Partitions of graphs with bounded maximum average degree

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Graph Theory 2011
Kolding,
Denmark
Thursday April 28 - Sunday May 1, 2011

Proper k-coloring

Let G be a graph and k ($k \geq 1$) an integer.
A proper k-coloring of G is a mapping $\phi : V(G) \longrightarrow \{1, \cdots, k\}$ such that:

- for every edge xy, $\phi(x) \neq \phi(y)$

In other words, a k-coloring of G is a partition $V_1, V_2, \cdots, V_k$ of $V(G)$ such that V_i is an independent set for every i, i.e., the subgraph induced by V_i has maximum degree zero.

d-improper k-coloring

Burr and Jacobson (1985), Cowen, Cowen, and Woodall (1986), Harary and Jones (1985).

d-improper k-coloring

Let G be a graph and k, d $(k, d \geq 1)$ integers.
A d-improper k-coloring of G is a mapping $\psi : V(G) \longrightarrow \{1, \cdots, k\}$ such that :

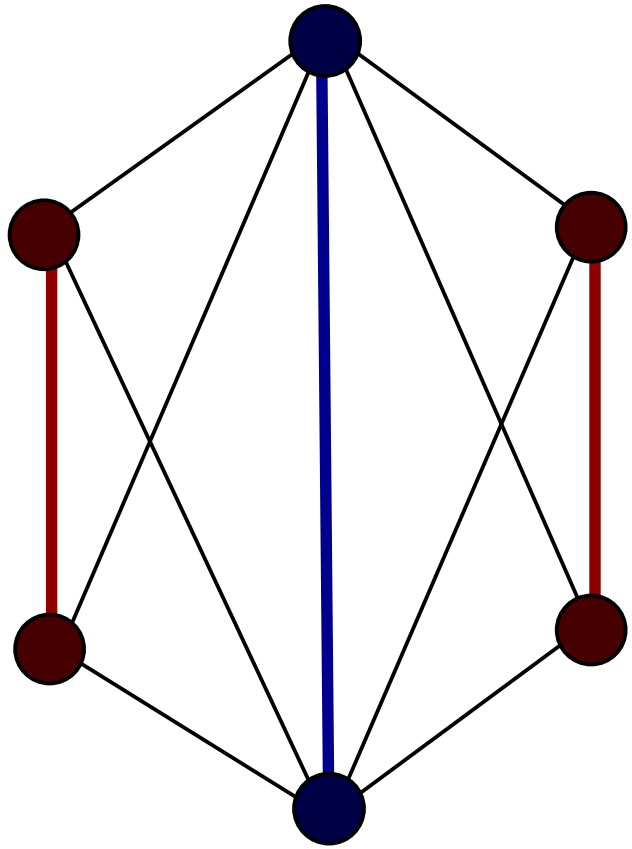
- $\forall i, 1 \leq i \leq k$, $G[i]$ has a maximum degree at most d
- $G[i]$ is the subgraph induced by color i.

Every vertex v has at most d neighbors receiving the same color as v. We

denote: $(d, d, \cdots, d)$-coloring = d-improper k-coloring

- a $(0, 0, 0, 0)$-coloring is a proper 4-coloring.
- a $(2, 2, 2)$-coloring is a 2-improper 3-coloring.

1-improper 2-coloring



$(1, 1)$-coloring

Known results

Appel and Haken, 1977

- Every planar graph is $(0, 0, 0, 0)$-colorable.

Cowen, Cowen, and Woodall, 1986

- Every planar graph is 2-improperly 3-colorable, i.e. $(2, 2, 2)$-colorable.

Xu, 2009

Every plane graph with neither adjacent triangles nor 5-cycles is $(1, 1, 1)$-colorable.

Known results-Choosability

Definition

A graph G is d-improper m-choosable, or simply $(m, d)^*$-choosable, if for every list assignment L, where $|L(v)| \geq m$ for every $v \in V(G)$, there exists an L-colouring of G such that each vertex of G has at most d neighbours coloured with the same colour as itself.

Eaton and Hull (1999), Škrekovski (1999)

- Every planar graph is 2-improper 3-choosable: $(3, 2)^*$-choosable.

