# Star Coloring of Graphs

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## Proper Coloring

A *k*-proper coloring of the vertices of a graph *G* is a mapping  $c : V(G) \rightarrow \{1, \dots, k\}$  such that  $\forall uv \in E(G), c(u) \neq c(v)$ .

## Star coloring (1)

A star coloring of a graph G is a proper coloring of G such that no path of length 3 in G is bicolored.

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#### Star coloring (1)

A star coloring of a graph G is a proper coloring of G such that no path of length 3 in G is bicolored.

#### Star coloring (2)

A star coloring of a graph G is a proper coloring of G such that the union of every two color classes induces a star forest

This notion was introduced by Grünbaum '73.

#### Star chromatic number

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 $\chi_s(P_{10})=5$ 









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#### Acyclic Chromatic number

The acyclic chromatic number, denoted by  $\chi_a(G)$ , of a graph G is the smallest integer k such that G has an acyclic k-coloring.

The acyclic coloring of graphs was introduced by Grünbaum in '73

#### Theorem - [Borodin'79]

# For any planar graph G, $\chi_a(G) \leq 5$



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A planar graph G with  $\chi_a(G) = 5$ 



$$\chi_a(G) \leq \chi_s(G)$$



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Theorem - [Grünbaum '73]

If  $\chi_{a}(G) \leq k$  then  $\chi_{s}(G) \leq k2^{k-1}$ .



$$\chi_a(G) \leq \chi_s(G)$$

Theorem - [Grünbaum '73]

If 
$$\chi_a(G) \leq k$$
 then  $\chi_s(G) \leq k2^{k-1}$ .

## Corollary

For any planar graph G,  $\chi_s(G) \leq 80$ 



## Theorem - [Albertson \* et al. '04]

# For any graph G, $\chi_s(G) \leq \chi_a(G)(2\chi_a(G)-1)$

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#### Corollary

For any planar graph G,  $\chi_{s}(G) \leq 45$ 

The girth of a graph G is the length of a shortest cycle in G.

#### Corollary

Let G be a planar graph with girth g.

- 1. If  $g \ge 7$  then  $\chi_s(G) \le 9$
- 2. If  $g \ge 5$  then,  $\chi_s(G) \le 16$

\*Albertson, Chapell, Kierstead, Kündgen, Ramamurthi



Theorem - [Nešetřil and Ossona de Mendez '03]

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For any planar graph G,  $\chi_s(G) \leq 20$ 



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#### Proposition

There is a planar graph G such that  $\chi_s(G) = 10$ 



#### Theorem - [Fertin, R., Reed '04]

If a graph G has a treewidth at most k, then  $\chi_s(G) \leq k(k+3)/2 + 1$ 



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If a graph G has a treewidth at most k, then  $\chi_s(G) \leq k(k+3)/2 + 1$ 

#### Corollary

If G is outerplanar, then  $\chi_s(G) \leq 6$  and this is best possible.

By using Lovász's Local Lemma:

Theorem -[Fertin, R., Reed '04]

Let G = (V, E) be a graph with maximum degree  $\Delta$  then  $\chi_s(G) \leq \lceil 20\Delta^{\frac{3}{2}} \rceil$ .



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Theorem - [Fertin, R., Reed '04]

There exists a graph G of maximum degree  $\Delta$  such that

$$\chi_s(G) \ge \epsilon . \frac{\Delta^{3/2}}{(\log \Delta)^{1/2}}$$

. . .

where  $\epsilon$  is an absolute constant.



# Degenerate and star coloring- Mojar and Špacapan

A graph G is k-degenerate if every subgraph of G has a vertex of degree less than k.

#### Definition

A coloring of a graph such that for every  $k \ge 1$ , the union of any k color classes induces a k-degenerate subgraph is a degenerate coloring.

# Degenerate and star coloring- Mojar and Špacapan

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#### Definition

The degenerate chromatic number of G, denoted as  $\chi_d(G)$ , is the least n such that G admits a degenerate n-coloring.

#### Definition

If a coloring is both, degenerate and star, it is a degenerate star coloring. The corresponding chromatic number is denoted as  $\chi_{sd}(G)$ .

# Theorem -[Mohar, Špacapan '08]

Let G be a simple graph embedded on a surface of Euler genus g. Then  $\chi_{sd}(G) \leq \lceil 1000g^{\frac{3}{5}} + 100000 \rceil$ .

