2$
2. $Forb(u) \leq \Delta - 1 + 3 = \Delta + 2$



A vertex of degree 4 cannot have two neighbours of degree 2 and two neighbours of degree 3

If $\deg(v) = 4$ and v has 2 neighbors of degree 2 and 2 neighbors of degree 3. We color $H \setminus u$



1. $\text{Forb}(v) \leq 6 < \Delta + 3$
2. $\text{Forb}(u) \leq \Delta - 1 + 3 = \Delta + 2$



Discharging rules

The initial charge is defined by $w(u) = \deg(u)$ for each $u \in V(H)$.

1. If $uv \in E(H)$ and $\deg(u) = 2$, $\deg(v) \geq 3$, v gives $\frac{2}{5}$ to u
2. If $uv \in E(H)$ and $\deg(u) = 3$, $\deg(v) \geq 4$, v gives $\frac{1}{10}$ to u .

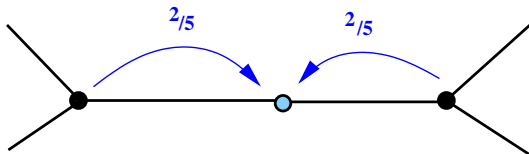
After the discharging procedure let $w^*(u)$ the new charge for each $u \in V(H)$.

We have $\sum_{u \in V(H)} w(u) = \sum_{u \in V(H)} w^*(u)$.



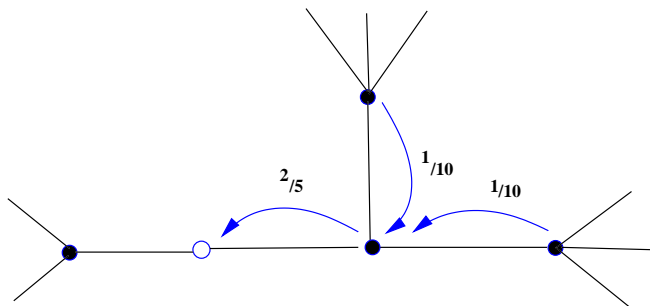
The new charges

- ▶ a vertex u of degree 2 has $w^*(u) = 2 + 2\frac{2}{5} = \frac{14}{5}$



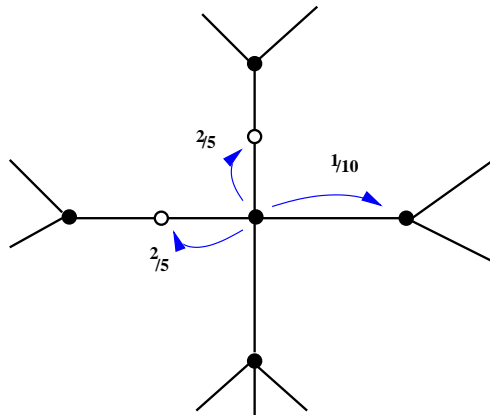
The new charges

- ▶ a vertex u of degree 3 has at most one neighbour of degree 2 and so $w^*(u) \geq 3 - \frac{2}{5} + 2 \cdot \frac{1}{10} = \frac{14}{5}$



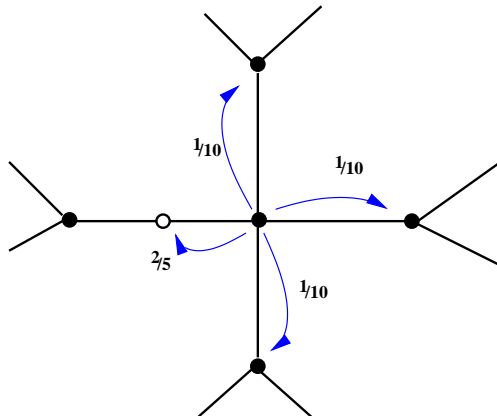
The new charges

- ▶ Let u be a vertex of degree 4.
If u has two neighbours of degree 2, it has at most one neighbour of degree 3, and then
 $w^*(u) \geq 4 - 2 \cdot \frac{2}{5} - \frac{1}{10} = \frac{31}{10} > \frac{14}{5}$.