If a graph G is 2-improper 3-choosable then it is $(2, 2, 2)$-colorable.

Škrekovski proved that for every k, there are planar graphs which are not k-improper 2-colorable.

Known results

Cushing and Kierstead (2009)

Every planar graph is 1-improper 4-choosable ($(4,1)^*$-choosable).

Dong and Xu (2009)

Let G be a plane graph without any cycles of length in $\{4,8\}$, then G is $(3,1)^*$-choosable.

Question (Xu and Zhang, 2007)

Is-it true that every planar graph without adjacent triangle is $(3,1)^*$-choosable.

Every planar graph without adjacent triangle is $(1,1,1)$-colorable?

Known results

Definition-Maximum average degree

$$\mathrm{Mad}(G) = max\left\{\frac{2\cdot|E(H)|}{|V(H)|}, H\subseteq G\right\}.$$

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$$\mathrm{Mad}(G) = max\left\{\frac{2\cdot|E(H)|}{|V(H)|}, H\subseteq G\right\}.$$

In 1995, Jensen and Toft showed that there is a polynomial algorithm to comput $Mad(G)$ for a given graph G.

T. R. Jensen and B. Toft, Choosability versus chromaticity, Geombinatorics 5(1995), 45-64.

if G is a planar graph with girth g, then $\mathrm{Mad}(G) < \frac{2g}{g-2}$.

Known results

Havet and Sereni, 2006

- For every $k \geq 0$, every graph G with $Mad(G) < \frac{4k+4}{k+2}$ is k-improperly 2-colorable (in fact k-improperly 2-choosable), i.e. (k, k)-colorable

- $k = 1$ $Mad(G) < \frac{8}{3}$: $(1, 1)$-colorable (planar, $g = 8$).
- $k = 2$ $Mad(G) < 3$: $(2, 2)$-colorable (planar, $g = 6$).

A more general result:

Theorem (Havet and Sereni)

For every $l \geq 2$ and every $k \geq 0$, all graphs of maximum average degree less than $\frac{l(l+2k)}{l+k}$ are k-improper l-choosable.

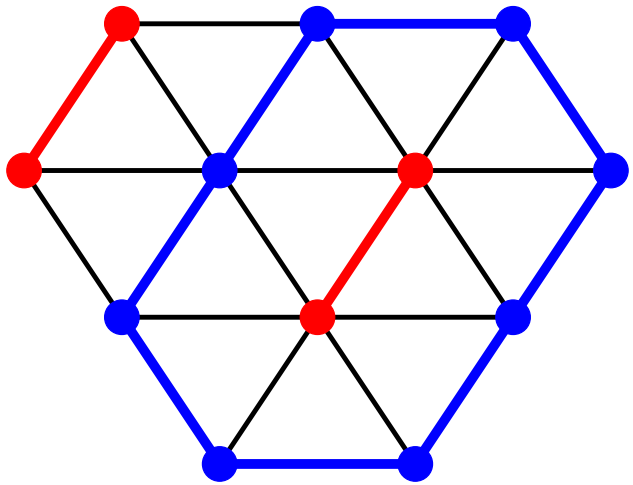
it implies $(k, \cdots, k)$-colorable.

$(d_1, d_2, \cdots, d_k)$-coloring

$(d_1, d_2, \cdots, d_k)$-coloring

A graph G is $(d_1, d_2, \cdots, d_k)$-colorable if and only if:

- it exists a partition of V: $V = V_1 \cup V_2 \cup \cdots \cup V_k$ such that $\forall i \in [1, k]$, $\Delta(G[V_i]) \leq d_i$



$(2, 1)$-coloring

In this talk: $(1, 0)$, $(k, 0)$, $(k, 1)$, (k, j), $(k, 0, \cdots, 0)$

$(1,0)$-colorable

Theorem (Glebov and Zambalaeva, 2007)

Every planar graph is $(1,0)$*-colorable if* $g(G) \geq 16$.