If G is a graph of genus g then  $\chi_{sd}(G) = O(g^{\frac{3}{5}})$ 

## Theorem -[Mohar, Špacapan '08]

For every large enough g, there is a graph G embeddable in a surface of genus g, such that  $\chi_{sd}(G) \geq \chi_s(G) \geq \frac{1}{32}g^{3/5}/(\log g)^{1/5}.$ 



#### Definition-Maximum average degree

$$\operatorname{Mad}(G) = \max\left\{\frac{2 \cdot |E(H)|}{|V(H)|}, H \subseteq G\right\}.$$

if G is a planar graph with girth g, then  $Mad(G) < \frac{2 \cdot g}{g-2}$ .



#### Theorem -[Bu, Cranston, Montassier, R., Wang '08]

- 1. If G is a graph with  $Mad(G) < \frac{26}{11}$ , then  $\chi_s(G) \leq 4$ .
- 2. If G is a graph with  $Mad(G) < \frac{18}{7}$  and girth at least 6, then  $\chi_s(G) \leq 5$ .
- 3. If G is a graph with  $Mad(G) < \frac{8}{3}$  and girth at least 6, then  $\chi_s(G) \le 6$ .



# $\operatorname{Mad}(G) < \frac{18}{7}$ and girth at least 6, then $\chi_s(G) \leq 5$ .

#### Lemma

A graph G with  $Mad(G) < \frac{18}{7}$  and  $g(G) \ge 6$  contains one of the following 14 configurations:







# Configurations





Let G be a counterexample, i.e. a graph with  $Mad(G) < \frac{18}{7}$ , with  $g(G) \ge 6$ , and containing none of the configurations.

#### k-thread

A k-thread is a (k + 1)-path whose k internal vertices are of degree 2.

#### $k_{i_1,\dots,i_k}$ -vertex

A  $k_{i_1,\dots,i_k}$ -vertex with  $i_1 \leq \dots \leq i_k$  is a k-vertex that is the initial vertex of k threads of lengths  $i_1,\dots,i_k$ .



Initial charge of a vertex v is:

$$\omega(v) = d(v)$$

Each  $\geq$ 3-vertex gives to its neighbors by using the following rules:

(R1)  $\frac{2}{7}$  to each 2-vertex that is not adjacent to a 2-vertex.

- (R2)  $\frac{4}{7}$  to each 2-vertex *u* that is adjacent to a 2-vertex (*u* is a  $2_{0,1}$ -vertex).
- (R3)  $\frac{1}{7}$  to each  $3_{0,1,1}$ -vertex.
- (R4)  $\frac{2}{7}$  to each  $4_{0,2,2,2}$ -vertex.



The new charge of a vertex v is denoted by :

$$\omega^*(v)$$

We verify that  $\omega^*(v) \geq \frac{18}{7}$  for all  $v \in V(G)$ .

Note that the discharging rules do not change the sum of the charges.

This leads to the following obvious contradiction:

$$\frac{18}{7} \le \frac{\frac{18}{7}|V(G)|}{|V(G)|} \le \frac{\sum_{v \in V(G)} \omega^*(v)}{|V(G)|} = \frac{\sum_{v \in V(G)} \omega(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|} \le \operatorname{Mad}(G) < \frac{18}{7}.$$

Hence no counterexample can exist.

#### Lemma

If G is a graph with  $Mad(G) < \frac{18}{7}$  and  $g(G) \ge 6$ , then there exists a vertex partition  $V(G) = F \cup I_1 \cup I_2$  such that:

- P1. F induces a forest,
- P2.  $I_1$  is an independent set such that for all  $x, y \in I_1$ ,  $d_{G[F \cup I_1]}(x, y) > 2$ , and
- P3.  $I_2$  is an independent set such that for all  $x, y \in I_2$ ,  $d_G(x, y) > 2$ .



# Lemma's proof

Let G be a counterexample with the fewest vertices. By the previous Lemma, G contains one of the 14 configurations.





We consider each of the 14 configurations. In each case we delete some part of the configuration, by minimality the obtained graph has the decomposition. We can then extend it to the whole graph. A contradiction.



Let G be a graph with  $Mad(G) < \frac{18}{7}$  and  $g(G) \ge 6$ . By the previous Lemma, there exists a vertex partition  $V(G) = F \cup I_1 \cup I_2$ . Now we color:

- G[F] with the colors 1, 2, and 3,
- $G[I_1]$  with the color 4, and
- $G[I_2]$  with the color 5.

This produces a star 5-coloring of G.







If G is a planar graph with girth g, then  $Mad(G) < \frac{2 \cdot g}{g-2}$ .

#### Corollary

Let G be a planar graph with girth g(G).