The new charges

- ▶ If u has fewer than two neighbours of degree 2, $w^*(u) \geq 4 - 3 \cdot \frac{1}{10} - \frac{2}{5} = \frac{33}{10} > \frac{14}{5}$.



- ▶ If u is of degree at least $k \geq 5$ then $w^*(u) \geq k - k \cdot \frac{2}{5} = k \cdot \frac{3}{5} \geq 3$



The new charges

Hence $w^*(u) \geq \frac{14}{5}$ for every $u \in V(G)$

$$Mad(H) \geq \frac{2|E(H)|}{|V(H)|} = \frac{\sum_{v \in V} d(v)}{|V(H)|} = \frac{\sum_{v \in V} w^*(v)}{|V(H)|} \geq \frac{\frac{14}{5}|V(H)|}{|V(H)|} \geq \frac{14}{5}$$

A contradiction.



Definition

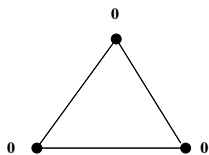
Let p, q be two nonnegative integers. An $L(p, q)$ -coloring of a graph G is a function $\phi : V(G) \rightarrow \{0, 1, \dots, k\}$ for some positive integer k such that:

1. $|\phi(x) - \phi(y)| \geq p$ if x and y are adjacent
2. $|\phi(x) - \phi(y)| \geq q$ if x and y are at distance 2

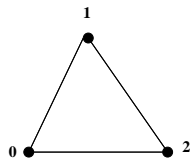
$\lambda(G; p, q)$ is the smallest k such that G has an $L(p, q)$ -coloring with $\max\{\phi(v) : v \in V(G)\} = k$



1. $\lambda(G; 1, 1) = \chi(G^2) - 1$
2. $\lambda(G; 0, 1) = \chi_i(G) - 1$ if G is triangle free.



$$\lambda(T; 0, 1) = 0$$



$$\chi_i(T) = 3$$



W.Wang and K. Lih proved

Theorem (W.Wang, K.Lih '03)

Let G be a planar graph and p and q two positive integers.

1. *If $g(G) \geq 7$ then $\lambda(G; p, q) \leq (2q - 1)\Delta(G) + 4p + 4q - 4$*
2. *If $g(G) \geq 6$ then $\lambda(G; p, q) \leq (2q - 1)\Delta(G) + 6p + 12q - 9$*
3. *If $g(G) \geq 5$ then $\lambda(G; p, q) \leq (2q - 1)\Delta(G) + 6p + 24q - 15$*

Corollary (W. Wang, K. Lih '03)

Let G be a planar graph

1. *If $g(G) \geq 7$ then $\chi(G^2) \leq \Delta(G) + 5$*
2. *If $g(G) \geq 6$ then $\chi(G^2) \leq \Delta(G) + 10$*
3. *If $g(G) \geq 5$ then $\chi(G^2) \leq \Delta(G) + 16$*

This gives upperbounds for $\chi_i(G)$



Proposition

If G is a planar graph with girth at least g then
$$\text{mad}(G) < 2 + \frac{4}{g-2}$$

Corollary

Let G be a planar graph

- 1. If $g(G) \geq 7$ then $\chi_i(G) \leq \Delta(G) + 3$*
- 2. If $g(G) \geq 6$ then $\chi_i(G) \leq \Delta(G) + 4$*
- 3. If $g(G) \geq 5$ then $\chi_i(G) \leq \Delta(G) + 8$*



Theorem (B.Lužar,R.Škrekovski, M.Tancer '06)

Let G be a planar graph

1. *If $g(G) \geq 19$ then $\chi_i(G) = \Delta(G)$, $\Delta \geq 4$*
2. *If $g(G) \geq 10$ then $\chi_i(G) \leq \Delta(G) + 1$, $\Delta \geq 4$*
3. *If $g(G) \geq 5$ then $\chi_i(G) \leq \Delta(G) + 4$, $\Delta \geq 139$*



K_4 -minor free Graphs

G is a minor of H if G can be obtained from H by a series of vertex deletions, edge deletions and/or edge contractions (replacing two adjacent vertices u, v by a vertex that is adjacent to all neighbours of u or v).