Theorem (Borodin and Ivanova, 2009)

Every graph is $(1,0)$*-colorable if* $Mad(G) < \frac{7}{3}$

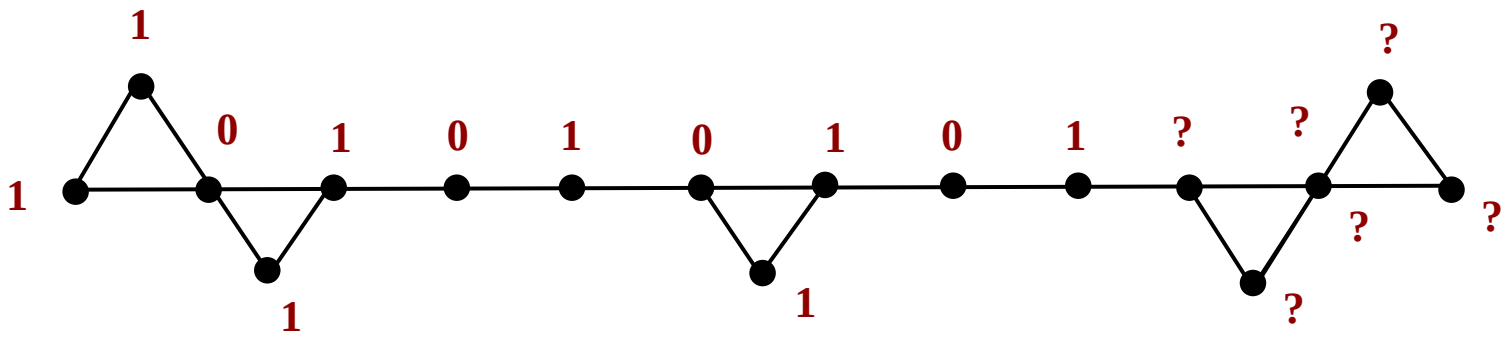
New technique first introduced by Borodin, Ivanova and Kostochka in 2006.

This implies: A planar graph is $(1,0)$-colorable if $g(G) \geq 14$

Improvement

Theorem (Borodin and Kostochka, 2010)

Every graph G with $Mad(G) \leq \frac{12}{5}$ is $(1,0)$-colorable and the restriction on $Mad(G)$ is sharp.



G_3

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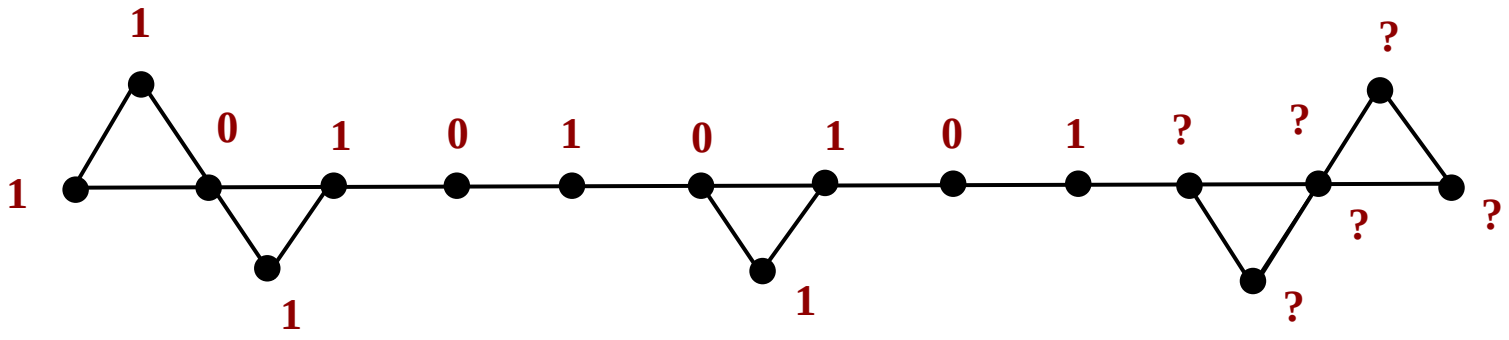


G_3

$$Mad(G_p) = \frac{2|E(G_p)|}{|V(G_p)|} = \frac{12p+6}{5p+2} = \frac{12}{5} + \frac{6}{5(5p+2)}$$

$(k, 0)$-coloring

$(k, 0)$-coloring

- Bipartition V_1,V_2 of $V(G)$
- $\Delta(G[V_1]) \leq k$ and $G[V_2]$ is a stable set.