1. If 
$$g(G) \geq 13$$
, then  $\chi_s(G) \leq 4$ .

2. If 
$$g(G) \geq$$
 9, then  $\chi_s(G) \leq$  5.

3. If 
$$g(G) \geq$$
 8, then  $\chi_{s}(G) \leq$  6.

2) and 3) proved independently by Timmons.



#### Star list coloring

- 1. G is L-star colorable if for a given list assignment L there is a star coloring  $\phi$  of the vertices such that  $\phi(v) \in L(v)$ .
- 2. If G is L-star colorable for any list assignment with  $|L(v)| \ge m$  for all  $v \in V(G)$ , then G is *m*-star choosable.

#### Star list chromatic number

The star list chromatic number of G, denoted by  $\chi_s^l(G)$ , is the smallest integer k such that G is k-star choosable.



## Definition

A proper coloring of an orientation of a graph G is called an *in-coloring* if for every 2-colored  $P_3$  in G, the edges are directed towards the midle vertex. A coloring of G is an in-coloring if it is an in-coloring of some orientation of G. An *L-in-coloring* of G is an in-coloring of G where the colors are chosen from the lists assigned to each vertex.

#### Lemma

An L-coloring of graph G is an L-star coloring if and only if it is an L-in-coloring of some orientation of G.

#### Lemma

An L-coloring of a graph G is an L-star-coloring if and only if it is an L-in-coloring of some orientation of G.

**Proof.** Given an *L*-star-coloring, we can construct an orientation by directing the edges towards the center of the star in each star-forest corresponding to the union of two color classes. Conversely, consider an *L*-in-coloring of  $\overrightarrow{G}$ , an orientation of *G*. Let  $P_4 = uvwz$  be any path on four vertices in *G*. We may assume the edge vw is directed towards w in  $\overrightarrow{G}$ . For the given coloring to be an *L*-in-coloring at v, we must have three different colors on u, v and w.



# Star list chromatic number





## Theorem-[Kierstead, Küngen, Timmons'08]

If G is bipartite planar graph then  $\chi'_s(G) \leq 14$ .

We can also prove that  $\chi'_{s}(G) \leq 14$  if G is a planar triangle free graph.

#### Theorem-[Küngen, Timmons'09]

Let G be a graph.

- 1. If Mad(G) < 8/3 then  $\chi'_s(G) \le 6$
- 2. If  $\operatorname{Mad}(G) < 14/5$  then  $\chi_s^l(G) \leq 7$



#### Corollary

#### Let G be a planar graph.

- 1. If  $g(G) \ge 7$ , then  $\chi_{s}^{l}(G) \le 7$ ;
- 2. If  $g(G) \ge 8$ , then  $\chi_{s}^{l}(G) \le 6$ ;

#### Theorem-[Küngen, Timmons,'09]

Let G be a planar graph. If  $g(G) \ge 6$ , then  $\chi'_s(G) \le 8$ ;

#### Theorem-[Chen, R., '09]

Let G be a graph. If Mad(G) < 3 then  $\chi'_s(G) \le 8$ 



For a family of graph  $\mathcal{F}$ , we denote:  $\chi_{s}(\mathcal{F}) = \max \{ \chi_{s}(\mathcal{G}), \mathcal{G} \in \mathcal{F} \}.$ 

Proposition-[Fertin, Reed, R., '04]

Let C denote the family of cubic graphs. We have  $6 \le \chi_s(C) \le 9$ 





# Star list chromatic number of planar subcubic graphs

## Theorem-[Chen, R., Wang, '09]

Let G be a planar subcubic graph.

- 1. If  $g(G) \ge 3$ , then  $\chi'_s(G) \le 6$ ;
- 2. If  $g(G) \ge 8$ , then  $\chi_s^l(G) \le 5$ ;
- 3. If  $g(G) \ge 12$ , then  $\chi'_{s}(G) = 4$ .

A 3-vertex v is said to be:

- *Type 1* if it is a (1, 1, 0)-vertex.
- ► Type 2 if it is a (0,0,0)-vertex and is adjacent to two Type 1 vertices.
- ► Type 3 if it is a (1,0,0)-vertex and is adjacent to one Type 1 vertex.