A graph G is K_4 -minor-free if K_4 isn't a minor of G .

Equivalent classes: partial 2-tree, series-parallel

Subclasses: 2-tree, outerplanar



K_4 -minor free Graphs

Theorem (K. Lih, W. Wang, X. Zhu '02)

Let G be a K_4 -minor free graph. Then

$$\chi(G^2) \leq \begin{cases} \Delta(G) + 3 & \text{if } 2 \leq \Delta(G) \leq 3; \\ \lfloor 3\Delta(G)/2 \rfloor + 1 & \text{if } \Delta(G) \geq 4. \end{cases}$$

Theorem (G. Hahn, A.R., W. Wang '05)

Let G be a K_4 -minor free graph. Then $\chi_i(G) \leq \lceil \frac{3}{2}\Delta(G) \rceil$.



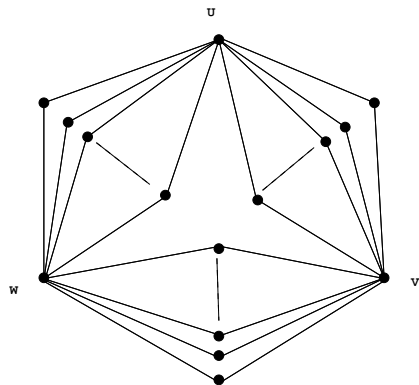
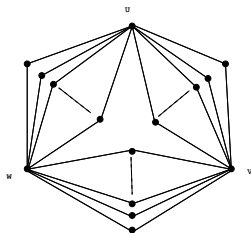


Figure: G_{2k}

k 2-paths between u and v
 k 2-paths between v and w
 k 2-paths between w and u





G_{2k} is a K_4 -minor free graph with $\Delta(G_{2k}) = 2k$.

$$\chi_i(G_{2k}) \leq \lceil \frac{3}{2} \Delta(G_{2k}) \rceil = 3k.$$

Any two vertices of degree 2 are at distance 2 in G_{2k} and there are exactly $3k$ such vertices in G_{2k} . Thus $\chi_i(G_{2k}) \geq 3k$

$$\chi_i(G_{2k}) = \lceil \frac{3}{2} \Delta(G_{2k}) \rceil = 3k.$$



Theorem (M. Montassier, A. R.'05)

Let G be a planar subcubic graph:

- 1. if $\text{girth}(G) \geq 14$ then $\chi(G^2) \leq 5$*
- 2. if $\text{girth}(G) \geq 10$ then $\chi(G^2) \leq 6$*
- 3. if $\text{girth}(G) \geq 8$ then $\chi(G^2) \leq 7$*

It was also proved independently by Z. Dvořak, R.Škrekovski and M. Tancer '05.



Lemma (G. Hahn et al. '02)

Let G be a graph of maximum degree Δ . Then
 $\chi_i(G) \leq \Delta^2 - \Delta + 1$.

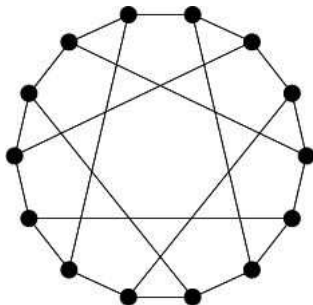
Theorem (G. Hahn et al. '02)

Let G be a connected graph of maximum degree $\Delta \geq 3$. Then
 $\chi_i(G) = \Delta^2 - \Delta + 1$ *if and only if there exists a projective plane*
 P *of order $\Delta - 1$ and G is isomorphic to $I(P)$.*



Proposition

Let G be a graph of maximum degree $\Delta \leq 3$. Then $\chi_i(G) \leq 7$; and $\chi_i(G) = 7$ if and only if G is isomorphic to the Heawood graph. In particular, when $|G| \neq 14$ or G is not bipartite, it holds that $\chi_i(G) \leq 6$.



Conjecture

Let G be a planar graph with maximum degree $\Delta(G)$. Then $\chi_i(G) \leq \lceil \frac{3}{2}\Delta(G) \rceil$.