Color k: vertices of V_1
Color 0: vertices of V_2.

$k = 1$

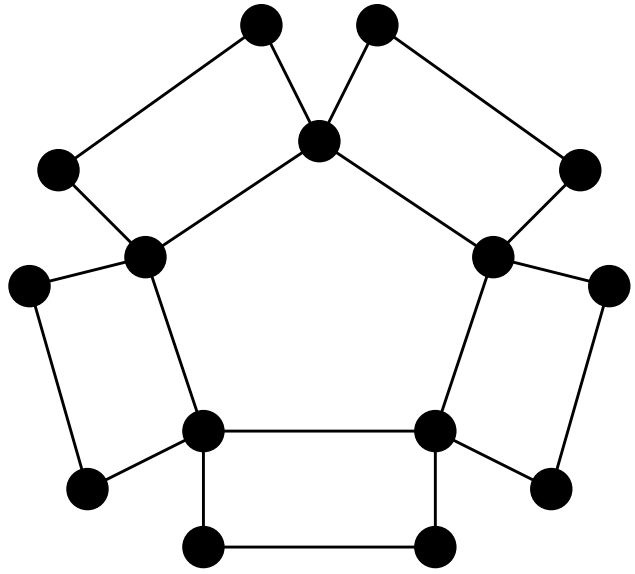
Theorem (Borodin and Kostochka, 2010)

Every graph is $(1, 0)$*-colorable if* $Mad(G) \leq \frac{12}{5}$

Corollary

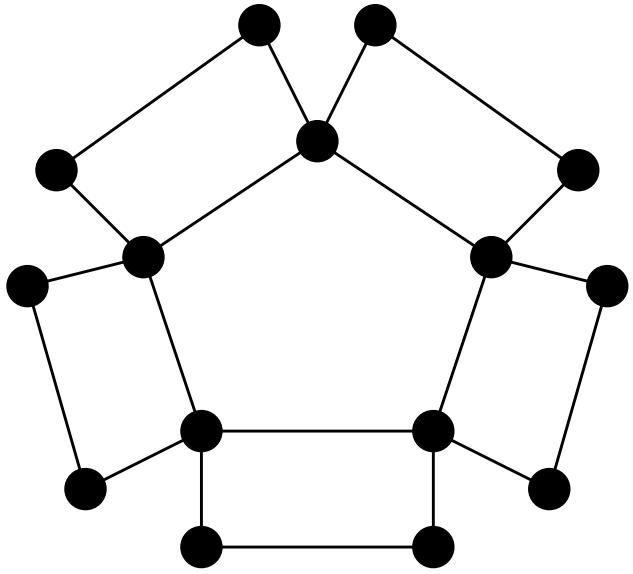
A planar graph is $(1, 0)$*-colorable if* $g(G) \geq 12$

Outerplanar graphs



Smallest value of k such that G admits a $(k, 0)$-coloring?

Outerplanar graphs



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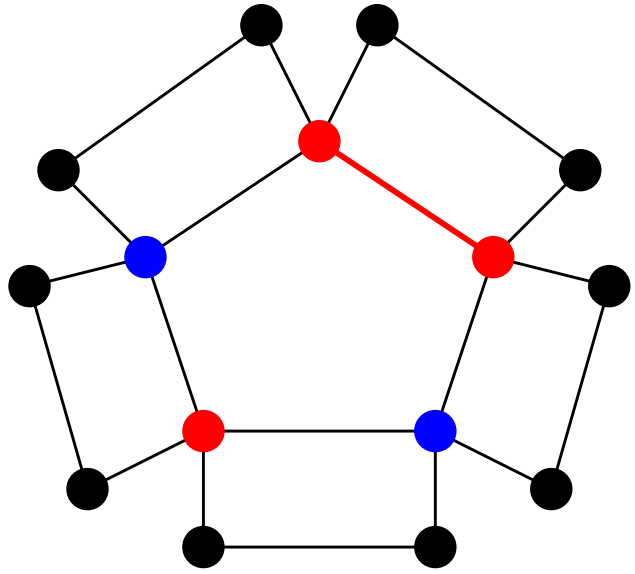