# If $g(G) \ge 8$ , then $\chi'_s(G) \le 5$

#### Lemma

A planar subcubic graph with  $g(G) \ge 8$  contains one of the following 11 configurations:

- (C1)  $A 1^{-}$ -vertex.
- (C2) A(1, 0)-vertex.
- (C3) A(1, 1, 1)-vertex.
- (C4) A (1,1,0)-vertex is adjacent to one (1,1,0)-vertex.
- (C5) A (1,0,0)-vertex is adjacent to one 2-vertex, one (1,1,0)-vertex and one (1,0,0)-vertex.
- (C6) A (0,0,0)-vertex is adjacent to two (1,1,0)-vertices and one (1,0,0)-vertex.
- (C7) A (0,0,0)-vertex is adjacent to two type 2 vertices.
- (C8) A(0,0,0)-vertex is adjacent to two type 3 vertices.
- (C9) A (0,0,0)-vertex is adjacent to one type 1 vertex and one type 2 vertex.

(C10) A (0,0,0)-vertex is adjacent to one type 1 vertex and one type 3 vertex.

(C11) A (0,0,0)-vertex is adjacent to one type 2 vertex and one type 3 vertex.



# If $g(G) \ge 8$ , then $\chi'_s(G) \le 5$



Figure 6: The 11 unavoidable configurations in Lemma 2.



Let G be a counterexample, i.e. a planar subcubic graph with  $g(G) \ge 8$  and containing none of the configurations of the previous Lemma as depicted in the previous Figure. Euler's formula |V(G)| - |E(G)| + |F(G)| = 2 gives

$$\sum_{v \in V(G)} (3d(v) - 8) + \sum_{f \in F(G)} (d(f) - 8) = -16.$$
(1)

For all  $v \in V(G), \omega(v)$  the initial charge is  $\omega(v) = 3d(v) - 8$  and to each face f an initial charge such that for all  $f \in F(G), \omega(f) = d(f) - 8$ 



- (R1) Each (0,0)-vertex gets a charge equal to 1 from each of its neighbor.
- (R2) Each type 1 vertex gets a charge equal to 1 from its neighbor of degree 3.
- (R3) Each type 2 vertex gets a charge equal to 1 from its neighbor of degree 3 which is not a type 1 vertex.
- (R4) Each type 3 vertex gets a charge equal to 1 from its neighbor of degree 3 which is not a type 1 vertex.



Once the discharging is finished, a new charge function  $\omega^*$  is produced. However, the total sum of charges is kept fixed when the discharging is in process.

We show that  $\omega^*(x) \ge 0$  for all  $x \in V(G) \cup F(G)$ 

$$-16 = \sum_{x \in V(G) \cup F(G)} \omega(x) = \sum_{x \in V(G) \cup F(G)} \omega^*(x) \ge 0$$



# If $g(G) \ge 8$ , then $\chi'_s(G) \le 5$

let G be a counterexample with the fewest number of vertices. G contains one of the following 11 configurations.



Figure 6: The 11 unavoidable configurations in Lemma 2.



 $\mathcal{F}$  is the set of all black vertices depicted in the previous Figure. We extend an *L*-in-coloring *c* of  $G - \mathcal{F}$  to  $\mathcal{F}$ , and finally we obtain an *L*-in-coloring of this orientation of *G*, a contradiction.



#### We have:

- 1. If G is a graph with  $Mad(G) < \frac{26}{14}$ , then  $\chi_s(G) \leq 4$ . 2. If G is a graph with  $Mad(G) < \frac{16}{7}$  and girth at least 6, then
- 2. If G is a graph with  $Mad(G) < \frac{18}{7}$  and girth at least 6, then  $\chi_s(G) \le 5$ .
- 3. If G is a graph with  $Mad(G) < \frac{8}{3}$  and girth at least 6, then  $\chi_s(G) \le 6$ .
- 4. If G is a graph with  $\operatorname{Mad}(G) < 14/5$  then  $\chi_{\mathfrak{s}}(G) \leq 7$

#### Definition

Let  $f : \mathbb{N} \to \mathbb{R}$  be defined by

$$f(n) = \inf \{ \operatorname{Mad}(H) \mid \chi_{s}(H) > n \}$$

$$f(1) = 1$$
.

If G contains an edge, then  $\chi_s(G) > 1$ ; otherwise, G is a independent set, so  $\chi_s(G) = 1$  and Mad(G) = 0



$$f(2)=\frac{3}{2}$$

If G contains a path of length 3, then  $\chi_s(G) > 2$  and  $Mad(G) \ge \frac{3}{2}$ . Otherwise, G is a star forest, so  $\chi_s(G) \le 2$  and Mad(G) < 2.



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$$f(3)=2.$$



$$f(n) = \inf \{ \operatorname{Mad}(H) \mid \chi_s(H) > n \}$$

Problem

What is the value of f(n) for  $n \ge 4$ ?



## Theorem - [Albertson et al. '04]

For any planar graph G,  $\chi_s(G) \leq 20$ 

#### Problem

Can we improve this upper bound?