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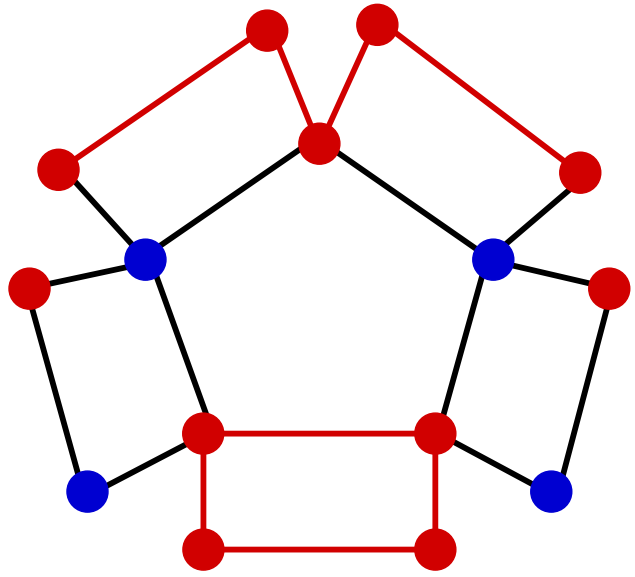


$(2, 0)$-coloring

Outerplanar graphs

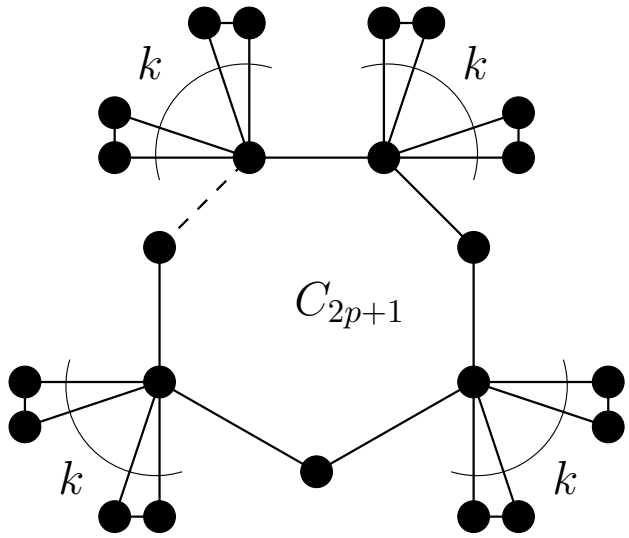
Claim

- Outerplanar graphs with girth 4 are (2,0)-colorable.
- Outerplanar graphs with girth 5 are (1,0)-colorable.

(the girth of a graph G is the length of a shortest cycle of G.)

Outerplanar graphs

For outerplanar graphs with girth 3, k is unbounded.



Non $(k, 0)$-colorable outerplanar graph with girth 3

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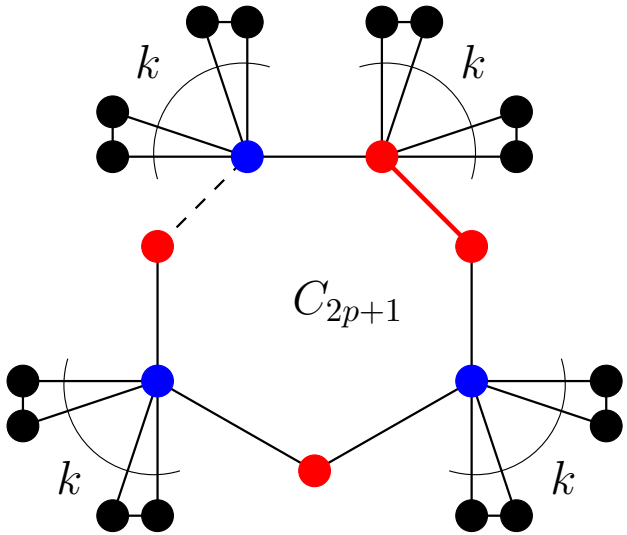

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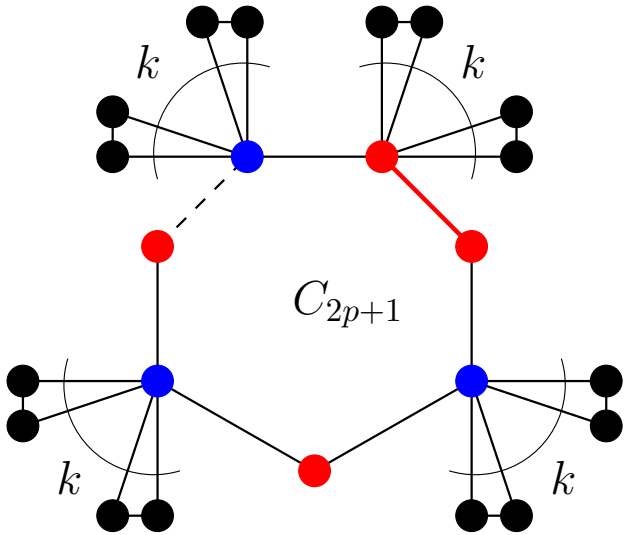

Non $(k, 0)$-colorable outerplanar graph with girth 3

$$Mad(G_{p,k}) = \frac{2|E(G_{p,k})|}{|V(G_{p,k})|} = \frac{2((3k+2)(p+1)-1)}{(2k+2)(p+1)-1} \xrightarrow[p\to\infty]{} \frac{3k+2}{k+1}$$

Sparse graphs

Key concepts:

soft components, feading area

Theorem (Borodin, Ivanova, Montassier, Ochem, R., 2009)

Let $k \geq 0$ be a integer. Every graph with maximum average degree smaller than $\frac{3k+4}{k+2}$ is $(k, 0)$-colorable.

- $Mad(G) < \frac{5}{2} \rightarrow (2, 0)$
- $Mad(G) < \frac{13}{5} \rightarrow (3, 0)$

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Optimality:

$$\lim_{p \to \infty} Mad(G_{p,k}) = \frac{3k+2}{k+1} < \frac{3k+4}{k+2} + \frac{1}{k+3}\,.$$

Planar graphs

Corollary

Every planar graph G is:

- $(1,0)$-colorable if $g(G) \geq 14$,
- $(2,0)$-colorable if $g(G) \geq 10$,
- $(3,0)$-colorable if $g(G) \geq 9$,
- $(4,0)$-colorable if $g(G) \geq 8$,
- $(8,0)$-colorable if $g(G) \geq 7$.

Planar graphs

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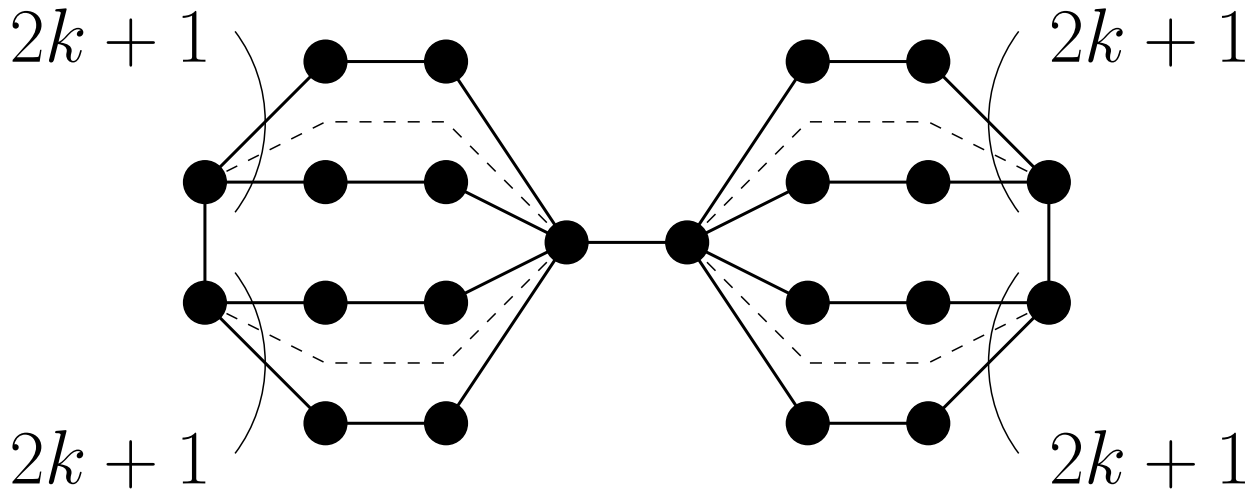
- $(8,0)$-colorable if $g(G) \geq 7$.

For planar graphs with girth 6, k is unbounded.



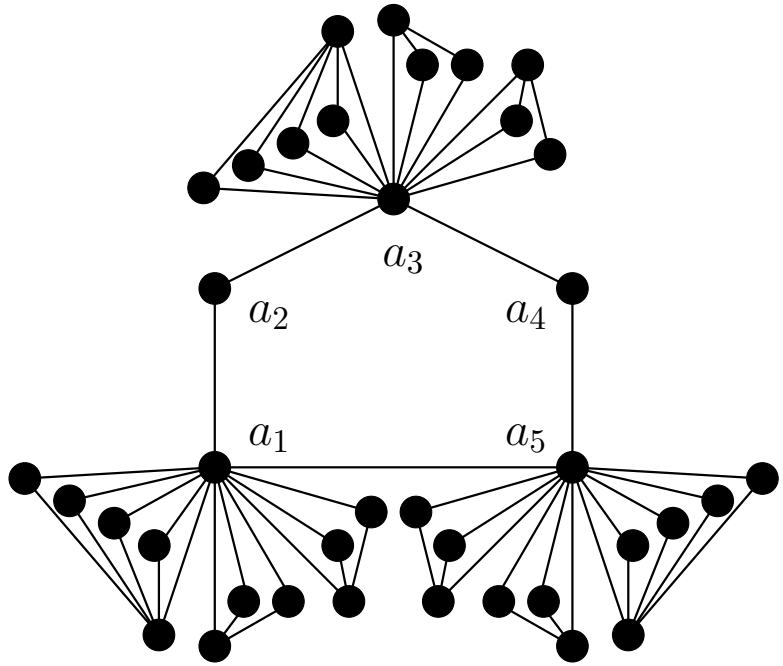
Non $(k, 0)$-colorable planar graph with girth 6

$(k, 1)$-coloring

Theorem (Borodin, Ivanova, Montassier, R., 2010)

Let $k \geq 2$ be a integer. Every graph with maximum average degree smaller than $\frac{10k+22}{3k+9}$ is $(k, 1)$-colorable.

$(k, 1)$-coloring



An example of $G_{n,k}$ with $n = 3$ and $k = 3$.

Non $(k, 1)$-colorable graph.

$$Mad(G_{n,k}) = \frac{2|E(G_{n,k})|}{|V(G_{n,k})|} = \frac{2(2n - 1 + 5(k-1)n + n(2k+3))}{2n - 1 + 3(k-1)n + n(k+2)} = \frac{2(7nk - 1)}{n(4k+1) - 1}$$

$$\lim_{n \to \infty} Mad(G_{n,k}) = \frac{14k}{4k+1}$$

(k, j)-coloring

Let k, j be two integers with $j \geq 2$, $k \geq j + 1$ and $r_1, r_2, r_3, r_4, r_{51}, r_{52}, m_{k,j}$ be seven non-negative reals. Given k, j, let $\mathcal{P}_{k,j}$ be the linear program described below.

Theorem (Borodin, Ivanova, Montassier, R., 2010)

Let $(r_1, r_2, r_3, r_4, r_{51}, r_{52}, m_{k,j})$ be a solution of $\mathcal{P}_{k,j}$, then every graph with $Mad(G) < m_{k,j}$ is (k, j)-colorable.

$\mathcal{P}_{k,j}$:

$$
\begin{aligned}
2 + r_1 + r_2 &\geq m_{k,j} && (1)\\
2 + 2 \cdot r_3 &\geq m_{k,j} && (2)\\
1 &\geq r_2 && (3)\\
j + 2 - (j+1) \cdot r_2 + r_1 &\geq m_{k,j} && (4)\\
j + 2 - j \cdot r_2 + 2 \cdot r_4 &\geq m_{k,j} && (5)\\
r_2 &\geq r_{51} && (6)\\
j + 2 - (j-1) \cdot r_2 - 2 \cdot r_{51} &\geq m_{k,j} && (7)\\
r_1 &\geq r_3 && (8)\\
r_1 &\geq r_4 + r_{52} && (9)\\
1 &\geq r_1 && (10)\\
k + 2 - (k+1) \cdot r_1 + r_{51} &\geq m_{k,j} && (11)\\
k + 2 - (k+1) \cdot r_1 - r_4 + r_{52} &\geq m_{k,j} && (12)\\
r_1 &\geq r_{51} && (13)\\
k + 2 - k \cdot r_1 - r_4 - r_{51} &\geq m_{k,j} && (14)\\
k + 2 - k \cdot r_1 - 2 \cdot r_4 - r_{52} &\geq m_{k,j} && (15)\\
k + j + 2 - (k+j+1) \cdot r_1 - r_4 - r_{52} &\geq m_{k,j} && (16)\\
k + j + 2 - (k+j+1) \cdot r_1 - r_{51} &\geq m_{k,j} && (17)\\
\text{Maximize } m_{k,j} & && (18)
\end{aligned}
$$

(k, j)-coloring

In particular:

Corollary

- Every graph G with $Mad(G) < \frac{10}{3}$ is $(13, 2)$-colorable.
- Every graph G with $Mad(G) < \frac{317}{95}$ is $(7, 3)$-colorable.

(k, j)-coloring

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For planar graphs:

$g(G)$	$(k, 0)$	$(k, 1)$	$(k, 2)$	$(k, 3)$	$(k, 4)$
3,4	$\times$	$\times$	$\times$	$\times$	$\times$
5	$\times$	?	$(13, 2)$	$(7, 3)$	$(4, 4)$ [HS06]
6	$\times$	$(5, 1)$	$(2, 2)$ [HS06]		
7	$(8, 0)$	$(2, 1)$			
8	$(4, 0)$	$(1, 1)$ [HS06]			
9	$(3, 0)$				
10	$(2, 0)$				
14	$(1, 0)$				

(k,j)-coloring: very last result

Let $F(j,k)$ denote the supremum of x such that every graph G with $Mad(G) < x$ is (k,j)-colorable. It is easy to see that $F(0,0) = 2$.

Theorem (Borodin and Kostochka, 2011)

Let $j \geq 0$ *and* $k \geq 2j+2$. *Then* $F(j,k) = 2\left(2 - \frac{k+2}{(j+2)(k+1)}\right)$.

- If $\mathsf{Mad(G)} < \frac{3k+2}{k+1}$ $(k \geq 2)$ then G is $(k,0)$-colorable.
- If $\mathsf{Mad}(G) < \frac{10k+8}{3k+3}$ $(k \geq 4)$ then G is $(k,1)$-colorable.

$(0, \cdots, 0, k)_l$-coloring

Definition

A $(0, \cdots, 0, k)_l$-coloring of G is a partition $(V_1, V_2, \ldots, V_l)$ $(l \geq 3)$ of the vertices of G such that for $1 \leq i \leq l-1$, $G[V_i]$ is a stable set, and $G[V_l]$ has maximum degree at most k.

Theorem (Kaiser, Montassier, R., Rucky, 2011)

Let $l \geq 3$ and $k \geq 0$ be integers. Every graph G with $\mathsf{Mad}(G) < l + \frac{2k}{2k+l+3}$ is $(0, \cdots, 0, k)_l$-colorable.

If $\mathsf{Mad}(G) < \frac{13}{4}$ then G is $(0, 0, 1)$-colorable.

Steinberg Conjecture

Conjecture (Steinberg, 1976)

Every planar graph without 4 *and* 5*-cycles is* 3*-colorable (*$(0,0,0)$*-colorable)*

Can we prove :

- $(0,0,1)$-colorable?

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- $(0,0,2)$-colorable?

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- $(0,0,2)$-colorable?

$(0,0,7)$ seems ok.....

Thank you for your attention!